Lec. 1. DIS

UV ↑ quarks, gluons

confinement, chiral symmetry breaking

IR hadrons

$$\mathcal{L}_{UV} = -\frac{1}{4g^2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \bar\psi i \not{D} \psi - m\bar\psi\psi$$

The goal of this lecture is to try to understand DIS at different level of rigorous, and explain why we believe QCD is the correct theory for strong interaction.

A. Elastic Lepton - Nucleon Scattering

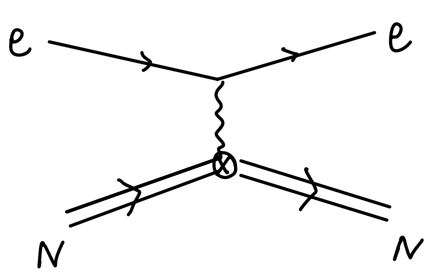
$$e(k) + N(p) \to e(k') + N(p')$$

$m_e = 0$

$m_N = m$

$q = p' - p$

$Q^2 = -q^2$



Nucleon Form factor:

$$\langle N(p')|\, J_\mu^{em}(0)\, |N(p)\rangle = \bar u(p')\left[\gamma_\mu F_1(Q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2\right] u(p)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{em}^2}{4E^2\sin^4\frac{\theta}{2}} \frac{1}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}} \left[\left(\underset{\substack{\Uparrow \\ \text{Dirac}}}{F_1^2} - \frac{q^2}{4m^2}\underset{\substack{\Uparrow \\ \text{Pauli}}}{F_2^2}\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]$$

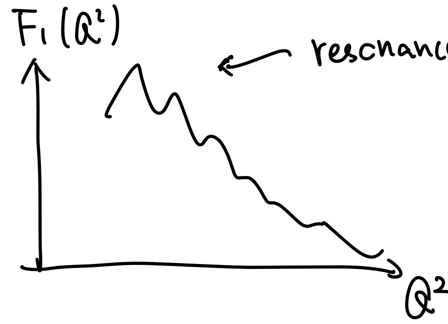
— Rosenbluth formula

$F_1^P(0) = 1, \quad F_1^N(0) = 0$

$F_2^P(0) \sim 2.8\mu_N \quad F_2^N(0) \sim -1.9 \qquad \mu_N = \frac{e\hbar}{2m_N}$

Q^2 dependence of F_1 and F_2 will probe the charge and magnetic distribution of nucleon.

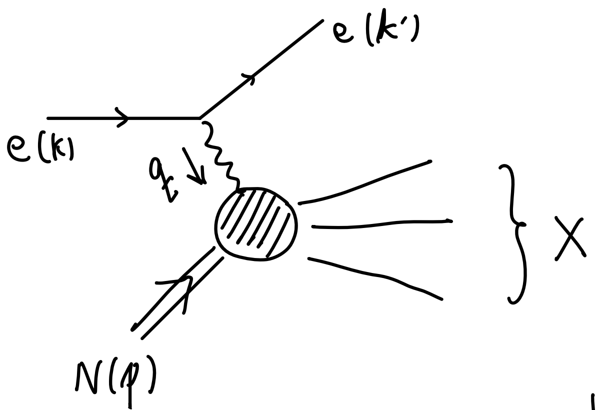
For $Q^2 < 1\ \text{GeV}^2$,



B. Deep Inelastic Scattering & Bjorken Scaling

$e(k) + N(p) \rightarrow e(k') + X$

⇑ Inclusive over X



Measurement : In the Nucleon rest frame.

$\begin{cases} E' : \text{scattered electron energy} \\ \theta : \text{scattering angle} \end{cases}$

⇓ rewrite in Lorentz invariant

$\begin{cases} p^\mu = (m, 0, 0, 0) \\ E' = \frac{k' \cdot p}{m} \qquad E = \frac{k \cdot p}{m} \\ \theta : \ q^2 = (k-k')^2 = -2EE'(1-\cos\theta) \end{cases}$

Bjorken propose to measure:

$$E - E' = \frac{p \cdot q}{m} = \nu$$

energy loss

$$Q^2 = -q^2$$

$$x_B = \frac{Q^2}{2p \cdot q}$$

kinematics: $\quad P_X^2 \geq m^2 \Rightarrow (q+p)^2 = -Q^2 + 2p\cdot q + m^2 \geq m^2$

$$\Rightarrow \quad 0 < x_B \leq 1$$

elastic limit: $\quad x_B = 1 \qquad$ High Energy Limit: $\quad x_B \ll 1$

- What can we say about differential Xsec for DIS?

$$\frac{d\sigma}{dx_B\, dQ^2} \propto \underbrace{L_{\mu\nu}}_{\text{leptonic tensor}} \; W^{\mu\nu}$$

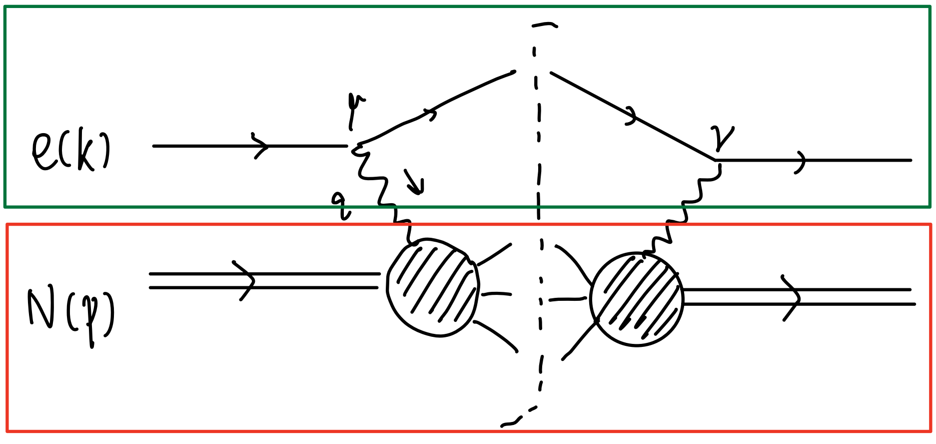
$W^{\mu\nu}$: hadronic tensor

$\boxed{\text{exercise}}$

$$W_{\mu\nu}(p,q) = \frac{1}{\pi} \operatorname{Im} T_{\mu\nu}(p,q)$$

$$T_{\mu\nu}(p,q) = i \int d^4x \, e^{iqx} \langle N(p) | T\{ J_\mu(x) J_\nu(0) \} | N(p) \rangle$$

