Three loop QCD corrections to massive form factors and its asymptotic behavior

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Plan of this talk

1. EPC and top-pair production

2. Heavy quark form factors

3. QCD corrections at three loops

4. Asymptotic behavior

5. Conclusion

EPC and top-pair production

Top-quark

✓ **Top quark is heavy**

- The heaviest SM particle - probes the Higgs sector most - plays unique role in understanding the EW symmetry breaking
- New physics potential : perfect place to manifest it

✓ **Top quark is short-lived**

- decays before hadronization - the only 'free' quark -
  This unique feature allows for direct measurement of its spin polarization and the study of spin correlations between the t and $\bar{t}$ without the complex effects of hadronization.

✓ High precision will be achieved at the future electron-positron colliders – In order to match the experimental accuracy, precise predictions are required on the theoretical side as well

Key motivation for Top @ future lepton colliders

✓ Precision Top Quark Mass and Width

- Perform a threshold scan of the cross-section to determine the top quark mass and width with a precision of tens of MeV.
- This method allows for a determination of a theoretically well-defined mass scheme (e.g., the $\overline{\text{MS}}$ mass) that is less ambiguous than methods used at hadron colliders.

✓ Precision Electroweak Couplings

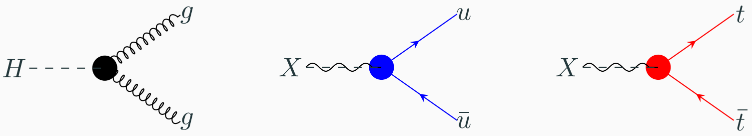
- Measure the $t\bar{t}\gamma$ and $t\bar{t}Z$ couplings with percent-level precision, which is significantly better than current or projected LHC capabilities.
- The possibility of beam polarization allow for a detailed and precise study of differential cross-sections and spin observables.

✓ Probing New Physics

- Precise measurements of the top quark's couplings and spin correlations act as indirect probes of BSM physics.

Form factors

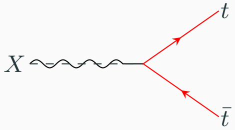
- ✓ The form factors are basic building blocks for many physical quantities
- ✓ They exhibit a universal infrared behavior - leads to information on anomalous dimensions
- ✓ The massive cusp anomalous dimension controls the infrared structure of massive form factors - studying the form factors helps in better understanding of the massive cusp
- ✓ Another important motive is to study high energy behavior of the massive form factors



Heavy quark form factors

The process

We consider decay of a color neutral massive particle to a pair of heavy quark of mass m.



Notation

$$X(q) \to t(q_1) + \bar{t}(q_2)$$

$$X = V, A, S, P$$

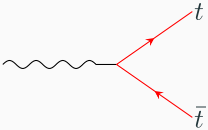
$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

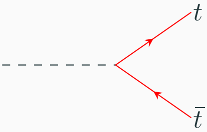
The general structure

Vector and Axial Vector

V: $-i\delta_{ij} v_Q \left(\gamma^\mu \, F_{V,1} + \frac{i}{2m} \sigma^{\mu\nu} q_\nu \, F_{V,2} \right)$

A: $-i\delta_{ij} a_Q \left(\gamma^\mu \gamma_5 \, F_{A,1} + \frac{1}{2m} q^\mu \gamma_5 \, F_{A,2} \right)$





Scalar and Pseudo Scalar

$-\frac{m}{v} \delta_{ij} \left[s_Q F_S + i p_Q \gamma_5 F_P \right]$

The form factors are expanded in the strong coupling constant as

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n F_I^{(n)}$$

To obtain $F_I^{(n)}$ $\Rightarrow$ appropriate projector on the amplitudes

$$P_{V,i} = \frac{i}{v_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu g^1_{V,i} + \frac{1}{2m}(q_{2\mu} - q_{1\mu}) g^2_{V,i}\right) \frac{\not{q}_1 + m}{m} \,,$$

$$P_{A,i} = \frac{i}{a_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu \gamma_5 g^1_{A,i} + \frac{1}{2m}(q_{1\mu} + q_{2\mu}) \gamma_5 g^2_{A,i}\right) \frac{\not{q}_1 + m}{m} \,,$$

$$P_S = \frac{v}{2ms_Q} \frac{\not{q}_2 - m}{m} \left(g_S\right) \frac{\not{q}_1 + m}{m} \,, \quad P_P = \frac{v}{2mp_Q} \frac{\not{q}_2 - m}{m} \left(i\gamma_5 g_P\right) \frac{\not{q}_1 + m}{m} \,,$$

$g_I \equiv g_I(s, m, d)$.

