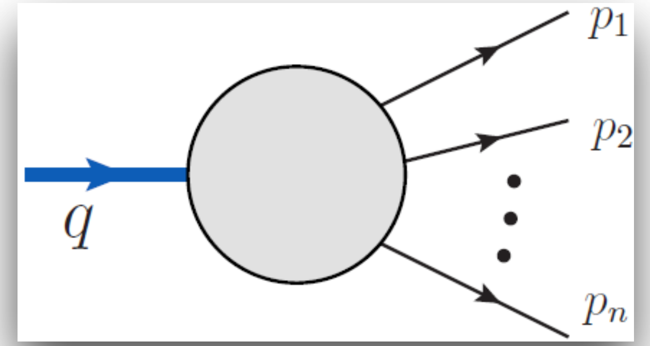
形状因子



杨刚

中国科学院理论物理研究所

2025 年“微扰量子场论及其应用”前沿讲习班暨前沿研讨会

济南，2025 年，7月6-21日

Outline

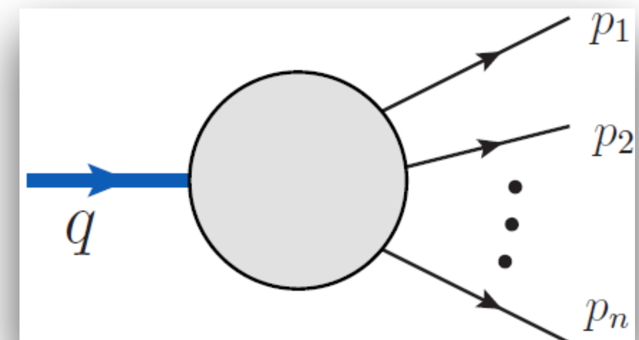
- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

What are form factors?

Form factors

Partially on-shell, partially off-shell:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\, \langle p_1 \ldots p_n|\mathcal{O}(x)|0\rangle$$
$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big)\, \langle p_1 \ldots p_n|\mathcal{O}(0)|0\rangle$$



$$q = \sum_i p_i, \quad q^2 \neq 0$$

form factors

$$\langle p_1 p_2 \ldots p_n|0\rangle$$

Scattering amplitude

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n\rangle$$

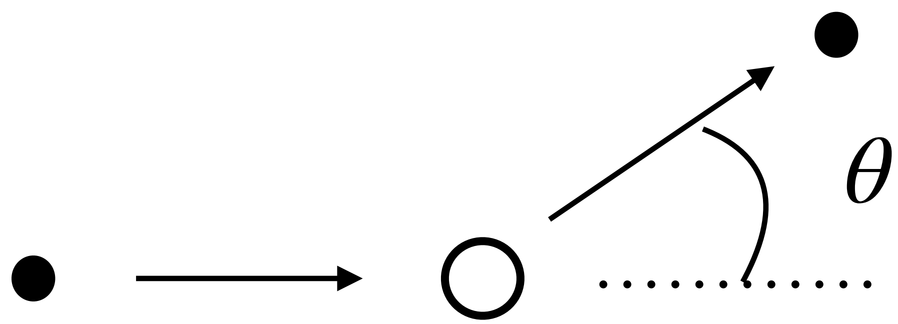
Correlation functions

What are form factors?

Some history

1) Nuclear "structure factor"

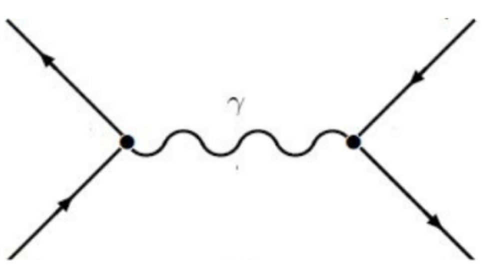
Point particle scattering



$$\sigma(\theta)$$

"Rutherford formula"

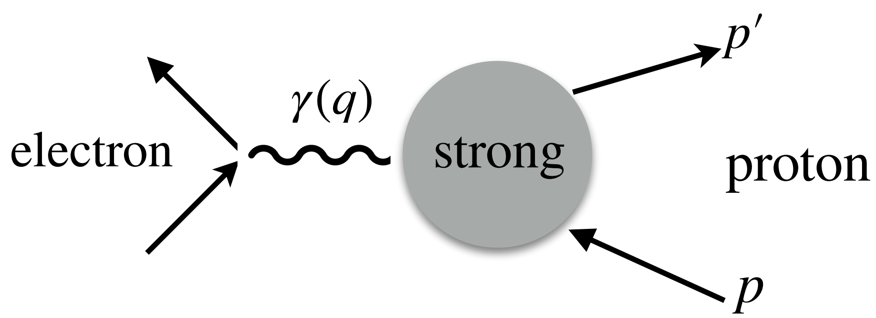
"Mott formula"



$$\sim e^2(\bar{u}\gamma^\mu u)\frac{\eta_{\mu\nu}}{q^2}(\bar{v}\gamma^\nu v)$$

1) Nuclear “structure factor”

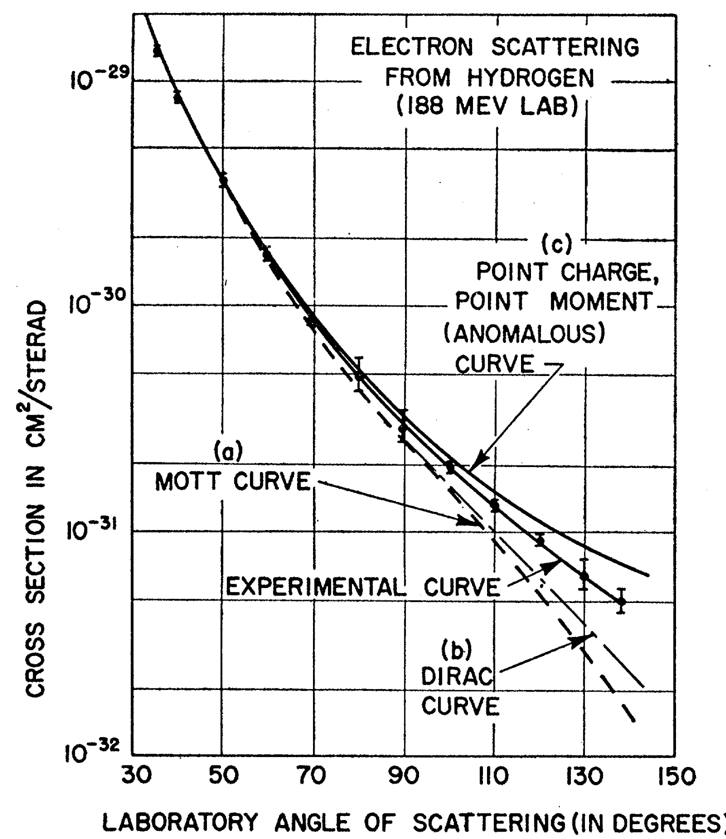
- Form factor characterizes the deviation from the point-particle picture.



$$\sigma_s(\theta)=\left(\frac{Ze^2}{2E}\right)^2\frac{\cos^2\frac{1}{2}\theta}{\sin^4\frac{1}{2}\theta}\left[\int_0^\infty \rho(r)\frac{\sin qr}{qr}4\pi r^2 dr\right]^2.$$

$$F=\frac{4\pi}{q}\int_0^\infty \rho(r)\sin(qr)r\,dr$$

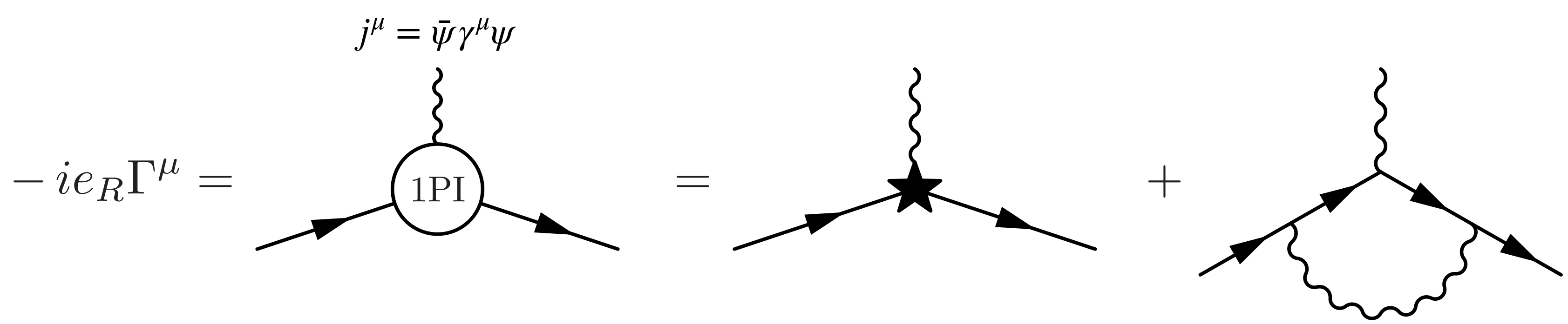
“form factor”



McAllister and Hofstadter, Phys.Rev. (1956)

Robert Hofstadter
(1915 – 1990)
Nobel laureate 1961

2) Form factor in text book



$$\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

Form factors

$$\bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}\Gamma^\mu\psi$$

"Rosenbluth formula"

Leading order: $F_1(p^2) = 1, \quad F_2(p^2) = 0$

One-loop order: $F_2(0) = \frac{\alpha}{2\pi}$ → $g - 2 = 2F_2(0) = \frac{\alpha}{\pi}$

Anomalous magnet moment

3) Sudakov form factor

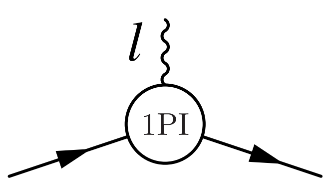
- Pioneer work by Vladimir Sudakov in 1954

Vladimir Sudakov

Vertex Parts at Very High Energies in Quantum Electrodynamics

V. V. SUDAKOV
(Submitted to JETP editor Nov. 4, 1954)
J. Exper. Theoret. Phys. USSR 30, 87-95 (January 1956)

A method is developed for calculating Feynman integrals with logarithmic accuracy, working to any order of perturbation theory. The method is applied to calculate the vertex part in quantum electrodynamics for a certain range of values of the momenta. The result is displayed as the sum of a perturbation series.

$-ie_R\Gamma^{\mu} =$ 

$$\Gamma_\sigma(p, q; l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right)^n = \gamma_\sigma \exp\left\{-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right\}.$$

A closed formula of summing up the leading-logarithm terms.

- "High-momentum electromagnetic vertex" was the PhD project of Roman Jackiw (1966), assigned by his advisor Ken Wilson. $\exp\left[-\frac{e^2}{16\pi^2}\log^2\frac{|k^2|}{\mu^2}\right]$

3) Sudakov form factor

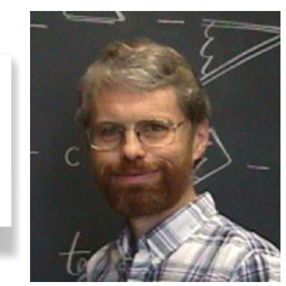
- Further development around 1980
  - **Alfred Mueller** (1979) and **John Collins** (1980) generalized Sudakov's result by including non-leading logarithms in QED.

Asymptotic behavior of the Sudakov form factor

A. H. Mueller
Department of Physics, Columbia University, New York, New York 10027
(Received 16 May 1979)

Algorithm to compute corrections to the Sudakov form factor

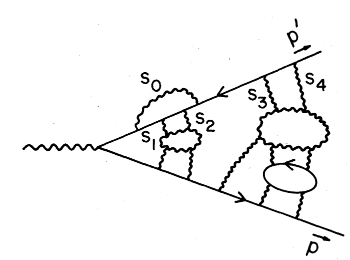
J. C. Collins
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 15 May 1980)



  - **Ashoke Sen** (1981) generalized the results to QCD.

Asymptotic behavior of the Sudakov form factor in quantum chromodynamics

Ashoke Sen
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 15 May 1981)



$$\exp\left(-\frac{16\pi^2 a_1}{\beta_0}\ln\frac{(q^2)^{1/2}}{\mu}\ln\ln\frac{(q^2)^{1/2}}{\mu}\right) \quad \text{in the limit } (q^2)^{1/2}\to\infty$$

IR divergences

Infrared structure of amplitudes:

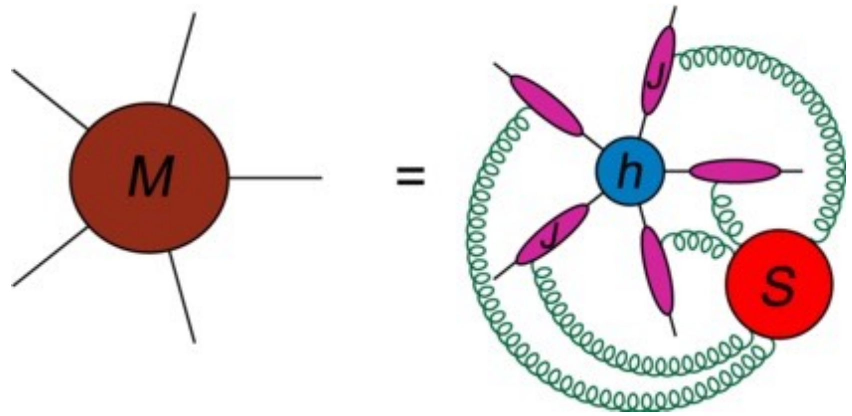


figure from L. Dixon 1105.0771

For modern dim-reg representation, see:
Magnea and Sterman 1990;
Catani 1998,
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n \left(k_i, \mu, \alpha_s, \epsilon \right)$$

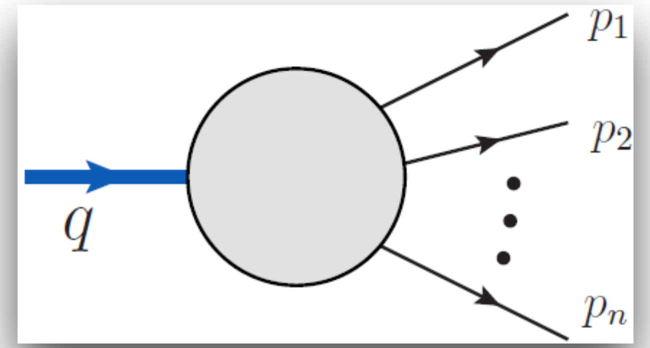
Sudakov form factor $= \exp\left[-\frac{1}{4} \sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2} \right)^{l\epsilon} \left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \right) + \text{finite} \right]$

Leading IR singularity -> Cusp anomalous dimension

4) "Modern" general form factors

Hybrids of on-shell states and off-shell operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x} \langle p_1 \ldots p_n|\mathcal{O}(x)|0\rangle$$
$$= \delta^{(4)}(\sum_{i=1}^{n} p_i - q)\, \langle p_1 \ldots p_n|\mathcal{O}(0)|0\rangle$$



$q = \sum_i p_i, \quad q^2 \neq 0$

form factors

$\langle p_1 p_2 \ldots p_n | 0 \rangle$

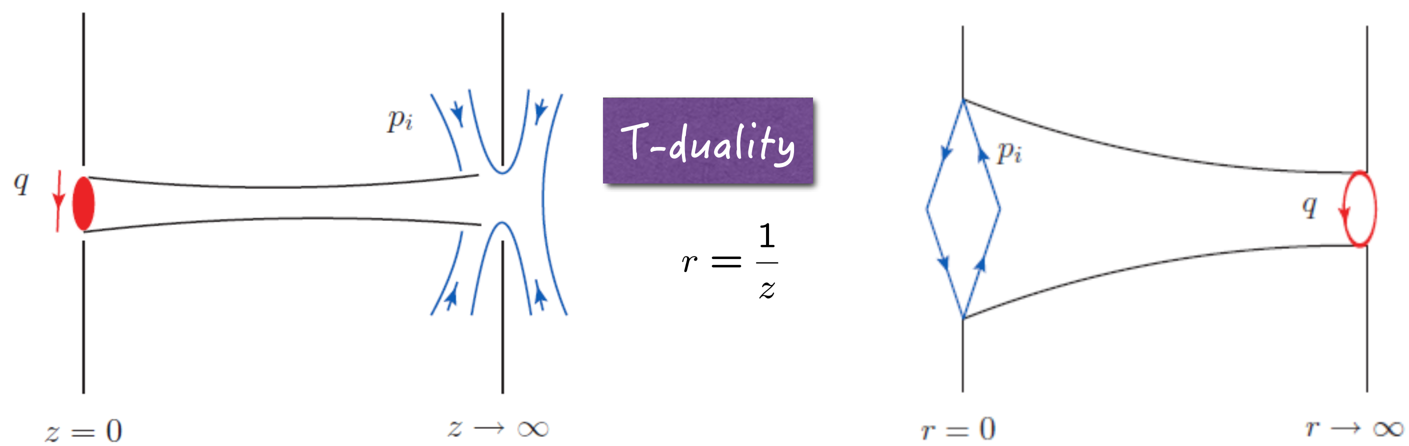
Scattering amplitude

$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$

Correlation functions

4) “Modern” general form factors

- **Maldacena** and **Zhiboedov** (2010) considered high-point form factors at strong coupling using AdS/CFT duality.



- **Brandhuber, Spence, Travaglini, GY** (2010) and **Bork, Kazakov, Vartanov** (2010) studied high-point form factors at weak coupling.

MHV structure of form factors: Brandhuber, Spence, Travaglini, GY 2010

$$F_n^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)) = \delta^4(\sum_{i=1}^{n} p_i - q) \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

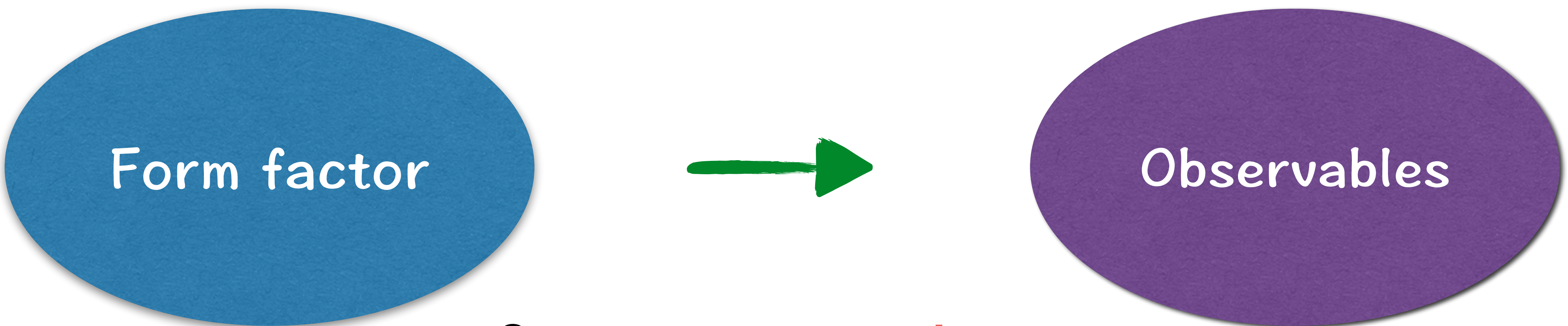
Applications of form factors

- Operator classification and spectrum
- EFT amplitudes
- IR divergences (Sudakov FF)
- Correlation functions (EEC, etc..)
- New hidden structures beyond amplitudes

$$\text{Loop form factor} = (\text{Universal IR div.}) + (\text{UV div.}) + (\text{Finite part})$$

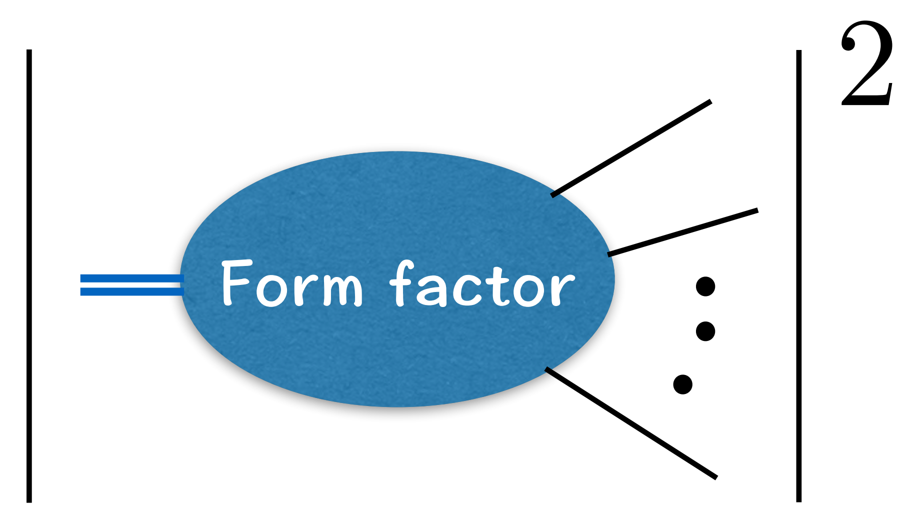
Sudakov form factor　　Renormalization　　EFT amplitudes

Applications of form factors



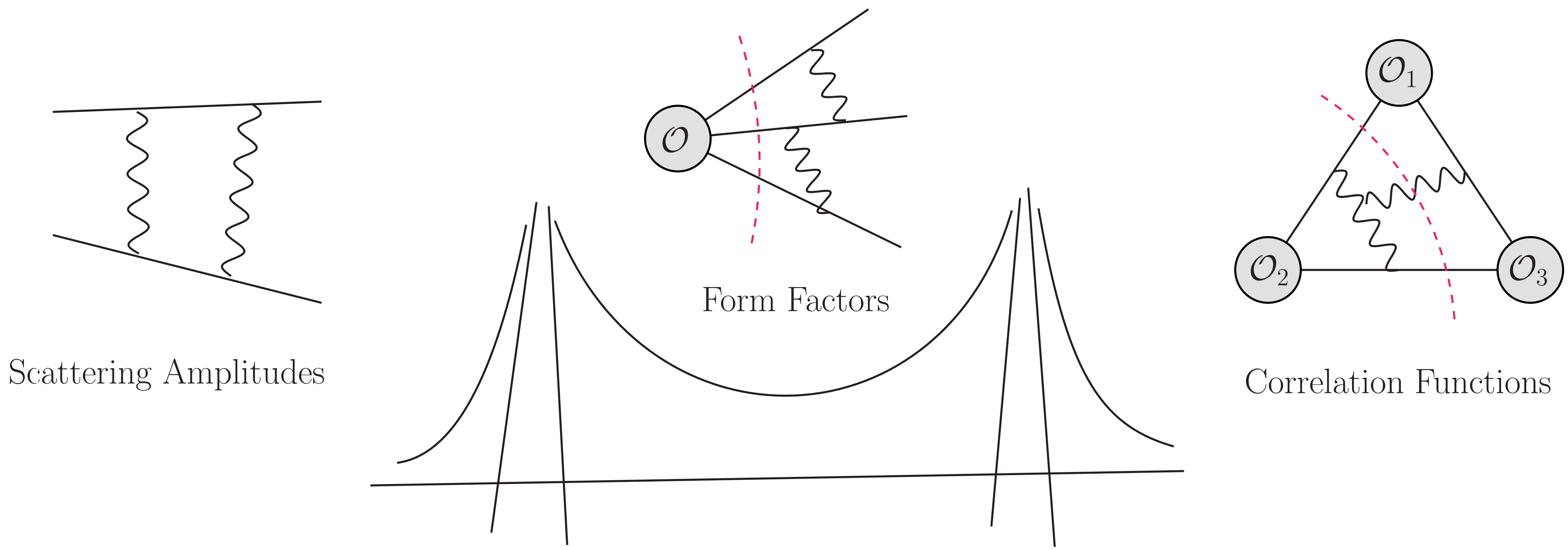
Compute **cross sections**

$$\int d\mathrm{PS}_n \times \text{(weight factor)}$$



Form factors serve a useful testing ground for such studies, and for also computing interesting observables such as EEC, etc.