Intutive explanation



Leptonic tensor

hadronic tensor

- Electromagnetic current conservation:

$$q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$$

$$\Rightarrow \quad W_{\mu\nu}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x_B, Q^2) + \left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) W_2$$

$$+ \underbrace{i\,\epsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta W_3}_{\text{parity violation}}$$

- Bjorken Scaling:

$$Q^2 \to \infty, \qquad W_1(x_B, Q^2) = W_1(x_B)$$

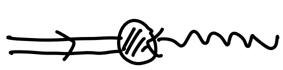
$$W_2(x_B, Q^2) \propto Q^2 \tilde{F}_2(x_B)$$

- Confirmed By SLAC experiment
- Implies the existence of pointlike particle in nucleon

C The Parton Model

Breit Frame:

$$q^\mu = (0, 0, 0, -Q), \qquad p^\mu = \left(\sqrt{P_z^2 + m^2}, 0, 0, P_z\right)$$



$$p\cdot q = +Q\cdot P_z \qquad x_B = \frac{Q^2}{2p\cdot q} \Rightarrow P_z = \frac{Q}{2x_B}$$

$$\Downarrow$$

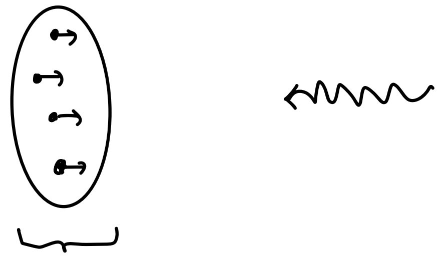
$$P_z \to \infty \quad \text{(Bjorken limit)}$$

- interaction time scale

  $t \simeq \frac{R}{c} \cdot \frac{P_z}{m} \to \infty$

  "free parton"

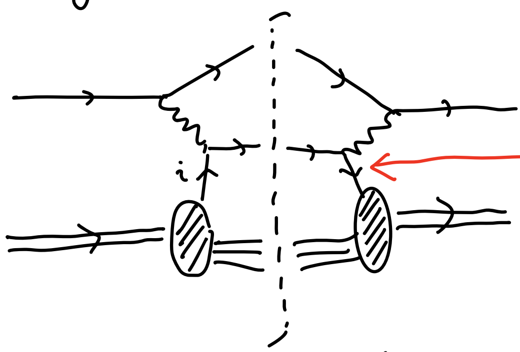
p



Length contraction

- one scattering each time
- Incoherent scattering

$\sum_i$



parton scattering with photon

- Parton Distribution Functions (PDFs):

$$\underbrace{f_i(\xi)\,d\xi}_{\text{probability with momentum fraction } \xi \text{ between } \xi \text{ an } \xi+d\xi} \qquad i = q, \bar{q}, g$$

parton momentum $\qquad \ell^\mu = (\xi P_z, 0, 0, \xi P_z)$

final-state parton mom: $\qquad \ell^\mu + q^\mu = (\xi P_z, 0, 0, \xi P_z - Q)$

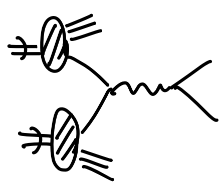
on-shellness : $\qquad \xi^2 P_z^2 - \xi^2 P_z^2 + 2\xi P_z Q - Q^2 = 0$

$$\Rightarrow \quad \boxed{\xi = \frac{Q^2}{2P_z Q} = x_B}$$

$$\sigma_{eN\to eX} = \sum_i \int_0^1 d\xi\, f_i(\xi)\, \sigma_{e(k)\,q_i(\xi p)\to e(k')\,q_i(\xi p+q)}$$

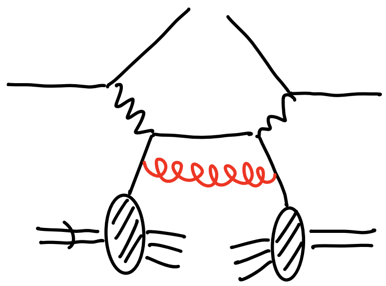
can be generalized to other processes

Drell-Yan:



(D) QCD Improved Parton Model

observation:



Should we put the gluonic correction into hard cross section or into PDFs ?

gluon loop : $\sim \alpha_s \int_m^Q \frac{d^4k}{k^4} \sim \alpha_s \ln \frac{Q^2}{m^2}$

Scale separation: $\ln \frac{Q^2}{m^2} = \ln \frac{Q^2}{\mu_F^2} + \ln \frac{\mu_F^2}{m^2}$ $\qquad Q^2 \gg \mu_F^2 \gg m^2$

$$1 + \alpha_s \ln \frac{Q^2}{m^2} + \cdots = \underbrace{\left(1 + \alpha_s \ln \frac{Q^2}{\mu_F^2} + \cdots\right)}_{\hat{\sigma}_{eq_i}(\mu_F)} \underbrace{\left(1 + \alpha_s \ln \frac{\mu_F^2}{m^2} + \cdots\right)}_{f_i(\xi, \mu_F)}$$