One-loop and beyond

$F_{V,i}^{(1)}, F_{A,i}^{(1)}$ [Arbuzov, Bardin, Leike '92; Djouadi, Lampe, Zerwas '95]

$F_S^{(1)}, F_P^{(1)}$ [Braaten, Leveille '80; Sakai '80; Drees, Hikasa '90]

$F_{V,i}^{(2)}, F_{A,i}^{(2)}$ [Altarelli, Lampe '93; Ravindran, van Neerven '98; Catani, Seymour '99]

$F_S^{(2)}, F_P^{(2)}$ [Gorishnii *et. al.* '91; Chetyrkin, Kwiatkowski '95; Harlander, Steinhauser '97]

Two-loop

$F_I^{(2)}$ [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04,'05]

$F_{V,i}^{(2)}(\mathcal{O}(\epsilon))$ [Gluza, Mitov, Moch, Riemann '09]

$F_I^{(2)}(\mathcal{O}(\epsilon^2))$ [Ablinger, Behring, Blümlein, Falcioni, Freitas, Marquard, Rana, Schneider '17]

Three-loop

$F_{V,i}^{(3)}|_{\text{large N}}$ [Henn, Smirnov, Smirnov, Steinhauser '16]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_S^{(3)}, F_P^{(3)}|_{\text{large N+full } n_l}$ [Lee, Smirnov, Smirnov, Steinhauser '18]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_S^{(3)}, F_P^{(3)}|_{\text{large N+full } n_l}$ [Ablinger, Blümlein, Marquard, Rana, Schneider '18]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_S^{(3)}, F_P^{(3)}|_{\text{heavy } n_h}$ [Blümlein, Marquard, Rana, Schneider '19]

$F_{V,i}^{(3)}, F_{A,i}^{(3)}, F_S^{(3)}, F_P^{(3)}|_{\text{full}}$ [Fael, Lange, Schoenwald, Steinhauser '22]

QCD corrections at three loops

J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, NR and C. Schneider,
Heavy quark form factors at two loops,
Phys.Rev. D97 (2018) 094022 (arXiv:1712.09889 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
Heavy Quark Form Factors at Three Loops in the Planar Limit, Phys.Lett. B782 (2018) 528-532 (arXiv:1804.07313 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable, Nucl.Phys. B939 (2019) 253-291 (arXiv:1810.12261 [hep-ph]).

J. Blümlein, P. Marquard, NR and C. Schneider,
The Heavy Fermion Contributions to the Massive Three Loop Form Factors,
Nucl.Phys. B949 (2019) 114751 (arXiv:1908.00357 [hep-ph]).

J. Blümlein, A. De Freitas, P. Marquard, NR and C. Schneider,
Analytic results on the massive three-loop form factors: Quarkonic contributions,
Phys.Rev.D 108 (2023) 094003 (arXiv:2307.02983 [hep-ph]).

Computational procedure

$$\boxed{d = 4 - 2\epsilon}$$

- Diagrammatic approach -> `QGRAF` to generate Feynman diagrams
- In-house `FORM` routines for algebraic manipulation : *Lorentz, Dirac and Color algebra*
- Projectors: different projectors for different currents
- Decomposition of the dot products to obtain scalar integrals
- Identity relations among scalar integrals : IBPs (using `Crusher` [Marquard, Seidel])
- Algebraic linear system of equations relating the integrals

$$\Downarrow$$

Master integrals (MIs)

- Computation of MIs : Method of differential equation (generic)
- UV renormalization and Infrared subtraction

Solving master integrals

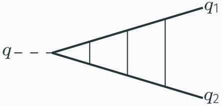
The main challenges are the MIs. We follow the method of differential equations to compute them.

The method of differential equations

An example of a three-loop scalar integral:

$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon\gamma_E}\right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

where, the $D_i^{\nu_i}$ can be from the following set of propagators



$$\begin{array}{l} l_1^2 - m^2, (l_1 - q)^2 - m^2, (l_1 - l_2)^2, \\ l_2^2 - m^2, (l_2 - q)^2 - m^2, (l_2 - l_3)^2, \\ l_3^2 - m^2, (l_3 - q)^2 - m^2, (l_1 - q_1)^2 \end{array}$$

These Feynman integrals are functions of spacetime dimension d and kinematic invariants x.

$$J(\nu_1, \ldots, \nu_n) = J_i \equiv f(d,\, x)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.\ x$ and solve it.

