Soft-collinear effective theory (SCET)

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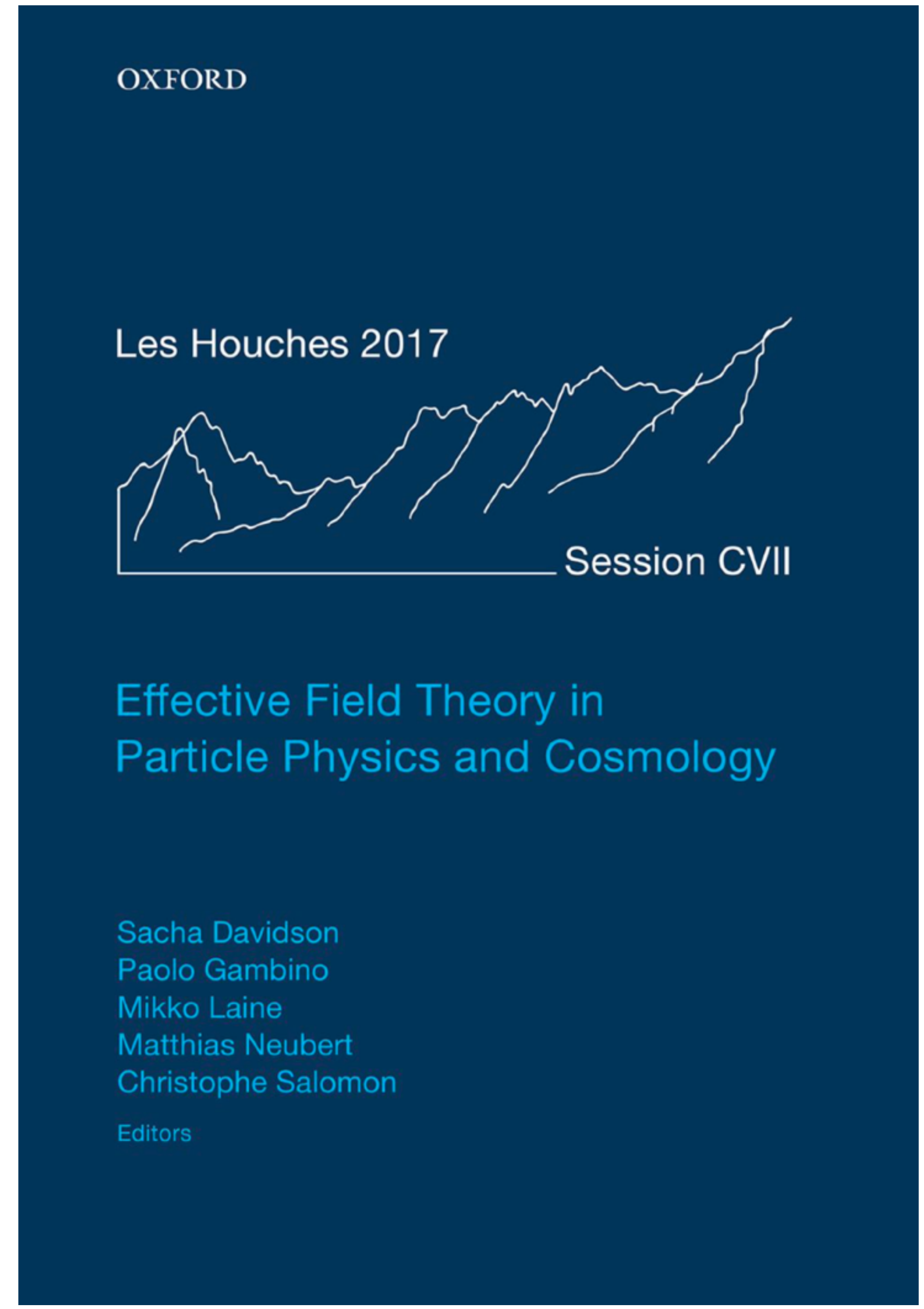
Effective Field Theories

- The intuitive idea behind effective theories is that you can calculate without knowing the exact theory.

- In some sense, the ideas of EFT are ‘obvious’. However, implementing them in a mathematically consistent way in an interacting QFT is not so obvious.

- An EFT is a quantum theory in its own right, and like any other QFT, it comes with a regularization and renormalization scheme necessary to obtain finite matrix elements.

主要参考资料



Historical note

Old - fashioned renormalization paradigm
($\lesssim$ 1980-1990)

Good theories are renormalizable.

Non-renormalizable theories are BAD (unpredictive ...)

Modern renormalization / EFT paradigm ($\gtrsim$ 1980-1990)

Most theories are probably effective theories and non-renormalizable.

Super-renormalizable interactions are BAD and should be forbidden by symmetries.

"Effective Field Theory: Concepts and Applications" Beneke

1. SMEFT

2. Soft effective theory in QED

3. Soft-collinear effective theory

4. Applications in event shapes

5. Applications in the jet physics

1. SMEFT

SMEFT

- If new, heavy particles exist beyond the Standard Model, with masses ***M*** much larger than the electroweak symmetry breaking scale ***v***, we can build their low-energy theory, called **SMEFT**, by enhancing the SM Lagrangian with high-dimensional local operators.

$$\mathcal{L}_{\mathrm{SMEFT}} = \mathcal{L}_{\mathrm{SM}} + \sum_{n\geq 1} \sum_i \frac{C_i^{(n)}}{M^n} \mathcal{O}_i^{(n)} .$$

- The new operators $\mathcal{O}_i^{(n)}$ with mass dimension ***D = 4 + n***. There is an infinite set of such operators, but importantly there exists only a finite set of operators for each dimension ***D***

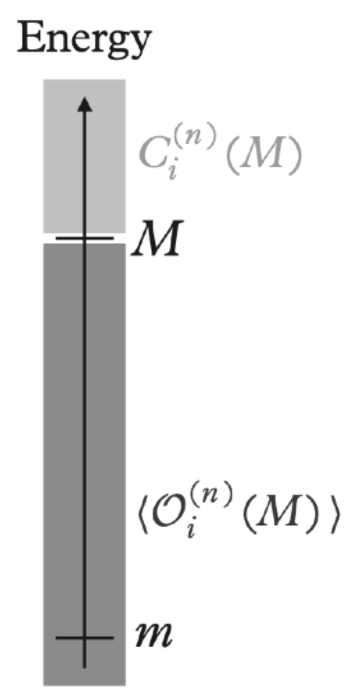
- The contributions of these operators to any given observable are suppressed by powers of $(v/M)^{D-4}$ relative to the contributions of the operators of the SM.

SMEFT and Wilsonian Approach

- In constructing the effective Lagrangian we split up the contributions from virtual particles into short- and long-distance modes:

$$\int_0^\infty \frac{d\omega}{\omega} = \int_M^\infty \frac{d\omega}{\omega} + \int_0^M \frac{d\omega}{\omega},$$

sensitive to UV physics and is absorbed into the Wilson coefficients



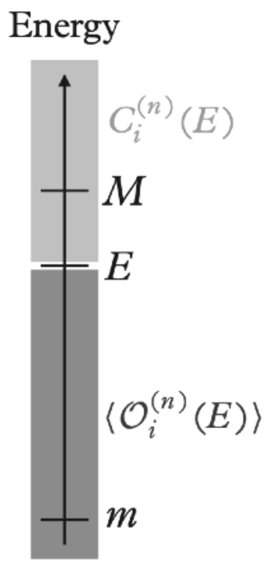
sensitive to IR physics and is absorbed into the matrix elements

Wilsonian Approach

- If we are performing a measurement at a characteristic energy scale E, such that m ≪ E < M, then we can integrate out the high-energy fluctuations (with frequencies ω > *E*) from the generating functional.

- This yields a different effective Lagrangian, but one in which the operators are the same as before

$$\int_0^\infty \frac{d\omega}{\omega} = \int_E^\infty \frac{d\omega}{\omega} + \int_0^E \frac{d\omega}{\omega},$$



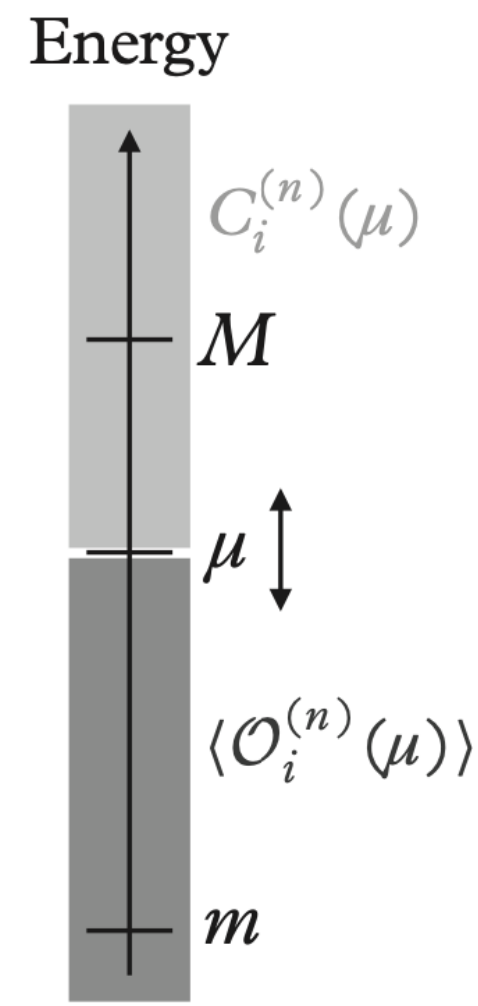
- The values of the Wilson coefficients and operators matrix elements need to be different

$$\mathcal{L}_{\mathrm{EFT}} = \sum_{n=0}^{\infty}\sum_i \frac{C_i^{(n)}(M)}{M^n}\,\mathcal{O}_i^{(n)}(M) = \sum_{n=0}^{\infty}\sum_i \frac{C_i^{(n)}(E)}{M^n}\,\mathcal{O}_i^{(n)}(E).$$

SMEFT and Wilsonian Approach

- We are thus led to study the effective Lagrangian

$$\mathcal{L}_{\mathrm{EFT}} = \sum_{n=0}^{\infty} \sum_{i} \frac{C_i^{(n)}(\mu)}{M^n} \mathcal{O}_i^{(n)}(\mu),$$



- Here $\mathcal{O}_i^{(n)}(\mu)$ are renormalized composite operators defined in dimensional regularization and the MS scheme

- $C_i^{(n)}(\mu)$ are the corresponding renormalized Wilson coefficients. These are nothing but the running couplings of the effective theory!

- The scale μ serves as the renormalization scale for these quantities, but at the same time it is the factorization scale which separates short-distance (high-energy) from long-distance (low-energy) contributions.

Collinear factorization in proton-proton collisions

For inclusive observables, sensitive only to a single high-energy scale *Q*, we have

$$\sigma = \sum_{a,b} \int_0^1 dx_1 dx_2 \hat{\sigma}_{ab}\left(Q, x_1, x_2, \mu_f\right) \left[f_a\left(x_1, \mu_f\right) f_b\left(x_2, \mu_f\right) + \mathcal{O}\left(\Lambda_{\mathrm{QCD}}/Q\right)\right]$$

partonic cross sections: perturbation theory	parton distribution functions (PDFs): nonperturbative	power corrections nonperturbative

The “right” way to look at this formula is EFT

$$\sigma = \sum_{a,b} \int_0^1 dx_1 dx_2 C_{ab}\left(Q, x_1, x_2, \mu\right) \left[\langle P\left(p_1\right) | O_a\left(x_1\right) | P\left(p_1\right)\rangle \langle P\left(p_2\right) | O_b\left(x_2\right) | P\left(p_2\right)\rangle + \mathcal{O}\left(\Lambda_{\mathrm{QCD}}/Q\right)\right]$$

Wilson coefficient:
matching at $\mu \approx Q$
perturbation theory

RG-evolution

low-energy matrix elements
nonperturbative

power suppressed operators

The matching coefficient C_{ab} is independent of external states and insensitive to physics below the matching scale μ.

Can use quark and gluon states to perform the matching.

- **Trivial matrix elements**

$$\langle q_{a'}(x'p)|O_a(x)|q_{a'}(x'\,p)\rangle = \delta_{aa'}\,\delta(x'-x)$$

- **Wilson coefficients are partonic cross section**

$$C_{ab}(Q,x_1,x_2) = \hat{\sigma}_{ab}(Q,x_1,x_2)$$

- **Bare Wilson coefficients have divergencies. Renormalization induces dependence on μ.**

Asymptotical expansion

- Consider an integral

$$I(m, M) = \int_0^M dx \frac{1}{x+m} = \ln \frac{m+M}{m}$$

- If $m \ll M$, we have $I(m, M) = -\ln \frac{m}{M} + \mathcal{O}\left(\frac{m}{M}\right)$

- Asymptotical expansion: not analytic in the expansion parameter because of presence of the logarithm.

- Our goal: obtain expanded results before carrying out the integral.

- Naive expansion breaks down: $\int_0^M dx \frac{1}{x} \qquad \int_0^\infty dx \frac{1}{x+m}$

Asymptotical expansion

- Cut-off regularisation: $\int_0^M \to \int_0^\Lambda + \int_\Lambda^M$

$$I_1(m, M) = \int_0^\Lambda dx \frac{1}{x+m} = \ln\frac{\Lambda}{m} + \mathcal{O}\left(\frac{m}{\Lambda}\right)$$

$$I_2(m, M) = \int_\Lambda^M dx \frac{1}{x+m} \sim \int_\Lambda^M dx \frac{1}{x} = \ln\frac{M}{\Lambda}$$

- Dimensional regularisation: $\int dx \to \int dx\, x^\epsilon$

$$I_1(m, M) = \int_0^M dx\, x^\epsilon \frac{1}{x+m} \sim \int_0^\infty dx\, x^\epsilon \frac{1}{x+m} = -\frac{1}{\epsilon} - \ln m$$

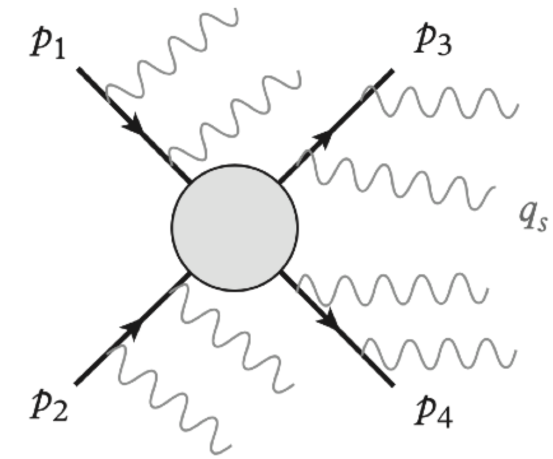
$$I_2(m, M) = \int_0^M dx\, x^\epsilon \frac{1}{x+m} \sim \int_0^M dx\, x^\epsilon \frac{1}{x} = \frac{1}{\epsilon} + \ln M$$

2. Soft effective theory in QED

Soft effective theory in QED

- When we talk about electron–electron scattering, we really measure the inclusive process

$$e^-(p_1) + e^-(p_2) \to e^-(p_3) + e^-(p_4) + X_s(q_s)$$



- It will be sufficient to assume that the total energy fulfills $E_s \ll m_e$

- We will now analyse the above process up to terms suppressed by powers of the expansion parameter $\lambda = E_\gamma / m_e$

- The effective Lagrangian $\mathcal{L}^\gamma_{\text{eff}} = \mathcal{L}^\gamma_4 + \frac{1}{m_e^2}\mathcal{L}^\gamma_6 + \frac{1}{m_e^4}\mathcal{L}^\gamma_8$

Effective Lagrangian

- The leading Lagrangian

$$\mathcal{L}_4^\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- The leading-power EFT Lagrangian is therefore simply the one for free photons, since the effective theory is obtained by integrating out the massive particles which leaves only the photons.

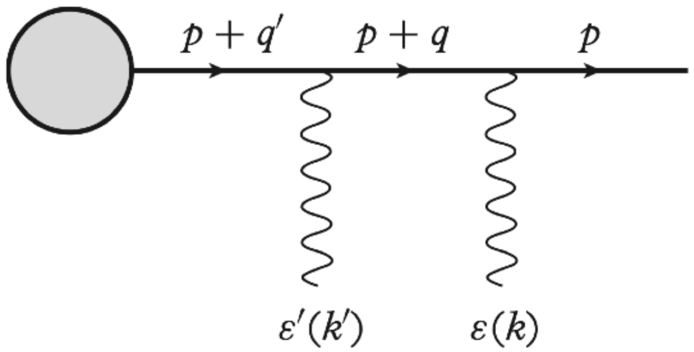
- Integrating out the electrons does induce higher-power operators which describe photon-photon interactions. While we will not need them, it is an interesting exercise to analyse these higher-power terms; the first non-trivial ones arise at dimension 8

- However, $\mathcal{L}_{\text{eff}}^\gamma$ is by itself not sufficient.