Applications of form factors



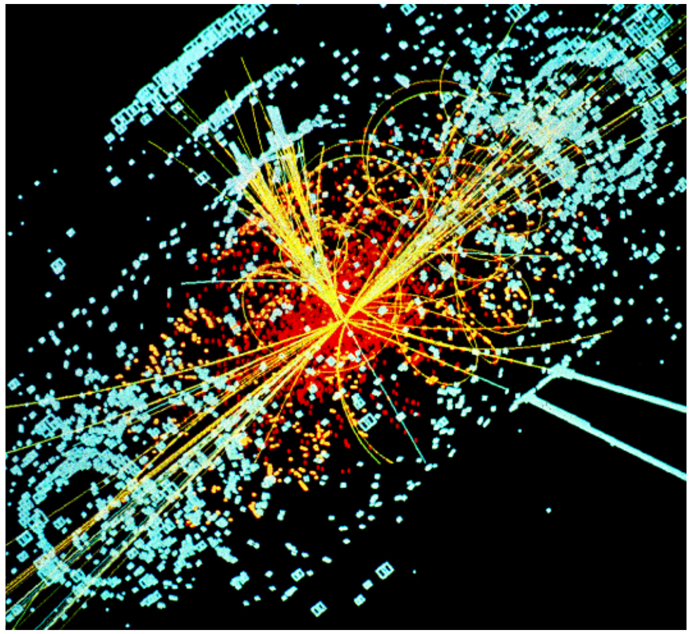
- Form factors provide a framework to study many operator quantities using powerful on-shell amplitude methods.

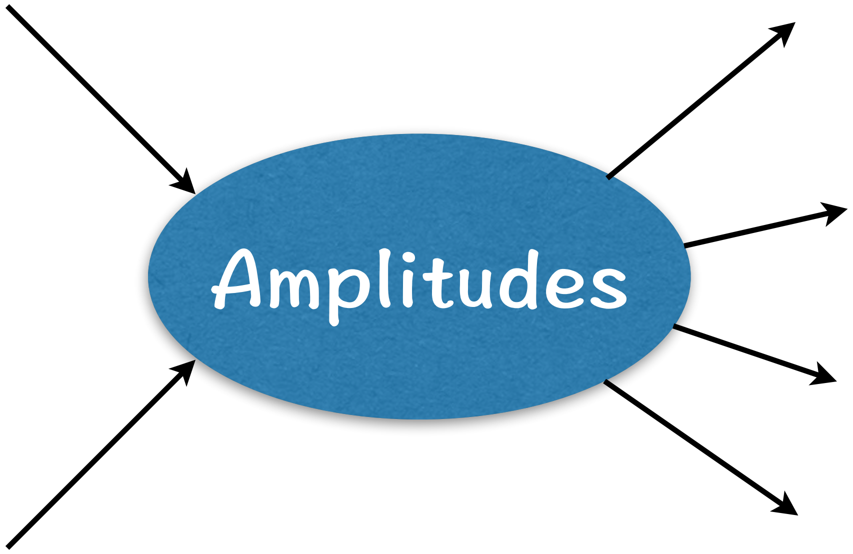
Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

On-shell methods for amplitudes

Scattering amplitudes



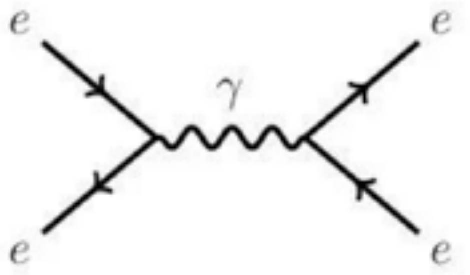


In past 30 years, significant progress has been made in the studies of scattering amplitudes.

Feynman diagram

Standard textbook method:

- universal
- simple rules
- intuitive picture



Feynman diagram

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

Loop amplitudes are even harder.

Surprising simplicity

by JAMES O'BRIEN FOR QUANTA MAGAZINE

Surprising simplicity

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

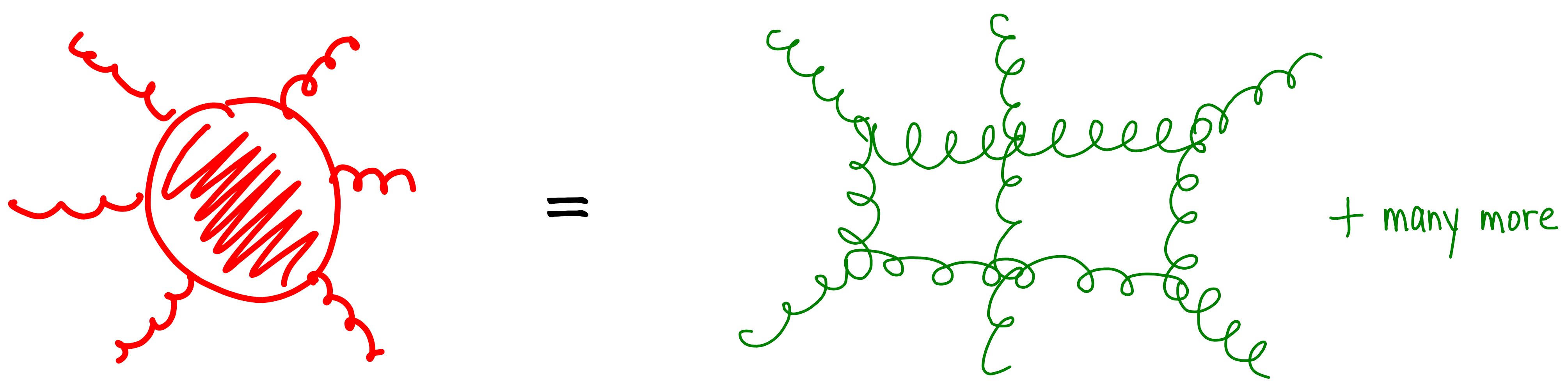
n-gluon MHV tree amplitudes: [Parke, Taylor, 1986]

$$A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

Written in spinor helicity formalism (Chinese Magic)
by Xu, Zhang, Chang 1984

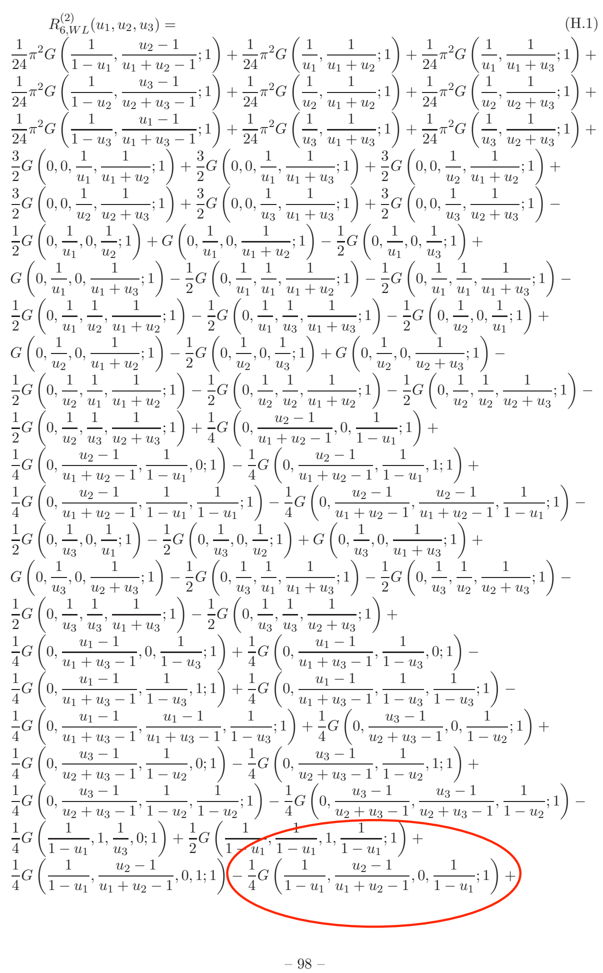
Surprising simplicity

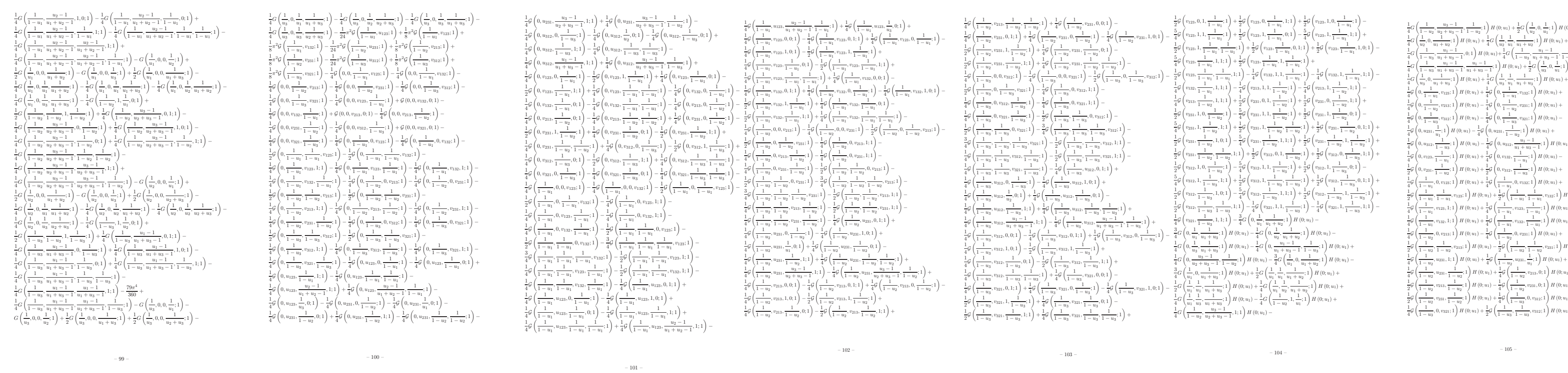
Six-gluon MHV amplitudes in N=4 SYM



[Del Duca, Duhr, Smirnov 2010] (heroic computation)

17 pages results



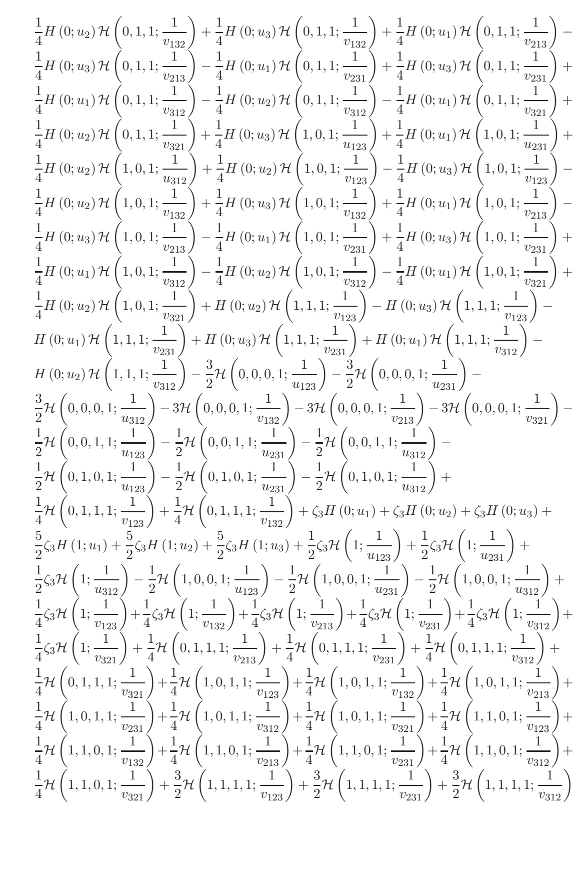


[Del Duca, Duhr, Smirnov 2010]

“multiple(Goncharov)-polylogrithm function”

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)$$

复杂的四重积分!



A much simpler form

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+, x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right)$$

$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-)).$$

a line result in terms of classical polylogarithms!

→ require advanced mathematical tools: "**Symbol**"

Alexander Goncharov

Lessons from modern amplitudes

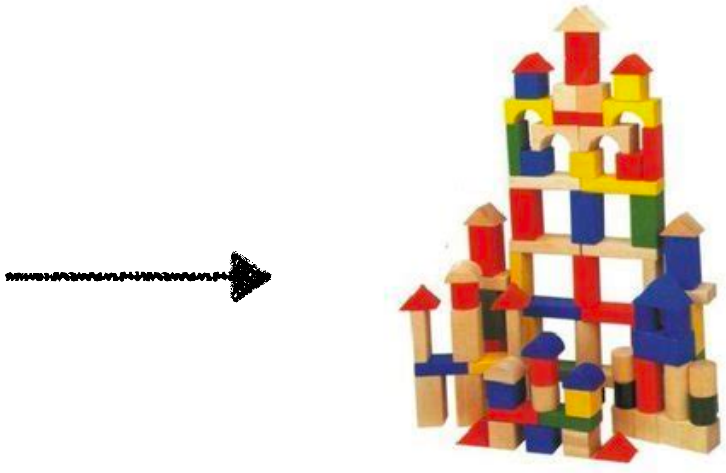
Such simplicity is totally unexpected using traditional Feynman diagrams.

Conceptually:	Methodologically:
New structures and new formulations	New powerful computational methods

Modern on-shell methods

In past 30 years, significant progress has been made in the studies of scattering amplitudes.

Using simple building blocks to construct more complicated ones:



"A Renaissance of the S-Matrix Program"

Modern amplitudes methods

S-matrix program

Wheeler 1937
Heisenberg 1943

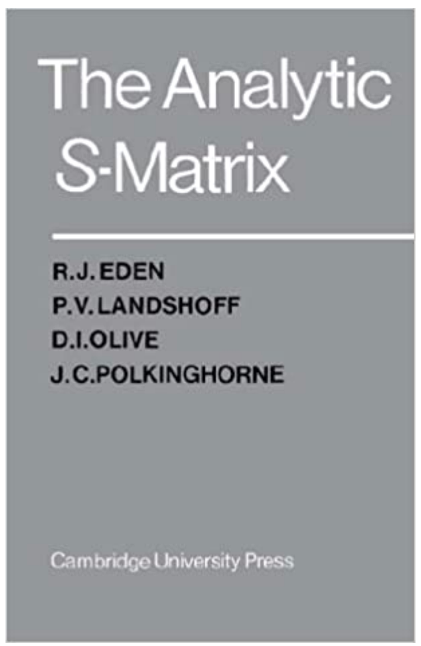
→

S-matrix bootstrap by
Chew, Mandelstam, etc
1950s-1960s

→

Modern amplitudes
On-shell methods

S-matrix program



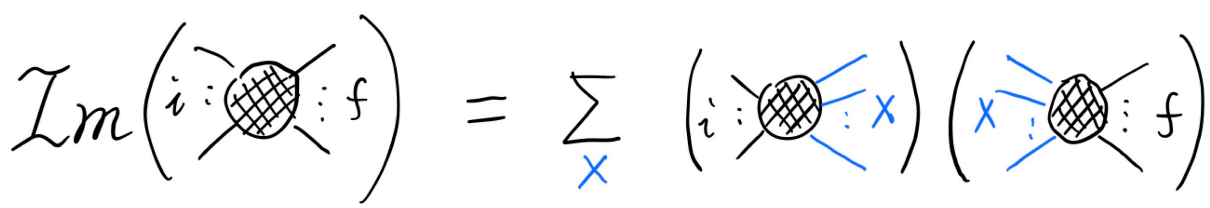
"The S-matrix is a Lorentz-invariant analytic function of all momentum variables with only those singularities required by unitarity."

"One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, …."

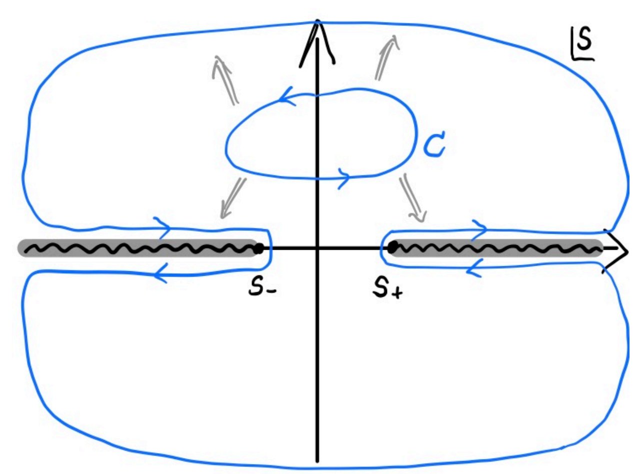
— Eden et.al, "The Analytic S-matrix", 1966

S-matrix bootstrap

Unitarity: $S^\dagger S = 1 = S S^\dagger \longrightarrow -\mathbb{i}(\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle) = \sum_X \langle f|T^\dagger|X\rangle\langle X|T|i\rangle$



Dispersion relation:



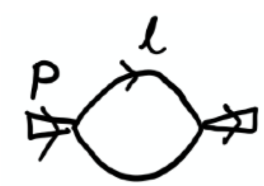
$$\mathcal{I}m[A] \Rightarrow A(s) \sim \int \frac{\mathcal{I}m[A]}{s'-s}\, ds'$$

(plus possible poles and asymptotic contributions)

A bubble-integral example

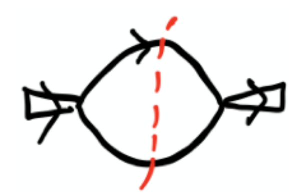
Let us compute this integral via S-matrix bootstrap:

$$I_2(P^2) = \int \frac{d^D l_1}{(2\pi)^D} \frac{1}{l^2 (l-P)^2}$$



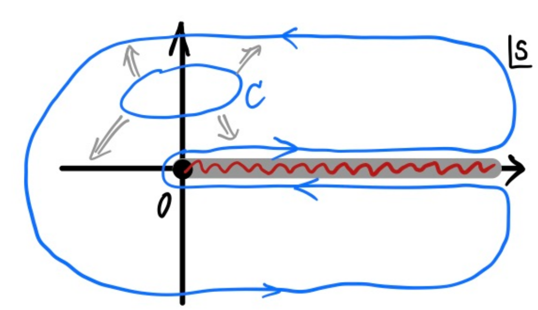
Step 1: compute discontinuity

$$\mathrm{Disc}[I_2(P^2)] = \int \frac{d^D l_1}{(2\pi)^D} (-2\pi \mathbb{i}) \delta(l^2) (-2\pi \mathbb{i}) \delta((l-P)^2) = -\frac{(P^2)^{-\epsilon}}{(4\pi)^{2-2\epsilon}} \frac{\pi^{\frac{3}{2}-\epsilon}}{\Gamma(\frac{3}{2}-\epsilon)}$$

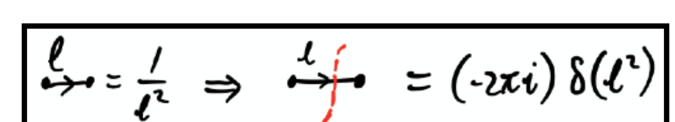


Step 2: apply dispersion relation $s = P^2 < 0,$

$$I_2(s) = \frac{1}{2\pi \mathbb{i}} \int_0^\infty \frac{dt}{t-s} \mathrm{Disc}[I_2(t)] = \frac{\mathbb{i}}{(4\pi)^{\frac{D}{2}}} (-s)^{-\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$



Cutkosky cutting rule:



Modern amplitudes methods

S-matrix program is replaced by the Standard Model since late1960s.

New ingredients in the modern on-shell methods:

- Working at perturbative level
- Generalized unitarity cuts
- Use of good variables, e.g. spinor helicity
- New mathematical functional structures (e.g. symbol)
- Using simple toy models (N=4 SYM) as testing ground

e.g. tree-level BCFW recursion relations, unitarity-cut methods

Bern, Dixon, Durban, Kosower 1994; Britto, Cachazo, Feng, Witten, 2004

Modern amplitudes methods

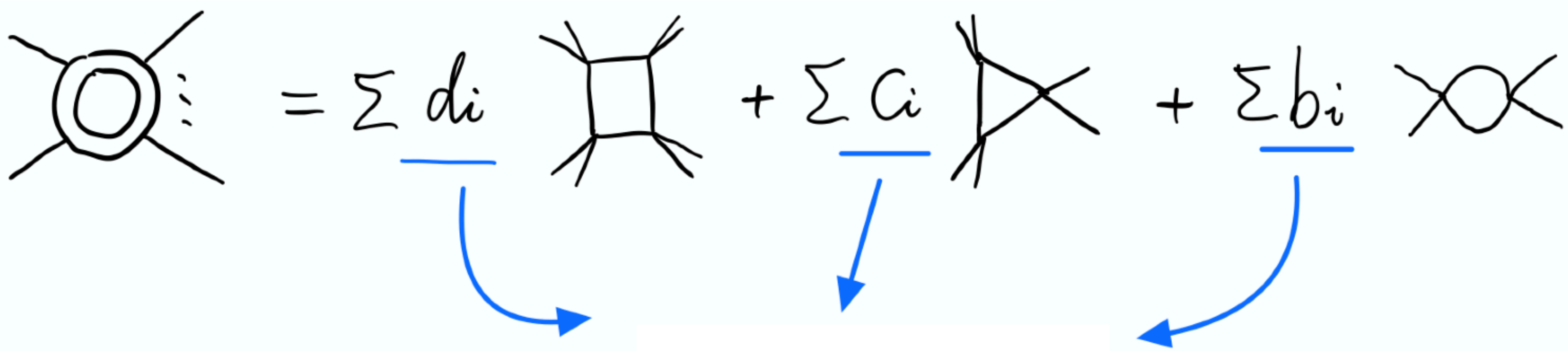
A question:

In the optical theorem, unitarity can be used to compute only the imaginary part.

How can the modern on-shell methods compute the full amplitudes via unitarity cuts?