The method of differential equations

An example of a three-loop scalar integral:

$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon \gamma_E}\right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

$$\frac{d}{dx} J_i = \text{some combinations of integrals}$$

$$\Downarrow \text{ IBP identities/reduction}$$

$$= \sum_{j=1}^{n} c_{ij} J_j$$

c_{ij}'s are rational function of d and x.

The method of differential equations

An example of a three-loop scalar integral:

$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon \gamma_E} \right)^3 \int \frac{d^d l_1}{(2\pi)^d} \, \frac{d^d l_2}{(2\pi)^d} \, \frac{d^d l_3}{(2\pi)^d} \, \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_x \mathbb{J} = \mathbb{A}(d, x) \mathbb{J}$$

The method of differential equations

An example of a three-loop scalar integral:

$$J(\nu_1, \ldots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon\gamma_E}\right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_{n-1} \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \cdots & 0 & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_{n-1} \\ J_n \end{pmatrix}$$

The bullets ($\bullet$) indicate a non-zero rational function of d and x.

To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form. Then, it can be solved using bottom-up approach.

The results are obtained in terms of iterated integrals (HPLs/GPLs).

An example of a system of differential equations

Ultraviolet renormalization

We consider a hybrid scheme for UV renormalization.

Heavy quark mass and wave function $(Z_{m,\mathrm{OS}}, Z_{2,\mathrm{OS}})$:	On-shell
QCD strong coupling constant (Z_{a_s}) :	$\overline{MS}$

The renormalization of F_I for these topologies, is straightforward

$$F_{V,i} = Z_{2,\mathrm{OS}} \hat{F}_{V,i} \qquad F_S = Z_{m,\mathrm{OS}} Z_{2,\mathrm{OS}} \hat{F}_S$$

$$F_{A,i} = Z_{2,\mathrm{OS}} \hat{F}_{A,i} \qquad F_P = Z_{m,\mathrm{OS}} Z_{2,\mathrm{OS}} \hat{F}_P$$

where $\hat{F}_I$ contains the counterterms from mass renormalization.

For non-singlet cases of Axial-Vector and Pseudo-Scalar currents, both the γ_5-matrices appear in the same chain of Dirac matrices, which allows us to use anti-commuting γ_5 in D space-time dimensions.

Ward identities

The UV-renormalized form factors satisfy the Ward identities, which also work as a check of the computation.

Chiral Ward identity $\boxed{q_\mu \Gamma_A^{\mu,ns} = 2m\Gamma_P^{ns}}$

$$2F_{A,1}^{ns} + \frac{1}{2}\left(-\frac{(1-x)^2}{x}\right) F_{A,2}^{ns} = 2mF_P^{ns}$$

Anomalous Ward identity $\boxed{q_\mu \Gamma_A^{\mu,s} = 2m\Gamma_P^{s} - i\frac{\alpha_s}{4\pi} T_F \langle G\tilde{G}\rangle_Q}$

$\langle G\tilde{G}\rangle_Q$ denotes the truncated matrix element of the gluonic operator $G\tilde{G}$ between the vacuum and an on-shell heavy quark pair ($Q\bar{Q}$).

Universal infrared structure

The structure of infrared singularities is universal and multiplicative

[Becher, Neubert '09]

$$F_I(\epsilon, x) = Z(\epsilon, x, \mu) F_I^{fin}(x, \mu)$$

$Z(\epsilon, x, \mu)$ is universal/independent of current
$F_I^{fin}(x, \mu)$ is finite as $\epsilon \to 0$

Renormalization group evolution of $Z(\epsilon, x, \mu)$ provides

$$Z(\epsilon, x, \mu) = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\epsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{\Gamma_0^2}{8} - \frac{\beta_0\Gamma_0}{4}\right) + \frac{1}{\epsilon}\left(\frac{\Gamma_1}{4}\right)\right]$$
$$+ \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{1}{\epsilon^3}\left(\frac{\Gamma_0^3}{48} - \frac{\beta_0\Gamma_0^2}{8} + \frac{\beta_0^2\Gamma_0}{6}\right) + \frac{1}{\epsilon^2}\left(\frac{\Gamma_0\Gamma_1}{8} - \frac{\beta_1\Gamma_0}{6}\right) + \frac{1}{\epsilon}\left(\frac{\Gamma_2}{6}\right)\right]$$

Γ_n is the n^{th} order massive cusp anomalous dimension.