- We do need to include the incoming and outgoing electrons in the effective theory

Soft Photons and eikonal approximation

- Consider an outgoing electron with momentum $p^\mu = m_e v^\mu$



Note that $q' = k + k'$ *and* $q = k$

- We can expand the internal fermion propagators in the small momentum

$$\Delta_F(p+q) = i\frac{\not p + \not q + m_e}{(p+q)^2 - m_e^2 + i0} = i\frac{\not p + m_e}{2p\cdot q + i0} = \frac{\not v + 1}{2}\,\frac{i}{v\cdot q + i0}$$
$$\equiv P_v\,\frac{i}{v\cdot q + i0},$$

- where we introduced the projection operator $P_v = \dfrac{1+\not v}{2}$

$$P_v^2 = P_v, \qquad\qquad P_v \not\varepsilon P_v = P_v\, \varepsilon\cdot v.$$ **HW**

Soft Photons in Electron–Electron Scattering

- This form of the expanded soft emissions is well known and called the eikonal approximation

$$\bar{u}(p)\,P_v\,\frac{i}{v\cdot q}(-ie\,\varepsilon\cdot v)\,P_v\,\frac{i}{v\cdot q'}(-ie\,\varepsilon'\cdot v)\ldots.$$

- Can we obtain the expanded expression from an effective Lagrangian?
  - View the expanded propagator as the propagator in the effective theory
  - The emissions in the expanded diagram must be resulting from a Feynman rule $-ie\,v^{\mu}$
  - Write down a Lagrangian which produces them!

Soft Photons in Electron–Electron Scattering

- Consider

$$\mathcal{L}_{\rm eff}^{v} = \bar{h}_v(x)\, iv\cdot D\, h_v(x) \qquad\qquad D_\mu = \partial_\mu + ieA_\mu$$

- h_v is an auxiliary fermion field and obtained by multiplying a regular fermion field with P_v

$$P_v\, h_v = \not{v} h_v = h_v$$

- The propagator can be obtained by inverting the quadratic part of the Lagrangian

- The propagator of the field only has a single pole in the energy corresponding to the fermion. The anti-fermion pole has been lost in the expansion.

    - In this situation anti-fermions cannot arise as external particles and their virtual effects can be absorbed into the Wilson coefficients of the effective theory.

Soft Photons in Electron–Electron Scattering

- We constructed the effective Lagrangian in such a way that it reproduces the expansion of the full-theory diagram

- In HQET, the same Lagrangian can also be derived in a path-integral method.

- The field h_v cannot describe other fermion lines which have different velocities. To account for all four fermion lines, we need to include four auxiliary fermion fields

$$\mathcal{L}_{\text{eff}} = \sum_{i=1}^{4} \bar{h}_{v_i}(x)\, iv_i \cdot D h_{v_i}(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Delta\mathcal{L}_{\text{int}}\,, \qquad v_i^\mu = p_i^\mu / m_e$$

- We need different fields to represent the electrons along the different directions in the effective theory, while all of these were described by a single field in QED

Soft Photons in Electron–Electron Scattering

- The interaction terms

$$\Delta\mathcal{L}_{\text{int}} = \sum_i C_i(v_1, v_2, v_3, v_4, m_e)\, \bar{h}_{v_3}(x)\Gamma_i h_{v_1}(x)\, \bar{h}_{v_4}(x)\Gamma_i h_{v_2}(x)\,,$$

- In principle we could also write down interaction terms involving only two fields such as

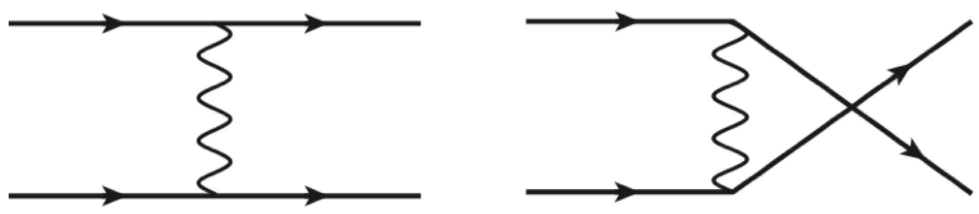
$$\Delta\mathcal{L}'_{\text{int}} = C_{\alpha\beta}(v_1, v_3)\, h^{\alpha}_{v_1}(x)\bar{h}^{\beta}_{v_3}(x)\,,$$

- Their Wilson coefficients are zero if the velocities are different, since the corresponding operator would describe a process in which a fermion spontaneously changes its velocity, which violates momentum conservation.

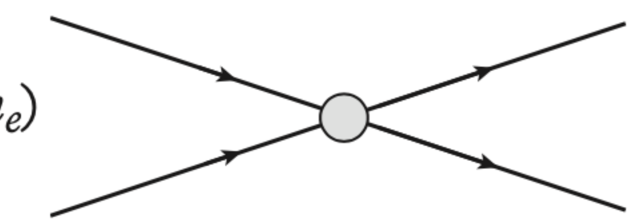
- We could also write interactions terms with covariant derivatives or more fields, but these are higher-dimensional operators, whose contributions are suppressed by powers of the electron mass

Wilson coefficients

- Compute the same quantity in QED and in the effective theory and then adjust the Wilson coefficient to reproduce the QED result. E.g. the amputated on-shell Green's function



$$= \quad C_{\alpha\beta\gamma\delta}(v_1, v_2, v_3, v_4, m_e)$$



- To reproduce the QED result, the Wilson coefficient must be set equal to the on-shell QED Green's function (which is the same as the scattering amplitude, up to the external spinors)

- At the moment, we are only discussing tree-level matching but the same simple relation also holds at loop level in dimensional regularization. The reason is that all loop corrections to the on-shell amplitude vanish in the effective theory because they are given by scaleless integrals.

Scaleless integrals

- Let us consider the x integration: $$\int_0^\infty dx \frac{1}{x^{1+\varepsilon}}$$
- it develops an ultraviolet divergence for $\varepsilon < 0$ and an infrared divergence for $\varepsilon > 0$
- In order to give a mathematical meaning to this integral we split the integration region into two parts using a regulator Λ

$$\int_0^\infty dx \frac{1}{x^{1+\varepsilon}} = \int_0^\Lambda dx \frac{1}{x^{1+\varepsilon}} + \int_\Lambda^\infty dx \frac{1}{x^{1+\varepsilon}}$$

- To distinguish the nature of the two divergences we can use two different regulators in the two different regions, by working out the integration for $\varepsilon_{IR} < 0$ and for $\varepsilon_{UV} > 0$

$$\int_0^\infty dx \frac{1}{x^{1+\varepsilon}} = -\frac{\Lambda^{-\varepsilon_{\rm IR}}}{\varepsilon_{\rm IR}} + \frac{\Lambda^{-\varepsilon_{\rm UV}}}{\varepsilon_{\rm UV}} \,,$$

- The r.h.s. can be analytically continued for arbitrary values of ε_{IR} and ε_{UV} without any constraint, therefore we are free to identify ε_{IR} and ε_{UV}. As a consequence of this, the integral vanishes

Wilson coefficients

- The on-shell amplitudes suffer from infrared singularities
- The Wilson coefficients have ultra-violet divergences
- The residual IR divergences in the on-shell amplitudes are identical to the UV divergences in the Wilson coefficient.
- The equality comes about since the (vanishing) on-shell loop integrals in the low energy effective theory suffer from both types of singularities. Schematically, the situation can be summarized by the following relation:

$$\boxed{\underbrace{\frac{1}{\varepsilon_{\mathrm{IR}}}}_{\text{on-shell amplitude}} = \underbrace{\frac{1}{\varepsilon_{\mathrm{UV}}}}_{\text{Wilson coeff.}} + \underbrace{\left(\frac{1}{\varepsilon_{\mathrm{IR}}} - \frac{1}{\varepsilon_{\mathrm{UV}}}\right)}_{\text{soft and coll. loop integrals}} .}$$

Soft Wilson line

- Our effective theory factorizes low- and high-energy physics: the hard scattering of the electrons is part of the Wilson coefficient, which depends on the high-energy scale m_e, while the low-energy diagrams in the effective theory only depend on photon-energy scales

- We can obtain a very elegant form of the low-energy matrix element by introducing the Wilson line

$$S_i(x) = \exp\left[-ie\int_{-\infty}^{0} ds\, v_i \cdot A(x + sv_i)\right].$$

- A point-like source which travels along the path $y^\mu(s) = x^\mu + sv_i^\mu$

- The energy of the outgoing electrons is much larger than the photon energies, they travel without recoiling when emitting photons.

Soft Wilson line

- Expand the Wilson line in the coupling

$$\begin{aligned}\langle \gamma(k)|S_i(0)|0\rangle &= -ie\int_{-\infty}^{0} ds\, v_i^{\mu}\langle \gamma(k)|A_{\mu}(sv_i)|0\rangle \\ &= -ie\int_{-\infty}^{0} ds\, v_i\cdot\varepsilon(k)e^{isv_i\cdot k} = e\frac{v_i\cdot\varepsilon(k)}{-v_i\cdot k+i0}\,.\end{aligned}$$

- We reproduce the eikonal structure

- To ensure the convergence of the integral at s = −∞, the exponent $v_i \cdot k$ must have a negative imaginary part, which amounts to the +i0 prescription in the eikonal propagator.

Decoupling transformation

- Perform a field redefinition

$$h_{v_i}(x) = S_i(x)\, h_{v_i}^{(0)}(x)\,,$$

- The fermion Lagrangian then takes the form

$$\begin{aligned}\bar{h}_{v_i}(x)\, iv_i \cdot D\, h_{v_i}(x) &= \bar{h}_{v_i}^{(0)}(x)\, S_i^\dagger(x)\, iv_i \cdot D\, S_i(x)\, h_{v_i}^{(0)}(x) \\ &= \bar{h}_{v_i}^{(0)}(x)\, S_i^\dagger(x)\, S_i(x)\, iv_i \cdot \partial\; h_{v_i}^{(0)}(x) \\ &= \bar{h}_{v_i}^{(0)}(x)\, iv_i \cdot \partial\; h_{v_i}^{(0)}(x)\,.\end{aligned}$$

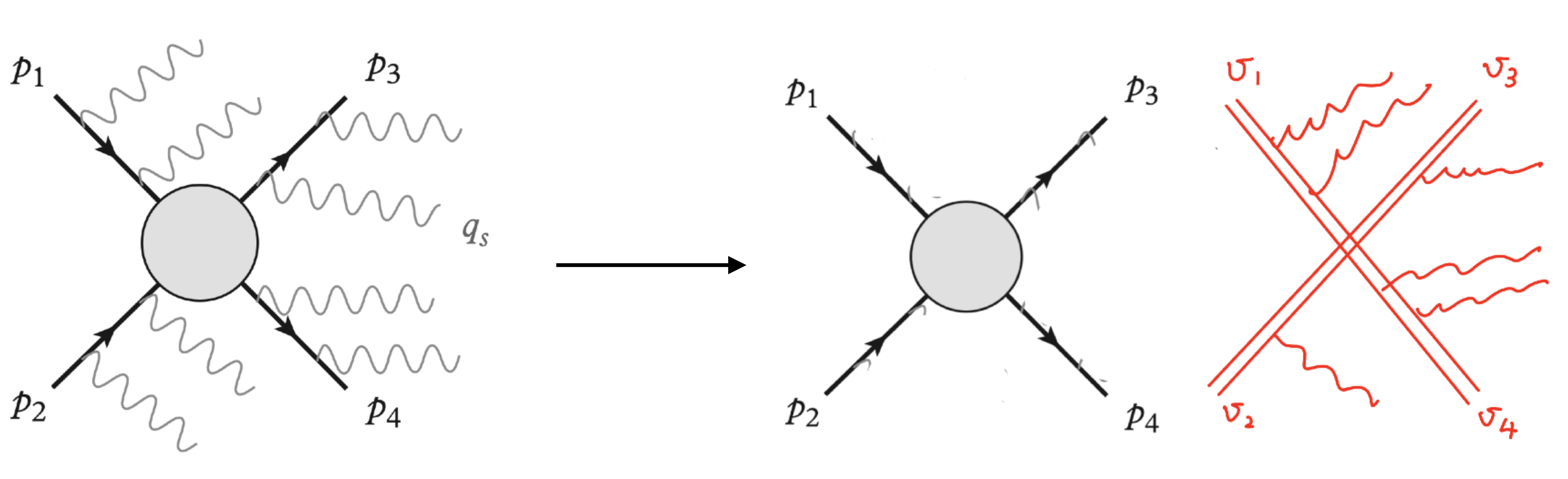
- Wilson line fulfills the equation $v_i \cdot D\, S_i(x) = 0$ **HW**

- The Wilson lines cancel in the fermion Lagrangian. We remove the interactions with the soft photons using the decoupling transformation

Effective Lagrangian

- We end up with Wilson lines along the directions of all particles in the scattering process.

$$\Delta\mathcal{L}_{\text{int}} = \sum_i C_i(v_1, v_2, v_3, v_4)\, \bar{h}_{v_3}^{(0)}\, \bar{S}_3^\dagger\, \Gamma_i\, S_1\, h_{v_1}^{(0)}\;\; \bar{h}_{v_4}^{(0)}\, \bar{S}_4^\dagger\, \Gamma_i\, S_2\, h_{v_2}^{(0)}$$



QED Factorization

- Since the photons no longer interact with the fermions after the decoupling, the relevant matrix element factorizes into a fermionic part times a photonic matrix element.

$$\begin{aligned}\mathcal{M} &= \sum_i C_i\,\bar{u}(v_3)\,\Gamma_i\,u(v_1)\,\bar{u}(v_4)\,\Gamma_i\,u(v_2)\,\langle X_s(k)|\bar{S}_3^\dagger\,S_1\,\bar{S}_4^\dagger\,S_2|0\rangle \\ &= \mathcal{M}_{ee}\,\langle X_s(k)|\bar{S}_3^\dagger\,S_1\,\bar{S}_4^\dagger\,S_2|0\rangle\,,\end{aligned}$$

- The amplitude factorizes into an amplitude without soft photons times a matrix element of Wilson lines.

- Analogous statements hold for soft gluon emissions in QCD, except that the Wilson lines will be matrices in colour space and we have to keep track of the colour indices.

Factorization

- The cross section takes the form

$$\sigma = \mathcal{H}(m_e, \{\underline{v}\})\, \mathcal{S}(E_s, \{\underline{v}\}),$$

- Hard function

$$\mathcal{H}(m_e, \{\underline{v}\}) = \frac{1}{2E_1 2E_2 |\vec{v}_1 - \vec{v}_2|} \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} |\mathcal{M}_{ee}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4),$$

- Soft function

$$\mathcal{S}(E_s, \{\underline{v}\}) = \sum\!\!\!\!\!\!\int_{X_s} \left| \langle X_s | \bar{S}_3^\dagger S_1 \bar{S}_4^\dagger S_2 | 0 \rangle \right|^2 \theta(E_s - E_{X_s}).$$

- Note that both the hard and soft functions depend on $\{\underline{v}\} = \{v_1, \ldots, v_4\}$
- We expanded the small soft momentum out of the momentum conservation δ function

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4 - k) = \delta^{(4)}(p_1 + p_2 - p_3 - p_4) + \mathcal{O}(\lambda),$$

Renormalization

- While the inclusive cross section is finite, the hard and soft functions individually suffer from divergences.

- The soft function suffers from UV divergences, which can be regularized using dimensional regularization. These UV divergences can be absorbed into the Wilson coefficients of the effective theory, which are encoded in the hard function.