One-loop structure

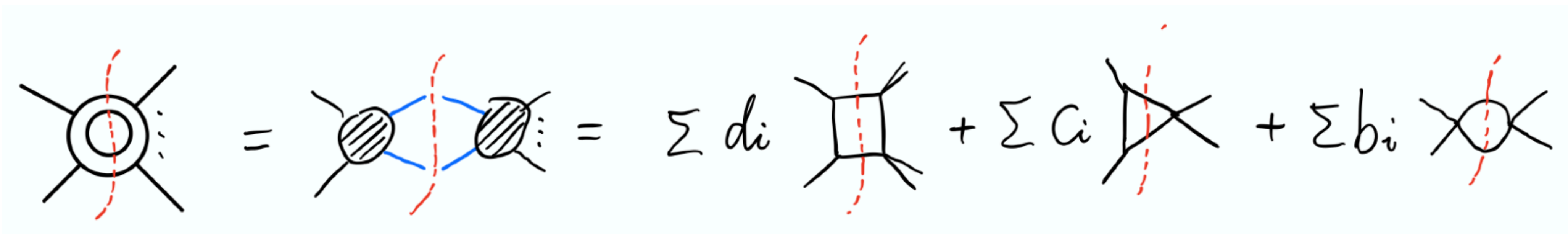
Consider one-loop amplitudes:



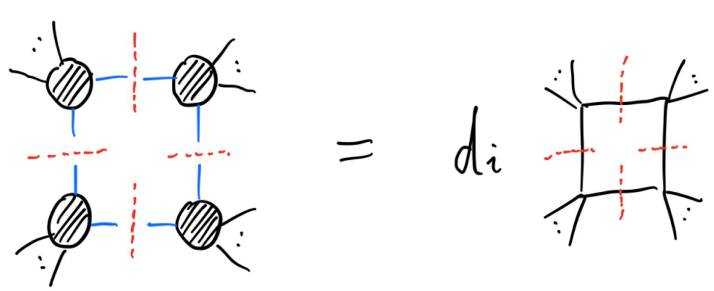
What we really want

Unitarity cuts

Using simpler tree-level blocks, one can derive the coefficients more efficiently:

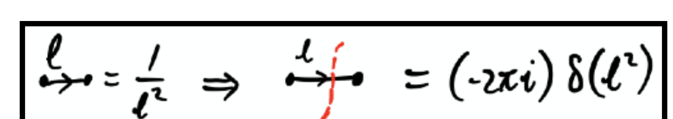


[Bern, Dixon, Dunbar, Kosower 1994]



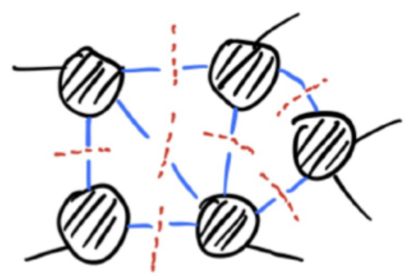
generalized multiple cuts [Britto, Cachazo, Feng 2004]

Cutkosky cutting rule:



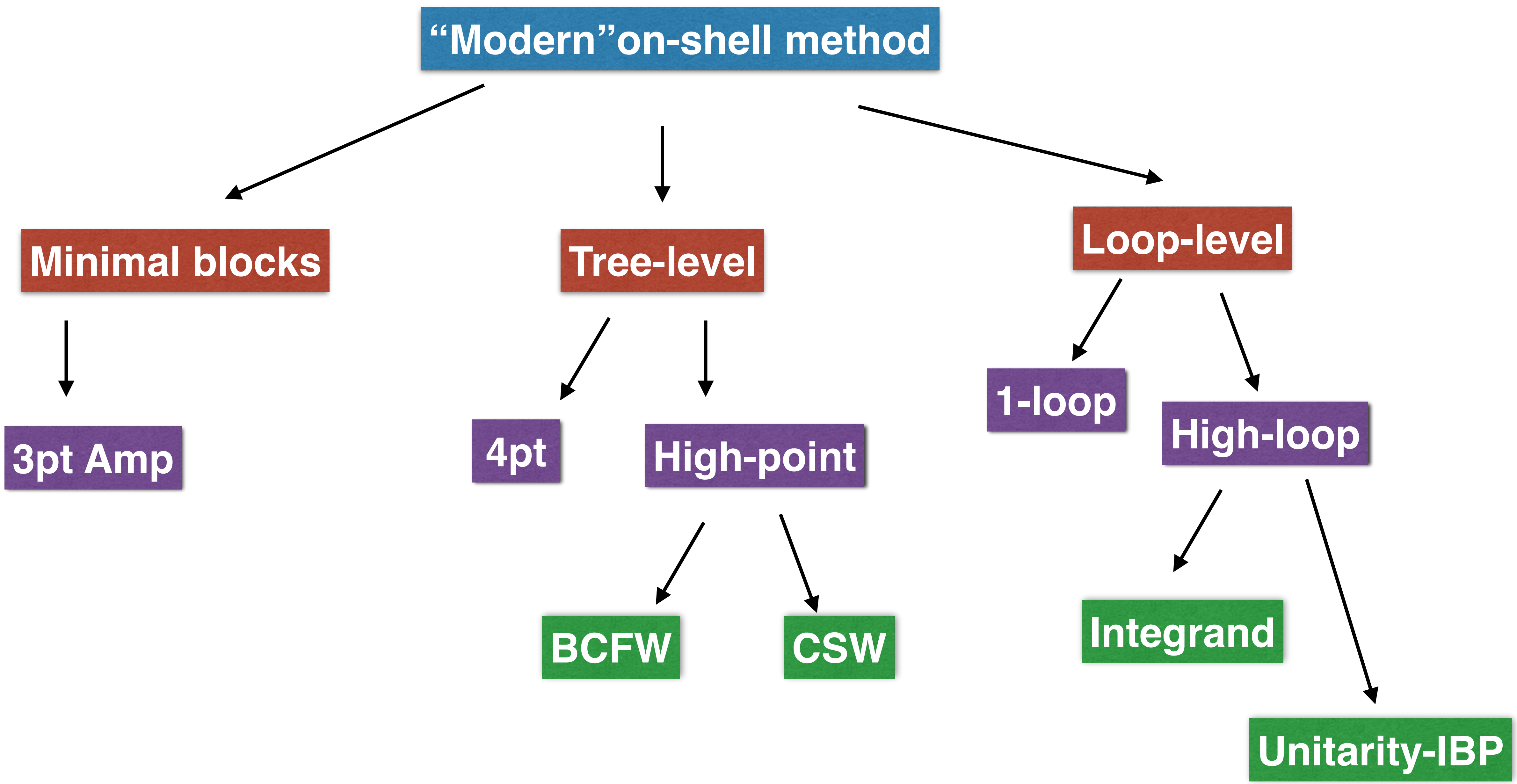
Loop integrands

Both the basis coefficients and integrand are rational functions, once they are obtained, one has the information for the full amplitudes.



$$= \text{Integrand}\Big|_{\text{multi-cuts}}$$

“On-shell” method

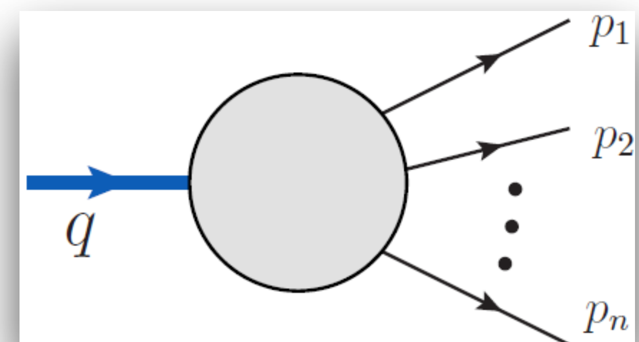


Towards form factors

Partially on-shell, partially off-shell:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x} \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big) \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$



$q = \sum_i p_i, \quad q^2 \neq 0$

form factors

$\langle p_1 p_2 \ldots p_n | 0 \rangle$

Scattering amplitude

$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$

Correlation functions

Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

Tree-level form factors

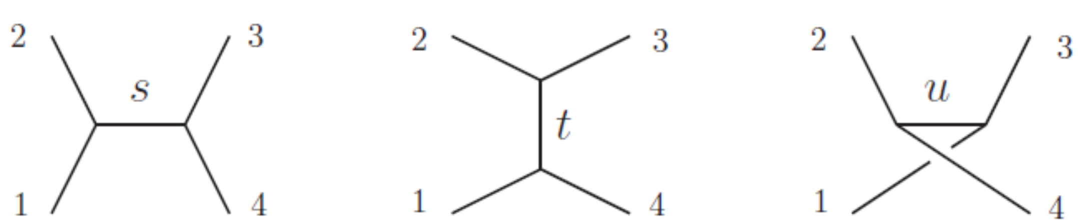
A warm-up example

Scattering amplitudes in massless scalar theory with ϕ^3 interaction:

$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$ → (LSZ reduction)

$$\left(\Pi_{i=1}^4 \int d^4x_i\, e^{ip_ix_i}\,\left(\partial^2_{x_i}\right)\right)\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = A_4(p_1,p_2,p_3,p_4)$$

$$\langle T\phi(x_1)\phi(x_2)\rangle = \Delta(x_1-x_2), \quad (\partial^2_{x_1})\Delta(x_1-x_2) = \delta^4(x_1-x_2)$$



$$0 = \sum_i p_i, \quad p_i^2 = 0$$

$$A_4 = \delta^4(p_1+p_2+p_3+p_4)\left(\frac{1}{(p_1+p_2)^2} + \frac{1}{(p_2+p_3)^2} + \frac{1}{(p_1+p_3)^2}\right)$$

A warm-up example

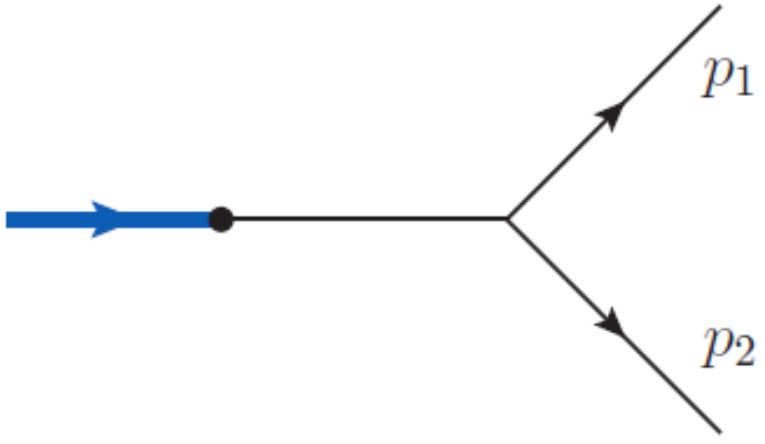
How about LSZ reduction for part of the fields?

$\langle 0|T\phi(x_1)\phi(x_2)\phi(y)|0\rangle$ → LSZ reduction for the first two fields, but only Fourier transformation for the third

$$\left(\int d^4y e^{-iqy} \prod_{i=1}^{2} \int d^4x_i e^{ip_i x_i} (\partial_{x_i}^2)\right) \langle 0|T\phi(x_1)\phi(x_2)\phi(y)|0\rangle$$

$$= \left(\int d^4z \int d^4y e^{-iqy+i(p_1+p_2)z}\right) \Delta(y-z)$$

$$= \delta^4(p_1+p_2-q)\frac{1}{(p_1+p_2)^2}$$



$$q = p_1 + p_2, \quad p_i^2 = 0$$

A warm-up example

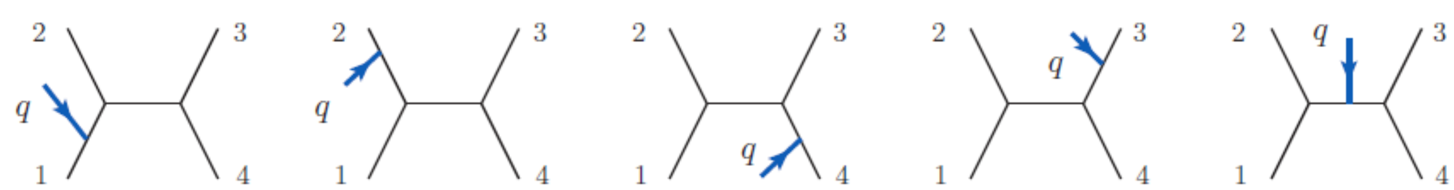
More interesting case with operator inserted:

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\text{tr}(\phi^2(y))|0\rangle$$ → LSZ reduction for elementary fields, Fourier transformation for the operator

$$\left(\int d^4y e^{-iqy}\prod_{i=1}^{4}\int d^4x_i e^{ip_ix_i}(\partial^2_{x_i})\right)\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\text{tr}(\phi^2(y))|0\rangle$$

→ $$F_4(p_1,p_2,p_3,p_4;q)=\delta^4\left(\sum_{i=1}^{4}p-q\right)\left(\frac{1}{s_{34}s_{234}}+\frac{1}{s_{34}s_{134}}+\cdots\right)$$

$$q=\sum_{i=1}^{4}p_i,\quad p^2=0,\quad q^2\neq 0$$



+ (t, u channel like diagrams)

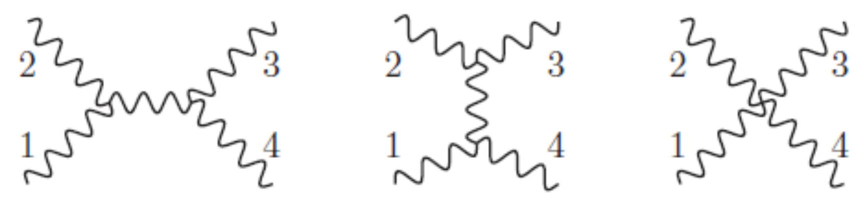
MHV form factor ?

MHV (color ordered) amplitudes (Parke-Taylor):

$$A_{\text{MHV}}(1^+,..,i^-,..,j^-,..,n^+) = \delta^4(\sum_i p_i)\,\frac{\langle i\ j\rangle^4}{\langle 1\ 2\rangle\cdots\langle n\ 1\rangle}$$

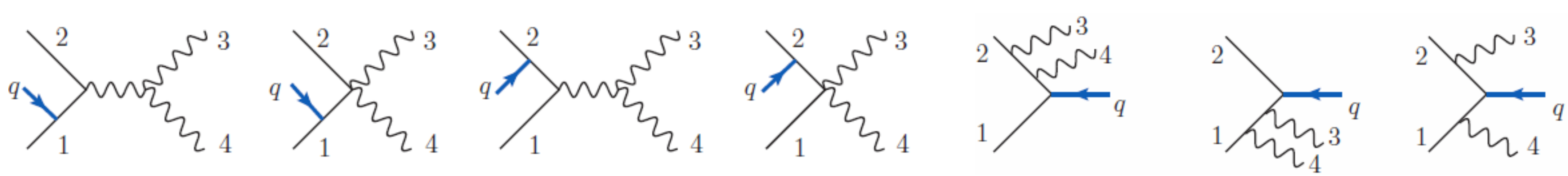
$$0 = \sum_i p_i, \quad p_i^2 = 0$$

(firstly found by computing Feynman diagrams)



Do we have MHV formula for (color ordered) form factor ?

A four-point example (in N=4): $F_4\Big(\phi(p_1),\phi(p_2),g^+(p_3),g^+(p_4);\text{tr}(\phi^2)(q)\Big)$



$$\longrightarrow \quad \delta^4\left(\sum_{i=1}^4 p_i - q\right)\,\frac{\langle 1\ 2\rangle^2}{\langle 1\ 2\rangle\cdots\langle 4\ 1\rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

MHV form factors !

$$F_n(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)(q))$$

$$= \delta^n \left(\sum_{i=1}^{n} p_i - q \right) \frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \cdots \langle n\ 1 \rangle}$$

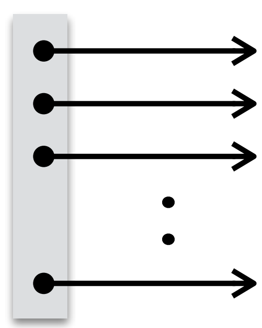
$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

MHV like structure implies the underlying simplicity of form factor !

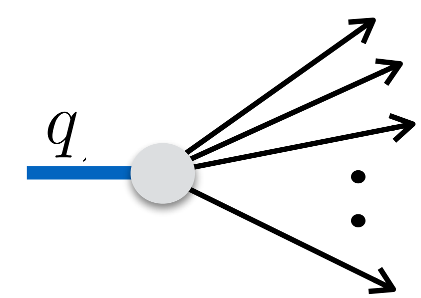
MHV rules, BCFW recursion relation, unitarity method can be applied efficiently.

Minimal tree form factors

One can translate any local operator into “on-shell” kinematics !



$$\mathcal{O}_L \;\Leftrightarrow\; \mathcal{F}_{\mathcal{O}_L,L}(1,\ldots,L)$$



These are called minimal form factors.

Spinor helicity formalism

Massless momentum:

$$p_\mu \quad \to \quad p_{\alpha\dot\alpha} = p_\mu \sigma^\mu_{\alpha\dot\alpha} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

$$p_\mu p^\mu = 0 \quad \to \quad p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}\,, \qquad \alpha, \dot\alpha = 1, 2$$

Polarisation vector:

$$\varepsilon^{(-)}_{i,\alpha\dot\alpha} = \frac{\lambda_i\,\tilde\xi}{[\tilde\lambda_i\,\tilde\xi]}\,, \qquad \varepsilon^{(+)}_{i,\alpha\dot\alpha} = \frac{\xi\,\tilde\lambda_i}{\langle\xi\lambda_i\rangle}$$

"Chinese Magic" [Xu, Zhang, Zhang, 84]

Spinor helicity formalism

For N=4 SYM, the superconformal group is:

$$PSU(2,2|4) \qquad \alpha, \dot{\alpha} = 1,2$$

$$\alpha, \dot{\alpha}|A \qquad A = 1,2,3,4$$

On-shell N=4 superfield (for all helicity states): [Nair 88]

$$\Phi(p,\eta) = g_+(p) + \eta^A \bar{\psi}_A(p) + \frac{\eta^A\eta^B}{2}\phi_{AB}(p) + \frac{\eta^A\eta^B\eta^D}{3!}\epsilon_{ABCD}\psi^D(p) + \eta^1\eta^2\eta^3\eta^4 g_-(p)$$

(Super) MHV amplitudes:

$$\mathcal{A}^{\mathrm{MHV}}(1,2,\ldots,n) = \frac{\delta^{(4)}(\sum_{i=1}^n p_i)\delta^{(8)}(\sum_{i=1}^n \lambda_i\eta_i)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$$

Gauge invariant operators

Local gauge invariant operators are constructed as traces of covariant fields.

gauge transformation

$$\mathcal{W} \to U\mathcal{W}U^\dagger$$

$$\mathcal{O}(x) = \mathrm{Tr}(\mathcal{W}_1^{(m_1)}\mathcal{W}_2^{(m_2)} \ldots \mathcal{W}_n^{(m_n)})(x)$$

$$\mathcal{W}^{(m)} := D^m \mathcal{W}\,, \qquad D_{\alpha\dot{\alpha}}\mathcal{W} = \partial_{\alpha\dot{\alpha}}\mathcal{W} - ig_{\mathrm{YM}}[A_{\alpha\dot{\alpha}},\, \mathcal{W}]$$

In N=4 SYM, there are following ‘letters’:

$$\mathcal{W}_i \in \{\phi_{AB}\,,\ F_{\alpha\beta}\,,\ \bar{F}_{\dot{\alpha}\dot{\beta}}\,,\ \bar{\psi}_{\dot{\alpha}A}\,,\ \psi_{\alpha ABC}\}$$

$$\alpha, \dot{\alpha} = 1, 2$$

$$A = 1, 2, 3, 4$$

Operator in terms of Oscillators

The operators may be represented through states of oscillators as follows: [Günaydin, Marcus 85]

$$
\begin{array}{ccc}
\bar{F}_{\dot\alpha\dot\beta} & \longrightarrow & \mathrm{b}^{\dagger\dot\alpha}\mathrm{b}^{\dagger\dot\beta}|0\rangle \\
\bar\psi_{\dot\alpha A} & \longrightarrow & \mathrm{b}^{\dagger\dot\alpha}\mathrm{d}^{\dagger A}|0\rangle \\
\phi_{AB} & \longrightarrow & \mathrm{d}^{\dagger A}\mathrm{d}^{\dagger B}|0\rangle \\
\psi_{\alpha ABC} & \longrightarrow & \mathrm{a}^{\dagger\alpha}\mathrm{d}^{\dagger A}\mathrm{d}^{\dagger B}\mathrm{d}^{\dagger C}|0\rangle \\
F_{\alpha\beta} & \longrightarrow & \mathrm{a}^{\dagger\alpha}\mathrm{a}^{\dagger\beta}\mathrm{d}^{\dagger 1}\mathrm{d}^{\dagger 2}\mathrm{d}^{\dagger 3}\mathrm{d}^{\dagger 4}|0\rangle \\
D_{\alpha\dot\alpha} & \longrightarrow & \mathrm{a}^{\dagger\alpha}\mathrm{b}^{\dagger\dot\alpha}|0\rangle
\end{array}
$$

For example: $\mathrm{tr}(F_{\alpha\beta}F^{\alpha\beta}) \to \mathrm{a}_1^{\dagger\alpha}\mathrm{a}_1^{\dagger\beta}(\mathrm{d}_1^\dagger)^4\mathrm{a}_{2\alpha}^\dagger\mathrm{a}_{2\beta}^\dagger(\mathrm{d}_2^\dagger)^4|0\rangle$

Operators and on-shell kinematics

In terms of spinor helicity variables:

[Beisert 10] [Zwiebel 11]
[Wilhelm 14]

$\bar{F}_{\dot{\alpha}\dot{\beta}}$	$\xrightarrow{g_+}$	$\tilde{\lambda}^{\dot{\alpha}}\tilde{\lambda}^{\dot{\beta}}$
$\bar{\psi}_{\dot{\alpha}A}$	$\xrightarrow{\bar{\psi}_{\dot{\alpha}A}}$	$\tilde{\lambda}^{\dot{\alpha}}\eta^{A}$
ϕ_{AB}	$\xrightarrow{\phi_{AB}}$	$\eta^{A}\eta^{B}$
$\psi_{\alpha ABC}$	$\xrightarrow{\psi_{\alpha ABC}}$	$\lambda^{\alpha}\eta^{A}\eta^{B}\eta^{C}$
$F_{\alpha\beta}$	$\xrightarrow{g_-}$	$\lambda^{\alpha}\lambda^{\beta}\eta^{1}\eta^{2}\eta^{3}\eta^{4}$
$D_{\alpha\dot{\alpha}}$	$\longrightarrow$	$\lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$

Compare to the oscillator picture: $\mathrm{a}^{\dagger} \sim \lambda\,,\mathrm{b}^{\dagger} \sim \tilde{\lambda}\,,\mathrm{d}^{\dagger} \sim \eta$

Operators and form factors

$\bar{F}_{\dot\alpha\dot\beta}$	$\xrightarrow{g+}$	$\tilde\lambda^{\dot\alpha}\tilde\lambda^{\dot\beta}$
$\bar\psi_{\dot\alpha A}$	$\xrightarrow{\bar\psi_{\dot\alpha A}}$	$\tilde\lambda^{\dot\alpha}\eta^{A}$
ϕ_{AB}	$\xrightarrow{\phi_{AB}}$	$\eta^A\eta^B$
$\psi_{\alpha ABC}$	$\xrightarrow{\psi_{\alpha ABC}}$	$\lambda^\alpha\eta^A\eta^B\eta^C$
$F_{\alpha\beta}$	$\xrightarrow{g-}$	$\lambda^\alpha\lambda^\beta\eta^1\eta^2\eta^3\eta^4$
$D_{\alpha\dot\alpha}$	$\longrightarrow$	$\lambda^\alpha\tilde\lambda^{\dot\alpha}$

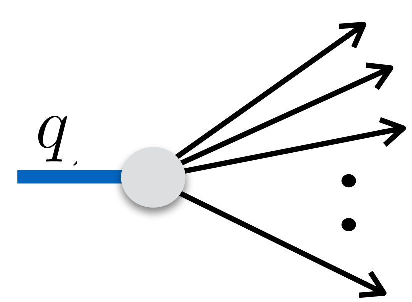
Applying the rules:

$$\mathrm{tr}(\bar{F}_{\alpha\beta}F^{\alpha\beta}) \to \lambda_1^\alpha\lambda_1^\beta\lambda_{2\alpha}\lambda_{2\beta}(\eta_1)^4(\eta_2)^4 = \langle 1\,2\rangle^2(\eta_1)^4(\eta_2)^4$$