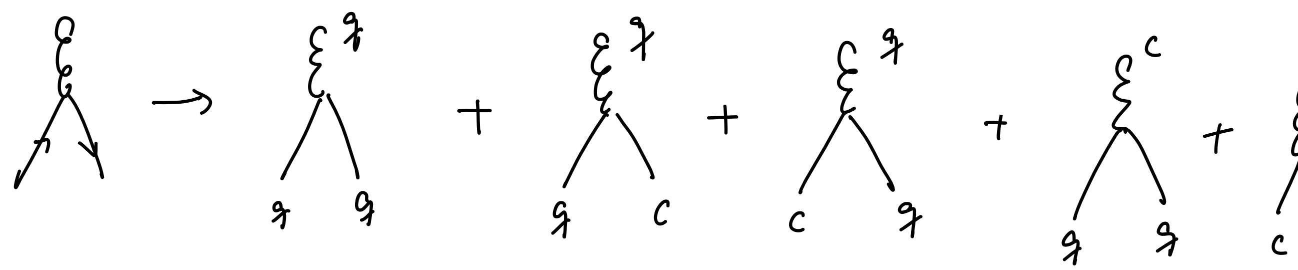
(D.1) Leading Order Calculation

- Scale separation at the level of fields:

$$\phi(x) = \left(\int_{|k|>\mu} \frac{d^3k}{(2\pi)^3 2E_k} + \int_{|k|<\mu} \frac{d^3k}{(2\pi)^3 2E_k} \right) \left[\hat{a}^+_k e^{ikx} + a_k e^{-ikx} \right] = \underbrace{\phi_q}_{\text{quantum}} + \underbrace{\phi_c}_{\text{classical}}$$

$$\langle 0 | T \{ A_q^\mu(x) A_q^\nu(0) \} | 0 \rangle = \int_{|k|>\mu} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i g^{\mu\nu}}{k^2 + i\varepsilon}$$

momentum conservation: $\langle 0 | T \{ A_q^\mu(x) A_c^\nu(0) \} | 0 \rangle = 0$



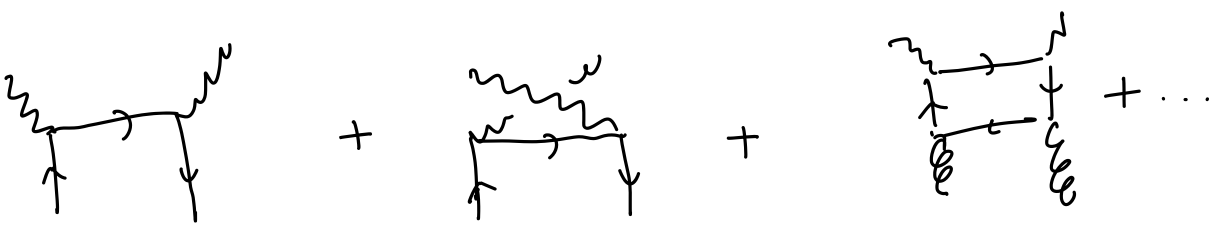
using $J_\mu = \bar\psi \gamma_\mu \psi$

$$T_{\mu\nu} = i\int d^4x\, e^{iqx} \langle N | T\{ \bar\psi(x)\gamma_\mu \psi(x)\, \bar\psi(0)\gamma_\nu \psi(0)\} | N\rangle$$

$$= i\int d^4x\, e^{iqx} \langle N | T\{ \bar\psi_c(x)\gamma_\mu \psi_q(x)\, \bar\psi_q(0)\gamma_\nu \psi_c(0)\} | N\rangle$$

$$+ i\int d^4x\, e^{iqx} \langle N | T\{ \bar\psi_q(x)\gamma_\mu \psi_c(x)\, \bar\psi_c(0)\gamma_\nu \psi_q(0)\} | N\rangle$$

$$+ i\int d^4x\, e^{iqx} \langle N | T\{ \bar\psi_q(x)\gamma_\mu \psi_q(x)\, \bar\psi_q(0)\gamma_\nu \psi_q(0)\} | N\rangle + \cdots$$



- quark propagator:

$$\langle 0 | T\{ \psi_q(x) \bar\psi_q(0)\} | 0\rangle = \int_{|k|>\mu} \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i\not{p}}{p^2+i\varepsilon} \sim \frac{i}{2\pi^2} \frac{\not{x}}{[-x^2+i\varepsilon]^2}$$

$$\Rightarrow T_{\mu\nu} = i\,\frac{i}{2\pi^2}\int d^4x\, \frac{e^{iqx}}{x^4} \langle N | \bar\psi_c(x)\gamma_\mu \not{x} \gamma_\nu \psi_c(0) - \bar\psi_c(0)\gamma_\nu \not{x}\gamma_\mu \psi_c(x) | N\rangle$$

- $$\gamma_\mu \not{x} \gamma_\nu = x_\mu \gamma_\nu + x_\nu \gamma_\mu - g_{\mu\nu} x^\rho \gamma_\rho + \underbrace{i\,\epsilon_{\mu\rho\nu\sigma} x^\rho \gamma_5 \gamma^\sigma}_{\text{spin dependent, ignored.}}$$

- The only relevant matrix element (Ignore subscript c)

$$\langle N | \bar{\psi}(x) \gamma^\sigma \psi(0) | N \rangle = p^\sigma f_1(px, x^2) + x^\sigma f_2(px, x^2)$$

- Bjorken limit

$$\langle N | \bar{\psi}(x) \gamma^\sigma \psi(0) | N \rangle \overset{x^2 \to 0}{\sim} p^\sigma f_1(px)$$

$$= 2p^\sigma \int_{-1}^{1} du \, e^{iupx} f(u)$$

$$\langle N | \bar{\psi}(0) \gamma^\sigma \psi(x) | N \rangle \Big|_{x^2=0} = 2p^\sigma \int_{-1}^{1} du \, e^{-iupx} f(u)$$