- After renormalization the factorization theorem takes the form

$$\sigma = \mathcal{H}(m_e, \{\underline{v}\}, \mu)\, \mathcal{S}(E_s, \{\underline{v}\}, \mu)\,,$$

- The μ dependence of the functions fulfills an RG equation

- Since the cross section is finite, the hard and soft anomalous dimensions must be equal and opposite

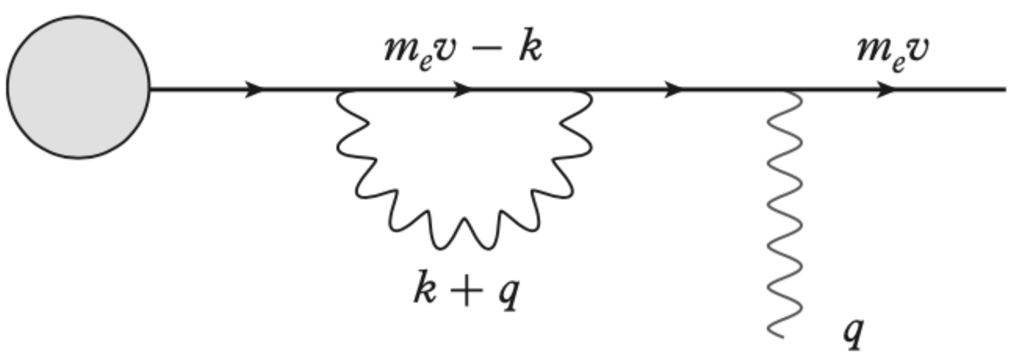
Method of Region

- When constructing EFT, we have expanded in the soft-photon momenta. This is fine for tree-level diagrams, but how about loops?

- The Taylor expansion does not commute with the loop integrations

- A general technique called the method of regions to expand loop integrals around various limits

- For simplicity we consider a scalar integral



$$F = \int d^d k \frac{1}{(k+q)^2} \frac{1}{(m_e v - k)^2 - m_e^2},$$

Method of Region

- In the low-energy theory, we assumed $k^\mu \sim q^\mu \ll m_e.$

- Expanding the integrand

$$F_{\text{low}} = \int d^d k \frac{1}{(k+q)^2} \frac{1}{-2m_e v \cdot k} \left\{ 1 + \frac{k^2}{2m_e v \cdot k} + \ldots \right\},$$

- The expansion produces exactly the linear propagators encountered in our tree-level discussion

- The expansion has produced ultraviolet divergences which are stronger than the one in the original integral

Method of Region

- To correct the problems from naively expanding the integrand, we consider

$$F_{\text{high}} \equiv F - F_{\text{low}}$$
$$= \int d^d k \frac{1}{(k+q)^2} \left[\frac{1}{(m_e v - k)^2 - m_e^2} - \frac{1}{-2m_e v \cdot k} \left\{ 1 + \frac{k^2}{2mv \cdot k} + \ldots \right\} \right]$$

- the integrand has only support for $k^\mu \gg q^\mu$

- We can therefore expand the integrand around $q^\mu = 0$

$$F_{\text{high}} = \int d^d k \frac{1}{k^2} \left\{ 1 - \frac{q^2}{2q \cdot k} + \ldots \right\} \left[\frac{1}{(m_e v - k)^2 - m_e^2} - \frac{1}{-2m_e v \cdot k} \left\{ 1 + \frac{k^2}{2mv \cdot k} + \ldots \right\} \right]$$

- Dropping the scaleless integrals, we get

$$F_{\text{high}} = \int d^d k \frac{1}{k^2} \left\{ 1 - \frac{q^2}{2q \cdot k} + \ldots \right\} \frac{1}{(m_e v - k)^2 - m_e^2},$$

- So we obtain the full result by performing the expansion of the integrand in two regions

Method of Region

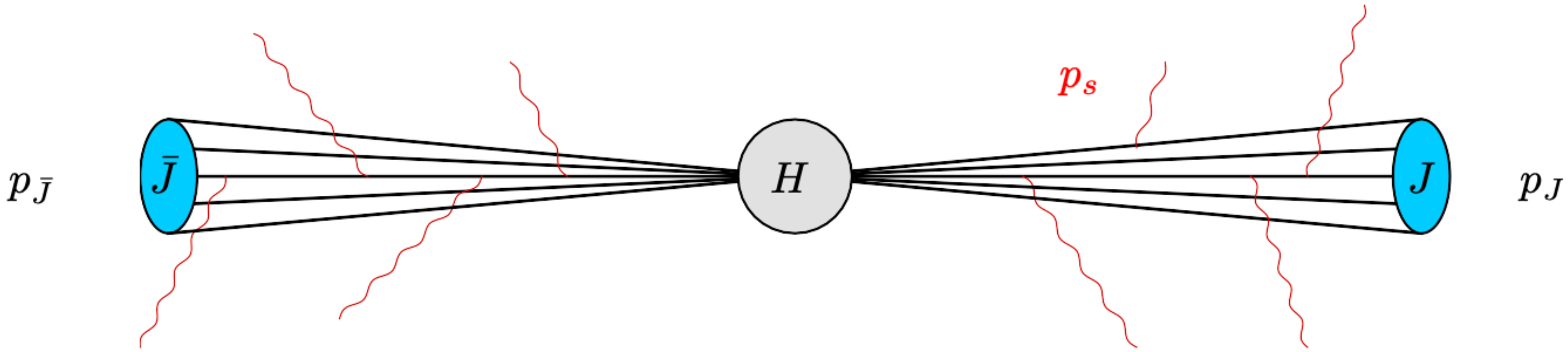
- We can summarize the method of regions expansion as follows:
  - Consider all relevant scalings (regions) of the loop momenta.
  - Expand the loop integral in each region.
  - Integrate each term over the full phase space
  - Add up the contributions.

- This technique provides a general method to expand loop integrals around different limits
- In some cases dim-reg alone is not sufficient. (e.g. TMD or small-x)
- The method of region technique has a close connection to EFTs in that the low-energy regions correspond to degrees of freedom in the EFT and the expanded full theory diagrams are equivalent to effective-theory diagrams.

3. Soft-collinear effective theory

Introduction

- Soft-Collinear Effective Theory (SCET) is the effective field theory for processes with energetic particles such as jet production at high-energy colliders.



- Typically such processes involve a scale hierarchy

$$Q^2 = (p_J + p_{\bar{J}})^2 \gg p_J^2 \sim p_{\bar{J}}^2 \gg p_s^2$$

- In SCET, the physics associated with the hard scale Q^2 is integrated out and absorbed into Wilson coefficients.

- SCET involves two different types of fields, collinear and soft fields to describe the physics associated with the two low-energy scales p_J^2 and p_s^2.

Introduction

- The result of a SCET analysis of a jet cross section is often a factorization theorem

$$\sigma = H \cdot J \otimes \bar{J} \otimes S \qquad Q^2 = (p_J + p_{\bar{J}})^2 \gg p_J^2 \sim p_{\bar{J}}^2 \gg p_s^2$$

  - Hard function; Jet function; Soft function

- The theorem is obtained after expanding in the ratios of the scales and holds at leading power.

- Each of the functions is only sensitive to a single scale.

- The individual functions furthermore fulfill renormalization group (RG) equations. By solving RG equations one can resum the large perturbative logarithms $\alpha^n \ln^m(Q^2/p_J^2)$

Introduction

- In certain cases, the soft or collinear scales can be so low that a perturbative expansion becomes unreliable

- Factorization theorems allow one to separate perturbative from non-perturbative physics, (e.g. PDFs). It is crucial to be able to make predictions.

- Traditionally, factorization theorems were derived purely diagrammatically

- An advantage of SCET is that effective theory provides an operator formulation of the low-energy physics, which simplifies and systematizes the analysis. This is especially important for complicated problems.

- Via the RG equations, SCET also provides a natural framework to perform resummations.

Introduction

- Compared to traditional effective field theories such as Fermi theory, SCET involves several complications.

- We cannot simply integrate out particles: quarks and gluons are still present in the low-energy theory. Instead, one splits the fields into modes

$$\begin{array}{ccccccccc} & & H & & J & & \bar{J} & & S \\ \phi & = & \phi_h & + & \phi_c & + & \phi_{\bar{c}} & + & \phi_s \end{array}$$

- An important and nontrivial element of the analysis is to identify the relevant momentum modes for the problem at hand, which are the degrees of freedom of the effective theory. This is done by analyzing full theory diagrams and provides the starting point of the effective theory construction.

Introduction

- A second complication is that the different momentum components of the fields scale differently.

  - E.g. Momentum components transverse to the jet direction are always small, but the components along the jet directions are large.

- To perform a derivative expansion of the effective Lagrangian, one therefore needs to split the momenta into different components.

- Introducing reference vectors

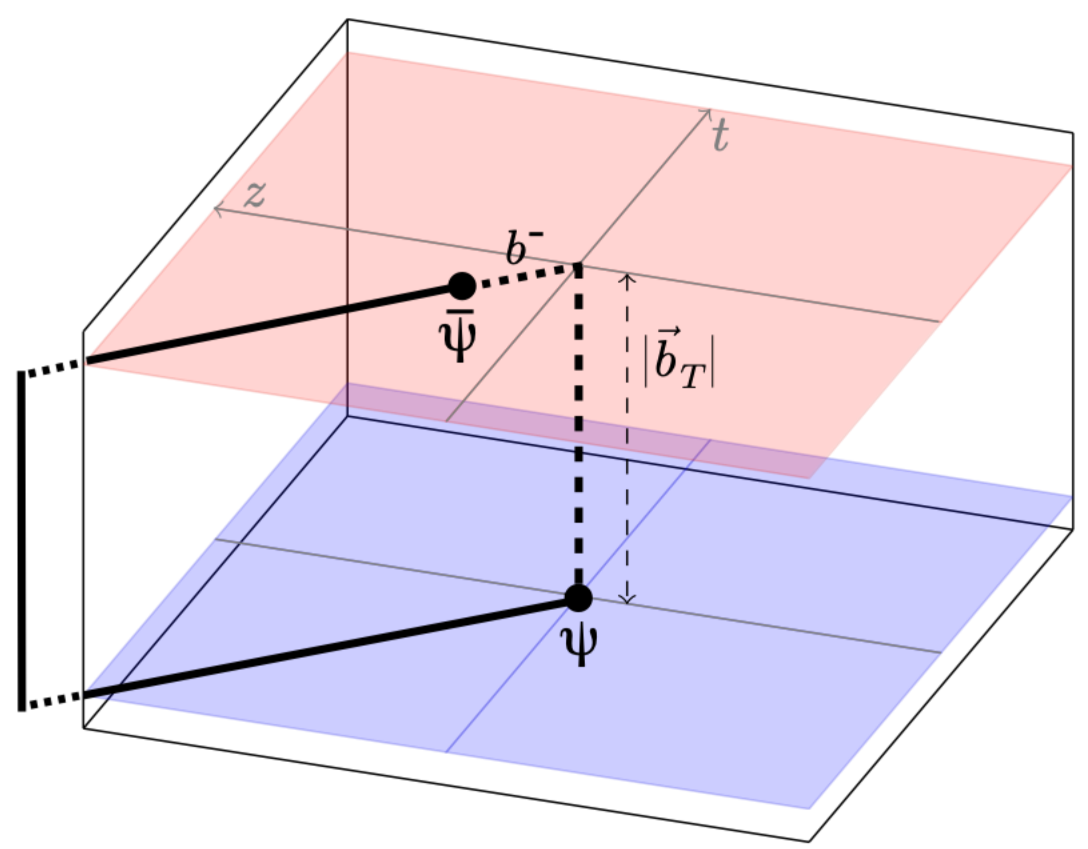
$$n^\mu \propto p_J^\mu \qquad\qquad \bar{n}^\mu \propto p_{\bar{J}}^\mu$$

Introduction

- The fact that the momentum components of the collinear particles along the jet are unsuppressed leads to a final complication, namely that one can write down operators with an arbitrary number of such derivatives.

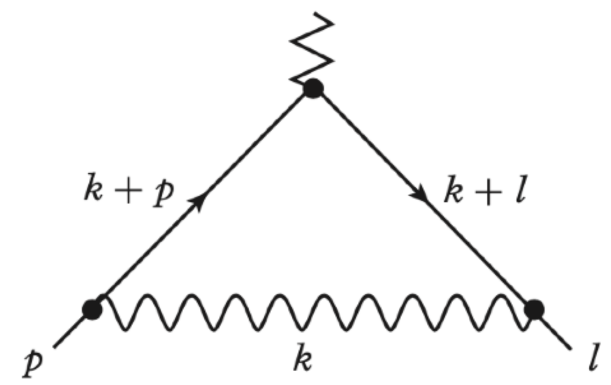
- One way to take all these operators into account is to make operators nonlocal along the corresponding light-cone directions

TMD PDF



The Sudakov Problem

- The one-loop contribution to the Sudakov form factor



- define $L^2 = -l^2 - i0$, $P^2 = -p^2 - i0$ and $Q^2 = -(l-p)^2 - i0$ and will analyse the form factor in the limit

$$L^2 \sim P^2 \ll Q^2 .$$

- This is the limit of large momentum transfer and small invariant mass, the same kinematics which is relevant for the jet process

The Sudakov Problem

- We want to find out which momentum modes are relevant in the Sudakov problem

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\left[(k+p)^2+i0\right]\left[(k+l)^2+i0\right]}.$$

- Introduce light-like reference vectors along p_μ and l_μ, in analogy to the vectors v_μ we introduced in our discussion of soft photons.

$$\begin{aligned} n^\mu &= (1,0,0,1) \approx p^\mu/p^0, \\ \bar{n}^\mu &= (1,0,0,-1) \approx l^\mu/l^0, \end{aligned} \qquad \text{with } n^2 = \bar{n}^2 = 0 \text{ and } n\cdot\bar{n} = 2.$$

- Any four vector can be decomposed in the form

$$\begin{aligned} p^\mu &= n\cdot p\frac{\bar{n}^\mu}{2} + \bar{n}\cdot p\frac{n^\mu}{2} + p_\perp^\mu \\ &= \quad p_+^\mu \quad + \quad p_-^\mu \quad + p_\perp^\mu. \end{aligned}$$

The Sudakov Problem

- Define a small expansion parameter $\lambda^2 \sim P^2/Q^2 \sim L^2/Q^2 \ll 1$.
- The following scalings yield non-zero contributions

		$(n\cdot k, \bar{n}\cdot k, k_\perp^\mu)$
hard	(h)	$(1,\ 1,\ 1)\,Q$,
collinear to p^μ	(c)	$(\lambda^2,\ 1,\ \lambda)\,Q$,
collinear to l^μ	$(\bar{c})$	$(1,\ \lambda^2,\ \lambda)\,Q$,
soft	(s)	$(\lambda^2,\ \lambda^2,\ \lambda^2)\,Q$.

- For some other observables, the soft mode scales as (λ,λ,λ). The version of SCET for this situation is called SCET2 to distinguish it from the one relevant for the Sudakov form factor which is also called SCET1.
- SCET2 involves so-called rapidity logarithms, which is related to the TMD physics
- Let us now expand the integrand in the different regions to leading power.

E.g. Collinear region

- In the collinear region the integration momentum scales as $k^\mu \sim (\lambda^2, 1, \lambda) Q$ and $k^2 \sim \lambda^2 Q^2$, we have

$$(k+l)^2 = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2)\,,$$

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^d k \frac{1}{\left(k^2+i0\right)\left(2k_- \cdot l_+ + i0\right)\left[(k+p)^2+i0\right]}\,.$$

- Using the Schwinger parametrization,

$$\frac{1}{A^n B^m} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\int_0^\infty d\eta \frac{\eta^{m-1}}{(A+\eta B)^{n+m}}\,,$$

- We have

$$I_c = -\frac{\Gamma(1+\varepsilon)}{2l_+ \cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^\varepsilon$$

All regions

- Having obtained the contributions from all the different momentum regions, we can now add them up and verify whether we reproduce the full integral.