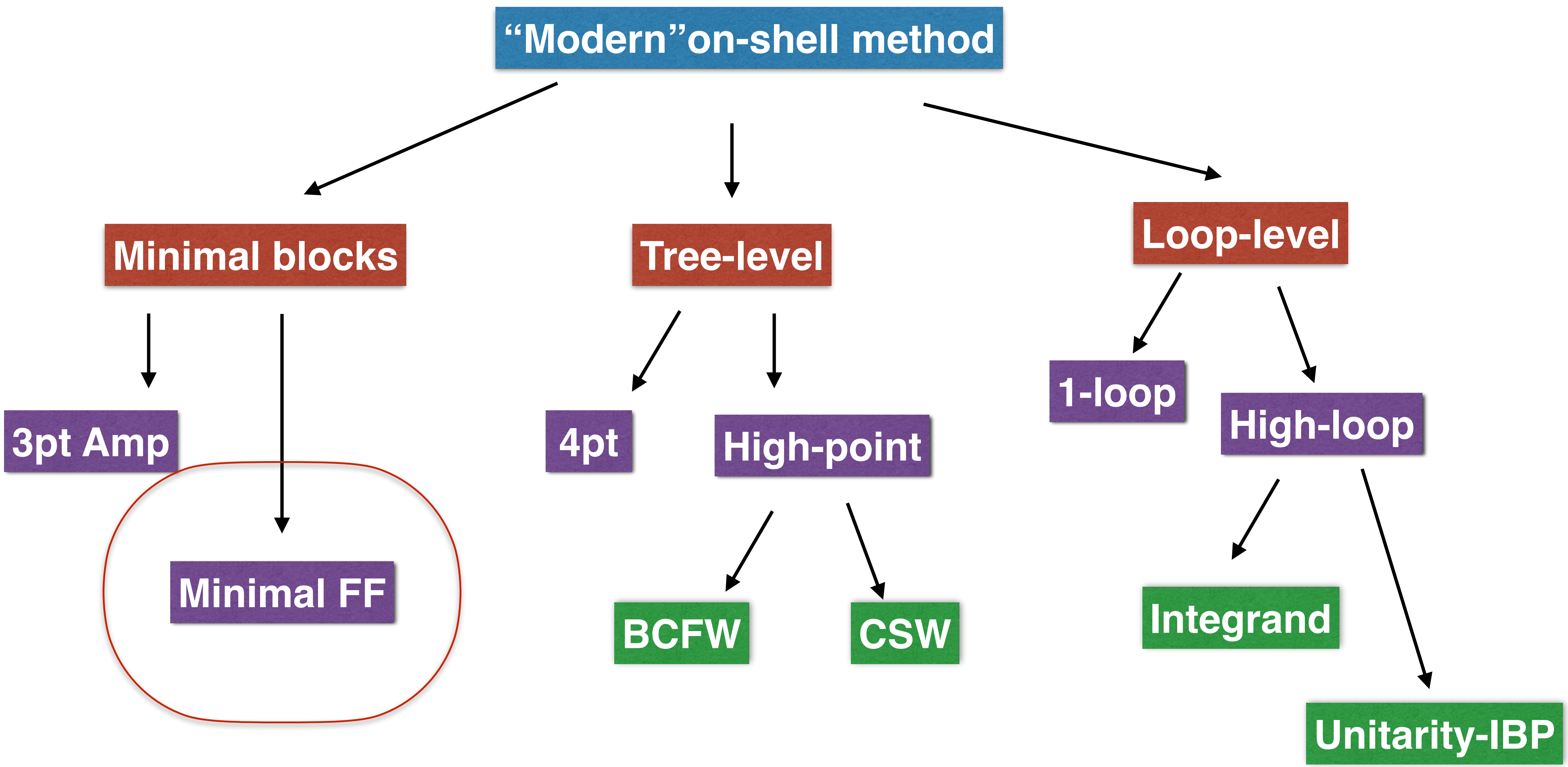
$$\mathrm{tr}(\bar{F}_{\dot\alpha}{}^{\dot\beta}\bar{F}_{\dot\beta}{}^{\dot\gamma}\bar{F}_{\dot\gamma}{}^{\dot\alpha}) \to \tilde\lambda_1^{\dot\alpha}\tilde\lambda_{1\dot\beta}\tilde\lambda_2^{\dot\beta}\tilde\lambda_{2\dot\gamma}\tilde\lambda_3^{\dot\gamma}\tilde\lambda_{3\dot\alpha} = [1\,2][2\,3][3\,1]$$

The RHS exactly reproduce the (minimal) form factor results:

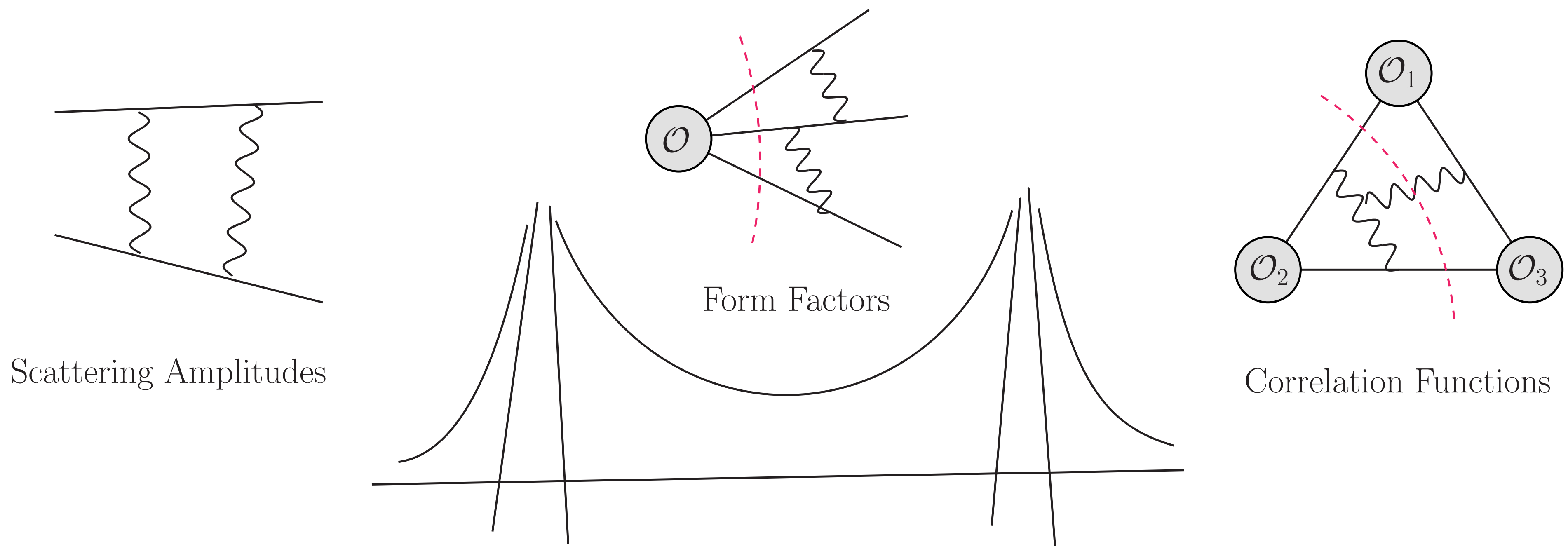
$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\,\langle p_1\ldots p_n|\mathcal{O}(x)|0\rangle = \delta^{(4)}(\sum_{i=1}^{n} p_i - q)\,\langle p_1\ldots p_n|\mathcal{O}(0)|0\rangle$$



“On-shell” method



Applications of form factors



Loop form factor = (Universal IR div.) + (UV div.) + (Finite part)

Sudakov form factor — Renormalization — EFT amplitudes

Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

IR divergences

Infrared structure of amplitudes:

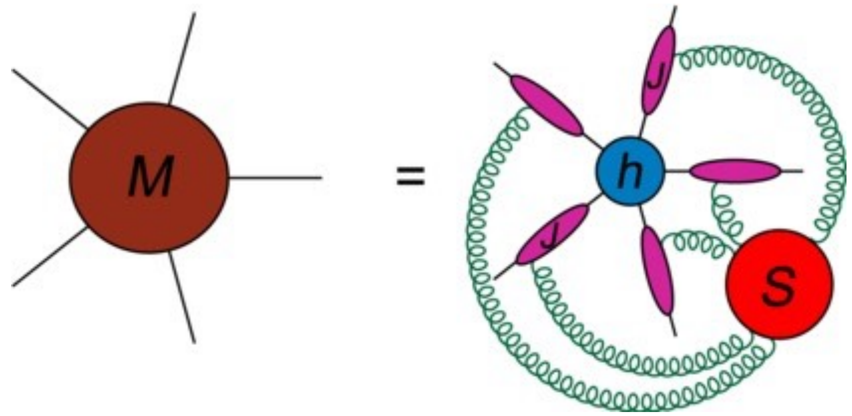


figure from L. Dixon 1105.0771

For modern dim-reg representation, see:
Magnea and Sterman 1990;
Catani 1998,
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

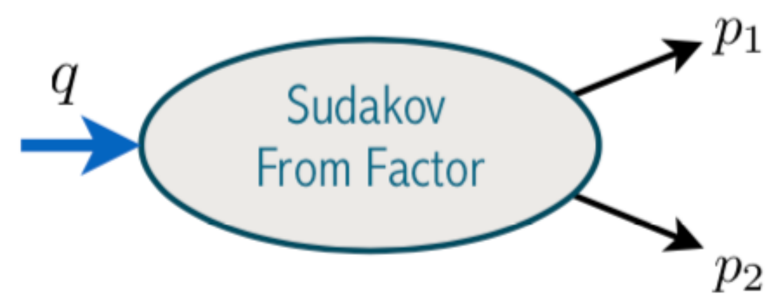
$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n \left(k_i, \mu, \alpha_s, \epsilon \right)$$

Sudakov form factor $= \exp\left[-\frac{1}{4} \sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2} \right)^{l\epsilon} \left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \right) + \text{finite} \right]$

Leading IR singularity -> Cusp anomalous dimension

Sudakov form factor

$$\mathcal{F} = \int d^4x \, e^{-ix\cdot q} \langle p_1, p_2 | \mathcal{O}(x) | 0 \rangle$$



$$p_1^2 = p_2^2 = 0$$

$$q^2 = (p_1 + p_2)^2$$

Logarithm behavior is well-understood:

For dim-reg representation, see:
Magnea and Sterman 1990;
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

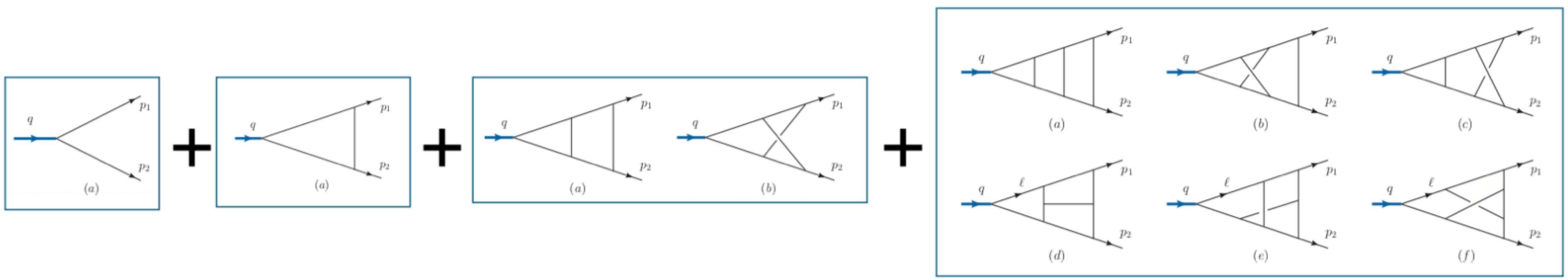
$$\log f = \sum g^{2l} (\log f)^{(l)} = -\sum_l g^{2l} (-q^2)^{-l\epsilon} \left[\frac{\gamma^{(l)}_{\text{cusp}}}{(2l\epsilon)^2} + \frac{\mathcal{G}^{(l)}_{\text{coll}}}{2l\epsilon} + \text{Fin}^{(l)} \right] + \mathcal{O}(\epsilon)$$

Leading IR singularity -> Cusp anomalous dimension

Loop form factors

Diagram-expansion up to 3 loops

$$\mathscr{F}^{(l)} = \mathscr{F}^{\text{tree}} \sum_{l=1}^{\infty} g^{2l}(-q^2)^{-l\epsilon} F^{(l)}$$

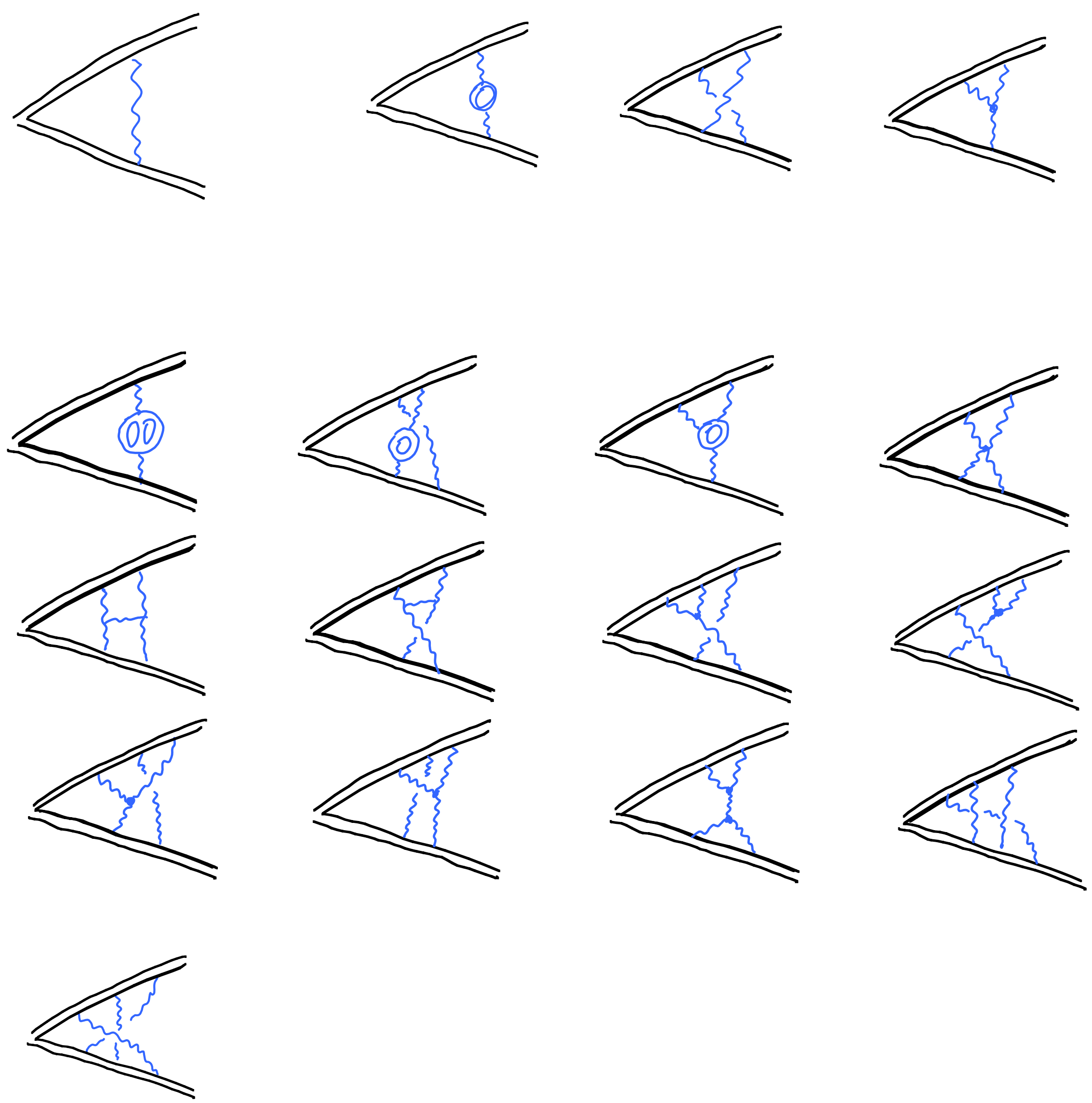


(N=4 SYM case)

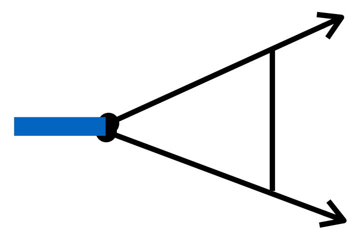
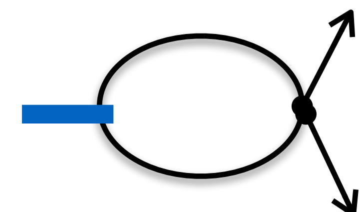
Basis	Numerator factor	Color factor	Symmetry factor
(a)	s_{12}^2	$8\,N_c^3\,\delta^{a_1a_2}$	2
(b)	s_{12}^2	$4\,N_c^3\,\delta^{a_1a_2}$	4
(c)	s_{12}^2	$4\,N_c^3\,\delta^{a_1a_2}$	4
(d)	$(p_2-p_1)\cdot\ell - p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1a_2}$	2
(e)	$-(p_2-p_1)\cdot\ell + p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1a_2}$	1
(f)	$(p_2-p_1)\cdot\ell - p_1\cdot p_2$	0	2

Wilson line computation

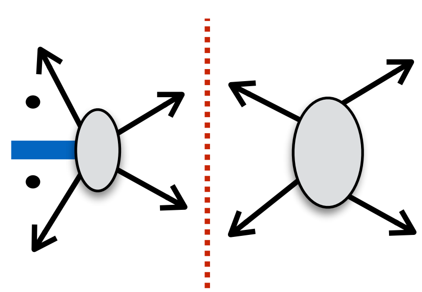
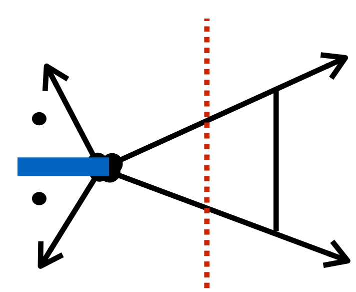
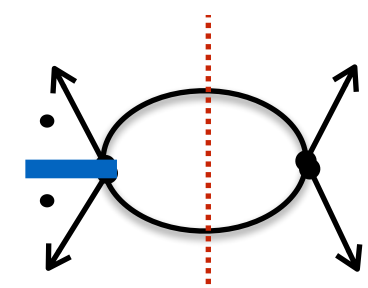
Web graphs via non-abelian exponential theorem: [Gatheral 1983; Frenkel and Taylor 1984]



Unitarity computation

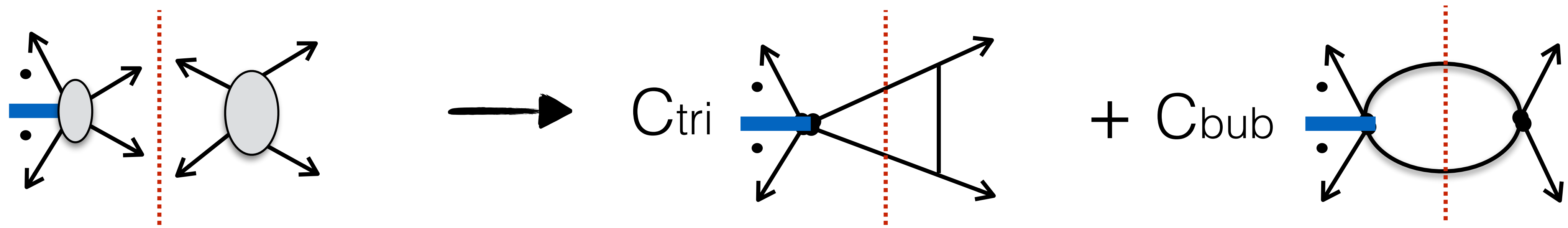
$F_2^{(1)} =$ C_{tri}  $+ \, C_{bub}$ 

The basis coefficient can be computed by cuts:

 → C_{tri}  $+ \, C_{bub}$ 

$$\mathcal{F}_2^{(1)}(1,2)\Big|_{s_{12}\text{-cut}} = \int d\mathrm{PS}_2\, \mathcal{F}_2^{(0)}(-l_1, -l_2)\mathcal{A}_4^{(0)}(1,2,l_2,l_1)$$

Unitarity computation



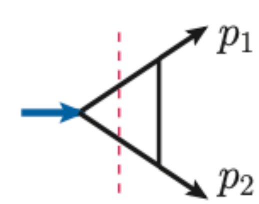
$$\mathcal{F}_2^{(1)}(1,2)\big|_{s_{12}\text{-cut}} = = \int d\mathrm{PS}_2 \sum_{\text{helicity of } l_i} \mathcal{F}_2^{(0)}(-l_1,-l_2) \times \mathcal{A}_4^{(0)}(1,2,l_2,l_1)$$

$$= \mathcal{F}_2^{(0)}(1,2)\, i \int d\mathrm{PS}_2\, 1 \times \frac{\langle l_1 l_2\rangle\langle 12\rangle}{\langle l_1 p_1\rangle\langle l_2 2\rangle}$$

$$\mathcal{O} = \mathrm{tr}(\phi_{12}^2)$$

$$= \mathcal{F}_2^{(0)}(1,2)\, i \int d\mathrm{PS}_2\, \frac{-s_{12}}{(l_1+p_1)^2}$$

$$= \mathcal{F}_2^{(0)}(1,2)(-s_{12})$$

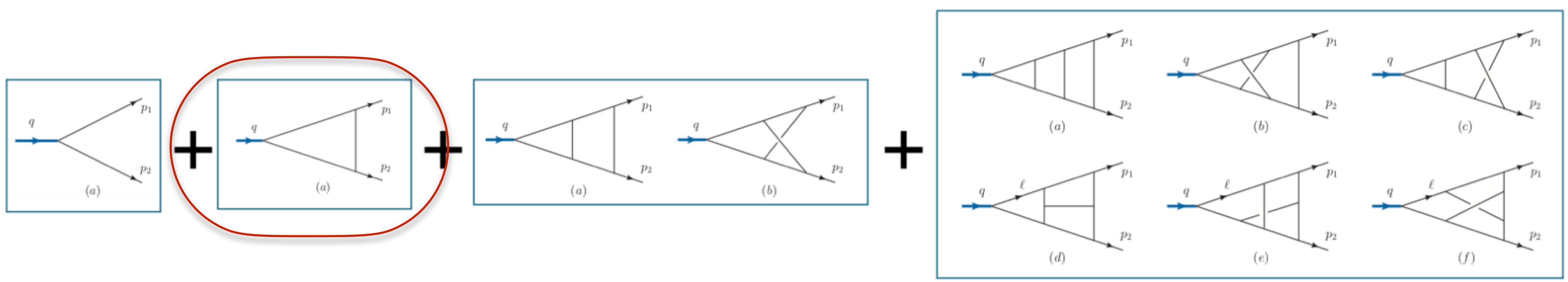


$$\longrightarrow \quad C_{\mathrm{tri}} = -\, s_{12}\,, \qquad C_{\mathrm{bub}} = 0$$

Loop form factors

Diagram-expansion up to 3 loops

$$\mathscr{F}^{(l)} = \mathscr{F}^{\text{tree}} \sum_{l=1}^{\infty} g^{2l} (-q^2)^{-l\epsilon} F^{(l)}$$