[Korchemsky, Radyushkin '87, '92; Grozin, Henn, Korchemsky, Marquard '14, '15]

Results & Checks

Results & Checks

- Computational : Automation to solve a single scale and first order factorizable system of differential equations.
- Phenomenological : We have performed UV renormalization and obtained all the form factors which are necessary to obtain N^{3}LO QCD corrections to top pair productions at electron-positron colliders.

$$F_{V,1}^{(3)}, F_{V,2}^{(3)}, F_{A,1}^{(3)}, F_{A,2}^{(3)}, F_{S}^{(3)}, F_{P}^{(3)}$$

✓ We agree with the results from Lee $et\ al.$ obtained using different method

✓ The results reproduce the universal infrared structure

✓ Chiral Ward identity is satisfied between $F_{A,i}^{(3)}$ and $F_{P}^{(3)}$

The functional structure of the form factors

The form factors contain $\mathrm{polynomials}$ of x, HPLs and cyclotomic HPLS.

$$F^{(3)} \equiv x, H[\_\_, x], \zeta_2, \zeta_3, \zeta_5, \mathrm{Li}_4\left(\frac{1}{2}\right), \mathrm{Li}_5\left(\frac{1}{2}\right)$$

The iterative integrals contribute up to weight 6, where 'weight' denotes the number of iteration. For example, a weight 3 HPL is

$$H[0, 1, \{6, 1\}, x] = \int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2} \int_0^{t_2} \frac{t_3 \; dt_3}{1 - t_3 + t_3^2}$$

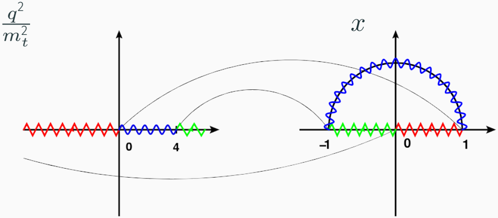
Goal

To obtain a numerical implementation for $x \in [-1, 1]$.

Outcome

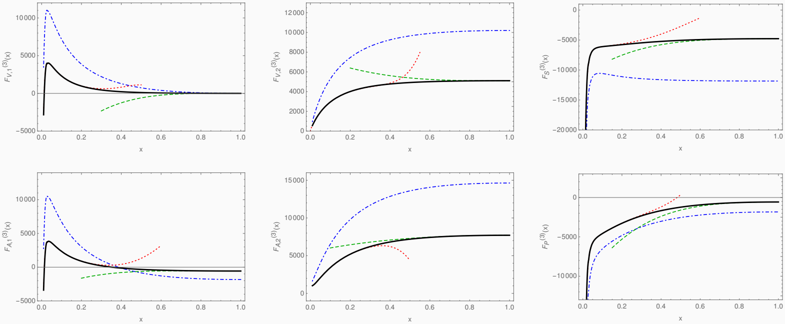
Fortran routines `HPOLY.f` and `CPOLY.f`

$$\boxed{\frac{q^2}{m_t^2} = -\frac{(1-x)^2}{x}}$$

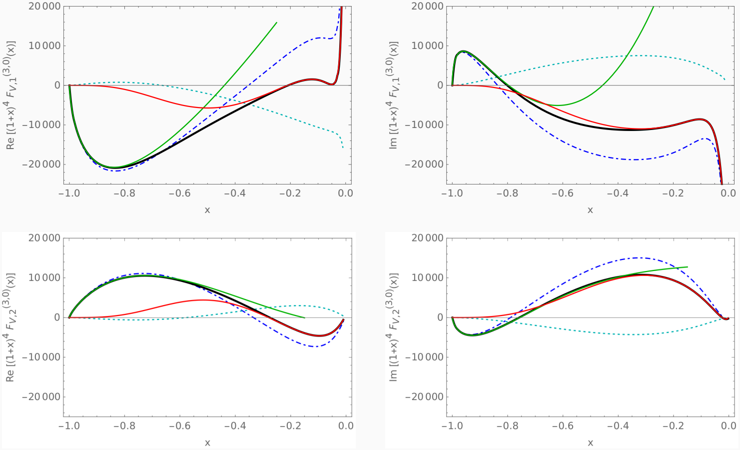


Low energy region $q^2 \ll m^2$ or $x \to 1$
High energy region $q^2 \gg m^2$ or $x \to 0$
Threshold region $q^2 \sim 4m^2$ or $x \to -1$