- This involves some non-trivial cancellations since the individual integrals are all divergent, while the full integral is finite in d=4

$$I_h = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right)$$

$$I_c = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \frac{\pi^2}{6}\right)$$

$$I_{\bar{c}} = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6}\right)$$

$$I_s = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2 Q^2}{L^2 P^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2 P^2} + \frac{\pi^2}{6}\right)$$

$$I_{\text{tot}} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3}\right).$$

SCET in the ϕ^3 theory

- We now construct a Lagrangian whose Feynman rules directly yield the expanded diagrams obtained using method of regions expansion

- Start from a toy model

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{g}{3!}\phi^3(x)$$

- Split the scalar field in the sum of a field collinear to the momentum *p*, a field collinear to the momentum *l*, and a soft field

$$\phi(x) \to \phi_c(x) + \phi_{\bar{c}}(x) + \phi_s(x)\,.$$

- It was not necessary to introduce in the sum above a field for the hard region, since these contributions are absorbed into Wilson coefficients

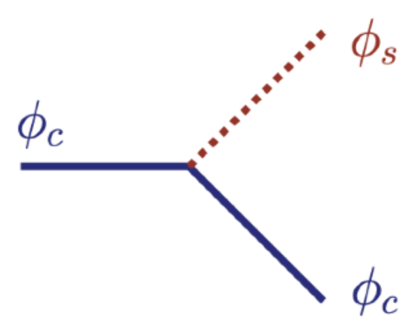
Effective Lagrangian

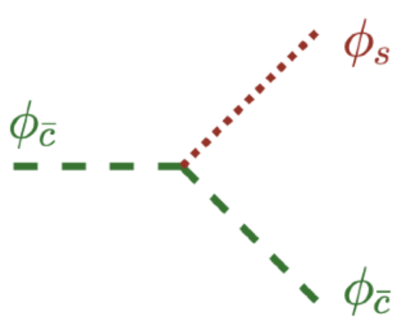
- the original Lagrangian can be written as the sum of four terms

$$\mathcal{L}(\phi) = \underbrace{\mathcal{L}(\phi_c)}_{\equiv \mathcal{L}_c} + \underbrace{\mathcal{L}(\phi_{\bar{c}})}_{\equiv \mathcal{L}_{\bar{c}}} + \underbrace{\mathcal{L}(\phi_s)}_{\equiv \mathcal{L}_s} + \mathcal{L}_{c+s}(\phi_c, \phi_{\bar{c}}, \phi_s) \ .$$

- The interaction of collinear and soft fields

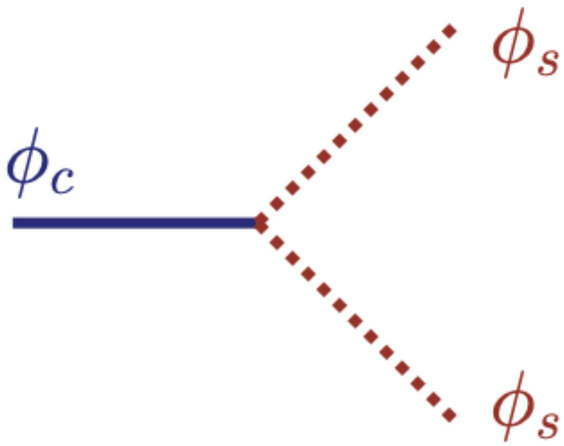
$$\mathcal{L}_{c+s}(\phi_c, \phi_{\bar{c}}, \phi_s) = -\frac{g}{2}{\phi_c}^2 \phi_s - \frac{g}{2}{\phi_{\bar{c}}}^2 \phi_s \ ,$$





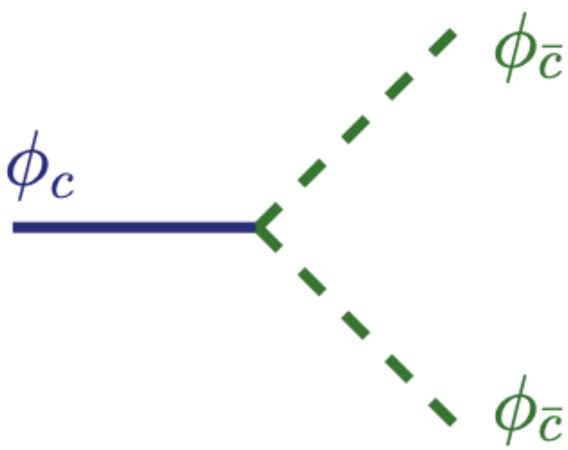
- It looks like there should be many additional interaction terms, but they are forbidden by momentum conservation

Interaction forbidden by momentum conservation



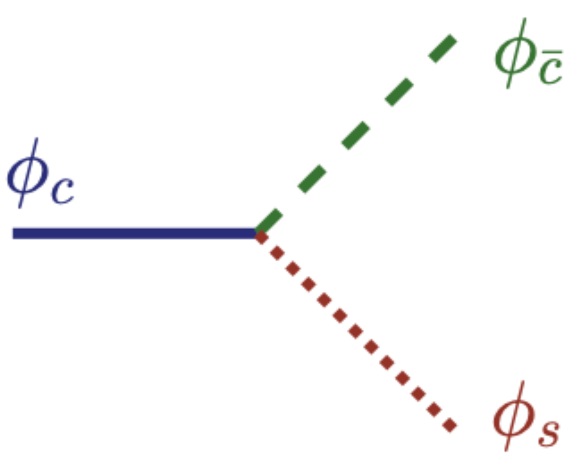
$$\phi_c(x)\phi_s^2(x)$$

An energetic particle cannot decay into two soft particles



$$\phi_c(x)\phi_{\bar{c}}^2(x)$$

A particle moving along the $+z$ direction cannot decay into two particle moving along the $-z$ direction



$$\phi_c(x)\phi_{\bar{c}}(x)\phi_s(x)$$

The "+ component" of the c field is of order λ^2, it cannot give rise to a field with a "+ component" of order 1, such as $\bar{c}$

Multipole expansion

- Consider the Fourier transform of the fields in a given interaction term

$$\int d^dx \phi_c^2(x)\phi_s(x) = \int d^dx \int \frac{d^dp_1}{(2\pi)^d} \int \frac{d^dp_2}{(2\pi)^d} \int \frac{d^dp_s}{(2\pi)^d} \tilde{\phi}_c(p_1)\tilde{\phi}_c(p_2)\tilde{\phi}_s(p_s) e^{-i(p_1+p_2+p_s)\cdot x},$$

- the sum of the three momenta scales as $p_1^\mu + p_2^\mu + p_s^\mu \sim (\lambda^2, 1, \lambda)\, Q$.

- Consequently the components of x must scale as $x^\mu \sim \left(1, \frac{1}{\lambda^2}, \frac{1}{\lambda}\right) \frac{1}{Q}$.

- the Taylor expansion of the soft field around the point x_-

$$\phi_s(x) = \phi_s(x_-) + \underbrace{x_\perp \cdot \partial_\perp}_{\mathcal{O}(\lambda)} \phi_s(x_-) + \underbrace{x_+ \cdot \partial_-}_{\mathcal{O}(\lambda^2)} \phi_s(x_-) + \frac{1}{2}\Big(\underbrace{x_{\mu\perp}x_{\nu\perp}\partial^\mu\partial^\nu}_{\mathcal{O}(\lambda^2)} \phi_s(x_-)\Big) + \dots$$

- The leading power scalar SCET Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} = {} & \frac{1}{2}\partial_\mu\phi_c(x)\partial^\mu\phi_c(x) - \frac{g}{3!}\phi_c^3(x) + \frac{1}{2}\partial_\mu\phi_{\bar{c}}(x)\partial^\mu\phi_{\bar{c}}(x) - \frac{g}{3!}\phi_{\bar{c}}^3(x) \\ & + \frac{1}{2}\partial_\mu\phi_s(x)\partial^\mu\phi_s(x) - \frac{g}{3!}\phi_s^3(x) - \frac{g}{2}\phi_c^2(x)\phi_s(x_-) - \frac{g}{2}\phi_{\bar{c}}^2(x)\phi_s(x_+)\,.\end{aligned}$$

Matching

- only the operators which involve collinear fields in different directions get matching corrections.

- in order to describe the Sudakov form factor, we introduce an external current coupling to two scalar fields

$$J = \phi^2 =$$  ,

- The most general form that the current operator can have in the effective theory is

$$J = J_2 + J_3 + \cdots = C_2 \phi_c \phi_{\bar{c}} + \frac{C_3}{2!} \left(\phi_c^2 \phi_{\bar{c}} + \phi_c \phi_{\bar{c}}^2 \right) + \cdots ,$$

- In addition to operators with multiple fields, one should also consider operators involving derivatives on the fields

Matching

- Even at leading power in λ, one needs to allow for the insertion of an arbitrary number of these derivatives in the current operators in the effective theory.

$$n \cdot \partial \phi_c(x) \sim \lambda^2 \phi_c(x)\,, \quad \partial_\perp^\mu \phi_c(x) \sim \lambda \phi_c(x)\,, \quad \bar{n} \cdot \partial \phi_c(x) \sim \lambda^0 \phi_c(x)\,;$$

- The expansion of a collinear field along the direction associated with the large momentum component can be written in terms of an infinite sum over the non-power suppressed derivatives

$$\phi_c(x + t\bar{n}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} (\bar{n} \cdot \partial)^i \phi_c(x)\,.$$

- to include terms with arbitrarily high derivatives is equivalent to allowing non-locality of the collinear fields along the collinear directions.

$$J_2(x) = \int ds dt\, C_2(s,t,\mu)\, \phi_c\,(x + s\bar{n})\, \phi_{\bar{c}}\,(x + tn)\ ,$$

Matching

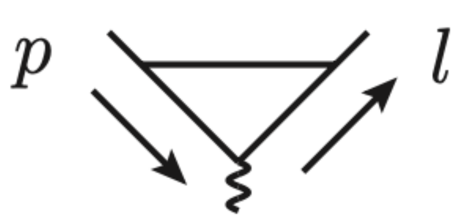
- the Fourier transform of the coefficient $C_2(s,t)$ will be

$$\tilde{C}_2\,(\bar{n}\cdot p\,, n\cdot l, \mu) = \int dsdt\, e^{is\bar{n}\cdot p} e^{-itn\cdot l} C_2(s,t,\mu)\,.$$

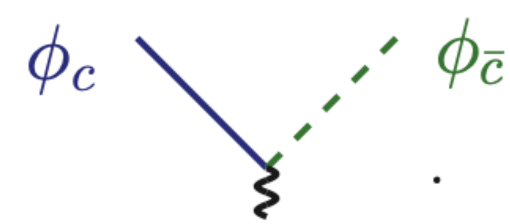
- The function C_2 must be expanded in powers of the coupling constant g as follows

$$\tilde{C}_2 = \tilde{C}_2^{(0)} + g^2\tilde{C}_2^{(1)} + g^4\tilde{C}_2^{(2)} + \cdots\,.$$

- the matching equation at the order of g^2



$$= \quad g^2\tilde{C}_2^{(1)}(\underbrace{\bar{n}\cdot p\, n\cdot l}_{=Q^2})$$



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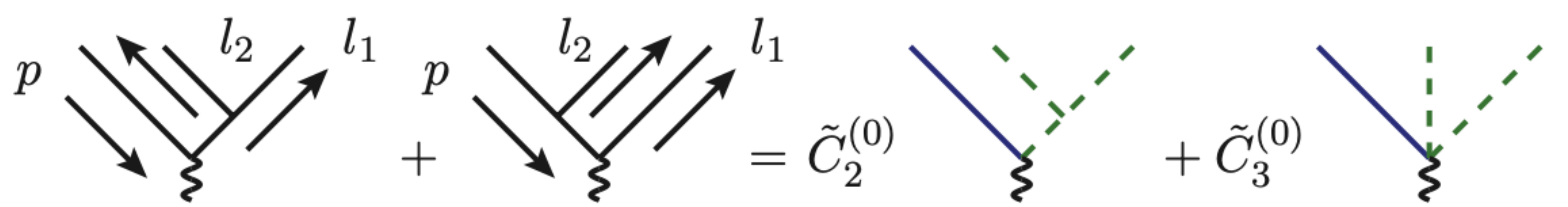
Matching

- match the Feynman diagrams involving a current operator J_3



- the first diagram on the l.h.s. and the first diagram on the r.h.s. give identical contributions

- the second diagrams give us

$$\tilde{C}_3^{(0)}\left(n\cdot l_1, n\cdot l_2, \bar{n}\cdot p, \mu\right) = \frac{g}{-\left(n\cdot l_2\right)\left(\bar{n}\cdot p\right) + i0}\,. \qquad (p-l_2)^2 = -2p\cdot l_2 + \mathcal{O}\left(\lambda^2\right) = -\left(n\cdot l_2\right)\left(\bar{n}\cdot p\right) + \mathcal{O}\left(\lambda^2\right)$$

- The inverse derivative of a field can be written as an integral

$$\frac{i}{in\cdot\partial + i0^+}\,\phi(x) = \int_{-\infty}^{0} ds\,\phi(x+sn)\,;$$

证明：$$\frac{i}{i\partial_x + i0^+}\phi(x) = \int_{-\infty}^{0} ds\, \phi(x+s)$$

Pr: $$\phi(x) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx}\, \tilde{\phi}(k)$$

Then $$\text{L.H.S.} = \frac{i}{i\partial_x + i0^+}\phi(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{i}{-k+i0^+} e^{ikx}\, \tilde{\phi}(k)$$

$$\begin{aligned}
\text{R.H.S.} &= \int_{-\infty}^{0} ds\, \phi(x+s) \\
&= \int_{-\infty}^{0} ds \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}(k)\, e^{ik\cdot(x+s)} \\
&= \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}(k)\, e^{ikx} \underbrace{\int_{-\infty}^{+\infty} ds\, \theta(-s)\, e^{iks}}
\end{aligned}$$

$$= \int_{-\infty}^{+\infty} ds\, \theta(s)\, e^{-iks}$$

$$= \sqrt{2\pi}\, \mathcal{F}[\theta(s)](k)$$: $\theta(s)$ 傅里叶变换

词卜2:

$$\theta(x) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \frac{e^{-i\omega x}}{\omega + i0^+}$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{-i\omega x} \left[\frac{i}{\sqrt{2\pi}}\, \frac{1}{\omega + i0^+} \right]$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{i\omega x} \left[\frac{i}{\sqrt{2\pi}}\, \frac{1}{-\omega + i0^+} \right]$$

$$= \mathcal{F}^{-1}\left[\frac{i}{\sqrt{2\pi}}\, \frac{1}{-\omega + i0^+} \right](x)$$

所以:

$$\mathcal{F}[\,\theta(s)\,](k) = \frac{1}{\sqrt{2\pi}}\, \frac{i}{-k + i0^+}$$

$$\text{R.H.S.} = \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}(k)\, e^{ikx}\, \frac{i}{-k + i0^+}$$

$$= \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}(k)\, \frac{i}{-k + i0^+}\, e^{ikx} = \text{L.H.S.}$$

Matching

- It is a characteristic feature of SCET that the operators are non-local along the directions of large light-cone momentum.

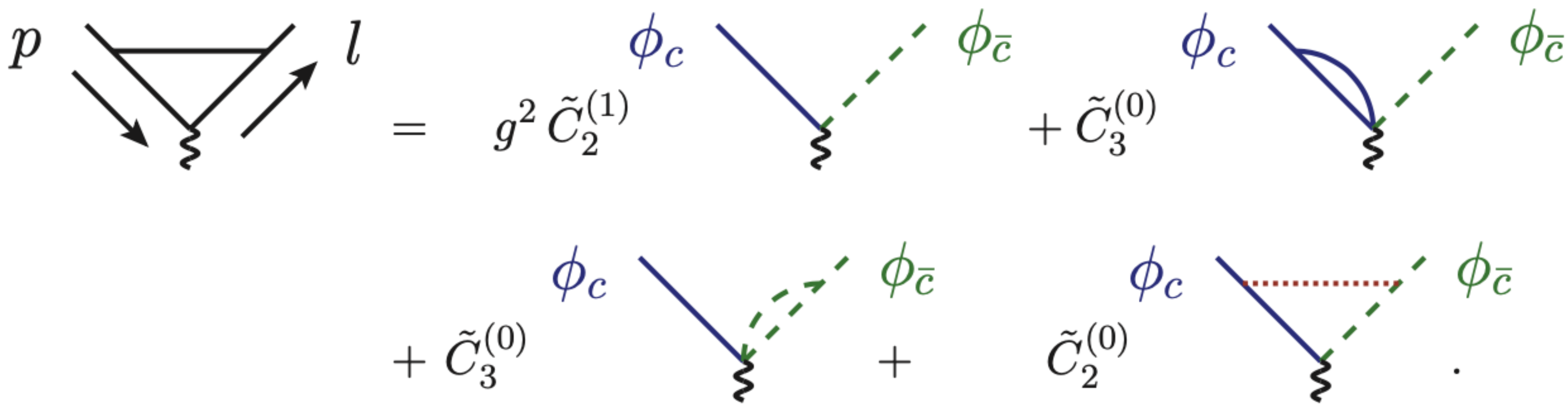
- In general, in order to write down the most general SCET operators, one smears the fields along the light cone.

$$J_2(x) = \int ds dt\, C_2(s,t,\mu)\, \phi_c\,(x+s\bar{n})\, \phi_{\bar{c}}\,(x+tn)\ ,$$

$$J_3(x) = \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, C_3(s,t_1,t_2,\mu)\phi_c(x+s\bar{n})\phi_{\bar{c}}(x+t_1 n)\phi_{\bar{c}}(x+t_2 n) + (c \leftrightarrow \bar{c})$$

Sudakov Form Factor in the ϕ^3 theory

- All of the elements needed for the calculation of the one-loop correction to the current operator in the ϕ^3 theory in the limit in which $\lambda \to 0$ are available