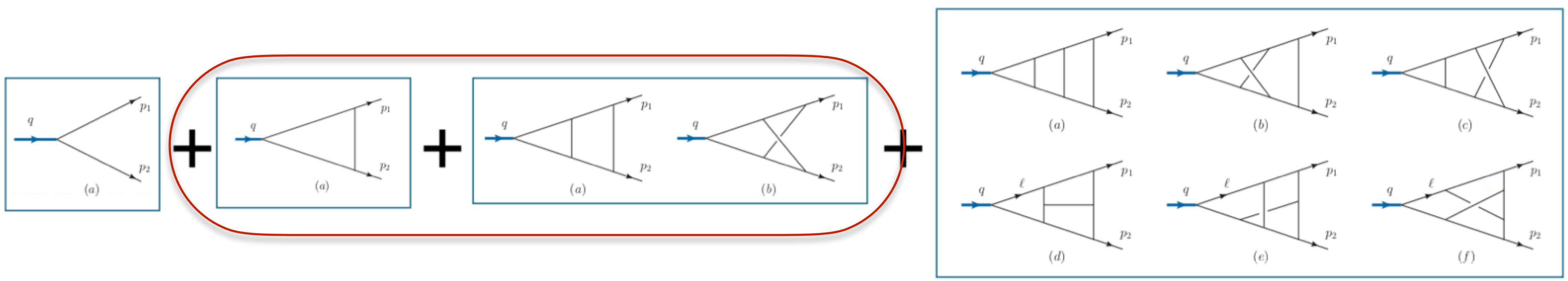
$$(\log f)^{(1)} = f^{(1)} = (-2s) I_3^{(1)} = (-s)^{-\epsilon} \left[-\frac{2}{\epsilon^2} + \mathcal{O}(\epsilon^0) \right]$$

$$\longrightarrow \qquad \gamma_{\text{cusp}}^{(1)} = 8 \,, \qquad \mathscr{G}_{\text{coll}}^{(1)} = 0$$

Loop form factors

Diagram-expansion up to 3 loops

$$\mathcal{F}^{(l)} = \mathcal{F}^{\text{tree}} \sum_{l=1}^{\infty} g^{2l} (-q^2)^{-l\epsilon} F^{(l)}$$



$$(\log f)^{(2)} = f^{(2)} - \frac{1}{2}(f^{(1)})^2 = s^2\left(4I^{(2)}_{\rm PL} + I^{(2)}_{\rm CL}\right) - \frac{1}{2}\left((-2s)I^{(1)}_3\right)^2$$
$$= (-s)^{-2\epsilon}\left[\frac{\zeta_2}{\epsilon^2} + \frac{\zeta_3}{\epsilon} + \mathcal{O}(\epsilon^0)\right]$$

$$\longrightarrow \qquad \gamma^{(2)}_{\rm cusp} = -16\zeta_2\,, \qquad G^{(2)}_{\rm coll} = -4\zeta_3$$

Color structure

Up to three loops, only quadratic Casimir appears:

L-loop	L=1	L=2	L=3
Color Factor	C_A	C_A^2	C_A^3

- Planar (Large Nc) limit is relatively well understood:

Known to all order in N=4 SYM

[Beisert, Eden, Staudacher 2006]

$$K_{ij} = j(-1)^{i(j+1)} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}$$

$$\Gamma_{\text{cusp}} = 4g^2 \left(\frac{1}{1+K}\right)_{11}$$

Casimir scaling conjecture

On the Structure of Infrared Singularities of Gauge-Theory Amplitudes

Thomas Becher [a] and Matthias Neubert [b]

[a] *Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510, U.S.A.*
[b] *Institut für Physik (THEP), Johannes Gutenberg-Universität, D-55099 Mainz, Germany*

Becher and Neubert (JHEP 2009)

Abstract

A closed formula is obtained for the infrared singularities of dimensionally regularized, massless gauge-theory scattering amplitudes with an arbitrary number of legs and loops. It follows from an all-order conjecture for the anomalous-dimension matrix of n-jet operators in soft-collinear effective theory. We show that the form of this anomalous dimension is severely constrained by soft-collinear factorization, non-abelian exponentiation, and the behavior of amplitudes in collinear limits. Using a diagrammatic analysis, we demonstrate that these constraints imply that to three-loop order the anomalous dimension involves only two-parton correlations, with the possible exception of a single color structure multiplying a function of conformal cross ratios depending on the momenta of four external partons, which would have to vanish in all two-particle collinear limits. We suggest that such a function does not appear at three-loop order, and that the same is true in higher orders. Our formula predicts Casimir scaling of the cusp anomalous dimension to all orders in perturbation theory, and we explicitly check that the constraints exclude the appearance of higher Casimir invariants at four loops. Using known results for the quark and gluon form factors, we derive the three-loop coefficients of the $1/\epsilon^n$ pole terms (with $n = 1, \ldots, 6$) for an arbitrary n-parton scattering amplitude in massless QCD. This generalizes Catani's two-loop formula proposed in 1998.

An explicit four-loop computation is needed.

Casimir scaling conjecture

Conjecture on quadratic Casimir scaling:

the non-planar corrections is zero to all orders in perturbative theory, based on lower loop results and effective theory (SCET) arguments.

[Becher, Neubert 2009]
see also [Gardi, Magnea 2009]

Counter arguments:

- expected to be violated [Alday, Maldacena 2009]
- break down at strong coupling [Armoni 2006]
- break down through instanton corrections [Korchemsky 2017]

An explicit perturbative computation is highly desired.

Why non-planar is difficult

- Leading order is at four loops!

At four-loop, there is a new quartic Casimir which contains non-planar part:

L-loop	L=1	L=2	L=3	L=4
Color Factor	C_A	C_A^2	C_A^3	$C_A^4\,, d_{44}$

For $SU(N)$:

$$C_A = N$$

$$d_{44} = \frac{N^2(N^2+36)}{24}$$

- Integrability methods is not applicable (yet)

We need to do an *"honest"* four-loop computation:

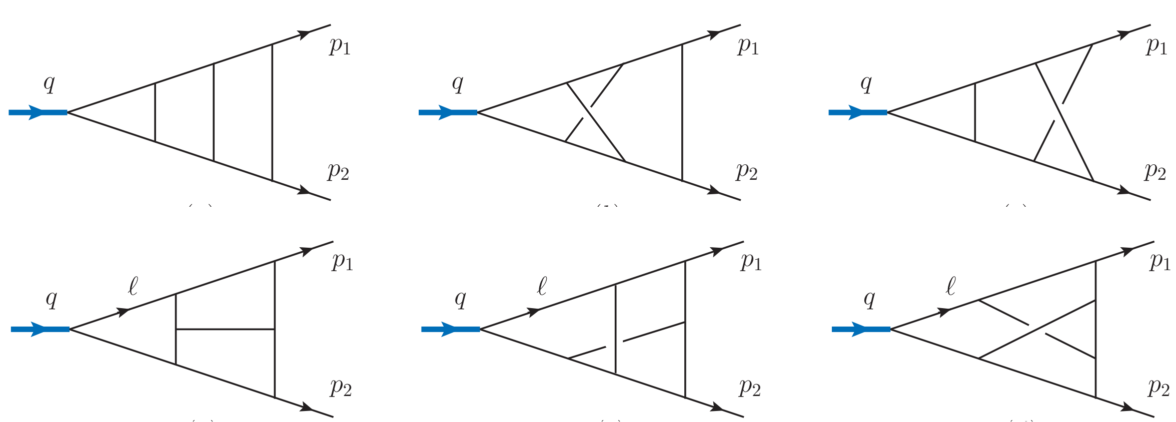
- both integrand and integrals are very complicated

Traditional approach

- generate an integrand, e.g. by Feynman graphs
- simplify the integrand, e.g. PV, IBP reduction methods
- compute the master integrals analytically or numerically

Three-loop Sudakov form factors in QCD were known since 2009:

[Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser 2009], …

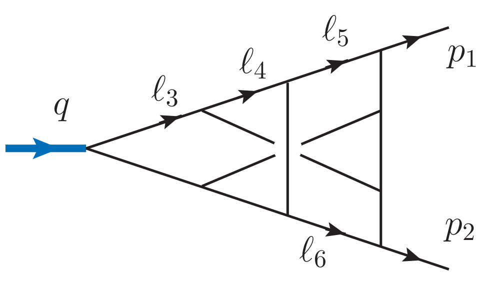


Sudakov form factor

Four-loop computation is much more challenging.

- Integrand

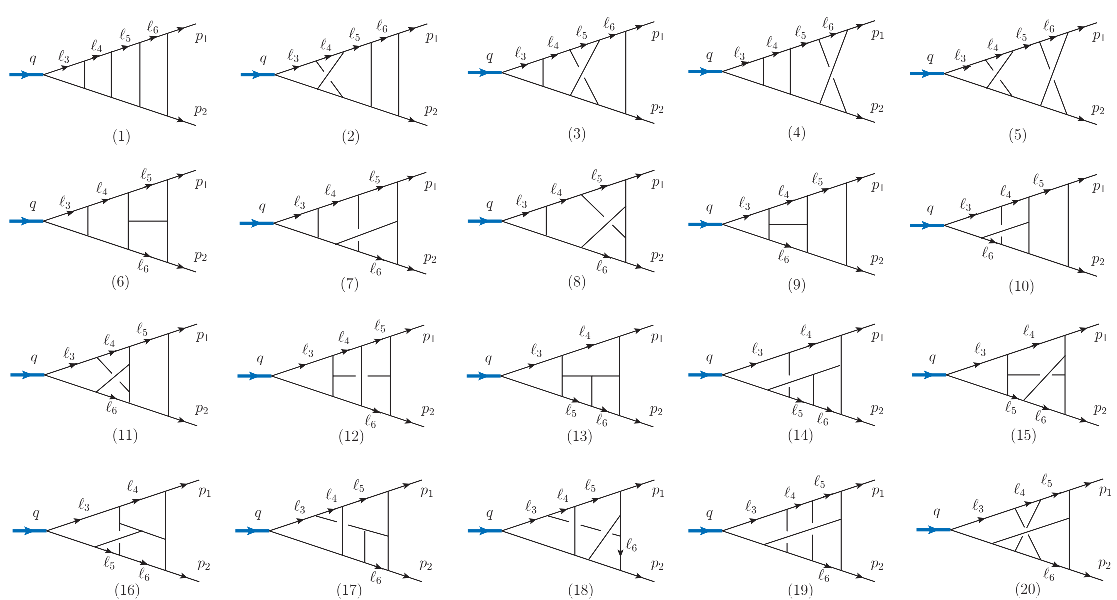
- Integration

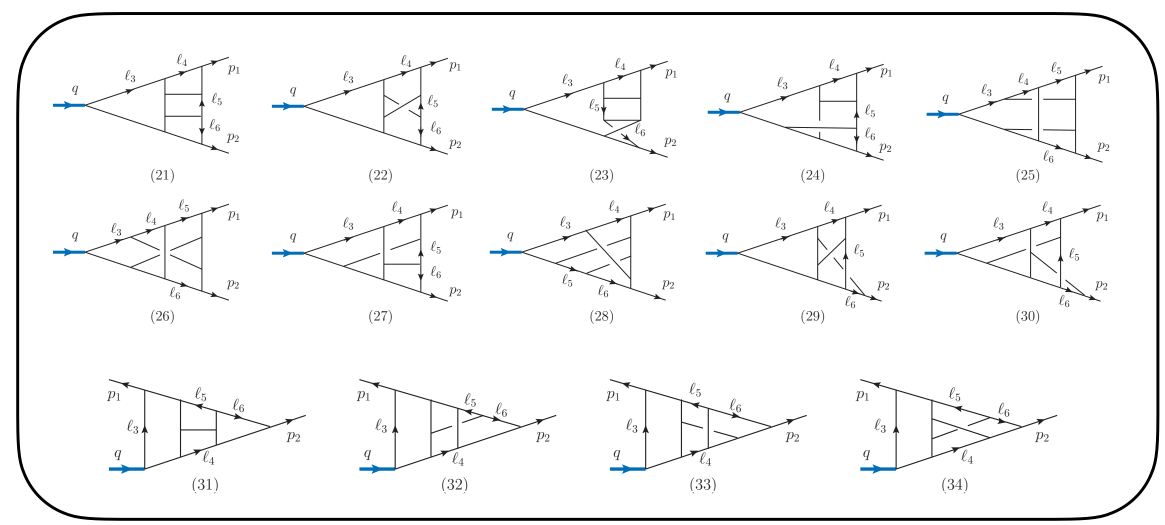


Four-loop integrand with NO Feynman diagrams

[Boels, Kniehl, Tarasov, GY 2012]

Four-loop form factor integrand was obtained by:
color-kinematics duality and unitarity:





non-planar

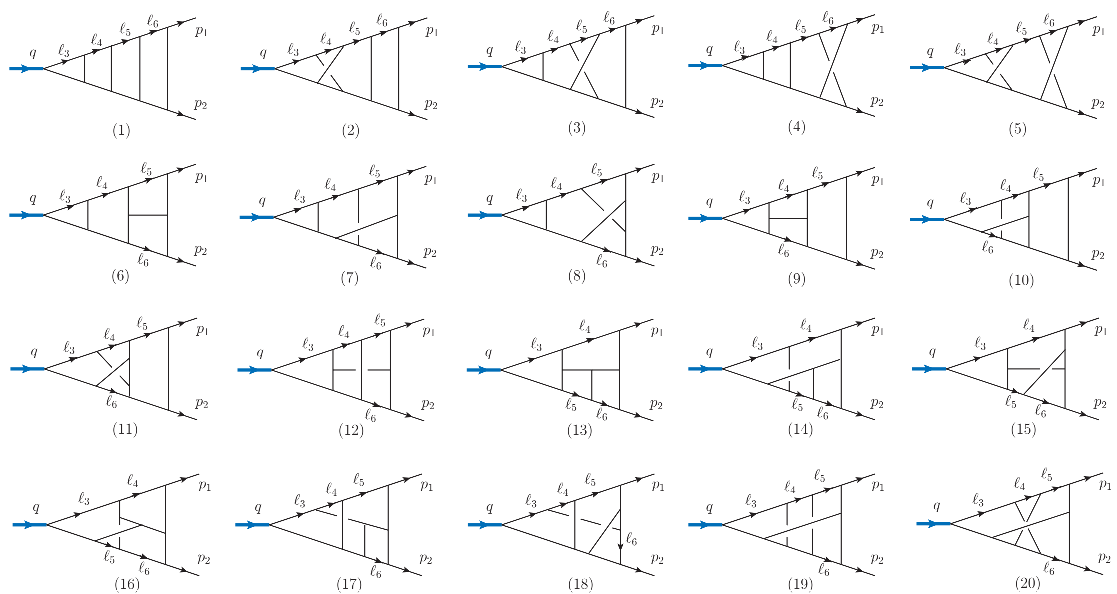
compact form and with only quadratic loop momenta in the numerator.

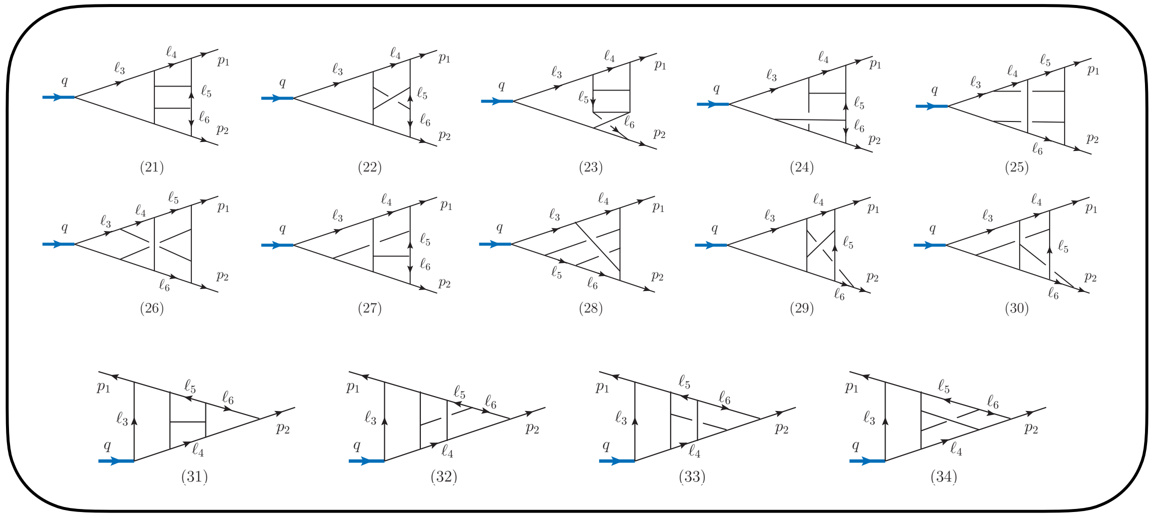
$$\begin{aligned} N_{21} = \; & -(\ell_3 \cdot p_1)^2 - (\ell_3 \cdot p_2)^2 - 6(\ell_3 \cdot p_1)(\ell_3 \cdot p_2) \\ & +(p_1 \cdot p_2)\big[2(\ell_3 \cdot \ell_3) + 4(\ell_3 \cdot p_1) + p_1 \cdot p_2\big] \\ & +(\alpha_1 + 1)\big[(\ell_3 \cdot p_{12} - p_1 \cdot p_2)^2 \\ & -\tfrac{2}{7}(\ell_3 \cdot (\ell_3 - p_{12}) + p_1 \cdot p_2)(p_1 \cdot p_2)\big] \end{aligned}$$

Four-loop integrand with NO Feynman diagrams

[Boels, Kniehl, Tarasov, GY 2012]

Four-loop form factor integrand was obtained by:
color-kinematics duality and unitarity:

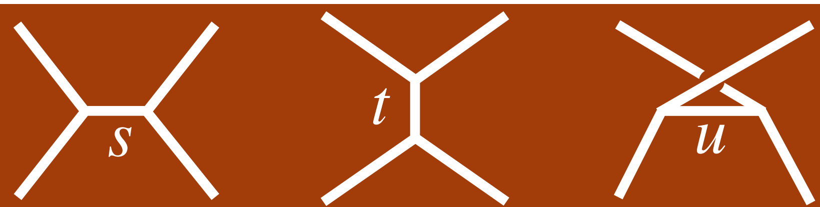




non-planar

Full five-loop integrand has also been obtained. >200 topologies [GY 2016]

Color-kinematics duality



$$A_4 = \frac{C_s N_s}{s} + \frac{C_t N_t}{t} + \frac{C_u N_u}{u}$$

Color factors

$$C_s = C_t + C_u$$

Jacobian identity

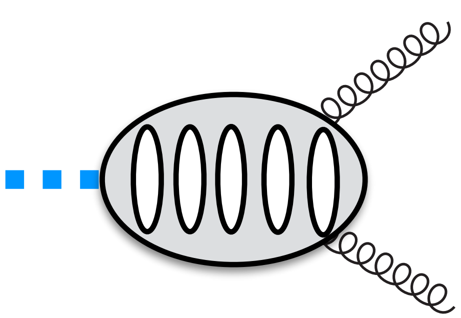
→

Momentum factors

$$N_s = N_t + N_u$$

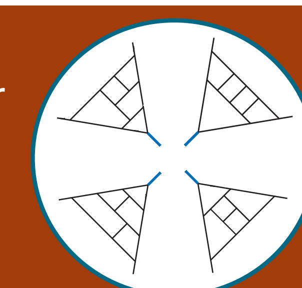
dual Jacobian relation

Large number of diagrams —CK-duality→ Very few “master” diagrams



L-loop	*L=1*	*L=2*	*L=3*	*L=4*	*L=5*
# of topologies	*1*	*2*	*6*	*34*	*306*
# of masters	*1*	*1*	*1*	*2*	*4*

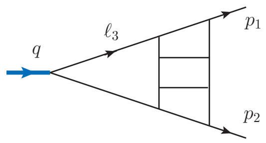
Four master graphs @ 5-loop:



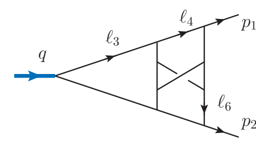
GY 2016

Four-loop non-planar cusp AD

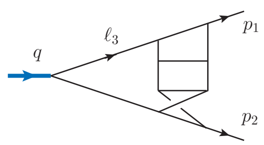
[Boels, Huber, GY 2017]

$I_1^{(21)} =$ 

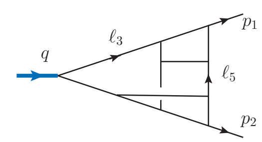
$\times\ [(\ell_3 - p_1)^2]^2$

$I_2^{(22)} =$ 

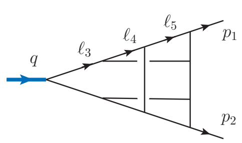
$\times\ (\ell_3 - p_1)^2\,[\ell_4^2 + \ell_6^2 - \ell_3^2 + (\ell_3 - \ell_4 + p_1)^2 + (\ell_3 - \ell_6 - p_1)^2]$

$I_3^{(23)} =$ 

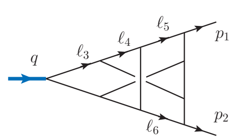
$\times\ [(\ell_3 - p_1)^2]^2$

$I_4^{(24)} =$ 

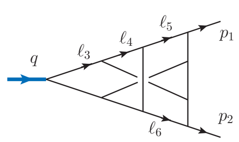
$\times\ (\ell_3 - p_1)^2\,[(q - \ell_3 - \ell_5)^2 + (\ell_5 + p_2)^2]$

$I_5^{(25)} =$ 

$\times\ \left\{\,\left[(p_1 - \ell_5)^2 + 2(\ell_4 - \ell_5)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - \ell_5)^2 - (p_1 - \ell_4)^2\right]^2 - 4\,(\ell_4 - \ell_5)^2\,(p_1 - \ell_3 + \ell_4 - \ell_5)^2\right\}$

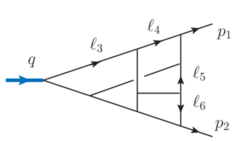
$I_6^{(26)} =$ 

$\times\ \left\{[(\ell_3 - \ell_4 - \ell_5)^2 - (\ell_3 - \ell_4 - p_1)^2 - (\ell_6 - p_2)^2 - \ell_5^2]\,[\ell_5^2 - \ell_4^2 - \ell_6^2 + (\ell_4 - \ell_6)^2] + 4\,\ell_5^2\,(\ell_6 - p_2)^2 + (\ell_4 - \ell_5)^2\,(\ell_3 - \ell_4 + \ell_6 - p_2)^2\right\}$

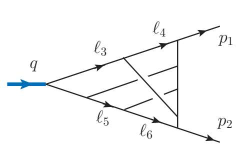
$I_7^{(26)} =$ 

$\times\ \Big\{4\,[(\ell_4 - \ell_5)(\ell_3 - \ell_4 + \ell_5 - p_1)]\,[(\ell_4 - \ell_6)(\ell_3 - \ell_4 + \ell_6 - p_2)]\ - \ell_5^2\,(\ell_6 - p_2)^2 - 4\,(\ell_4 - \ell_5)^2\,(\ell_3 - \ell_4 + \ell_6 - p_2)^2 - \ell_6^2\,(\ell_5 - p_1)^2$

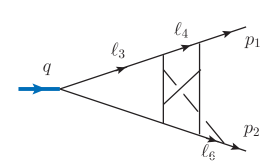
$- (\ell_3 - \ell_4)^2\,(\ell_5 + \ell_6 - \ell_4)^2 - \ell_4^2\,(\ell_3 - \ell_4 + \ell_5 + \ell_6 - p_1 - p_2)^2\Big\}$

$I_8^{(27)} =$ 

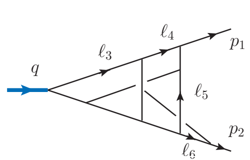
$\times\ \frac{1}{2}\,[\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2]\ \ [(\ell_3 - \ell_4 - \ell_5)^2 + (\ell_5 + p_2)^2]$

$I_9^{(28)} =$ 

$\times\ (\ell_3 - \ell_4 - p_2)^2\ \ [(\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]$