THE FINITE COMPONENT OF THE FORM FACTORS and THEIR EXPANSIONS IN DIFFERENT REGION



The finite component of the VECTOR FORM FACTORS
threshold Expansions

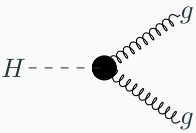


Asymptotic behavior

J. Blümlein, P. Marquard and NR,
The asymptotic behavior of the heavy quark form factors at higher order,
Phys.Rev. D99 (2019) 016013 (arXiv:1810.08943 [hep-ph]).

The three different variant of form factors

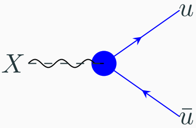
Gluon FF



massless gluons

$F^{(n)} \sim (Q^2)^{n\epsilon}$

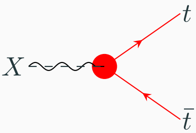
Quark FF



massless quarks

$F^{(n)} \sim (Q^2)^{n\epsilon}$

Massive FF



Massive quarks

$F^{(n)} \sim F^{(n)}(Q^2, m_t^2)$

In a first glance, there is no similarity between massless and massive form factors. But in the asymptotic limit ($q^2 \gg m_t^2$), the leading term of massive form factors can be related to the massless form factors, proving the universality of QCD corrections.

Sudakov behavior of MASSLESS form factors

[Mueller, Collins, Sen, Sudakov]

The integro-differential Sudakov equation

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}_i(\hat{a}_s, Q^2, \mu^2) = \frac{1}{2}\left[K_i\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}\right) + G_i\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}\right)\right]$$

$\ln \Rightarrow$ the exponentiation of massless two-point correlation function
$Q^2 \frac{d}{dQ^2} \Rightarrow$ studying each point of Q^2 phase-space available to virtual quanta
$r.h.s. \Rightarrow$ factorization of QCD amplitudes

$K_i \equiv$ contains all the singularities and universal
$G_i \equiv$ finite and contains information of the process

Renormalization group equation :

$$\mu_R^2 \frac{d}{d\mu_R^2} \hat{F}_i = 0 \quad \Rightarrow \quad \mu_R^2 \frac{d}{d\mu_R^2} \hat{K}_i = -\mu_R^2 \frac{d}{d\mu_R^2} \hat{G}_i = -A_i(a_s(\mu_R^2))$$

$A_i \equiv$ cusp anomalous dimension

Sudakov behavior of MASSLESS form factors

One can systematically solve the Sudakov equation and RGE

$$\ln \hat{F}_i = \sum_{n=1}^{\infty} \hat{a}_s^n \mathcal{F}_i^{(n)}$$

$$\hat{\mathcal{F}}_i^{(1)}(\epsilon) = \frac{1}{\epsilon^2}\left\{-2A_i^{(1)}\right\} + \frac{1}{\epsilon}\left\{G_i^{(1)}(\epsilon)\right\}$$

$$\hat{\mathcal{F}}_i^{(2)}(\epsilon) = \frac{1}{\epsilon^3}\left\{\beta_0 A_i^{(1)}\right\} + \frac{1}{\epsilon^2}\left\{-\frac{1}{2}A_i^{(2)} - \beta_0 G_i^{(1)}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}G_i^{(2)}(\epsilon)\right\}$$

$$\hat{\mathcal{F}}_i^{(3)}(\epsilon) = \frac{1}{\epsilon^4}\left\{-\frac{8}{9}\beta_0^2 A_i^{(1)}\right\} + \frac{1}{\epsilon^3}\left\{\frac{2}{9}\beta_1 A_i^{(1)} + \frac{8}{9}\beta_0 A_i^{(2)} + \frac{4}{3}\beta_0^2 G_i^{(1)}(\epsilon)\right\}$$
$$+ \frac{1}{\epsilon^2}\left\{-\frac{2}{9}A_i^{(3)} - \frac{1}{3}\beta_1 G_i^{(1)}(\epsilon) - \frac{4}{3}\beta_0 G_i^{(2)}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{3}G_i^{(3)}(\epsilon)\right\}$$