- The squares of the external momenta p and l are small but not exactly equal to zero

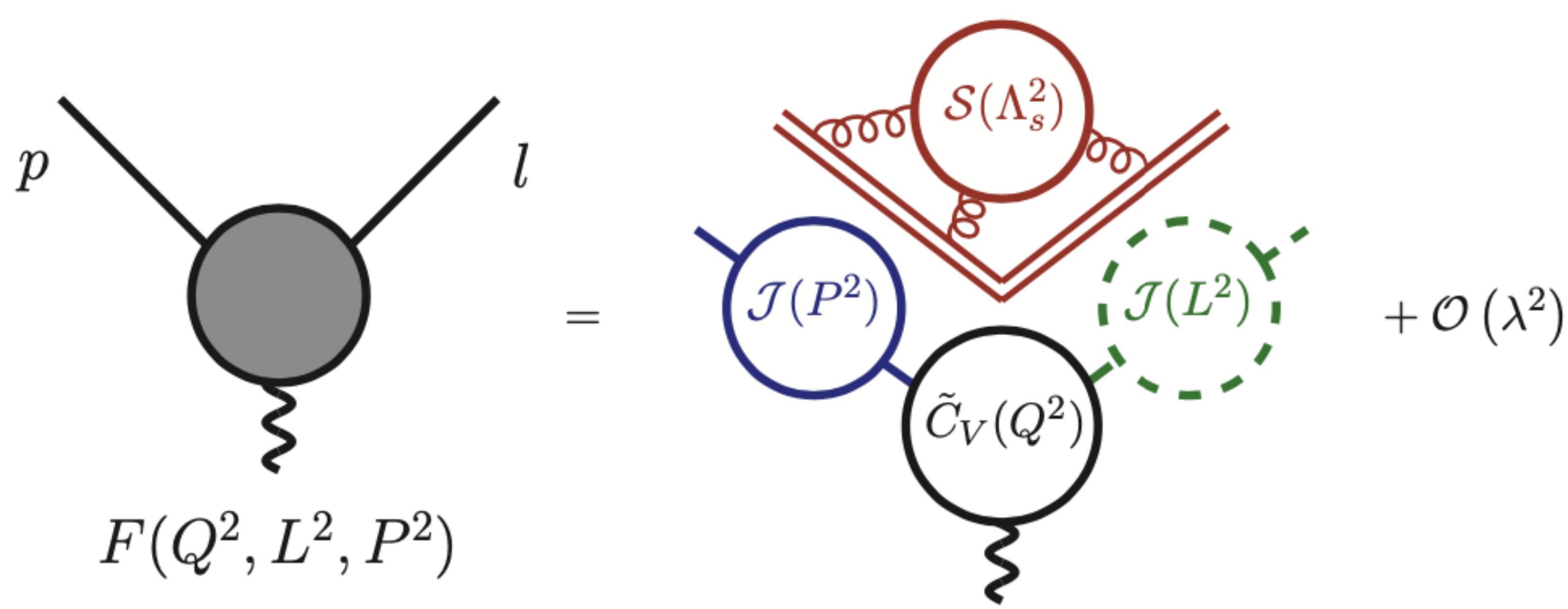
- For order-by-order calculations, the direct application of the strategy of regions is more efficient. However, SCET allows one to study all-order properties of scattering amplitudes, such as factorization theorems.

QCD Sudakov form factor within SCET

- In QCD the most general leading-power SCET current operator.

$$J^\mu(x) = \int ds \int dt\, C_V(s,t) \bar{\chi}_c^{(0)}\left(x_+ + x_\perp + s\bar{n}\right) S_n^\dagger(0)\, S_{\bar{n}}(0)\, \gamma_\perp^\mu \chi_{\bar{c}}^{(0)}(x_- + x_\perp + tn)\,.$$

- The soft interactions do not cancel, and the Sudakov form factor receives low-energy contributions which describe a long-range interaction between the fast moving ingoing and outgoing quarks.



Resummation by RG Evolution

- The one-loop correction to the Wilson coefficient

$$\tilde{C}_V^{\text{bare}}(\varepsilon, Q^2) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{\pi^2}{6} - 8 + \mathcal{O}(\varepsilon) \right) \left(\frac{Q^2}{\mu^2} \right)^{-\varepsilon} + \mathcal{O}\left(\alpha_s^2\right)$$

- We have added a label bare to the Wilson coefficient to indicate that we still need to renormalize it, which is done by absorbing the divergences into a multiplicative Z-factor

$$\tilde{C}_V(Q^2, \mu) = \lim_{\varepsilon \to 0} Z^{-1}\left(\varepsilon, Q^2, \mu\right) \tilde{C}_V^{\text{bare}}(\varepsilon, Q^2).$$

- Doing so, leaves us with the renormalized Wilson coefficient

$$\tilde{C}_V(Q^2, \mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} + \frac{\pi^2}{6} - 8 \right) + \mathcal{O}(\alpha_s^2).$$

- The whole procedure is the same as renormalization in standard quantum field theory, up to the fact that we had to deal with $1/\varepsilon^2$ divergences, which arise because we have both soft and collinear divergences.

Resummation by RG Evolution

- Due to the presence of the double logarithms, the anomalous dimension governing the RG equation for the Wilson coefficient has a logarithmic piece

$$\frac{d}{d\ln\mu}\tilde{C}_V(Q^2,\mu) = \left[C_F\gamma_{\text{cusp}}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]\tilde{C}_V(Q^2,\mu),$$

- The complete form factor can then be written as

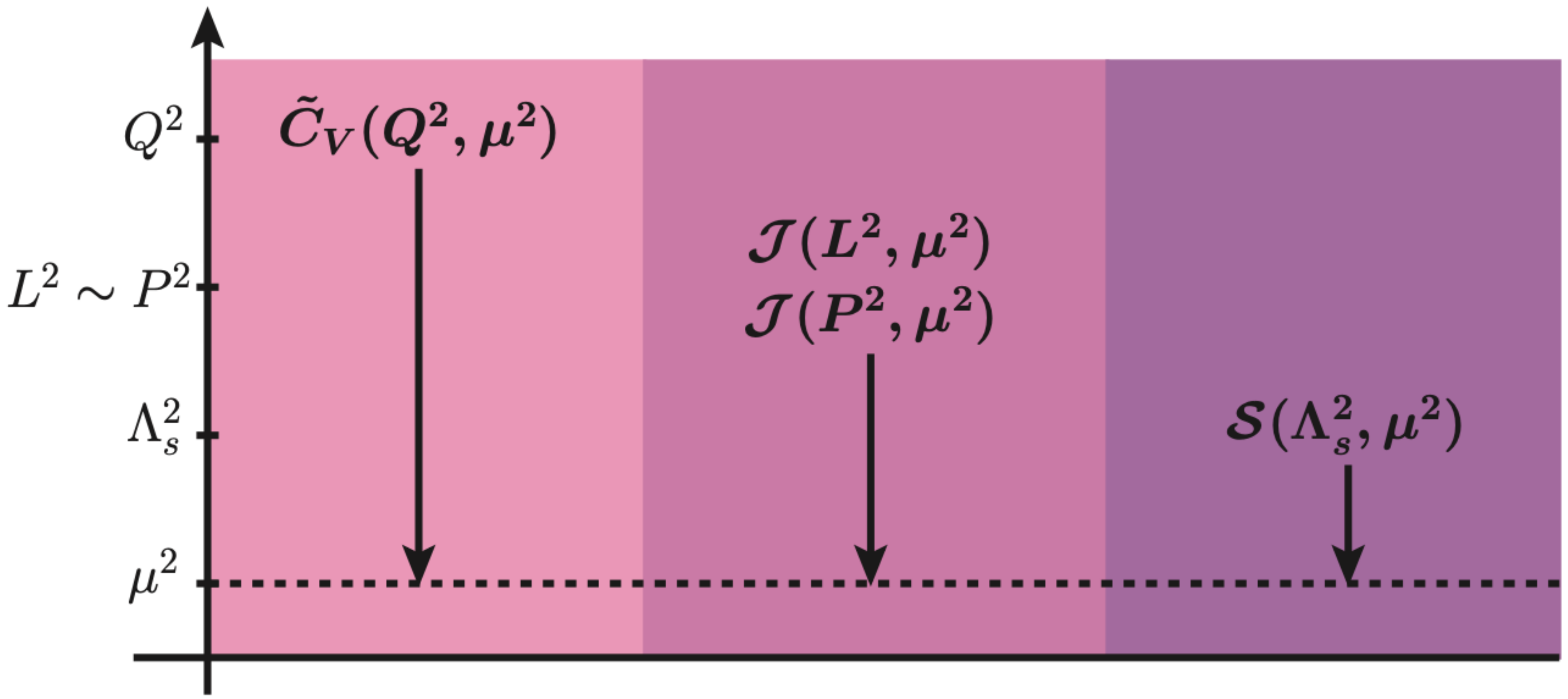
$$F\left(Q^2,L^2,P^2\right) = \tilde{C}_V\left(Q^2,\mu^2\right)\mathcal{J}\left(L^2,\mu^2\right)\mathcal{J}\left(P^2,\mu^2\right)\mathcal{S}\left(\Lambda_s^2,\mu^2\right)$$

- The factorization formula puts constraints on the anomalous dimensions governing the RG equation of the various factors

$$\frac{d}{d\ln\mu}\left[\tilde{C}_V\left(Q^2,\mu^2\right)\mathcal{J}\left(L^2,\mu^2\right)\mathcal{J}\left(P^2,\mu^2\right)\mathcal{S}\left(\Lambda_s^2,\mu^2\right)\right] = 0\,.$$

Resummation by RG

- Schematic representation of the scale separation and of the calculational procedure in renormalization group improved perturbation theory.

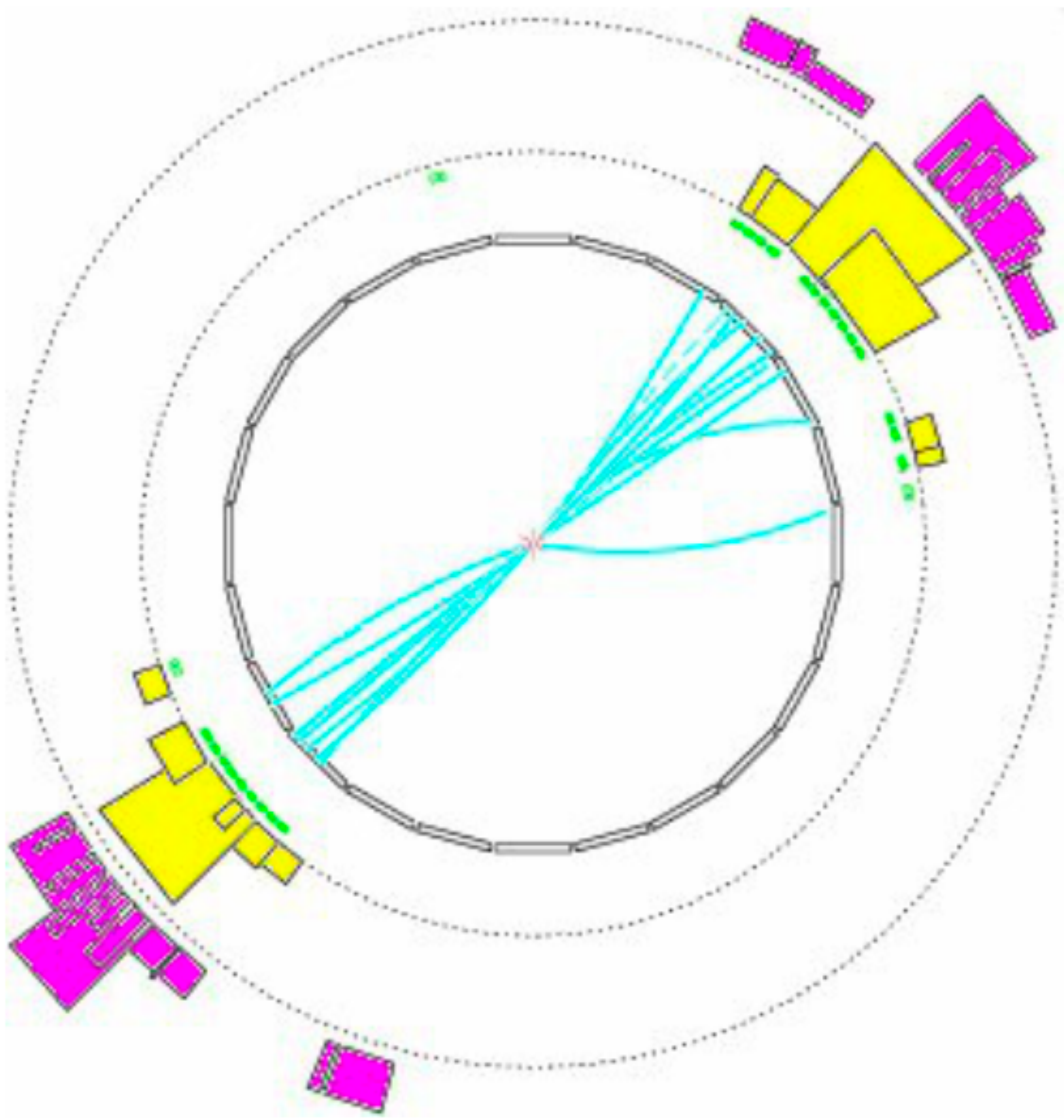


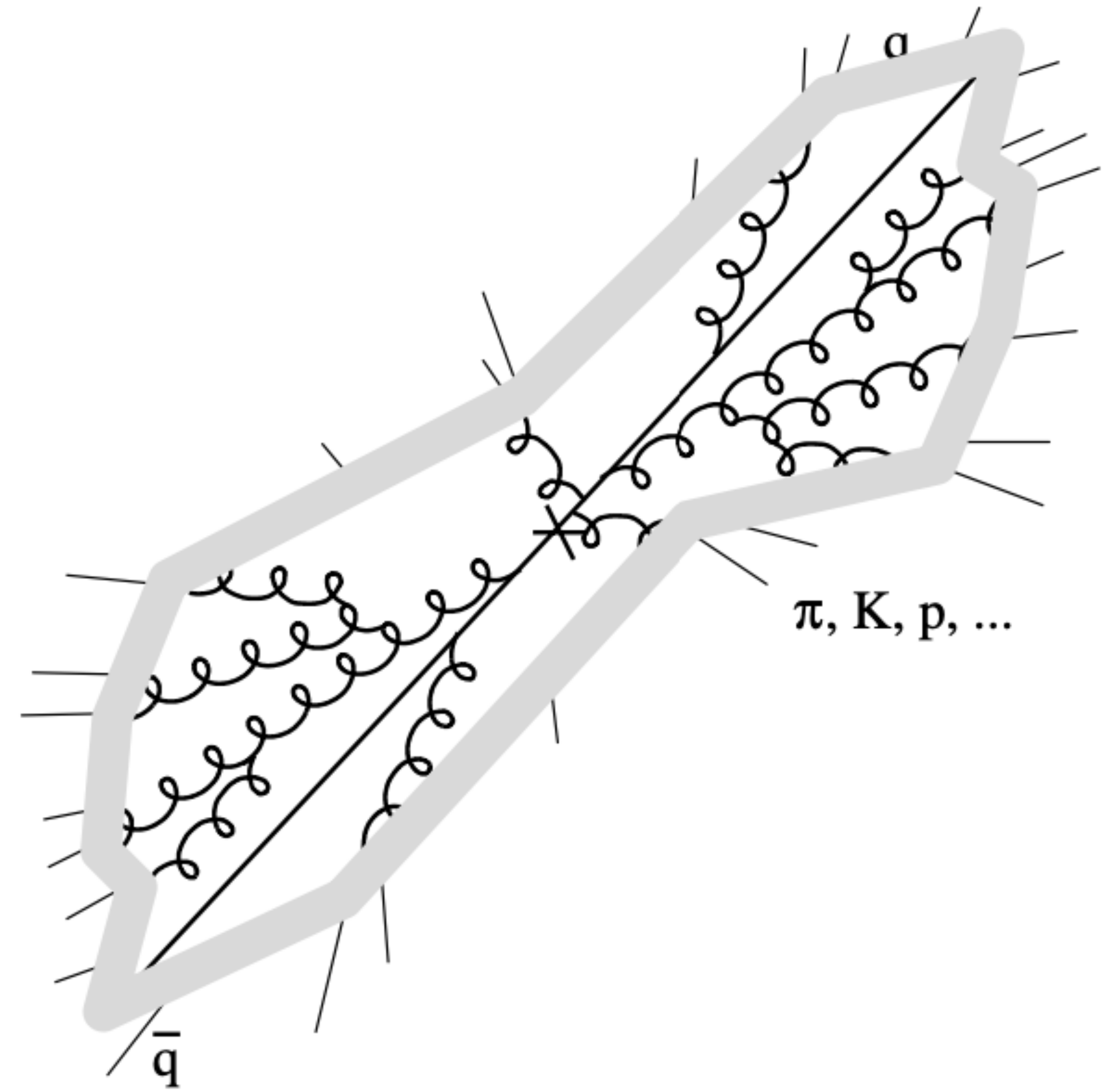
$$\frac{d}{d\ln\mu}\tilde{C}_V(Q^2,\mu) = \left[C_F\gamma_{\text{cusp}}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]\tilde{C}_V(Q^2,\mu)$$

$$\frac{d}{d\ln\mu}\mathcal{J}\left(L^2,\mu^2\right) = -\left[C_F\gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{L^2}{\mu^2} + \gamma_J\left(\alpha_s\right)\right]\mathcal{J}\left(L^2,\mu^2\right)\ ,$$

$$\frac{d}{d\ln\mu}\mathcal{S}\left(\Lambda_s^2,\mu^2\right) = \left[C_F\gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{\Lambda_s^2}{\mu^2} + \gamma_S\left(\alpha_s\right)\right]\mathcal{S}\left(\Lambda_s^2,\mu^2\right)\ ;$$