After some Algebra (exercise)

$$\frac{1}{\pi} \operatorname{Im} T_{\mu\nu} = [f(x_B) - f(-x_B)] \left\{ \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{2x_B}{p \cdot q} \left(p_\mu + \frac{q_\mu}{2x_B}\right)\left(p_\nu + \frac{q_\nu}{2x_B}\right) \right\}$$

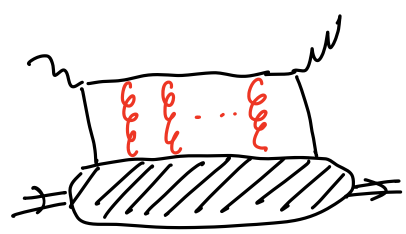
Reproduce parton model result.

(D.2) Definition of PDFs.

Our calculation above suggest that PDF can be defined as inverse Fourier tranform of

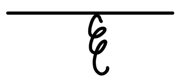
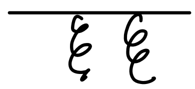
$$\langle N | \bar{\psi}(x) \gamma^\rho \psi(0) | N \rangle$$

- Only LO approximation
- Not gauge invariant

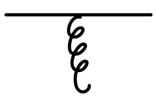


a quantum field propagate in a classical background.

- (Quantum) quark propagator in background field. (only keep $\sim \frac{1}{x^4}$)

 +  +  + ...

 $= \overbrace{\psi(x)\bar{\psi}(0)} = \frac{i}{2\pi^2}\frac{\not{x}}{x^4}$

 $= \psi(x)\; ig\int d^4z\, \bar{\psi}(z) A_\mu(z)\gamma^\mu \psi(z)\; \bar{\psi}(0)$

$$= ig\left(\frac{i}{2\pi^2}\right)^2 \int d^4z\, \frac{\not{x}-\not{z}}{(x-z)^4} A_\mu(z)\gamma^\mu \frac{\not{z}}{z^4}$$

$\bar{u} = 1-u$ Feyn trick

$$= -\frac{ig}{4\pi^4}\int_0^1 du\, u(1-u)\int d^4z\, A_\mu(z)(\not{x}-\not{z})\gamma^\mu \not{z}\,\frac{\Gamma(4)}{\left[(z-ux)^2+u\bar{u}x^2\right]^4}$$

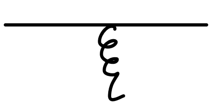
shift variable

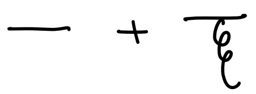
$$= -\frac{ig}{4\pi^4}\int_0^1 du\, u\bar{u}\int d^4z\, A_\mu(z+ux)(\bar{u}\not{x}-\not{z})\gamma^\mu(\not{z}+u\not{x})$$

$$\times \frac{\Gamma(4)}{\left[z^2+u\bar{u}x^2\right]^4}$$

- expansion on light cone

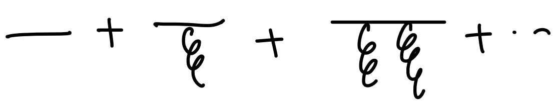
$$A_\mu(z+ux) = A_\mu(ux) + \underbrace{z^\nu\partial_\nu A_\mu(ux) + \cdots}_{\text{power suppressed in } x^2}.$$

 $= -\frac{g}{4\pi^2}\int_0^1 du\left\{\frac{2x^\mu}{x^4}A_\mu(ux)\not{x} - \frac{1}{x^2}\left[\cdots\right]\right\}$



$$= \frac{i\not{x}}{2\pi^2 x^4}\left[1 + ig\int_0^1 du\, x_\mu A^\mu(ux)\right]$$

Summing all diagrams:



$$= \frac{i\not{x}}{2\pi^2 x^4}\, P\exp\left[ig\int_0^1 du\, x_\mu A^\mu(ux)\right]$$

$$\equiv \frac{i\not{x}}{2\pi^2 x^4}\,[x,0]_c \qquad \text{(gauge link)}$$

The quark contribution to $T_{\mu\nu}$ now becomes

$$T_{\mu\nu} = i\int d^4x\, e^{iqx}\langle N|\, T\{\bar\psi_c(x)\gamma_\mu \overbrace{\psi_q(x)\,\bar\psi_q(0)}^{\text{background}}\gamma_\nu \psi_c(0)\}\,|N\rangle$$

$$= i\,\frac{i}{2\pi^2}\int d^4x\,\frac{e^{iqx}}{x^4}\langle N|\,\bar\psi_c(x)\gamma_\mu\not{x}\gamma_\nu\,[x,0]_c\,\psi_c(0)\}\,|N\rangle$$

$$\Rightarrow \boxed{\langle N(p)|\,\bar\psi(x)\gamma^\sigma[x,0]\psi(0)\,|N(p)\rangle\Big|_{x^2=0} = 2p^\sigma\int_{-1}^{1} du\, e^{iupx} f(u)}$$

The matrix element requires renormalization, leads to DGLAP equation:

$$\mu^2\frac{d}{d\mu^2} f_q(u,\mu^2) = \int_u^1 \frac{d\xi}{\xi}\, P_{qq}(\xi)\, f_q\!\left(\frac{u}{\xi},\mu^2\right) + \cdots$$