Points to note :

- K_i is exactly known in terms of A_i
- RGE for G_i is solved up to a boundary constant $G_i|_{b.c.} = \sum_{n=1}^{\infty} a_s^n(Q^2) G_i^{(n)}$.
  The structure of $G_i^{(n)}$ was conjectured in [Ravindran, Smith, van Neerven]

$$G_i^{(n)} = 2(B_i^{(n)} - \gamma_i^{(n)}) + f_i^{(n)} + C_i^{(n)} + \sum_{k=1}^{\infty} \epsilon^k g_i^{(n,k)}$$

- All the poles can be predicted

Sudakov behavior of massless form factors

Anomalous dimensions

Cusp	A_i	the Wilson cusp
Collinear	B_i	the behavior of a massless collinear parton
Soft	f_i	the behavior of a soft gluon
UV	γ_i	the ultraviolet divergences

SUDAKOV BEHAVIOR OF MASSLESS FORM FACTORS

Anomalous dimensions

Cusp	A_i	the Wilson cusp
Collinear	B_i	the behavior of a massless collinear parton
Soft	f_i	the behavior of a soft gluon
UV	γ_i	the ultraviolet divergences

Dissection of the divergences

$\frac{1}{\epsilon}$ collinear $(B_i^{(n)}) \oplus$ soft $(f_i^{(n)}) \oplus$ UV $(\gamma_i^{(n)})$

$\frac{1}{\epsilon^2}$ double collinear $(B_i^{(n_1)} B_i^{(n_2)}) \oplus$ double soft $(f_i^{(n_1)} f_i^{(n_2)}) \oplus$ soft and collinear $(B_i^{(n_1)} f_i^{(n_2)}) \oplus \cdots \oplus$ cusp $(A_i^{(n)})$

$\frac{1}{\epsilon^3}$ contributions from lower orders in α_s

Sudakov behavior of MASSIVE form factors

- Does the FF exponentiate?
  NO!
- Does the FF exponentiate in high energy limit?
  Only the massless QCD corrections!
- Do the divergences factorize in that scenario?
  Yes!

Sudakov behavior of MASSIVE form factors

- Does the FF exponentiate?
  NO!
- Does the FF exponentiate in high energy limit?
  Only the massless QCD corrections!
- Do the divergences factorize in that scenario?
  Yes!

Topological difference from massless : `massive quark line`

$\Downarrow$

Intuitively, some collinear divergences coming from massless quark in massless FF, will appear as $\ln m_t$ *in case of massive FF*

SUDAKOV BEHAVIOR OF MASSIVE FORM FACTORS

Now one can write the Sudakov equation for massive case

$$m^2 \frac{d}{dm^2} \ln \hat{F}_m(\hat{a}_s, Q^2, \mu^2) = \frac{1}{2} \left[K_m \left(\hat{a}_s, \frac{m_t^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2} \right) + G_m \left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2} \right) \right]$$

[Mitov, Moch; Ahmed, Henn, Steinhauser; Blümlein, Marquard, NR]

$K_i \equiv$ contains contribution from m_t

$G_i \equiv$ finite and contains information of the process only

The RGE is still the same

$$\mu_R^2 \frac{d}{d\mu_R^2} \hat{F}_m = 0 \quad \Rightarrow \quad \mu_R^2 \frac{d}{d\mu_R^2} \hat{K}_m = -\mu_R^2 \frac{d}{d\mu_R^2} \hat{G}_m = -A_q(a_s(\mu_R^2))$$

but now, K_m can not be solved exactly. Instead, both the differential equations can be solved up to the boundary constants

$$K_m|_{b.c.} = \sum_{n=1}^{\infty} a_s^n(Q^2) K_m^{(n)} \text{ and } G_m|_{b.c.} = \sum_{n=1}^{\infty} a_s^n(Q^2) G_m^{(n)}.$$