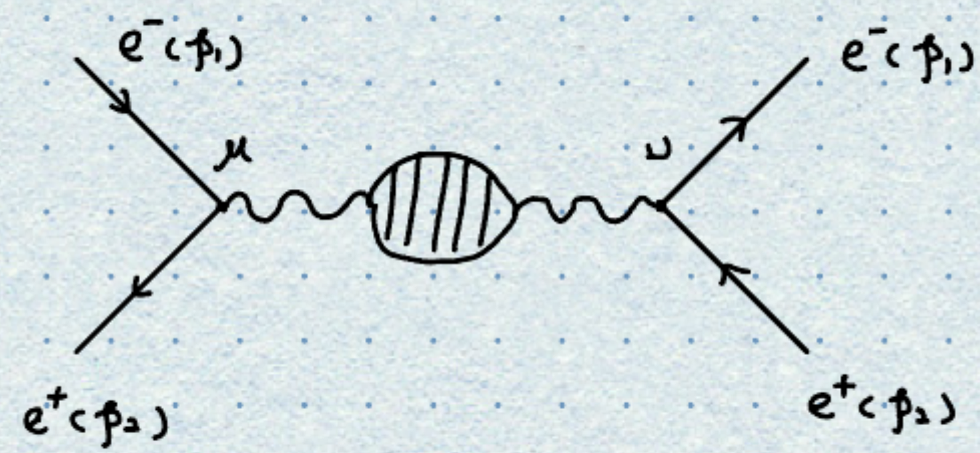
4. Applications in event shapes




q
π, K, p, ...
q̄

Operator analysis of e^+e^-

The diagrams for $e^+e^- \to e^+e^-$ whose imaginary part gives $e^+e^- \to$ hadrons



The amplitude is

$$i\mathcal{M} = (-ie)^2\, \bar{v}(p_2)\gamma^\mu u(p_1)\, \bar{u}(p_1)\gamma^\nu v(p_2)\, \frac{-i}{s}\, i\Pi^h_{\mu\nu}(q)\, \frac{-i}{s}$$

$\Pi^h_{\mu\nu}$ is the hadronic part of the self energy and has the form

$$\Pi_h^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi_h(q^2)$$

$$\hookrightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\, \mathrm{Im}\, \Pi_h(s)$$

As a check, we should now evaluate the photon self energy:

one finds $$\Pi(s) = -\frac{\alpha}{3} N_c \sum_q e_q^2 \left[-\frac{1}{\pi} \ln(-s-i\varepsilon) \right]$$

$$\mathrm{Im}\, \Pi(s+i\varepsilon) = -\frac{\alpha}{3} N_c \sum_q e_q^2$$

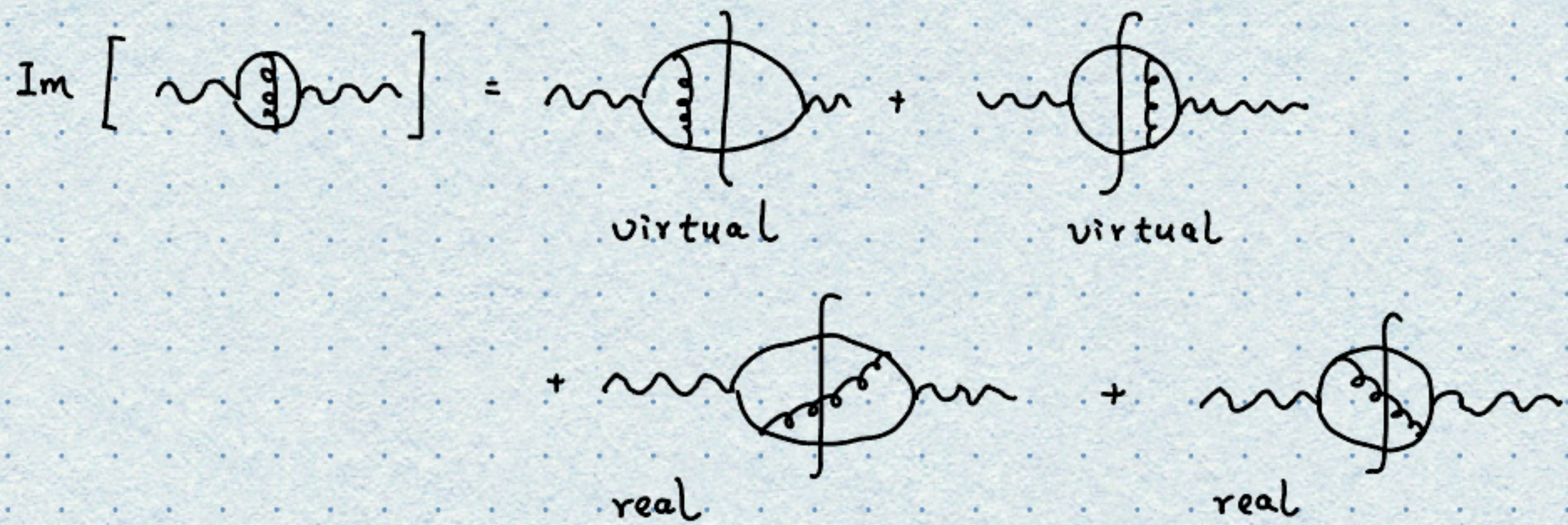
$$\Rightarrow \quad \sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} N_c \sum_q e_q^2$$

So we now get the cross section from a loop calculation <u>alone</u>.

At $O(\alpha_s)$

 +  + 

The IR div. cancellation between real and virtual is manifest:



"The imaginary part arises when particles in the loop go on the mass shell."

The electromagnetic coupling of quarks has the form

$$J^{\mu} = \sum_{q} e_q \, \bar{q}_f \gamma^{\mu} q_f$$

So the self energy:

$$i \Pi_h^{\mu\nu}(q) = (ie)^2 \int d^4x \, e^{iq\cdot x} \langle \Omega | T\{ J^{\mu}(x) J^{\nu}(0) \} | \Omega \rangle$$

In the limit $x \to 0$, we can expand the product of currents:

$$J_\mu(x) J_\nu(0) = C^{1}_{\mu\nu}(x)\, \mathbb{1} + C^{q\bar{q}}_{\mu\nu}(x)\, m_q \bar{q} q(0) + C^{G^2}_{\mu\nu}(x) \left[G^a_{\alpha\beta}(0) \right]^2$$

with $C^{1}_{\mu\nu} \sim (x^2)^{-3}$, $C^{q\bar{q}}_{\mu\nu} \sim (x^2)^{-1}$, $C^{G^2}_{\mu\nu} \sim (x^2)^{-1}$

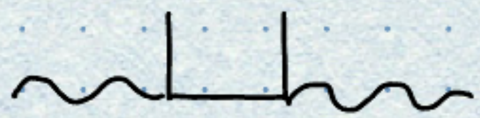
If the Fourier transform is dominated by $x^2 \approx 0$, we get

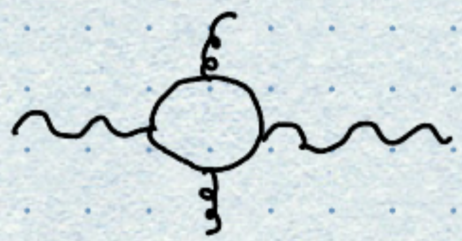
$$i \Pi^{\mu\nu}_h(q) = -i e^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) \left[C^1(q^2) + C^{q\bar{q}}(q^2)\, m \bar{q} q + C^{G^2}(q^2) (G^a_{\mu\nu})^2 \right]$$

$$C^1 \sim (q^2)^0 \quad , \quad \underbrace{C^{q\bar{q}} \sim 1/q^2 \, , \quad C^{G^2} \sim 1/q^2}_{\text{suppressed at high energy}}$$

Since the OPE is an operator relation, we can take an arbitrary matrix element to obtain the Wilson coefficients. In particular, we can work with quark and gluon states.

 $\longrightarrow \quad C'(q^2)$

 $\longrightarrow \quad C^{q\bar{q}}(q^2)$

 $\longrightarrow \quad C^{G^2}(q^2)$

Then, with the coefficients determined from perturbative calculations, we take the physical vaccum matrix element.

$$\hookrightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s}\,[\;\mathrm{Im}\, C'(q^2)$$

$$+ \;\mathrm{Im}\, C^{q\bar{q}}(q^2)\, \underbrace{\langle \Omega | m\bar{q}q | \Omega\rangle}_{\sim (0.3\,\mathrm{MeV})^4}$$

quark condensate

$$+ \;\mathrm{Im}\, C^{G^2}(q^2)\, \langle \Omega | (G^a_{\mu\nu})^2 | \Omega \rangle$$

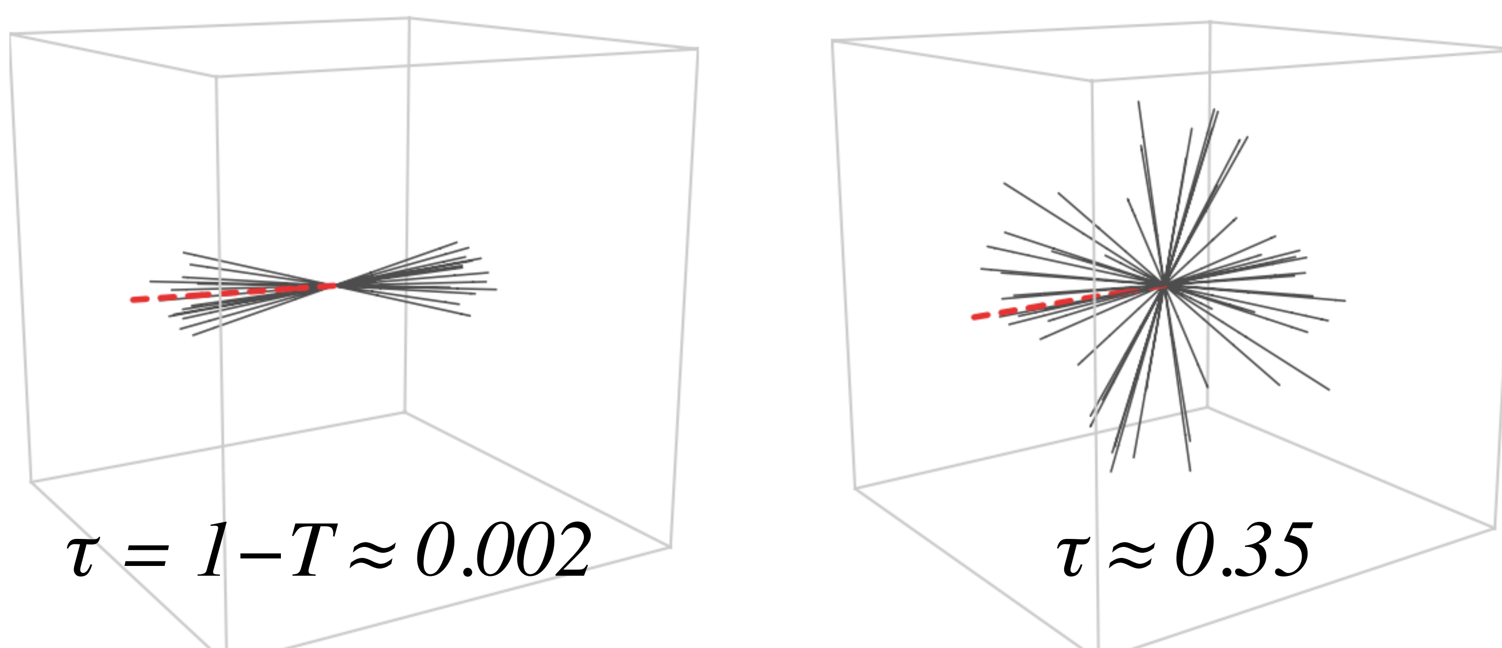
$\sim$ Gluon condensate $(0.5\,\mathrm{GeV})^4$

Factorization for the event-shape variable thrust

- The definition of thrust

$$T = \frac{1}{Q} \max_{\vec{n}_T} \sum_i |\vec{n}_T \cdot \vec{p}_i|$$

- A sum over all particles in the event and one sums the projections of their momenta along the thrust axis n_T which must be chosen to maximize the sum.
- The thrust T ($\tau = 1 - T$) thus measures the fraction of momentum flowing along the thrust axis.



$\tau = 1 - T \approx 0.002$ $\tau \approx 0.35$

IRC safe

- Thrust is soft and collinear safe, i.e. its value does not change under exactly collinear splittings or infinitely soft emissions.

- This property makes it possible to compute it perturbatively.

$$e^+e^- \to \bar{q}qg \qquad\qquad \frac{1}{\sigma_0}\frac{d\sigma}{ds\,dt} = C_F\frac{\alpha_s}{2\pi}\frac{s^2+t^2+2u}{st}$$

$$\begin{aligned}
\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} &= \frac{1}{\sigma_0}\int ds\,dt\frac{d\sigma}{ds\,dt}\Big[\delta(\tau-s)\,\theta(t-s)\,\theta(u-s) \\
&\qquad + \delta(\tau-t)\,\theta(s-t)\,\theta(u-t) + \delta(\tau-u)\,\theta(t-u)\,\theta(s-u)\Big] \\
&= \frac{2}{\sigma_0}\int_\tau^{1-2\tau} ds\int dt\frac{d\sigma}{ds\,dt}\delta(\tau-t) + \int_\tau^{1-2\tau} ds\int dt\frac{d\sigma}{ds\,dt}\delta(\tau-u) \\
&= C_F\frac{\alpha_s}{2\pi}\left\{\frac{3(1+\tau)(3\tau-1)}{\tau} + \frac{[4+6\tau(\tau-1)]\ln\frac{1-2\tau}{\tau}}{\tau(1-\tau)}\right\}.
\end{aligned}$$

- However, for small $\tau \ll 1$ we encounter large logarithms.

IRC limit

- Choose the SCET reference vectors

$$n^\mu \;=\; (1, \vec{n}_T) \qquad\qquad \bar{n}^\mu \;=\; (1, -\vec{n}_T).$$

- We separate the sum over particles into individual sums in the soft and collinear sectors

$$\begin{aligned}
\tau Q &= \sum_i |\vec{p}_i| - |\vec{n}_T \cdot \vec{p}_i| \\
&= \sum_i n \cdot p_{ci} + \sum_i \bar{n} \cdot p_{\bar{c}i} + \sum_i n \cdot p_{si}^R + \sum_i \bar{n} \cdot p_{si}^L \\
&= n \cdot p_{X_c} + n \cdot p_{X_s}^R + \bar{n} \cdot p_{X_{\bar{c}}} + \bar{n} \cdot p_{X_s}^L \,,
\end{aligned}$$

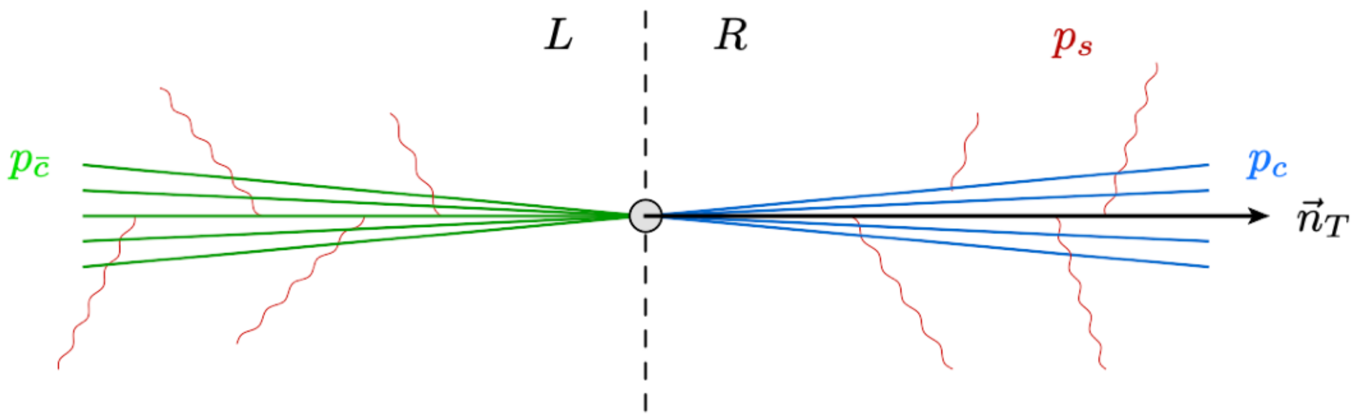
- We split the soft particles into left- and right-moving ones in order to be able write the sums in terms of light-cone components.