(E) Solve DGLAP Evolution

Mellin Space: $M_q^N = \int_0^1 dx\, x^{N-1} f_q(x)$

$$\mu^2 \frac{d}{d\mu^2} M_q^N(\mu^2) = \frac{\alpha_s}{2\pi} \int_0^1 du\, u^{N-1} \int_0^1 d\xi_1 d\xi_2\, P_{qq}^{(0)}(\xi_1) f_q(\xi_2) \delta(u - \xi_1 \xi_2)$$

$$= -\gamma_{qq}^{(0)}(N) M_q^N \frac{\alpha_s}{2\pi}$$

$$\gamma_{qq}^{(0)} = -\int_0^1 du\, u^{N-1} P_{qq}^{(0)}(u)$$

$$\overset{\text{1-loop}}{=} -C_F \cdot \left\{ 2 \underbrace{\sum_{j=2}^{N} \frac{1}{j}}_{\text{related to } \psi(x) = \frac{dT(x)}{T(x)}} - \frac{1}{N(N+1)} + \frac{1}{2} \right\}$$

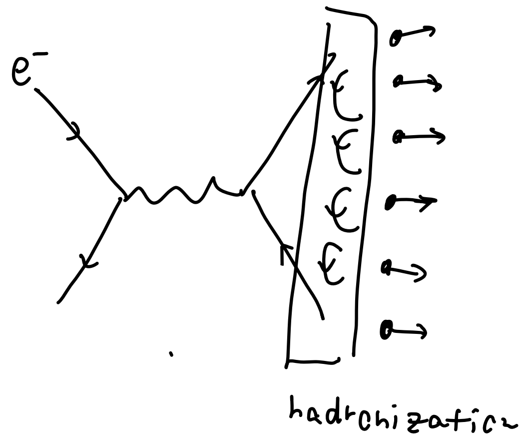
RG solution: $M_q^N(Q^2) = \left(\dfrac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)} \right)^{2\gamma_{qq}^{(0)}(N)/\beta_0} M_q^N(\mu_0)$

PDF can be reconstructed from Mellin moment

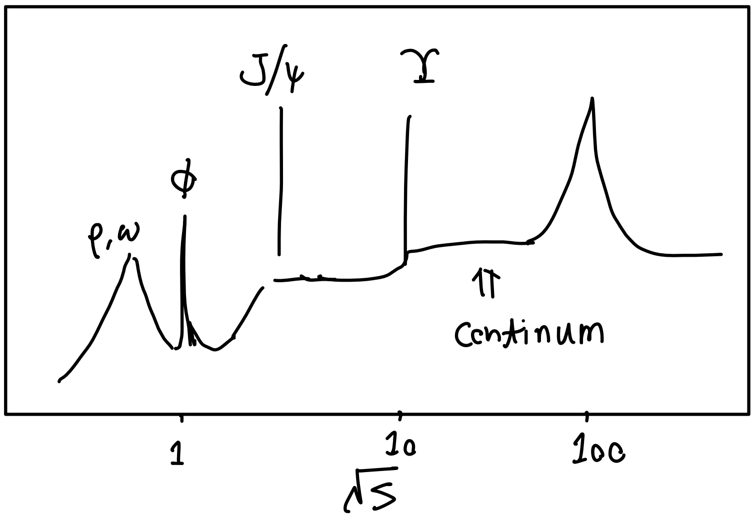
$$F_q(x) = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} dN\, x^{-N} M_q^N$$

Lecture 2: e^+e^- annihilation

$e^+e^- \to$ hadrons



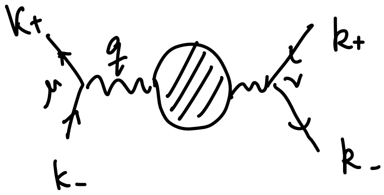
$$R(s) = \frac{\sigma_{tot}(e^+e^- \to \text{hadrons})}{\sigma_{tot}(e^+e^- \to \mu^+\mu^-)}$$

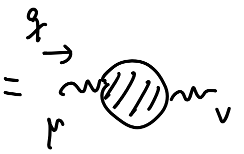


(A) R-ratio at away from resonance

(A.1) Optical theorem $\quad S^\dagger S = \mathbb{1}$.

$$\sigma_{tot}(e^+e^- \to \text{hadrons}) = \frac{1}{s} \operatorname{Im} M (e^+e^- \to e^+e^-)$$

$M =$  $= e^4 \bar{u}(k_-)\gamma^\mu v(k_+) \frac{1}{s} \Pi_{\mu\nu}(q) \frac{1}{s} \bar{v}(k_+)\gamma^\nu u(k_-)$

$\Pi_{\mu\nu}(q) =$  $= i\int d^4x\, e^{iqx} \langle 0 | T\, J_\mu(x) J_\nu(0) | 0 \rangle$

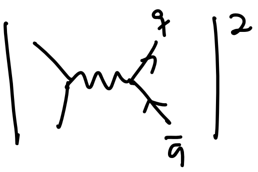
$$= (q_\mu q_\nu - q^2)\,\Pi(s) \qquad s = q^2$$

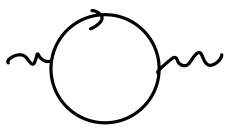
After some manipulation, one obtains

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{1}{s}(4\pi\alpha)^2\, \mathrm{Im}\,\Pi(s)$$

$$R(s) = 12\pi\, \Pi(s)$$

- One-loop calculation



Im 

- Can you guess the result for $R(s)$ for one quark flavor with charge e?