$I_{10}^{(29)} =$ 

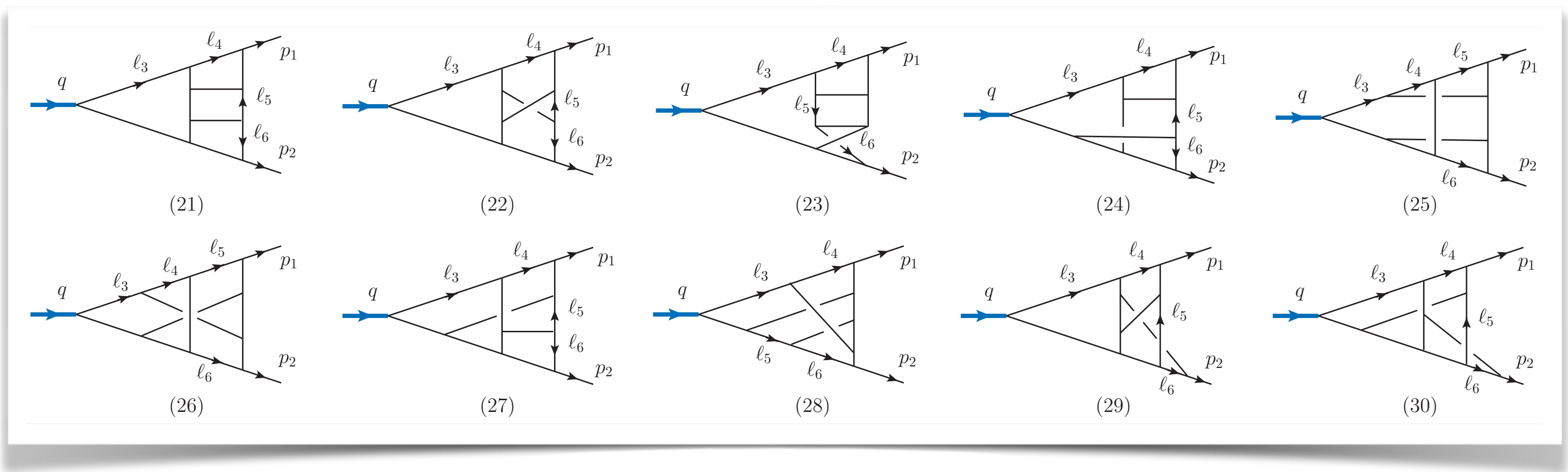
$\times\ \frac{1}{2}\,[\ell_3^2 - \ell_4^2 - (\ell_4 - \ell_3 - p_1)^2]\ \ [\ell_6 \cdot (\ell_6 - \ell_4 + \ell_3 - p_2)]$

$I_{11}^{(30)} =$ 

$\times\ (\ell_3 - \ell_4 - p_2)^2\,[(p_1 - \ell_4)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]$

Plus 12 simpler 11- and 10-line integrals

Four-loop non-planar cusp AD



- Four-loop integration took five years (using UT basis):

$$\gamma_{\text{cusp}} = 8g^2 - 16\zeta_2 g^4 + 176\zeta_4 g^6 + \Big(+ \gamma^{(4)}_{\text{cusp,P}} + \gamma^{(4)}_{\text{cusp,NP}} \Big) g^8 + \mathcal{O}(g^{10}) \qquad \gamma^{(4)}_{\text{cusp,P}} = -1752\zeta_6 - 64\zeta_3^2$$

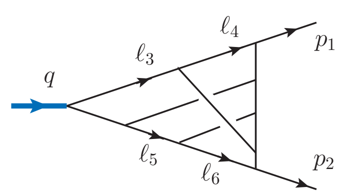
$$\boxed{\gamma^{(4)}_{\text{cusp, NP}} = -3072 \times (1.60 \pm 0.19) \frac{1}{N_c^2}} \longrightarrow$$ Casimir scaling conjecture is wrong

Boels, Huber, GY 2017

Non-trivial consistency check

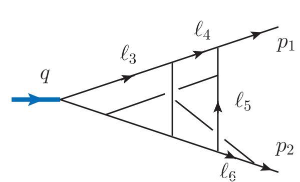
$$F^{(4)}_{\mathrm{NP}} = -\frac{\gamma^{(4)}_{\mathrm{cusp,\ NP}}}{(8\epsilon)^2} - \frac{\mathcal{G}^{(4)}_{\mathrm{coll,NP}}}{8\epsilon} - \mathrm{Fin}^{(4)}_{\mathrm{NP}} + \mathcal{O}(\epsilon)$$ starts at ϵ^{-2}

Most basis integrals start at ϵ^{-8} order, e.g.

$I^{(28)}_{9} =$ 

$\times\ (\ell_3 - \ell_4 - p_2)^2\ \left[(\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2\right]$

$$= -\frac{0.0104167}{\epsilon^8} + \frac{0.000000002(253)}{\epsilon^7} + \frac{0.554023(5)}{\epsilon^6} + \frac{2.26219(5)}{\epsilon^5} - \frac{3.56367(64)}{\epsilon^4} - \frac{60.6800(73)}{\epsilon^3} - \frac{182.180(84)}{\epsilon^2}$$

$I^{(30)}_{11} =$ 

$\times\ (\ell_3 - \ell_4 - p_2)^2\ [(p_1 - \ell_4)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2]$

$$= \frac{0.00347222}{\epsilon^8} - \frac{0.05140419}{\epsilon^6} - \frac{0.2601674}{\epsilon^5} - \frac{1.5145009}{\epsilon^4} - \frac{17.34721164(4)}{\epsilon^3} - \frac{133.31287(3)}{\epsilon^2}$$

All poles up to ϵ^{-3} order should cancel !

Results

The full form factor result is:

ϵ order	-8	-7	-6	-5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	-4	-3	-2	-1
result	$+3.5 \times 10^{-6}$	$+\ 0.0007$	$+1.60$	-17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	$\pm\ 0.0186$	± 0.19	$\pm\ 3.25$

Indeed, cancellation for all poles up to ϵ^{-3} order

Results

The full form factor result is:

ϵ order	-8	-7	-6	-5
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}
uncertainty	–	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$

ϵ order	-4	-3	-2	-1
result	$+3.5 \times 10^{-6}$	$+\ 0.0007$	$+1.60$	-17.98
uncertainty	$\pm 1.5 \times 10^{-3}$	$\pm\ 0.0186$	± 0.19	$\pm\ 3.25$

- Four-loop non-planar cusp AD:

Boels, Huber, GY 2017

$$\gamma^{(4)}_{\text{cusp, NP}} = -3072 \times (1.60 \pm 0.19)\frac{1}{N_c^2}$$

- Analytic result in 2019:

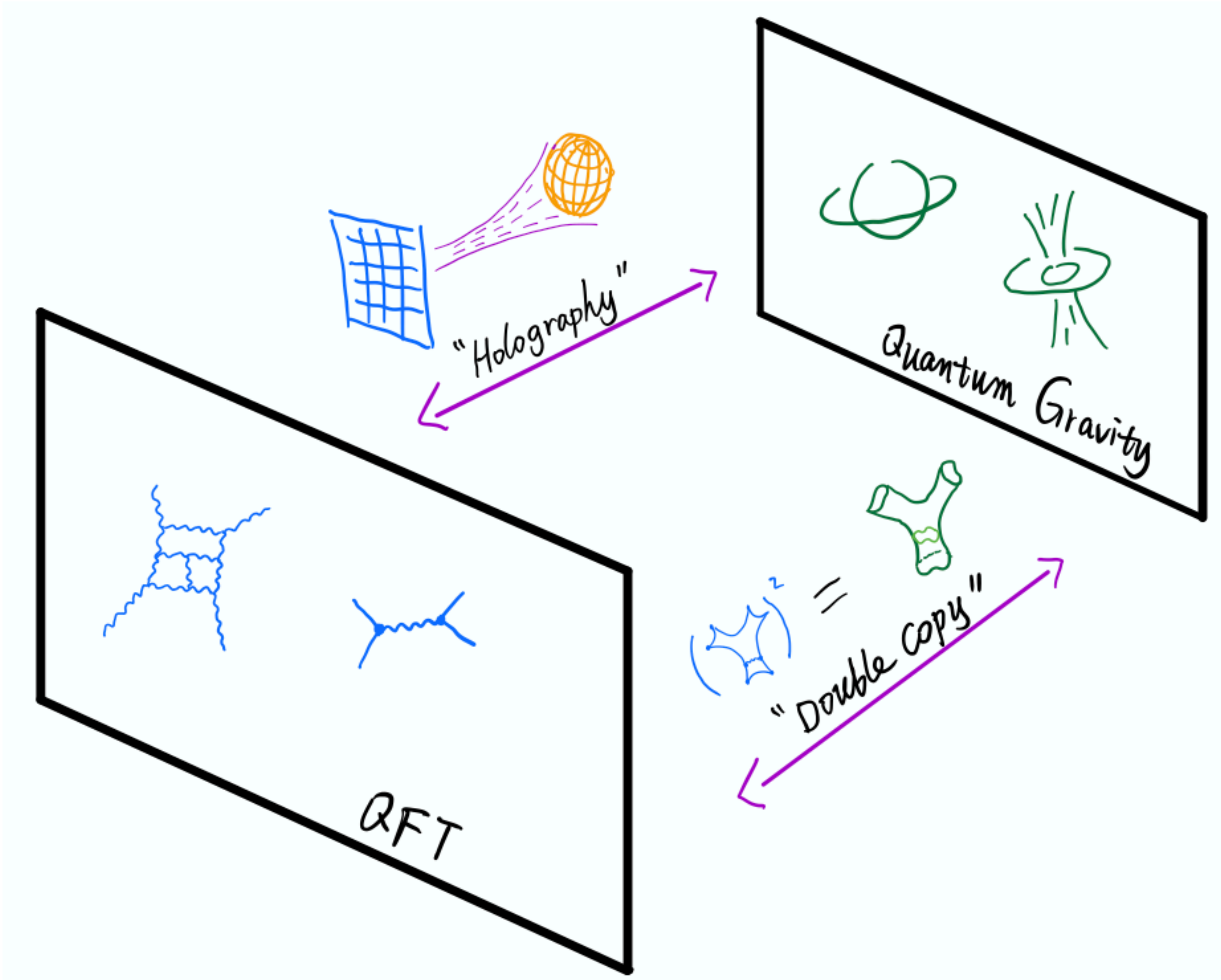
Huber, von Manteuffel, Panzer, Schabinger, GY 2019; Henn, Korchemsky, Mistlberger 2019

$$\gamma^{(4)}_{\text{cusp,NP}} = -3072 \times (\frac{3}{8}\zeta_3^2 + \frac{31}{140}\zeta_2^3)\frac{1}{N_c^2} = -3072 \times 1.52\frac{1}{N_c^2}$$

Outline

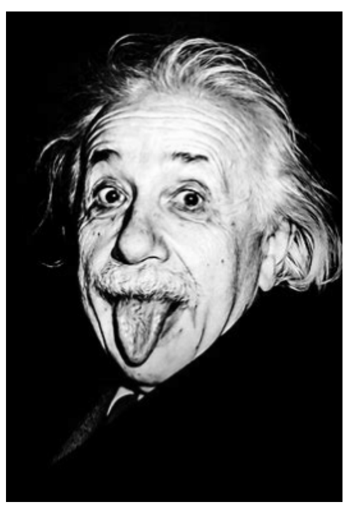
- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

Gauge-gravity correspondence



Double copy

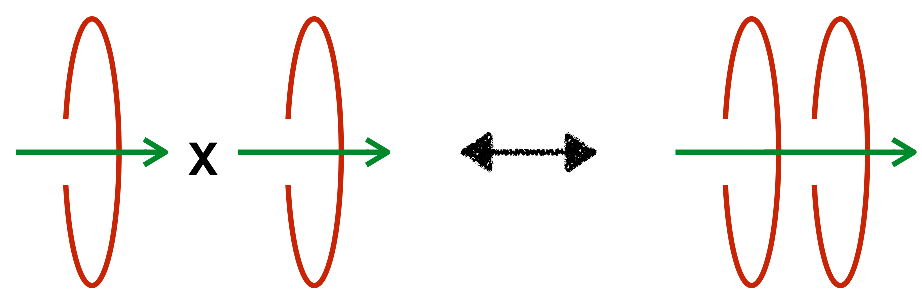
Gravity ⟷ (Yang-Mills)^2



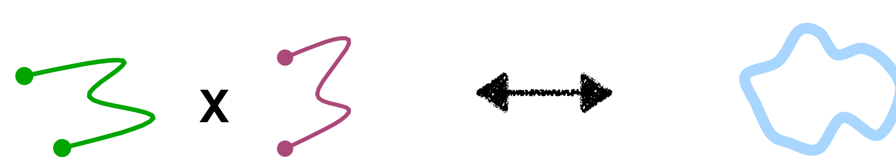
x

Double copy

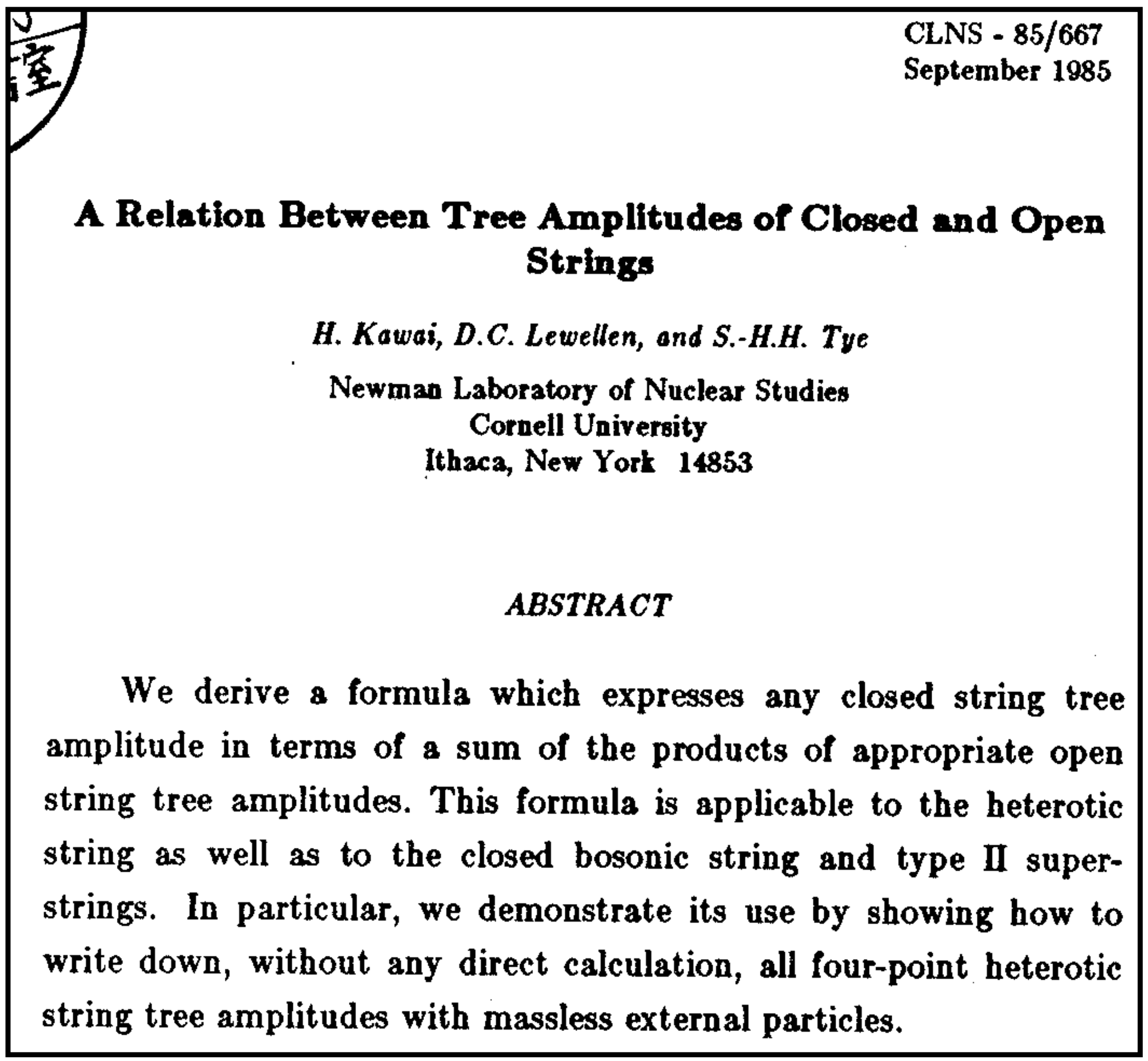
Gravity ⟷ (Yang-Mills)^2



(spin-1)^2 Spin-2



KLT relation



CLNS - 85/667
September 1985

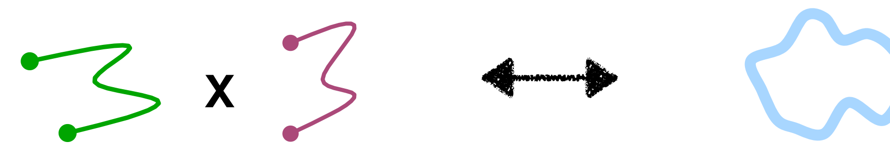
A Relation Between Tree Amplitudes of Closed and Open Strings

H. Kawai, D.C. Lewellen, and S.-H.H. Tye

Newman Laboratory of Nuclear Studies
Cornell University
Ithaca, New York 14853

ABSTRACT

We derive a formula which expresses any closed string tree amplitude in terms of a sum of the products of appropriate open string tree amplitudes. This formula is applicable to the heterotic string as well as to the closed bosonic string and type II superstrings. In particular, we demonstrate its use by showing how to write down, without any direct calculation, all four-point heterotic string tree amplitudes with massless external particles.



$$A^{(4)}_{closed} = -\pi\kappa^2 \sin(\pi\kappa_1\cdot\kappa_2) A^{(4)}_{open}(s,t)\ \bar{A}^{(4)}_{open}(s,u)$$

$$A^{(5)}_{closed} = \pi\kappa^3 A^{(5)}_{open}(12345)\bar{A}^{(5)}_{open}(21435)\sin(\pi\kappa_1\cdot\kappa_2)\sin(\pi\kappa_3\cdot\kappa_4) + \pi\kappa^3\ A^{(5)}_{open}(13245)\bar{A}^{(5)}_{open}(31425)\sin(\pi k_1\cdot k_3)\sin(\pi k_2\cdot k_4).$$

↓ Field theory limit

$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)A_5^{\rm tree}(2,1,4,3,5) + is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

KLT works at tree level. New ideas are needed for loop level.

Color-kinematics duality

An intriguing duality between color and kinematic factors for gauge amplitudes was discovered in 2008: [Bern, Carrasco, Johansson 2008]

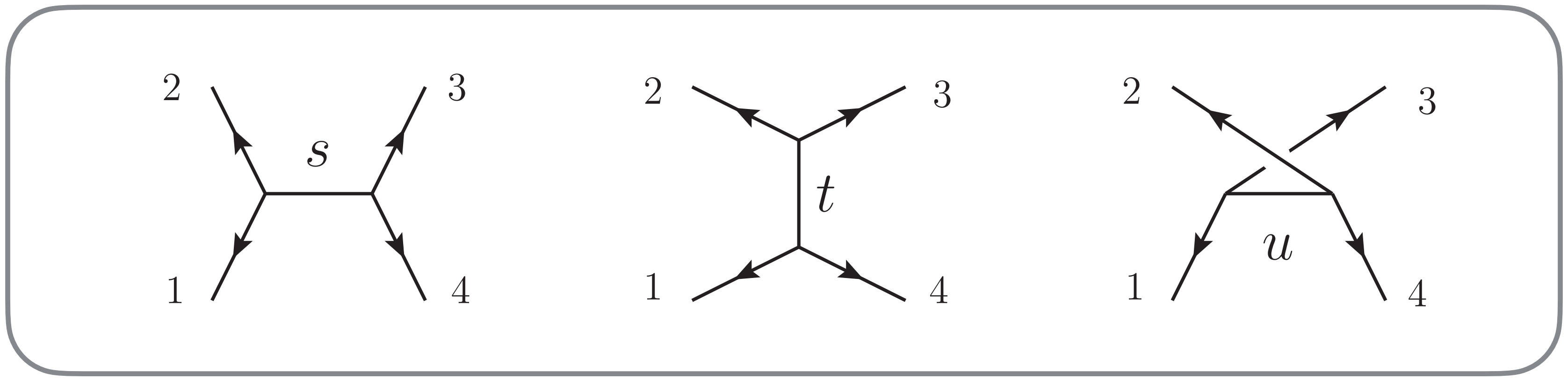
Color factor — Duality (conjecture) — Kinematic factor

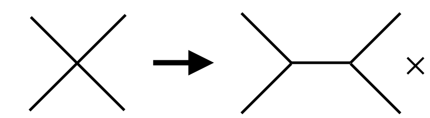
$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c)$$

$$s_{ij} = (p_i + p_j)^2$$

Gauge symmetry — **Spacetime symmetry**

Example: 4-pt amplitude





$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_2 a_4}$$

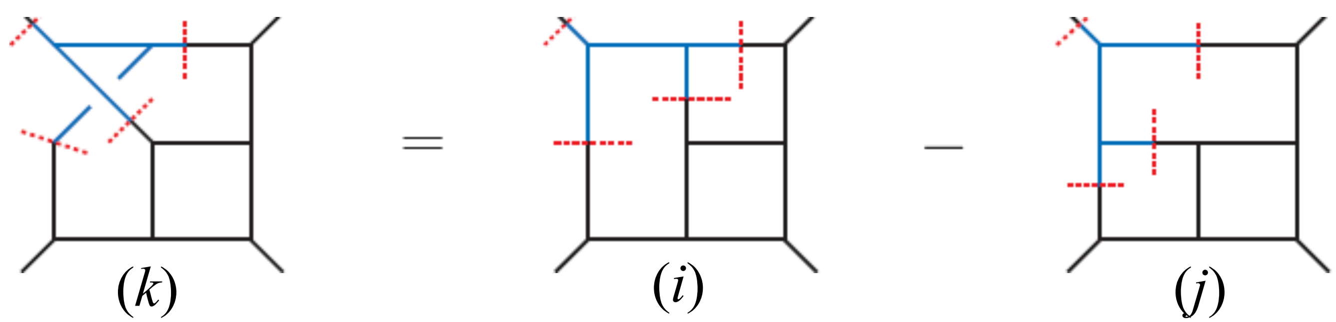
$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity

dual Jacobi relation

CK duality at loop level

$$A^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$ = Sum of many trivalent topologies



$$C_k = C_i - C_j \longrightarrow N_k = N_i - N_j$$

Jacobi identity

dual Jacobi relation

From YM to gravity

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \quad \longrightarrow \quad M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

From YM to gravity

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

Gauge invariance $\xrightarrow{\text{CK-duality}}$ Diffeomorphism invariance

$$\varepsilon_i^\mu \to \varepsilon_i^\mu + p_i^\mu \qquad \varepsilon_i^{\mu\nu} \to \varepsilon_i^{\mu\nu} + p_i^{(\mu} q^{\nu)}$$