- In the last line, we have introduced the total momentum in each category.

IRC limit

- Due to the definition of the thrust axis, the total transverse momentum is zero in each hemisphere

- Up to power corrections, we therefore write the invariant mass of all particles in the right hemisphere as



$$
\begin{aligned}
M_R^2 &= (p_{X_c} + p_{X_s}^R)^2 \\
&= p_{X_c}^2 + \bar{n}\cdot p_{X_c} n\cdot p_{X_s}^R \\
&= \bar{n}\cdot p_{X_c} n\cdot p_{X_c} + \bar{n}\cdot p_{X_c} n\cdot p_{X_s}^R \\
&= Q(n\cdot p_{X_c} + p_{X_s}^R)\,.
\end{aligned}
$$

- Up to power corrections, we obtain

$$
\tau Q^2 = M_L^2 + M_R^2 = p_{X_c}^2 + p_{X_{\bar{c}}}^2 + Q\left(n\cdot p_{X_s}^R + \bar{n}\cdot p_{X_s}^L\right)
$$

Factorization

- The cross section

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2} \sum\!\!\!\!\!\!\int_X \left|\mathcal{M}(e^+e^- \to \gamma^* \to X)\right|^2 (2\pi)^4 \delta^{(4)}(q - p_X)\,\delta(\tau - \tau(X))$$

- hadronic tensor

$$H_{\mu\nu}(q,\tau) = \sum\!\!\!\!\!\!\int_X \langle 0|\, J^\dagger_\nu(0)\, |X\rangle\langle X|\, J_\mu(0)\, |0\rangle (2\pi)^4 \delta^{(4)}(q - p_X)\,\delta(\tau - \tau(X))$$

- leptonic tensor

$$L^{\mu\nu}(q_1, q_2) = \frac{e^4 Q_q^2}{Q^4}\, \bar{v}(q_2)\,\gamma^\mu\, u(q_1)\, \bar{u}(q_1)\,\gamma^\nu\, v(q_2)$$

- Averaging over the spins of the incoming leptons

$$L^{\mu\nu}(q_1, q_2) = \frac{e^4 Q_q^2}{Q^4}\, \left(q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - q_1 \cdot q_2\, g^{\mu\nu}\right)$$

Factorization

- Introduce the dummy integration

$$1 = \int d^3\vec{n}\, \delta^{(3)}(\vec{n} - \vec{n}_T)$$

- Using the fact that momentum conservation fixes $|\vec{p}_{X_c}| = Q/2$

$$\int d^3\vec{n}\, \delta^{(3)}(\vec{n} - \vec{n}_T) = (2\pi) \int d\cos\theta \left(\frac{Q}{2}\right)^2 \delta^{(2)}(p^{\perp}_{X_c})$$

- The cross section involves a δ-function which fixes the total transverse momentum of the collinear radiation to be zero
- Combining it with the momentum conservation δ-functions and expanding away small momentum components

$$\begin{aligned}
&\delta^{(4)}(q - p_{X_c} - p_{X_{\bar{c}}} - p_{X_s})\, \delta^{(2)}(p^{\perp}_{X_c}) \\
&\qquad\qquad = 2\, \delta(\bar{n} \cdot p_{X_c} - Q)\, \delta(n \cdot p_{X_{\bar{c}}} - Q) \delta^{(2)}(p^{\perp}_{X_c}) \delta^{(2)}(p^{\perp}_{X_{\bar{c}}})
\end{aligned}$$

- The factor of 2 is the Jacobian for converting to light-cone components.

Factorization

- To separate the individual contributions to thrust, we introduce three more integrations

$$1 = \int dM_c^2\, \delta(M_c^2 - p_{X_c}^2) \int dM_{\bar{c}}^2\, \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2) \int d\omega\, \delta(\omega - n\cdot p_{X_s}^R - \bar{n}\cdot p_{X_s}^L)$$

- Plug in the factorized SCET current into the hadron tensor, we obtain the cross section in the factorized form

$$\begin{aligned}\frac{d\sigma}{d\tau d\cos\theta} =& \frac{\pi}{2} L_{\mu\nu} |\tilde{C}_V(-Q^2 - i0, \mu)|^2 \int dM_c^2 \int dM_{\bar{c}}^2 \int d\omega\, \delta(\tau - \frac{M_c^2 + M_{\bar{c}}^2 + Q\omega}{Q^2}) \\ &\times \sum\!\!\!\!\!\!\!\int_{X_c} \langle 0|\, \chi_{c,\delta}^a(0) |X_c\rangle\langle X_c|\, \bar{\chi}_{c,\alpha}^b\, |0\rangle\, \delta(M_c^2 - p_{X_c}^2)\, \delta^{(2)}(p_{X_c}^\perp)\delta(\bar{n}\cdot p_{X_c} - Q) \\ &\times \sum\!\!\!\!\!\!\!\int_{X_{\bar{c}}} \langle 0|\, \bar{\chi}_{\bar{c},\gamma}^d(0) |X_{\bar{c}}\rangle\langle X_{\bar{c}}|\, \chi_{\bar{c},\beta}^e\, |0\rangle\, \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2)\, \delta^{(2)}(p_{X_{\bar{c}}}^\perp)\delta(n\cdot p_{X_{\bar{c}}} - Q) \\ &\times \sum\!\!\!\!\!\!\!\int_{X_s} \langle 0| \left[S_n^\dagger S_{\bar{n}}\right]_{da} |X_s\rangle\langle X_s|\, \left[S_{\bar{n}}^\dagger S_n\right]_{be} |0\rangle\, \delta(\omega - n\cdot p_{X_s}^R - \bar{n}\cdot p_{X_s}^L) \\ &\times (2\pi)^4\, (\gamma_\perp^\mu)_{\alpha\beta}\, (\gamma_\perp^\nu)_{\gamma\delta}\,.\end{aligned}$$

Factorization

- Collinear matrix elements are color diagonal, proportional to $\delta_{ab}\delta_{de}$
- Soft function

$$S(\omega) = \frac{1}{N_c} \sum\!\!\!\!\!\!\int_{X_s} \langle 0| \left[S_n^\dagger S_{\bar{n}}\right]_{ab} |X_s\rangle\langle X_s| \left[S_{\bar{n}}^\dagger S_n\right]_{ba} |0\rangle\, \delta(\omega - n\cdot p_{X_s}^R - \bar{n}\cdot p_{X_s}^L)$$

- Jet function

$$\frac{\delta^{ab}}{2(2\pi)^3}\left[\frac{\not{n}}{2}\right]_{\delta\alpha} J(M^2) = \sum\!\!\!\!\!\!\int_{X_c} \langle 0|\, \chi^a_{c,\delta}(0)\,|X_c\rangle\,\langle X_c|\,\bar{\chi}^b_{c,\alpha}(0)\,|0\rangle$$
$$\times\, \delta(M^2 - p^2_{X_c})\, \delta^{(2)}(p^\perp_{X_c})\delta(\bar{n}\cdot p_{X_c} - Q)\,,$$
$$\frac{\delta^{de}}{2(2\pi)^3}\left[\frac{\not{\bar{n}}}{2}\right]_{\beta\gamma} J(M^2) = \sum\!\!\!\!\!\!\int_{X_{\bar{c}}} \langle 0|\, \bar{\chi}^d_{\bar{c},\gamma}(0)\,|X_{\bar{c}}\rangle\,\langle X_{\bar{c}}|\,\chi^e_{\bar{c},\beta}(0)\,|0\rangle$$
$$\times\, \delta(M^2 - p^2_{X_{\bar{c}}})\, \delta^{(2)}(p^\perp_{X_{\bar{c}}})\delta(n\cdot p_{X_{\bar{c}}} - Q)\,.$$

Factorization

- Factorization formula

$$\frac{d\sigma}{d\tau\, d\cos\theta} = \frac{\pi N_c Q_f^2 \alpha^2}{2Q^2}(1+\cos^2\theta)|\tilde{C}_V(-Q^2-i0,\mu)|^2 \int dM_c^2 \int dM_{\bar{c}}^2 \int d\omega$$
$$\delta\left(\tau - \frac{M_c^2 + M_{\bar{c}}^2 + Q\omega}{Q^2}\right) J(M_c^2,\mu)\, J(M_{\bar{c}}^2,\mu)\, S(\omega,\mu)\,,$$

- Hard, jet and soft functions depend on μ. To resum large logarithms, one can again solve the RG equations and evolve to a common reference scale
- The soft and jet functions are convolved; This complication can be avoided by working in Laplace space since the transformation turns the convolution into a product

$$h(\omega) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\, \delta(\omega-\omega_1-\omega_2) f(\omega_1) g(\omega_2)$$

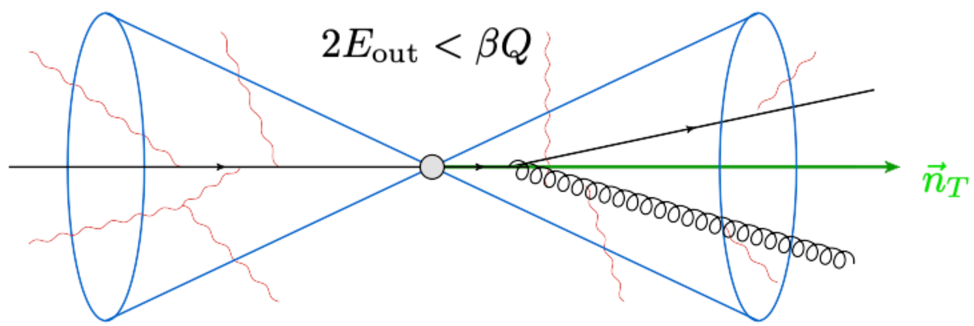
$$\tilde{h}(s) = \int_0^\infty d\omega e^{-\omega s} h(\omega) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\, e^{-\omega_1 s - \omega_2 s}\, f(\omega_1) g(\omega_2) = \tilde{f}(s)\, \tilde{g}(s)$$

5. Applications in the jet physics

An effective field theory for jet processes

Becher, Neubert, Rothen, DYS '15

EFT contains two modes:



$$\text{hard:}\quad p_h \sim \quad Q\,(1,1,1)$$
$$\text{soft:}\quad p_s \sim Q\beta\,(1,1,1)$$

NB:

1. no collinear singularity, only single logs

2. method of region to verify at two-loop level

Hard parton described by collinear field $\Phi_i \in \{\chi_i, \bar{\chi}_i, \mathcal{A}^{\mu}_{i\perp}\}$

gauge invariant: $\chi_i(0) = W_i^{\dagger}(\bar{n}_i)\,\frac{\not{n}_i\bar{\not{n}}_i}{4}\,\psi_i(0)$

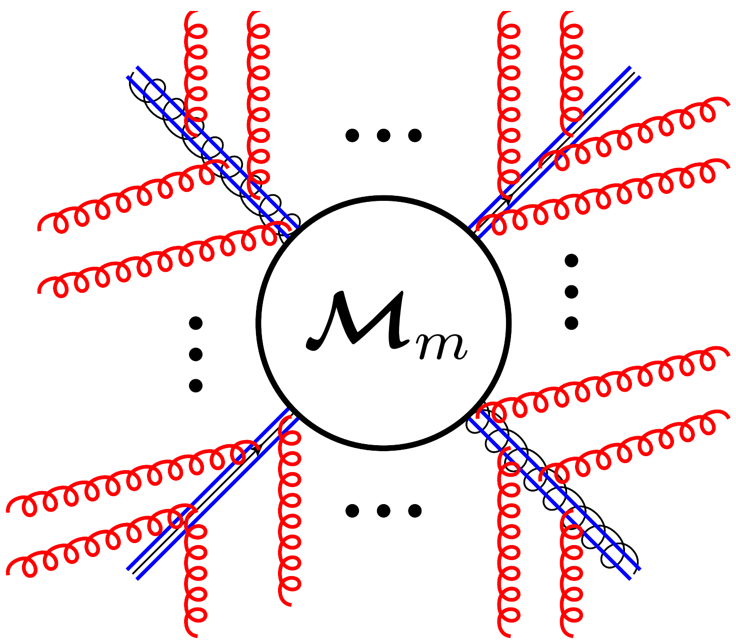
Perform decoupling transformation: $\Phi_i = \boldsymbol{S}_i(n_i)\,\Phi_i^{(0)}$ $\quad \boldsymbol{S}_i(n_i) = \mathbf{P}\exp\left(ig_s\int_0^{\infty} ds\, n_i\cdot A_s^a(sn_i)\,\boldsymbol{T}_i^a\right)$

Evaluates the matrix element of the operator with one collinear particle

$$\langle 0|\,\chi_j^{(0)}(0)\,|p_i\rangle = \delta_{ij}\,u(p_i)$$

Factorization

- **The operator for the emission from an amplitude with m hard partons**



hard scattering amplitude with m particles (vector in color space)

$$\underbrace{\boldsymbol{S}_1(n_1)\, \boldsymbol{S}_2(n_2)\, \ldots\, \boldsymbol{S}_m(n_m)}\, \underbrace{|\mathcal{M}_m(\{\underline{p}\})\rangle}$$

soft Wilson lines along the directions of the energetic particles (color matrices)

NB: No jet function, since no collinear scales

To get the cross section, we need to square & integrate over phase space

Factorization for gap between jets in e+e-

(Becher, Neubert, Rothen, DYS, ’15 PRL, ’16 JHEP; Caron-Huot ’15 JHEP)

Hard function
m hard partons along fixed directions $\{n_1, \ldots, n_m\}$
$$\boldsymbol{\mathcal{H}}_m \propto |\mathcal{M}_m\rangle\langle\mathcal{M}_m|$$

$$\mathrm{Tr}[\rho\,\mathcal{O}]$$

Soft function
squared amplitude with m Wilson lines

Color Trace **Hard scale** **Soft scale**

$$\sigma(Q, Q_\Omega) \sim \sum_{m=2}^{\infty} \underbrace{\prod_{i=1}^{m} \int \frac{d\Omega(\vec{n}_i)}{4\pi}} \mathrm{Tr}_c \left[\mathcal{H}_m(\{\underline{\vec{n}_1, \cdots, \vec{n}_m}\}, Q, \mu)\mathcal{S}_m(\{\underline{\vec{n}_1, \cdots, \vec{n}_m}\}, Q_\Omega, \mu)\right]$$

# of jet not fixed

Integrate the angles for hard partons

Renormalization

UV poles inside hard function removed by renormalizing the hard function as

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \delta, \epsilon) = \sum_{l=2}^{m} \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu)\, \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)$$

1. **obtain the bare hard function from on-shell matching. The IR poles are in one-to-one correspondence to UV div, since the EFT loop-integrals are scaleless.**
2. **We can understand the UV div. of hard function from the structure of the IR div. in the real and virtual diagrams**
3. **lower multiplicity virtual diagrams are needed to cancel the div. of real emission diagrams**

$$\boldsymbol{\mathcal{H}}_3^{(1)}(Q,\mu) = \boldsymbol{\mathcal{H}}_3^{(1)}(Q,\epsilon) - Z_{23}^{(1)}(Q,\epsilon,\mu)\boldsymbol{\mathcal{H}}_2^{(0)}(Q,\mu)$$

the renormalization matrix must have the form:

$$\boldsymbol{Z}^H(\{\underline{n}\}, Q, \delta, \epsilon, \mu) \sim \begin{pmatrix} 1 & \alpha_s & \alpha_s^2 & \alpha_s^3 & \ldots \\ 0 & 1 & \alpha_s & \alpha_s^2 & \ldots \\ 0 & 0 & 1 & \alpha_s & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

At each higher order in perturbation theory, more off-diagonal contributions fill in

By consistency, the matrix Z^H must render the soft functions finite

$$\boldsymbol{S}_l(\{\underline{n}\}, Q\beta, \delta, \mu) = \sum_{m=l}^{\infty} \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu) \,\hat{\otimes}\, \boldsymbol{S}_m(\{\underline{n}\}, Q\beta, \delta, \epsilon)$$

Higher multiplicity soft functions are needed to absorb the div. of matrix elements with fewer Wilson lines

$$S_2(\mu) = Z^H_{22}\, S_2(\varepsilon) + Z^H_{23} \,\hat{\otimes}\, S_3(\varepsilon) + Z^H_{24} \,\hat{\otimes}\, 1$$

Test at two-loop level !!!