UV pole

$$\Pi_{\mu\nu}(q) = \frac{N_c}{12\pi^2}(q_\mu q_\nu - q^2 g_{\mu\nu})\left(\#\frac{1}{\epsilon} + \ln\frac{\mu^2}{-q^2 - i\varepsilon} + \cdots\right)$$

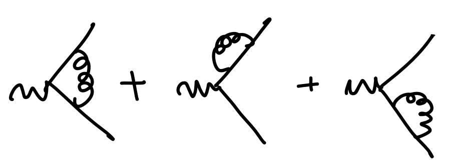
$$\mathrm{Im}\ln\frac{\mu^2}{-q^2-i\varepsilon} = \pi\,\Theta(q^2) \Rightarrow \mathrm{Im}\,\Pi(s) = \frac{N_c}{12\pi}\,\Theta(s)$$

$\Rightarrow R(s) = N_c$ as expected

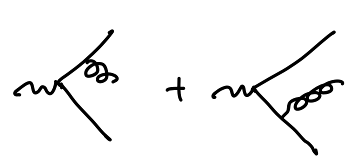
(A.2) Amplitude approach @ NLO (with a small gluon mass)

tree

NLO virtual



NLO real



$$\sigma_{e^+e^-\to q\bar{q}} = \sigma_0 + \sigma_0 C_F \frac{\alpha_s}{\pi}\left[-2\ln^2\frac{Q}{\lambda} + 3\ln\frac{Q}{\lambda} - \frac{7}{4} + \frac{\pi^2}{6}\right]$$

$$\sigma_{e^+e^-\to q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{\pi}\left[2\ln^2\frac{Q}{\lambda} - 3\ln\frac{Q}{\lambda} + \frac{5}{2} - \frac{\pi^2}{6}\right]$$

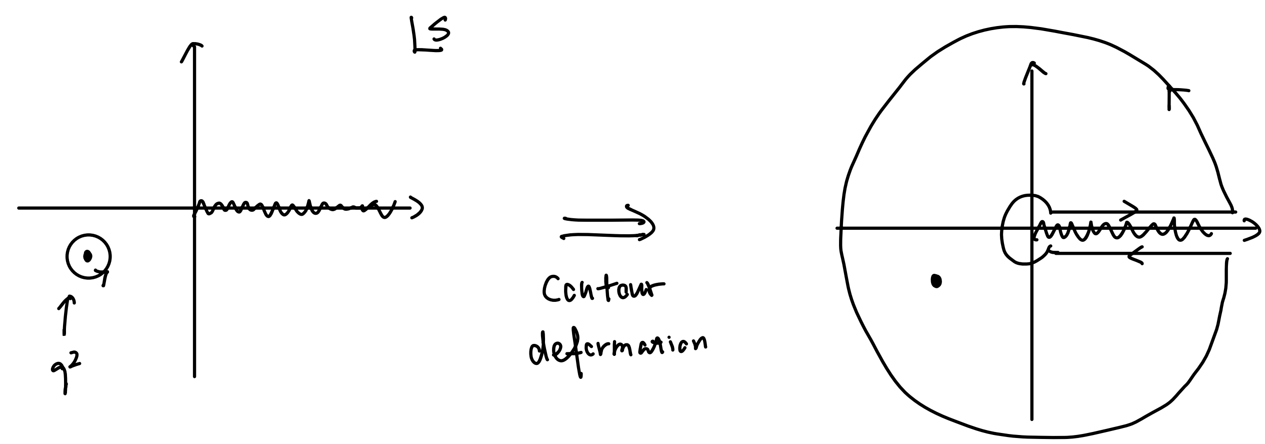
$$\sigma_{tot} = \sigma_0\left(1 + \frac{\alpha_s}{\pi} + \cdots\right)$$

(B) Resonance region and QCD sum rule.

- Dispersion Relation

Causality $\Rightarrow$ Analyticity in s for $\Pi(s)$

In the massless quark loop example above.



$$\underbrace{\Pi(q^2) = \frac{1}{2\pi i}\oint ds\,\frac{\Pi(s)}{s-q^2}}_{\text{Cauchy formula}} = \frac{1}{2\pi i}\int_0^\infty \frac{ds}{s-q^2}\underbrace{\left[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\right]}_{2i\,\Pi(s+i\varepsilon) = \frac{2i}{12\pi}R(s)}$$

+ contour at infinity

(If ignore contour at ∞) $\boxed{\Pi(q^2) = \frac{1}{12\pi^2}\int_0^\infty \frac{ds}{s-q^2}R(s)}$

In some cases, $\Pi(q^2)$ might have bad behavior at ∞.

- one-time subtraction dispersion relation

$$\frac{\Pi(s)-\Pi(0)}{s} = f(s)$$ no pole at $s=0$, good behavior at ∞.

$$f(q^2) = \frac{1}{2\pi i}\int ds \frac{f(s)}{s-q^2} = \frac{1}{2\pi i}\int_0^\infty \frac{2i \, \mathrm{Im}\, f(s+i\epsilon)}{s-q^2}$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{2i \, \mathrm{Im}\, \Pi(s+i\varepsilon)}{(s-q^2)\, s}$$

$$\Rightarrow \quad \Pi(q^2) = \Pi(0) + \frac{q^2}{12\pi^2}\int_0^\infty ds \frac{R(s)}{(s-q^2)\, s}$$

- How to use it? SVZ QCD sum rule

(1) Choose $q^2=-Q^2$ in deep Euclidean space

(2) Calculate LHS with P.T. and OPE

(3) Calculate RHS with Experiment data

(4) Compare LHS & RHS and draw conclusion.