$n_i \to n_i + \delta_i\,,$

$\delta_i = n_i|_{\varepsilon_j \to p_j}$

$$\sum_i \frac{c_i \delta_i}{D_i} = 0 \longrightarrow \sum_i \frac{n_i \delta_i}{D_i} = 0$$

$$c_i = c_j + c_k \qquad n_i = n_j + n_k$$

Double copy

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

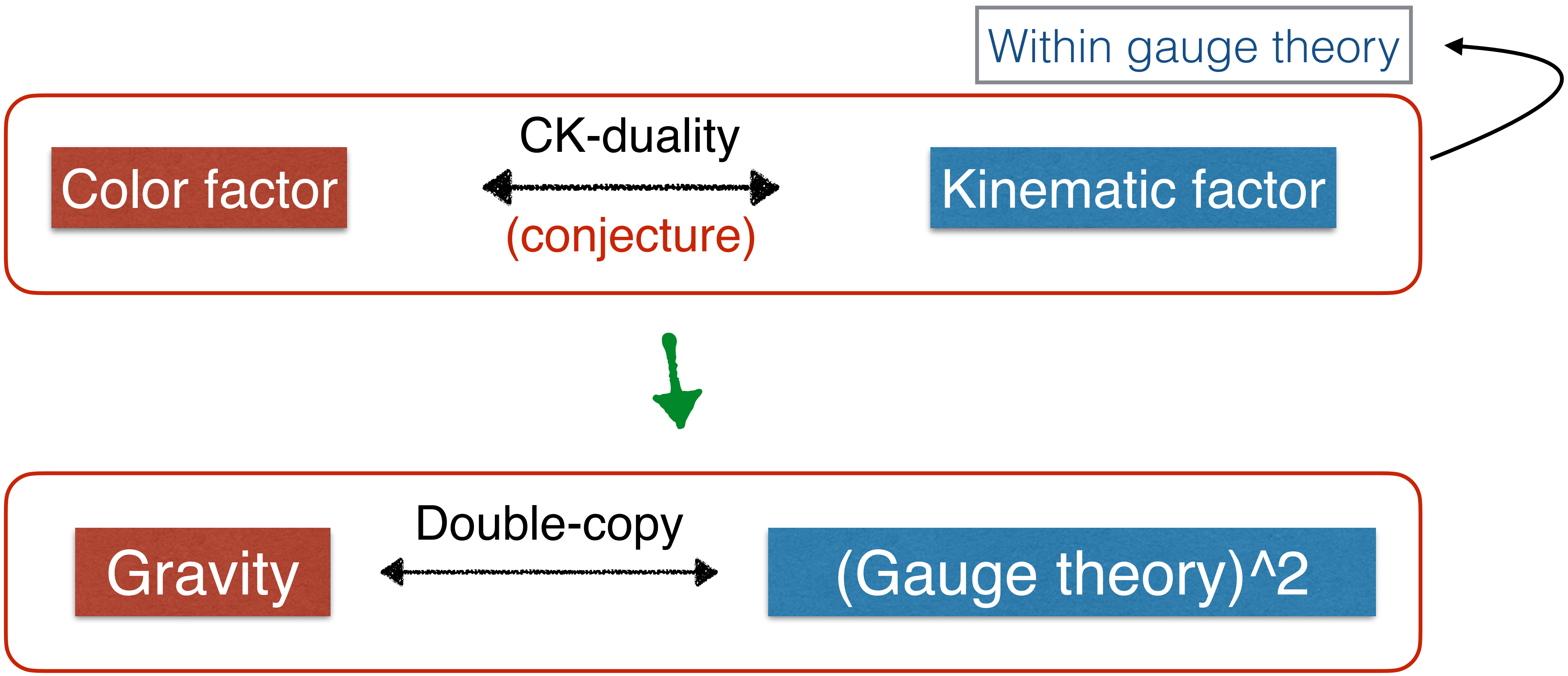
$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

It can be generalized to high loops:

$$A^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D} \longrightarrow M^{(\ell)} \sim \sum_i \int \frac{N_i \times N_i}{\prod D}$$

Gauge x Gauge $\xrightarrow{\text{CK-duality}}$ Gravity

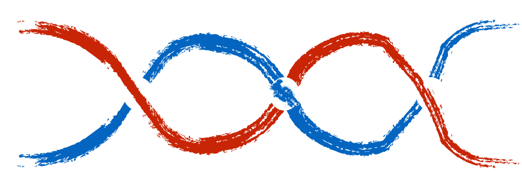
CK-duality v.s. Double-copy



By studying the simpler gauge theory, one may understand the far more complicated gravity theory.

How to construct color-kinematics duality ?

CK-duality



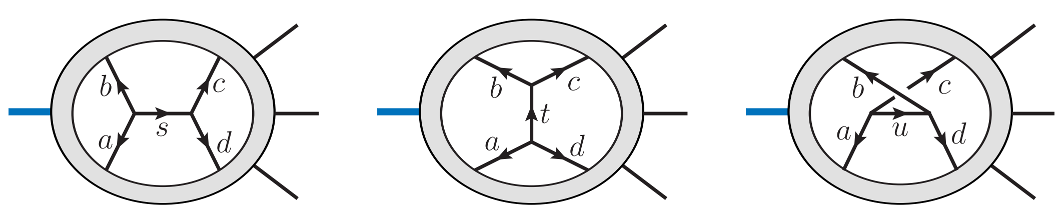
Unitarity cuts

Strategy of loop computation

CK-duality

$$\mathscr{F}^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$

Compact ansatz of the loop integrand

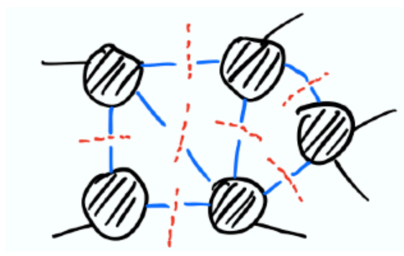


$$C_s = C_t + C_u \longrightarrow N_s = N_t + N_u$$

Strategy of loop computation

CK-duality

Compact ansatz of the loop integrand



Unitarity cuts

$$\text{Loop-ansatz}\,|_{\text{cut}} = \prod \text{tree-blocks}$$

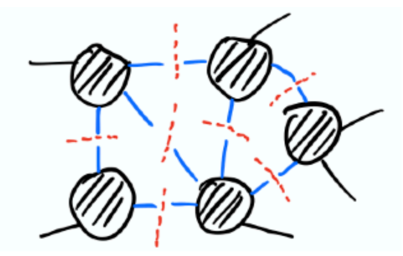
Solving linear equations

Strategy of loop computation

CK-duality

Conjecture !

Compact ansatz of the loop integrand



Unitarity cuts

$$\text{Loop-ansatz}\,|_{\text{cut}} = \prod \text{tree-blocks}$$

Solving linear equations

Main challenge: it is a priori not known whether the solution exists

Color-kinematics duality

Proved at tree-level:

- String Monodromy relation
- BCFW recursion

Bjerrum-Bohr et.al 2009; Stieberger 2009
Mafra, Schlotterer and Stieberger 2011

Feng, Huang, Jia 2010

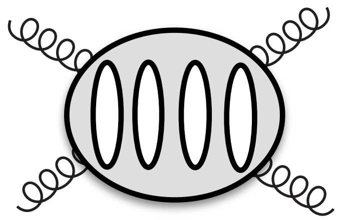
Still a conjecture at loop level, relying on explicit constructions.

Loop-level CK duality

For N=4 SYM, there are high loop examples that manifest **global CK-dual Jacobi relations**:

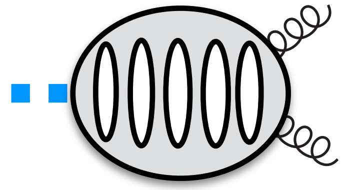
- 4-loop 4-point amplitude in N=4

  Bern, Carrasco, Dixon, Johansson, Roiban, 2012



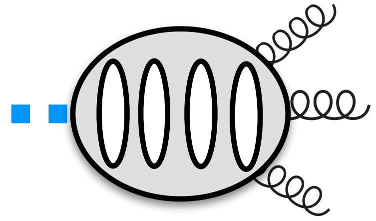
- 5-loop Sudakov form factor in N=4

  GY 2016

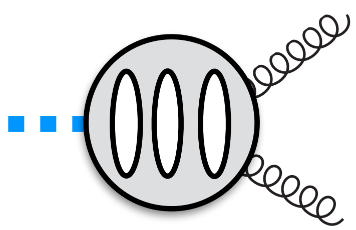


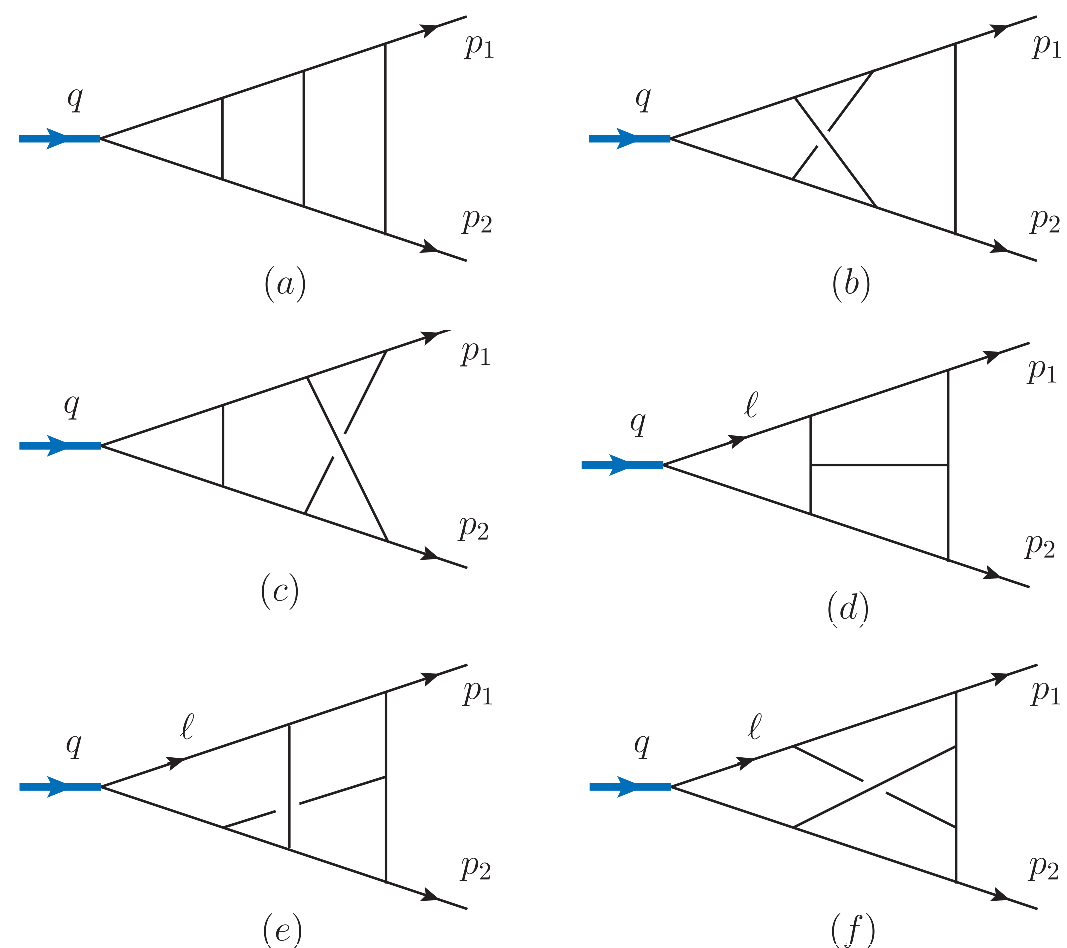
- 4-loop three-point form factor in N=4

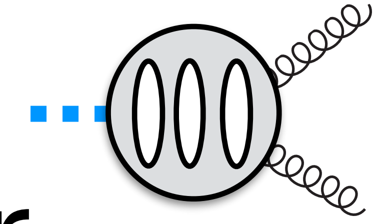
  Lin, GY, Zhang, 2112.09123



3-loop Sudakov form factor



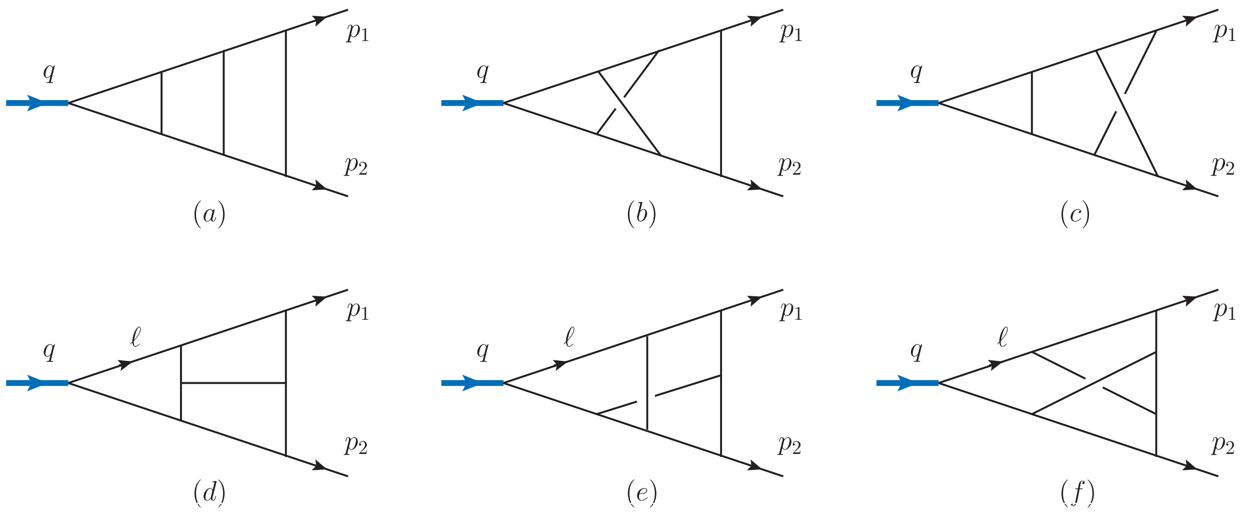


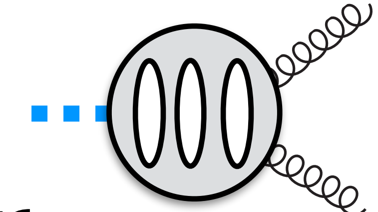


3-loop Sudakov form factor

[Boels, Kniehl, Tarasov, GY 2013]

- Generate all topologies (no-triangle property)

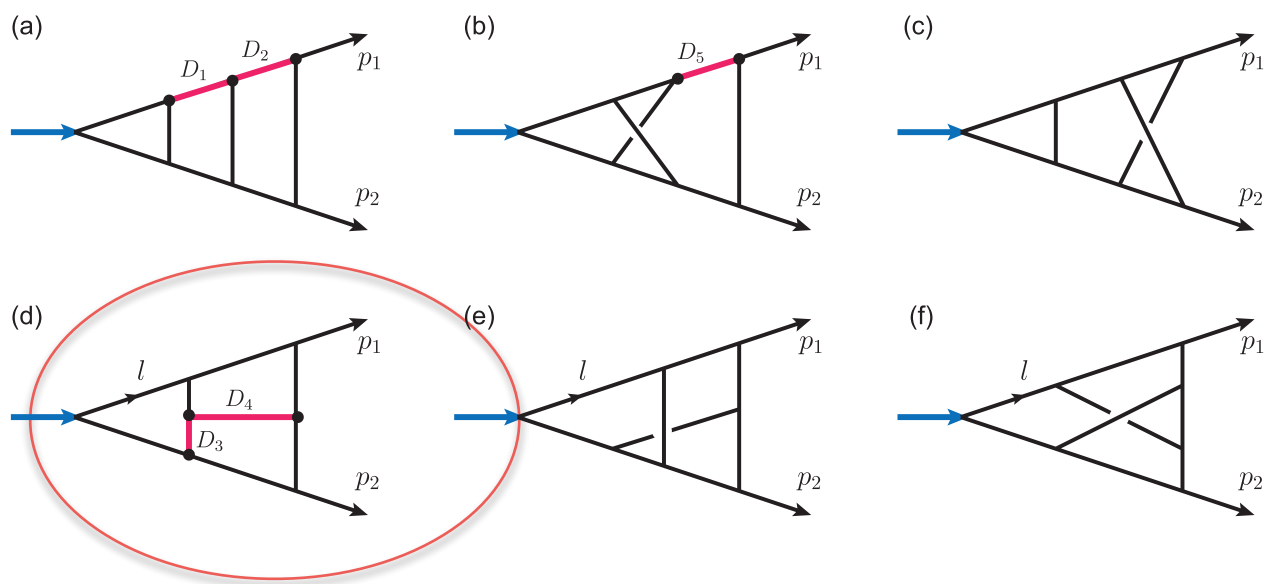




3-loop Sudakov form factor

[Boels, Kniehl, Tarasov, GY 2013]

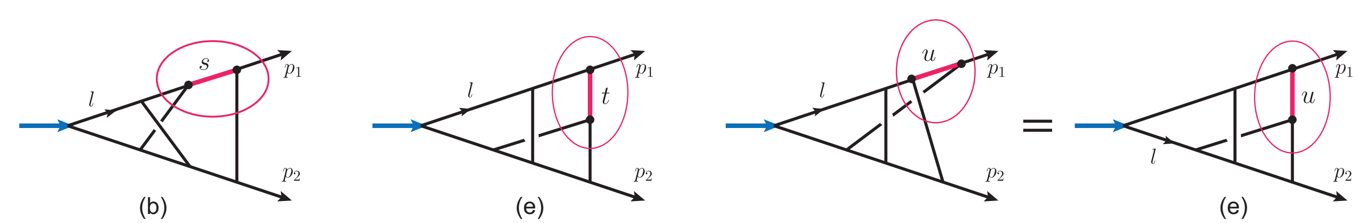
- Generate all topologies (no-triangle property)
- Find master numerator via CK duality

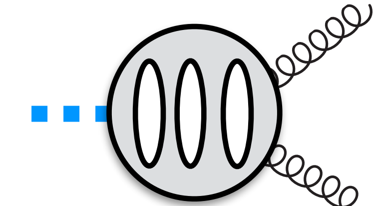


master integral

$$N_a \overset{D_1}{=} N_b\,,\; N_a \overset{D_2}{=} N_c\,,\; N_d \overset{D_3}{=} -N_e\,,\; N_d \overset{D_4}{=} N_f\,,$$

$$N_b(p_1, p_2, l) \overset{D_5}{=} N_e(p_1, p_2, l) + N_e(p_1, p_2, p_1 + p_2 - l)\,.$$



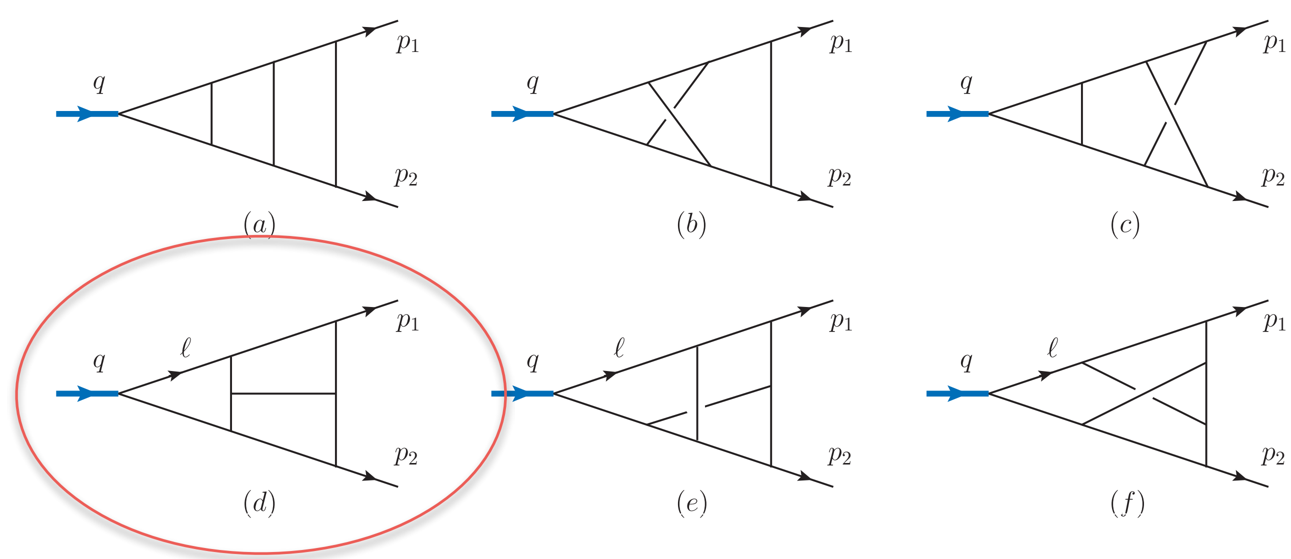


3-loop Sudakov form factor

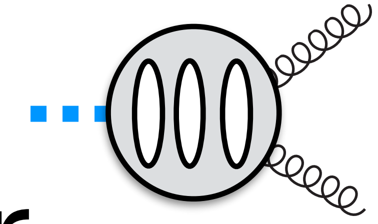
[Boels, Kniehl, Tarasov, GY 2013]

- Generate all topologies (no-triangle property)
- Find master numerator via CK duality
- Make ansatz for the master numerator

$$N_d^{\text{ansatz}}(p_1, p_2, \ell) = \alpha_1 \ell \cdot p_1 + \alpha_2 \ell \cdot p_2 + \alpha_3 p_1 \cdot p_2$$



master integral



3-loop Sudakov form factor

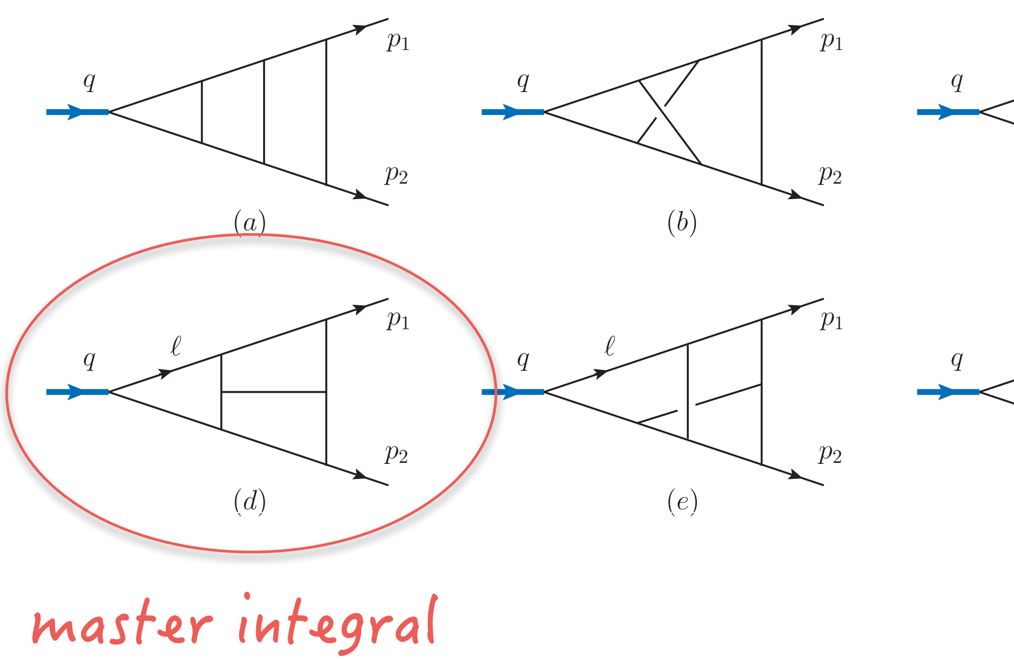
[Boels, Kniehl, Tarasov, GY 2013]

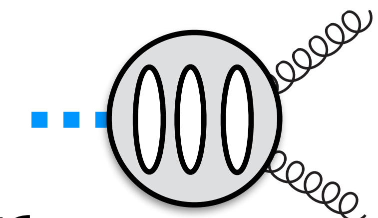
- Generate all topologies (no-triangle property)
- Find master numerator via CK duality
- Make ansatz for the master numerator

$$N_d^{\text{ansatz}}(p_1, p_2, \ell) = \alpha_1 \ell \cdot p_1 + \alpha_2 \ell \cdot p_2 + \alpha_3 p_1 \cdot p_2$$