Leading Log Resummation = master equation under Markovian approximation

$$\sigma^{\rm LL}(Q,Q_0) = \sum_{m=2}^{\infty} \left\langle \boldsymbol{\mathcal{H}}_2(\{n_1,n_2\},Q,\mu_h) \otimes \boldsymbol{U}_{2m}(\{\underline{n}\},\mu_s,\mu_h)\,\hat{\otimes}\,\mathbf{1} \right\rangle$$

One-loop anomalous dim. : $$\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\boldsymbol{U}(\{\underline{n}\},\mu_s,\mu_h) = \mathbf{P}\exp\left[\int_{\mu_s}^{\mu_h}\frac{d\mu}{\mu}\boldsymbol{\Gamma}^H(\{\underline{n}\},\mu)\right]$$

$$\boldsymbol{V}_m = 2\sum_{(ij)} (\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}) \int \frac{d\Omega(n_k)}{4\pi} W_{ij}^k,$$

$$\boldsymbol{R}_m = -4\sum_{(ij)} \boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,R} W_{ij}^{m+1}\,\Theta_{\rm in}(n_{m+1}),$$

dipole: $$W_{ij}^k = \frac{n_i\cdot n_j}{(n_i\cdot n_k)(n_j\cdot n_k)}$$

virtual: $$\boldsymbol{V}_m\mathcal{H}_m \sim \sum_a \boldsymbol{T}_i^a\cdot\boldsymbol{T}_j^a|\mathcal{M}_m\rangle\langle\mathcal{M}_m| + |\mathcal{M}_m\rangle\langle\mathcal{M}_m|\sum_a \boldsymbol{T}_i^a\cdot\boldsymbol{T}_j^a$$

real: $$\boldsymbol{R}_m\mathcal{H}_m \sim \boldsymbol{T}_i^a|\mathcal{M}_m\rangle\langle\mathcal{M}_m|\,\boldsymbol{T}_j^a$$

RG equation: $$\frac{d}{dt}\boldsymbol{\mathcal{H}}_m(t) = \boldsymbol{\mathcal{H}}_m(t)\boldsymbol{V}_m + \boldsymbol{\mathcal{H}}_{m-1}(t)\boldsymbol{R}_{m-1}$$ $$t = \int_{\alpha(\mu)}^{\alpha(Q)}\frac{d\alpha}{\beta(\alpha)}\frac{\alpha}{4\pi}$$

Lindblad equation: $$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_{k=1}^{M-1}\left(L_k\rho L_k^\dagger - \tfrac{1}{2}L_k^\dagger L_k\rho - \tfrac{1}{2}\rho L_k^\dagger L_k\right)$$

Super-leading logs = Lindblad eqn in color space

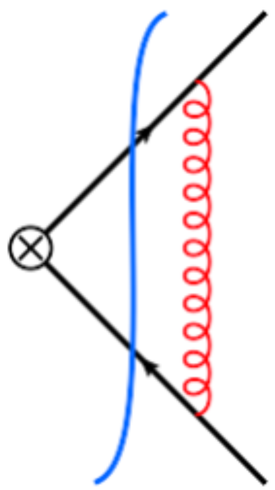
One-loop anomalous dimension:

$$\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$W_{ij}^k = \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_j \cdot n_k}$$

$$\boldsymbol{V}_m = -2 \sum_{(ij)} \int \frac{d\Omega(n_k)}{4\pi} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right) W_{ij}^k \left[\Theta_{\text{in}}^{n\bar{n}}(k) + \Theta_{\text{out}}^{n\bar{n}}(k)\right]$$
$$+ 2\, i\pi \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right) \Pi_{ij},$$
$$\boldsymbol{R}_m = 4 \sum_{(ij)} \boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,R} \, W_{ij}^{m+1} \, \Theta_{\text{in}}(n_{m+1}) \, .$$

$\Pi_{ij} = 1$ **if both incoming or outgoing**



Individually R_m and V_m contain singularities when emitted gluon k gets collinear to parsons i or j.

- **Expect cancellation in inclusive soft observables such as gaps between jets at lepton colliders**
- **Collinear factorization violation: Glauber phases spoil this cancellation : soft+collinear double logs! "Super-leading logs"**

Extracting the collinear singularities: $\overline{W}_{ij}^k = \dfrac{n_i \cdot n_j}{n_i \cdot n_k \, n_j \cdot n_k} - \dfrac{\delta(n_k - n_i)}{n_i \cdot n_k} - \dfrac{\delta(n_k - n_j)}{n_j \cdot n_k}$

The one-loop anomalous dimension is

$$\boldsymbol{V}_m = \overline{\boldsymbol{V}}_m + \boldsymbol{V}^G + \sum_{i=1,2} \boldsymbol{V}_i^c \, \ln \frac{\mu^2}{\hat{s}}$$

$$\boldsymbol{R}_m = \overline{\boldsymbol{R}}_m + \sum_{i=1,2} \boldsymbol{R}_i^c \, \ln \frac{\mu^2}{\hat{s}} \, ,$$

with

$$\overline{\boldsymbol{V}}_m = 2 \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int \frac{d\Omega\,(n_k)}{4\pi} \bar{W}_{ij}^k$$

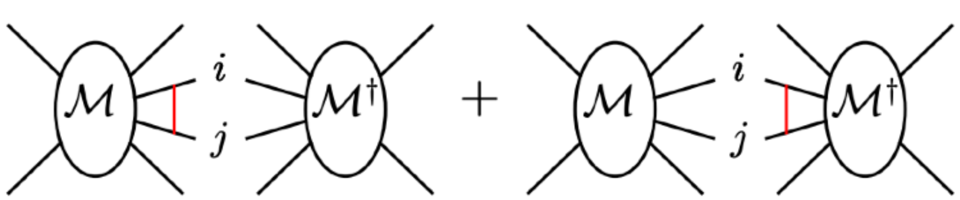
$$\boldsymbol{V}_i^c = 4 C_i \mathbf{1}$$

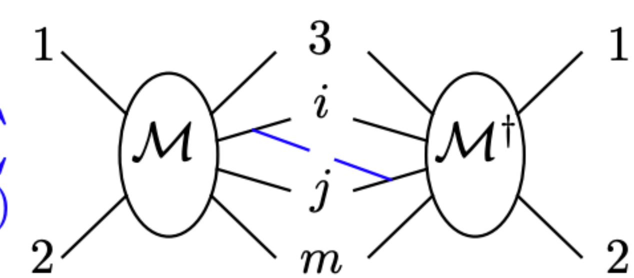
$$\boldsymbol{V}^G = -8 i \pi \left(\boldsymbol{T}_{1,L} \cdot \boldsymbol{T}_{2,L} - \boldsymbol{T}_{1,R} \cdot \boldsymbol{T}_{2,R} \right)$$

$$\overline{\boldsymbol{R}}_m = -4 \sum_{(ij)} \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{j,R} \bar{W}_{ij}^{m+1} \Theta_{\text{hard}}\left(n_{m+1}\right)$$

$$\boldsymbol{R}_i^c = -4 \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{i,R} \delta\left(n_k - n_i\right)$$

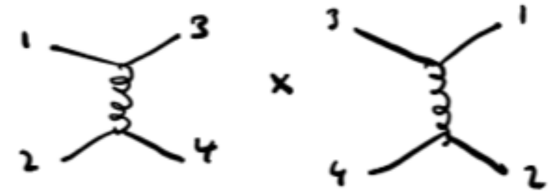
$$\boxed{\boldsymbol{\mathcal{H}}_m \, \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{j,R} = \boldsymbol{T}_i^a \, \boldsymbol{\mathcal{H}}_m \, \boldsymbol{T}_j^{\tilde{a}}}$$

$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{V}}_m = \sum_{(ij)}$ 

$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{R}}_m = \sum_{(ij)}$ 

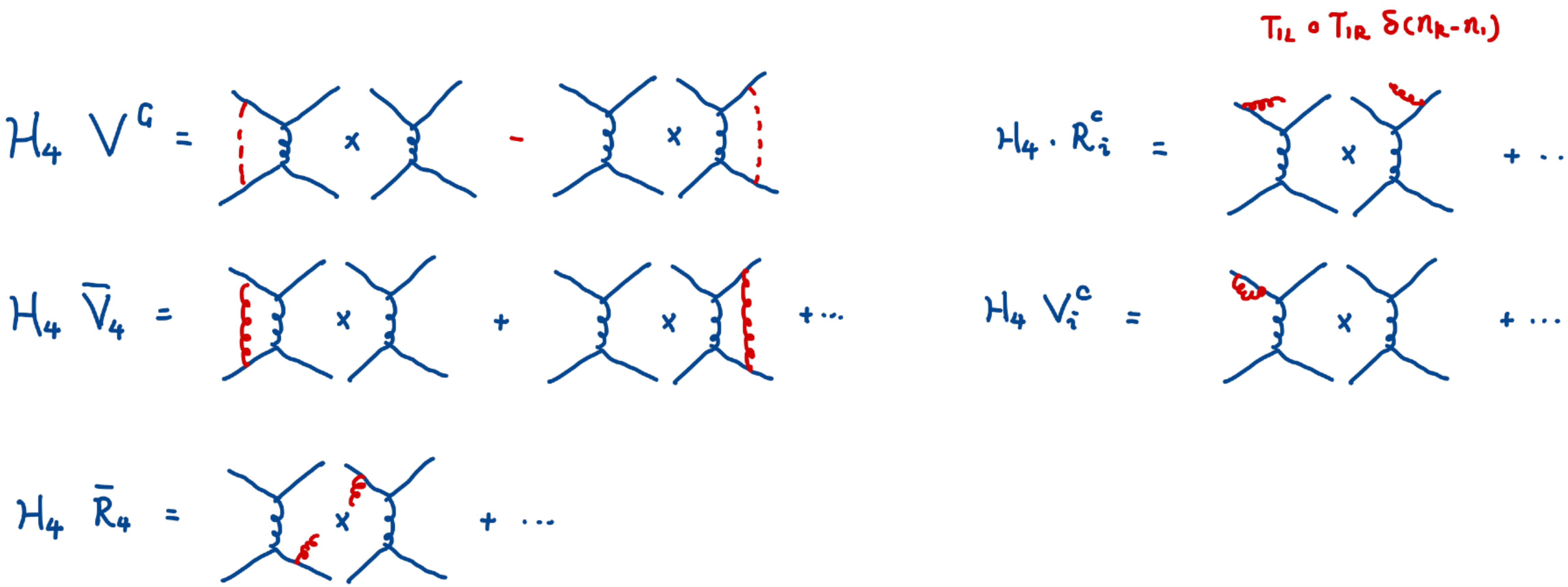
Hard function for octet exchange:

$\mathcal{H}_4 =$



$$\sim \; t^a_{\alpha_3\alpha_1} t^a_{\alpha_4\alpha_2} t^b_{\beta_1\beta_3} t^b_{\beta_2\beta_4} \sigma_0$$

Action of the anomalous dimension



Compute $\mathcal{H}_4\, \boldsymbol{U}(\mu_s,\mu_h) = \mathcal{H}_4\, \mathbf{P}\exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu}\, \mathbf{\Gamma}^H(Q,\mu)\right]$

$$= \mathcal{H}_4 + \int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu}\, \mathcal{H}_4\, \mathbf{\Gamma}^H(Q,\mu) + \int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \int_{\mu}^{\mu_h} \frac{d\mu'}{\mu'} \mathcal{H}_4\, \mathbf{\Gamma}^H(Q,\mu')\, \mathbf{\Gamma}^H(Q,\mu)$$

All-order evolution of leading Super-Leading Logs

(Becher, Neubert, DYS ’21 PRL + Stillger’23 JHEP)

All-order structure: Kampe de Feriet function (a two-variable generalization of the generalized hypergeometric series, the general sextic equation can be solved in terms of it)

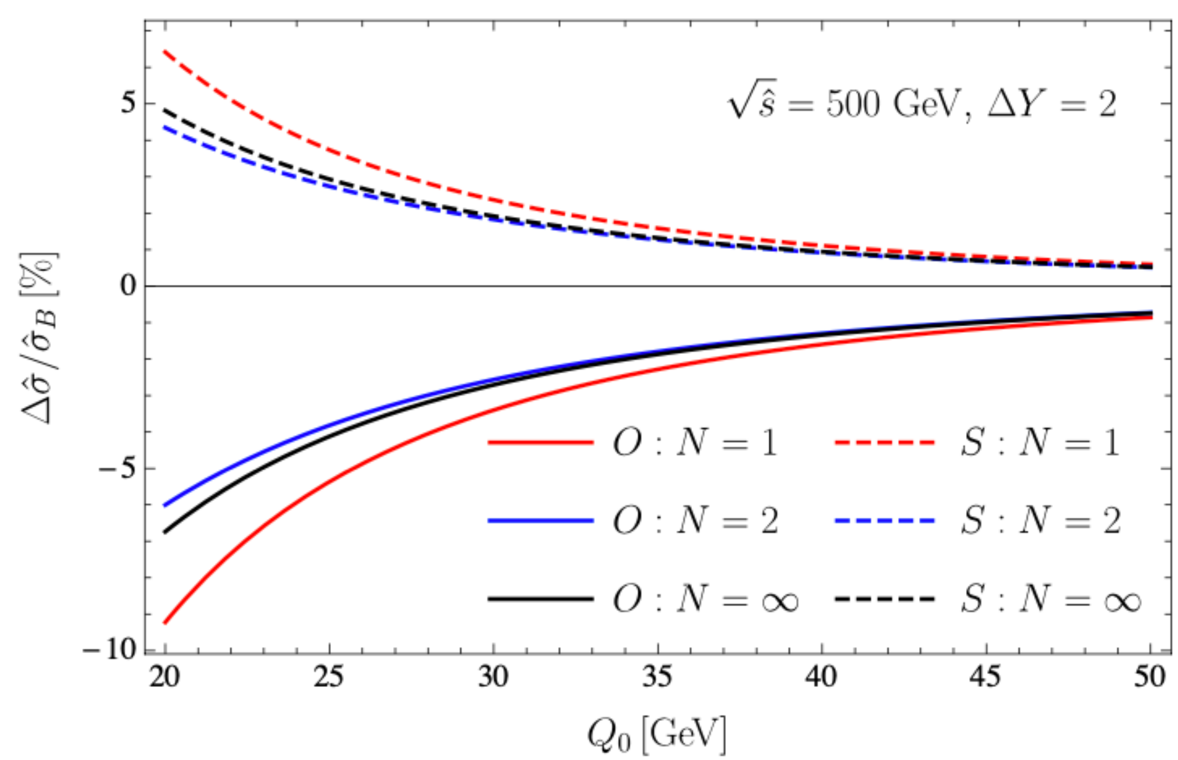
$$\Sigma(v,w) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1)_{m+r}(1)_m\left(\frac{1}{2}\right)_r}{(2)_{m+r}\left(\frac{5}{2}\right)_{m+r}} \frac{(-w)^m(-vw)^r}{m!\,r!}$$

$$= {}^{1+1}F_{2+0}\left(\begin{matrix} 1:1,\frac{1}{2}; \\ 2,\frac{5}{2}: \quad ; \end{matrix}, -w, -vw\right) \qquad w = \frac{N_c\alpha_s(\bar\mu)}{\pi}\ln^2\left(\frac{\mu_h}{\mu_s}\right)$$

Sudakov suppression of the superleading logarithms is weaker than the one present for global observables

Global logs $\longrightarrow e^{-\omega}$

Superleading logs $\xrightarrow{\omega\to\infty} \dfrac{1}{\omega}$



Red: Four loop Blue: Five loop Black: all order