(C) Operator Product Expansion (OPE)

$$T\{J_\mu(x) J_\nu(0)\} \underset{|x|\to 0}{=} \sum_n C^n_{\mu\nu}(x)\, O_n(0)$$ Wilson

$\Uparrow$

Gauge and Lorentz invariant operator

$d_{J_\mu} = 3$

$[\mathrm{LHS}] = 6$

- List of gauge invariant scalar operator in QCD.

		d	C	
1) Unity Operator	$\mathbb{1}$	0	6	$\sim \frac{1}{x^6}$
2) Quark operator	$\bar{\psi}\psi$	3	3	$\sim \frac{1}{x^3}$
3) Gluon	$G^A_{\mu\nu} G^{\mu\nu,A}$	4	2	$\sim \frac{1}{x^2}$
4) mixed	$\bar{\psi} g \sigma_{\mu\nu} G^{\mu\nu} \psi$	5	1	$\sim \frac{1}{x}$
5)	$(\bar{\psi}\Gamma\psi)^2$	6	0	$\sim \ln x$

? How does the condensate of these operator contribute to $\Pi(q^2)$?

$$\int d^4x\, e^{iqx} C_{\mathbb{1}}(x) \sim (q^\mu q^\nu - q^2 g^{\mu\nu}) \cdot \ln q^2$$

which has the structure of perturbation theory

$$\int d^4x\, e^{iqx} C_{\bar{\psi}\psi}(x) \sim \frac{1}{|q|}$$

$$\int d^4x\, e^{iqx} C_{GG}(x) \sim \frac{1}{|q|^2}$$

Why has to be propotional to quark mass?

$$\Pi(q^2) = \tilde{C}_{\mathbb{1}} + \frac{m_\psi}{q^4} \tilde{C}_{\psi\psi} \langle \bar{\psi}\psi \rangle + \frac{1}{q^4} \tilde{C}_{GG} \langle GG \rangle + \cdots$$

Perturbation theory

(Shifman, Vainstein, Zakharov, 1979)

$$\Pi(q^2) = \frac{1}{4\pi^2} \ln \frac{\mu^2}{-q^2} [1 + \alpha_s + \cdots] + \boxed{\frac{2m_\psi}{q^4} \langle \bar{\psi}\psi \rangle} + \frac{1}{12 q^4} \langle \frac{\alpha_s}{\pi} GG \rangle$$

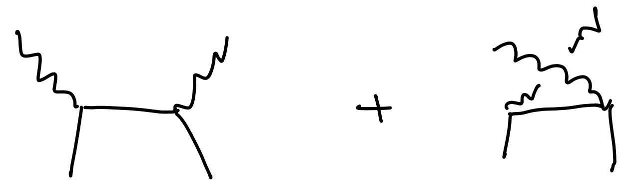
$$+ \frac{224}{81 q^6} \alpha_s \pi \langle \bar{\psi}\psi \rangle^2 + \cdots$$

(C.1) Calculate of the OPE coefficient (Example)

OPE are operator equation, holds in any state.

OPE coefficients are calculable in perturbation theory.

Example: The $\langle\bar{\psi}\psi\rangle$ condensate (one flavor ψ)



$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iq\cdot x}\langle 0|T\,\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(0)\gamma_\nu\psi(0)$$

$$+\ \bar{\psi}(0)\gamma_\nu\psi(0)\bar{\psi}(x)\gamma_\mu\psi(x)|0\rangle$$

$$\psi(x) = \psi(0) + x^\rho \overrightarrow{D}_\rho\psi(0) + \cdots$$

$$\bar{\psi}(x) = \bar{\psi}(0) + \bar{\psi}(0)\overleftarrow{D}_\rho x^\rho + \cdots$$

$$\langle 0|\bar{\psi}^i_\alpha \psi^j_\beta|0\rangle = A\,\delta^{ij}\delta_{\alpha\beta}$$

$$\langle 0|\bar{\psi}^i_\alpha \overrightarrow{D}_\rho \psi^j_\beta|0\rangle = B\,\delta^{ij}(\gamma_\rho)_{\alpha\beta}\,,\quad \langle 0|\bar{\psi}^i_\alpha \overleftarrow{D}_\rho \psi^j_\beta|0\rangle = \bar{B}\,\delta^{ij}(\gamma_\rho)_{\beta\alpha}$$

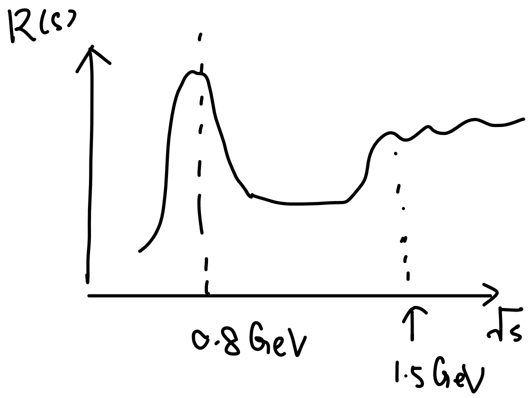
contract the indicies with appropriate tensor

and use EOM $i\not{D}\psi = m\psi$

$$A = \frac{1}{12}\langle 0|\bar{\psi}\psi|0\rangle\,,\quad B = \frac{1}{48}\langle 0|\bar{\psi}\not{D}\psi|0\rangle = -\frac{im}{48}\langle 0|\bar{\psi}\psi|0\rangle$$

(C.2) QCD sum rule example: ρ meson.

Experiment:

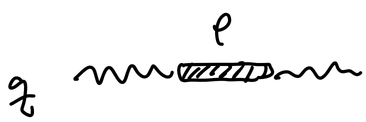


we're interested in the ρ meson contribution.