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Solving the Sudakov equation and RGE

$$\ln \hat{F}_m = \sum_{n=1}^{\infty} \hat{a}_s^n \mathcal{F}_m^{(n)} \qquad L = \ln\left(\frac{Q^2}{m_t^2}\right)$$

$$\begin{aligned}
\mathcal{F}_m^{(1)} &= \frac{1}{\epsilon}\Big[\Big(-\frac{1}{2}K_m^{(1)} - \frac{1}{2}G_m^{(1)}\Big) + L\Big(\frac{1}{2}A_q^{(1)}\Big)\Big] \\
&\quad + L\Big(\frac{1}{2}G_m^{(1)}\Big) + L^2\Big(-\frac{1}{4}A_q^{(1)}\Big) + \epsilon\Big[L^2\Big(-\frac{1}{4}G_m^{(1)}\Big) + L^3\Big(\frac{1}{12}A_q^{(1)}\Big)\Big] \\
\mathcal{F}_m^{(2)} &= \frac{1}{\epsilon^2}\Big[\Big(-\frac{1}{4}K_m^{(1)}\beta_0 - \frac{1}{4}G_m^{(1)}\beta_0\Big) + L\Big(\frac{1}{4}A_q^{(1)}\beta_0\Big)\Big] \\
&\quad + \frac{1}{\epsilon}\Big[\Big(-\frac{1}{4}K_m^{(2)} - \frac{1}{4}G_m^{(2)}\Big) + L\Big(\frac{1}{4}A_q^{(2)} + \frac{1}{2}G_m^{(1)}\beta_0\Big) + L^2\Big(-\frac{1}{4}A_q^{(1)}\beta_0\Big)\Big] \\
&\quad + L\Big(\frac{1}{2}G_m^{(2)}\Big) + L^2\Big(-\frac{1}{4}A_q^{(2)} - \frac{1}{2}G_m^{(1)}\beta_0\Big) + L^3\Big(\frac{1}{6}A_q^{(1)}\beta_0\Big) \\
&\quad + \epsilon\Big[L^2\Big(-\frac{1}{2}G_m^{(2)}\Big) + L^3\Big(\frac{1}{6}A_q^{(2)} + \frac{1}{3}G_m^{(1)}\beta_0\Big) + L^4\Big(-\frac{1}{12}A_q^{(1)}\beta_0\Big)\Big] \\
\mathcal{F}_m^{(3)} &= \dots \\
\mathcal{F}_m^{(4)} &= \dots
\end{aligned}$$

We need $K_m^{(n)}$ and $G_m^{(n)}$

Sudakov behavior of MASSIVE form factors

G_m depends only on Q^2 (no dependence on m_t^2). Hence,

$$G_m = G_q = 2(B_q - \gamma_q) + f_q + C_q + \mathcal{O}(\epsilon^2)$$

Following soft-collinear effective theory (SCET), it can be shown that the two–loop anomalous dimension matrix for two heavy parton correlations, which controls the singular structure, contains a heavy quark anomalous dimension γ_Q. Hence, we proposed [Blümlein, Marquard, NR]

$$K_m = -2(B_q - \gamma_q) - f_q - 2\gamma_Q$$

γ_Q is the heavy quark anomalous dimension.
(was known up to two-loop. We obtained it at three-loop level using Γ_2).

SUDAKOV BEHAVIOR OF MASSIVE FORM FACTORS

Finally, solving the Sudakov equation

$$F_m\left(a_s, \frac{Q^2}{\mu_R^2}, \frac{m_t^2}{\mu_R^2}\right) = \mathcal{C}_m(a_s(\mu_R^2))\Big| \exp\left[\sum_{n=1}^{\infty} \hat{a}_s^n \mathcal{F}_m^{(n)}\right]\Big|_{\hat{a}_s \to a_s(\mu_R^2)}$$

$\mathcal{C}_m$ is the matching coefficient.

Outcome

Inputs :

- Massive form factors at one and two loops in the asymptotic limit
- Massless form factors up to three loops
- All anomalous dimensions at three loop

Output :

- All the poles of massive form factors at three loops in the asymptotic limit
- Also all the $\ln(Q^2/m_t^2)$ coefficients for finite part

$\mathcal{C}_m$ can only be obtained through exact computation.

Conclusion

- *In the first part of the talk, we presented an overview on the automatization of solving first order factorizable system of differential equations. This method has been used to obtain complete color-planar and full light and heavy quark contributions to the three loop massive form factors.*
- *In the second part of the talk, we presented a Sudakov equation for massive form factors in the asymptotic limit, which allowed us to predict all logarithmic contributions at three loop level and partial four loop results in the corresponding limit.*

Thank You for your attention!