$$\Pi_{\mu\nu}(q) = \int d^4x \, e^{iqx} \Big[\theta(x^0) \langle 0| J_\mu(x) |\rho\rangle\langle\rho| J_\nu(0)|0\rangle$$
$$+ \theta(-x^0) \langle 0| J_\nu(0) |\rho\rangle\langle\rho| J_\mu(x)|0\rangle \Big]$$

let $\quad \langle 0| J_\mu(0) |\rho^{(\lambda)}\rangle = m_\rho^2 \dfrac{\sqrt{2}}{g_\rho} \epsilon_\rho^{(\lambda)} \qquad \lambda = 1, 2, 3$

corresponds to the diagram

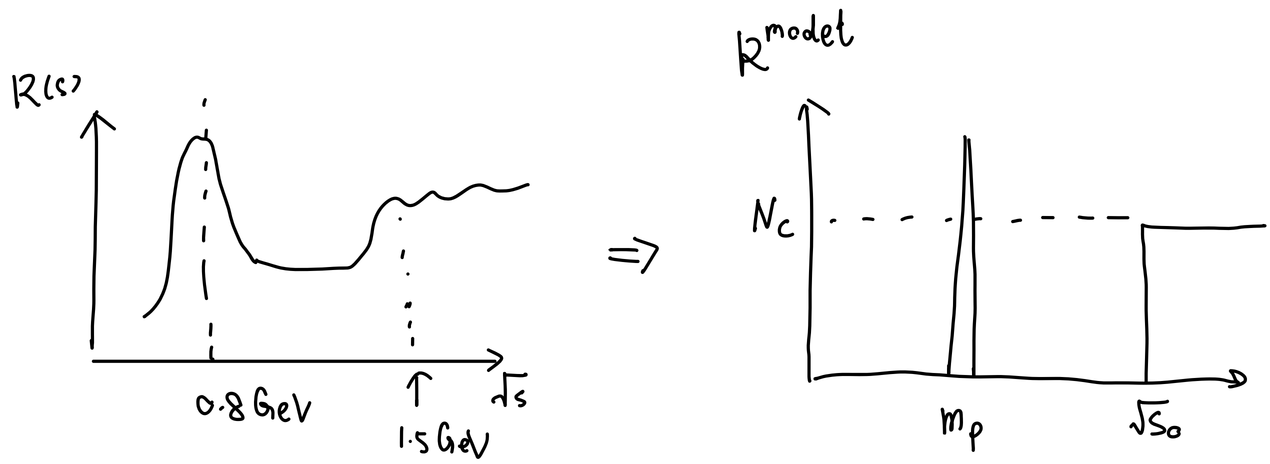


$$= \sum_\lambda m_\rho^2 \frac{\sqrt{2}}{g_\rho} \epsilon_\mu^{(\lambda)} \cdot \frac{1}{m_\rho^2 - q^2} \epsilon_\nu^{(\lambda)*} m_\rho^2 \frac{\sqrt{2}}{g_\rho}$$

$$\sum_\lambda \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)*} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}$$

$$\Rightarrow \quad \Pi(q^2)\Big|_{\rho\text{-meson}} = \frac{2m_\rho^2}{g_\rho^2} \frac{1}{m_\rho^2 - q^2 - i\varepsilon}$$

$$R_\rho(s) = 12\pi \, \mathrm{Im}\, \Pi(q^2)\Big|_\rho = 12\pi^2 \frac{2m_\rho^2}{g_\rho^2} \delta(s - m_\rho^2)$$



Model Exp. Data

$$\Pi^{\text{model}}(q^2) = \frac{1}{12\pi^2}\int_0^{\mu^2}\frac{ds}{s-q^2}\,R^{\text{model}}(s) = \frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} + \frac{1}{12\pi^2}\int_{s_0}^{\mu^2}\frac{ds}{s-q^2}N_c$$

QCD OPE

too small

$$\Pi^{\text{QCD}}(q^2) = \frac{N_c}{12\pi^2}\ln\frac{\mu^2}{-q^2} + \cancel{\frac{2m_q}{q^4}}\underbrace{\langle\bar{\psi}\psi\rangle}_{-(250\,\text{MeV})^3} + \frac{1}{12q^4}\underbrace{\langle\frac{\alpha_s}{\pi}G^2\rangle}_{0.012\,\text{GeV}^4} + \frac{224}{81q^6}\alpha_s\pi\langle\bar{\psi}\psi\rangle^2$$

trick #1 :

$$\ln\frac{\mu^2}{-q^2} = \int_0^{\mu^2}\frac{ds}{s-q^2} = \int_0^{s_0}\frac{ds}{s-q^2} + \int_{s_0}^{\mu^2}\frac{ds}{s-q^2}$$

$$\Rightarrow \frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} = \frac{N_c}{12\pi^2}\int_0^{s_0}\frac{ds}{s-q^2} + \frac{1}{12q^4}\langle\frac{\alpha_s}{\pi}G^2\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar{\psi}\psi\rangle^2$$

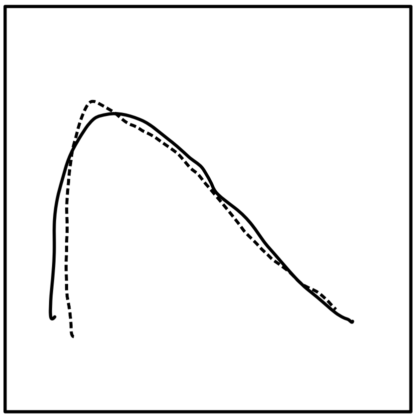
trick #2: Borel transformation

$$B\left[\frac{1}{m^2-q^2}\right] \Rightarrow \frac{1}{M^2}e^{-m^2/M^2}$$

$$B\left[\frac{1}{(-q^2)^n}\right] \Rightarrow \frac{1}{(n-1)!}\frac{1}{M^{2n}}$$

$\Rightarrow$ SVZ sum rule

$$\frac{2m_\rho^2}{g_\rho^2}\frac{e^{-m_\rho^2/M^2}}{M^2} = \frac{N_c}{12\pi^2 M^2}\int_0^{s_0} ds\, e^{-s/M^2} + \frac{1}{12M^4}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle - \frac{112}{81M^6}\alpha_s\pi\langle\bar{\psi}\psi\rangle^2$$



$m_\rho^2 \sim 0.5\ \mathrm{GeV}^2$

$g_\rho^2 \sim 28$

$s_0 \sim 1.5\ \mathrm{GeV}^2$