- Apply symmetry property

$$\{p_1, p_2, \ell\} \quad \Longleftrightarrow \quad \{p_2, p_1, q - \ell\} \quad \Rightarrow \quad \alpha_2 = -\alpha_1$$





3-loop Sudakov form factor

[Boels, Kniehl, Tarasov, GY 2013]

- Generate all topologies (no-triangle property)
- Find master numerator via CK duality
- Make ansatz for the master numerator

$$\boxed{N_d^{\text{ansatz}}(p_1, p_2, \ell) = \alpha_1 \ell \cdot p_1 + \alpha_2 \ell \cdot p_2 + \alpha_3 p_1 \cdot p_2}$$

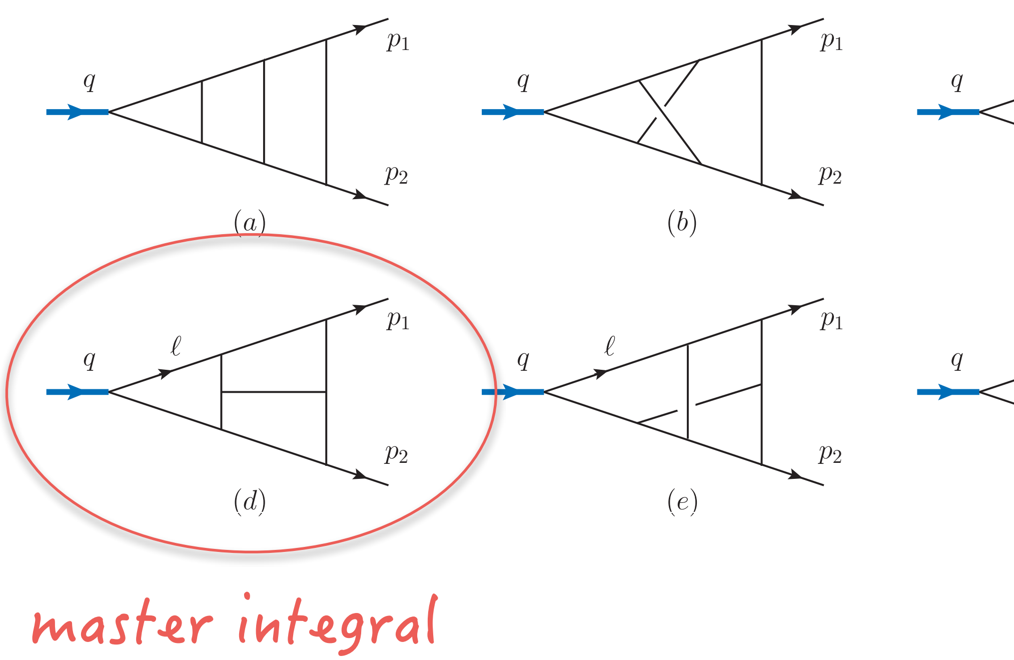
- Apply symmetry property

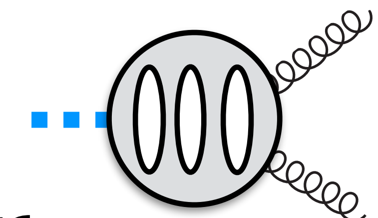
$$\boxed{\{p_1, p_2, \ell\} \quad \Longleftrightarrow \quad \{p_2, p_1, q-\ell\}} \;\Rightarrow\; \boxed{\alpha_2 = -\alpha_1}$$

- Apply a simple cut

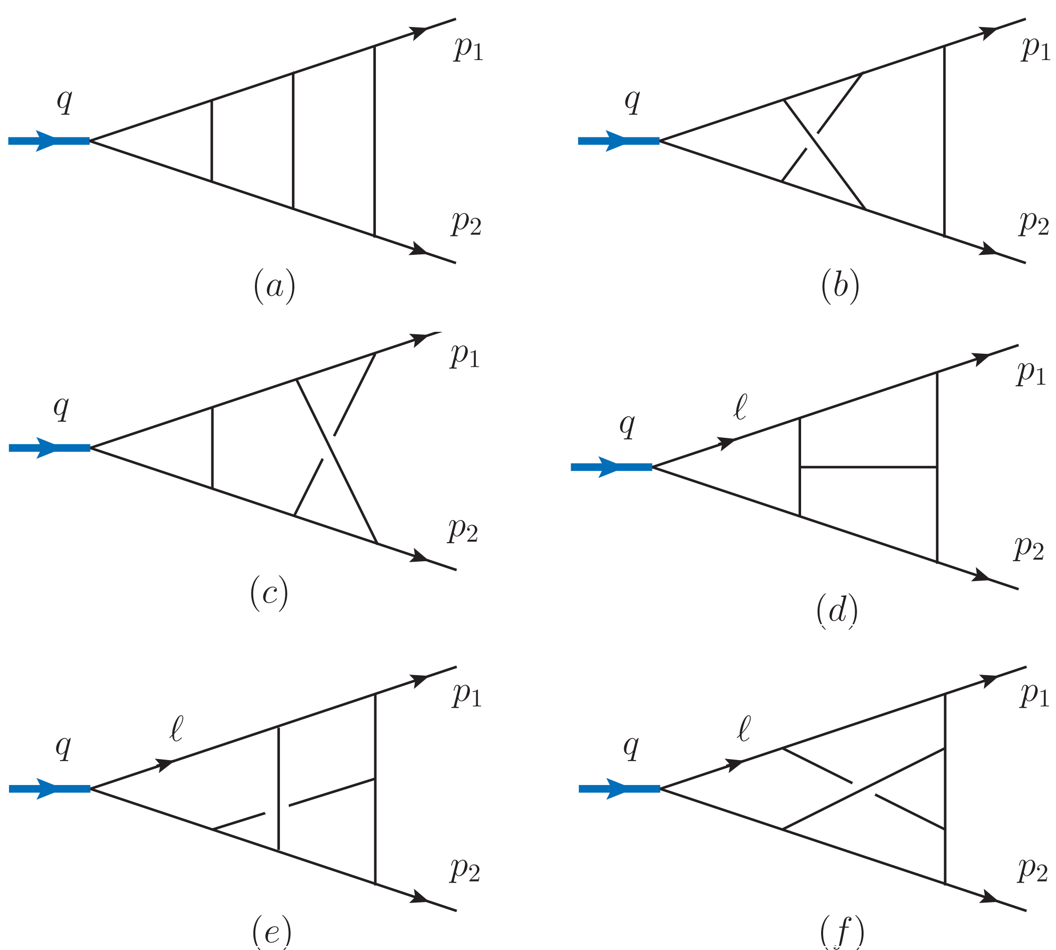
$$\boxed{\left[N_d(p_1, p_2, \ell) - (\ell - p_1)^2\right]\Big|_{\text{maximal cut}} = 0}$$

$$\Rightarrow \quad \boxed{\alpha_1 = -1, \qquad \alpha_3 = -1}$$



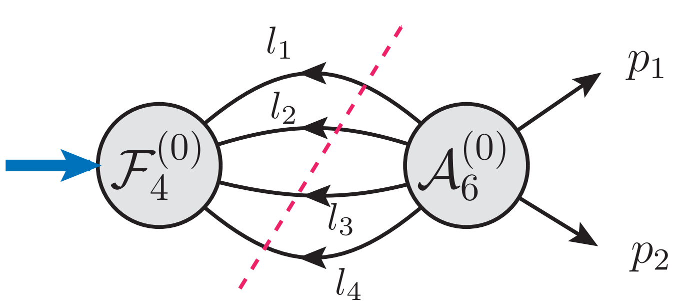


3-loop Sudakov form factor



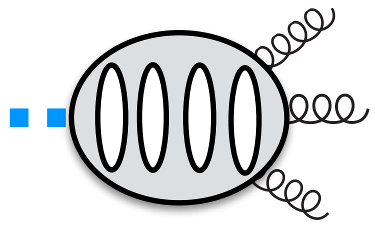
Basis	Numerator factor	Color factor	Symmetry factor
(a)	s_{12}^2	$8\,N_c^3\,\delta^{a_1 a_2}$	2
(b)	s_{12}^2	$4\,N_c^3\,\delta^{a_1 a_2}$	4
(c)	s_{12}^2	$4\,N_c^3\,\delta^{a_1 a_2}$	4
(d)	$(p_2 - p_1)\cdot\ell - p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1 a_2}$	2
(e)	$-(p_2 - p_1)\cdot\ell + p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1 a_2}$	1
(f)	$(p_2 - p_1)\cdot\ell - p_1\cdot p_2$	0	2

- Finally, check all other cuts are satisfied

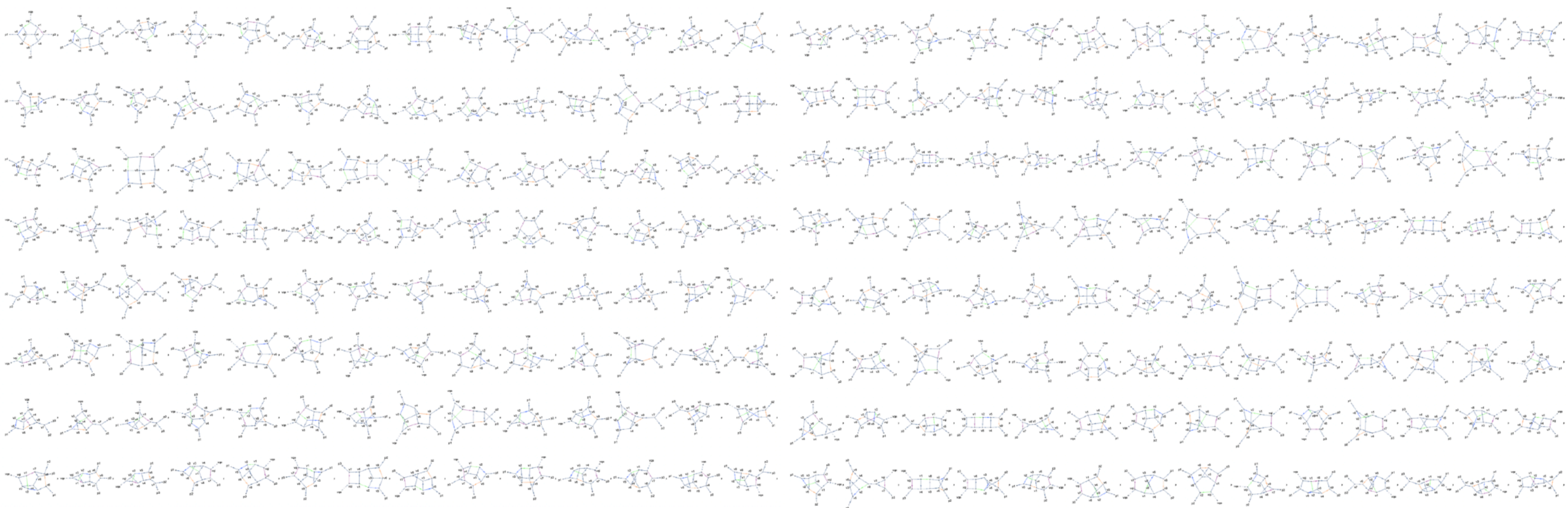


4-loop 3-point form factor

A more complicated example:

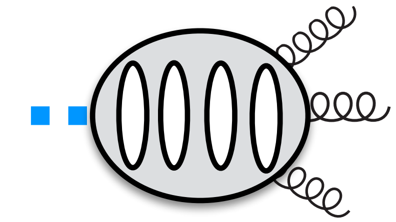


229 trivalent graphs

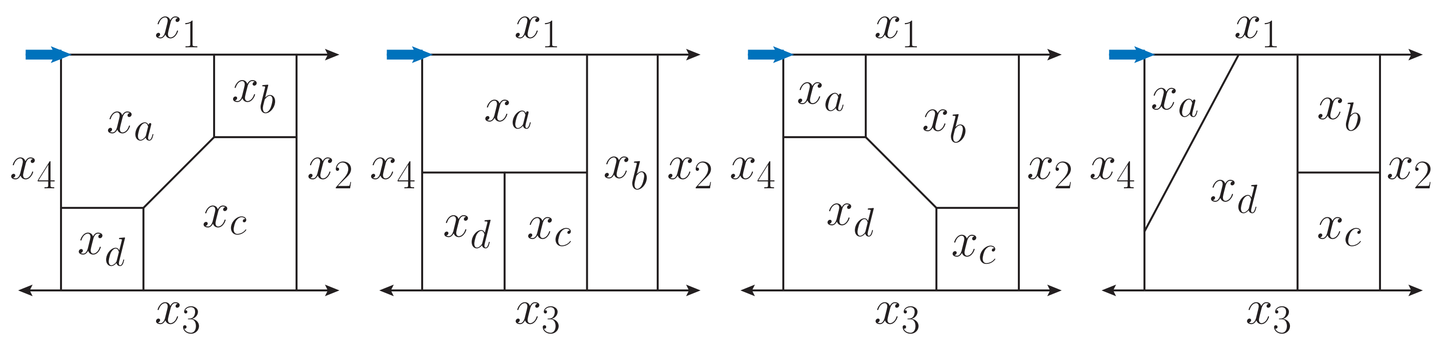


G. Lin, GY, S. Zhang, 2112.09123

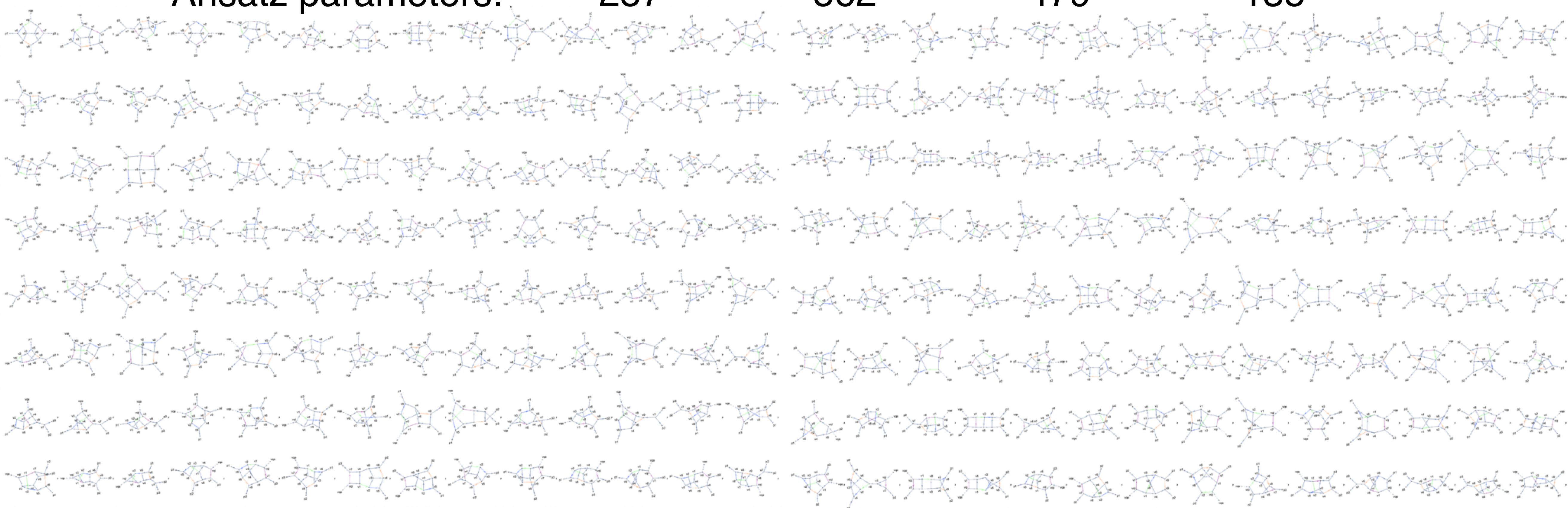
4-loop 3-point form factor



Master graphs



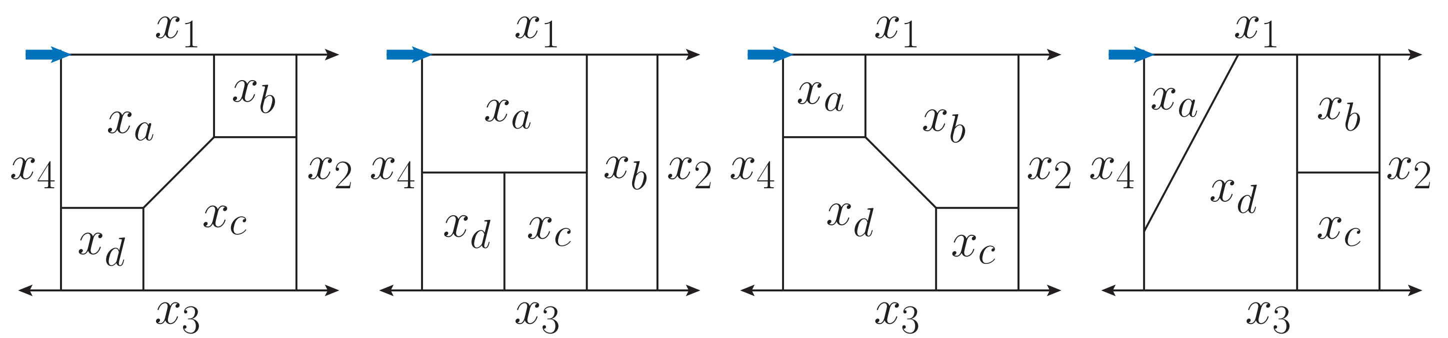
Ansatz parameters: 257 562 479 135



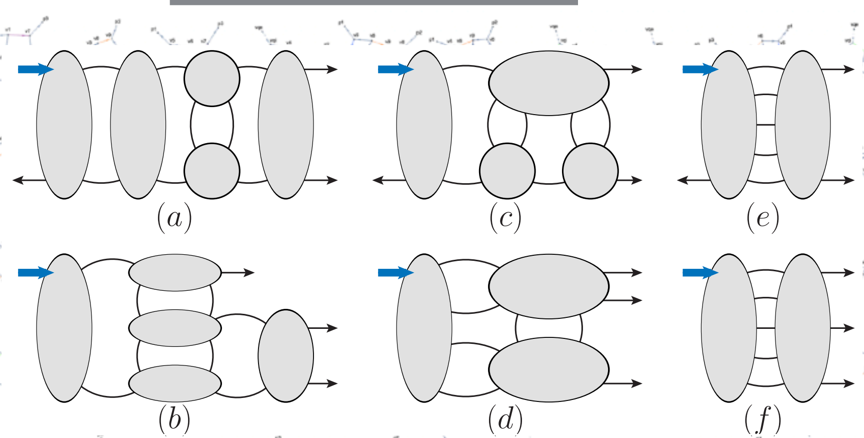
G. Lin, GY, S. Zhang, 2112.09123

4-loop 3-point form factor

Master graphs



Unitarity cuts

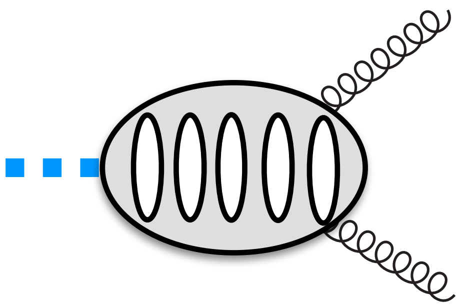


Final solution with 133 free parameters!

$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^{4} d^D \ell_j \frac{1}{S_i} \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

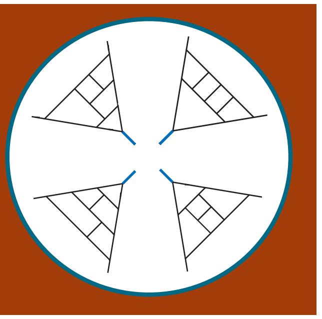
G. Lin, GY, S. Zhang, 2112.09123

CK-duality of form factor



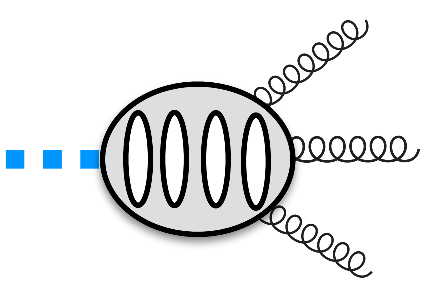
L-loop	L=1	L=2	L=3	L=4	L=5
# of topologies	1	2	6	34	306
# of masters	1	1	1	2	4

Four master graphs @ 5-loop:



Boels, Kniehl, Tarasov, GY 2012

GY, 2016



L loops	L=1	L=2	L=3	L=4
# of cubic graphs	2	6	29	229
# of planar masters	1	2	2	4
# of free parameters	1	4	24	133

Lin, GY, Zhang, 2021

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

How about double-copy of the form factors ?

Double copy of form factor

Gauge x Gauge → (Double-copy) → Gravity

- An surprising new mechanism for form factors:

Spurious poles (Gauge theory) → (Double-copy) → Real propagators (Gravity theory)

- Hidden “factorization” relations of gauge form factors

$$\vec{v} \cdot \vec{\mathcal{F}}_n \big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

2111.12719, 2211.01386, 2306.04672 with Guanda Lin

Two physical requirements

Physical amplitudes should preserve

"gauge symmetry" in gauge theory: $A_n|_{\varepsilon_i^\mu \to p_i^\mu} = 0$

"diffeomorphism invariance" in gravity: $M_n|_{\varepsilon_i^{\mu\nu} \to p_i^{(\mu}\xi_i^{\nu)}} = 0$

Physical amplitudes should also have correct "factorization property".
For example, the four-point tree amplitude:

$$\lim_{s\to 0} \mathcal{M}_4 \times s = \sum_{\varepsilon_P^h} \mathcal{M}_3 \times \mathcal{M}_3$$

For form factors: challenge 1

The double-copy of a local operator is not obvious: a “local” operator would break the diffeomorphism invariance in gravity.

$$\mathcal{O}(x) \overset{?}{\rightarrow} \int d^4x \mathcal{O}(x)$$

Solution: view operator as a scalar Higgs particle and impose CK duality.

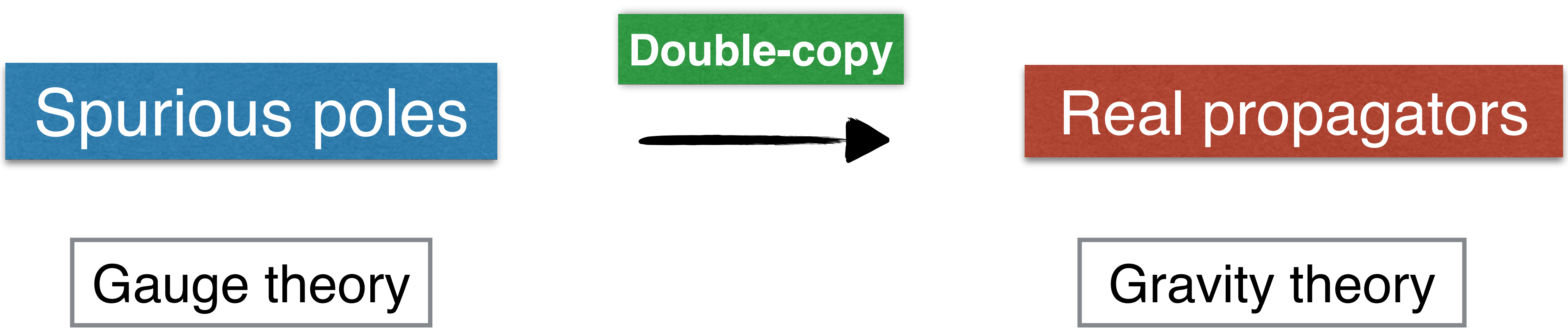
$$\sum_a \frac{c_a (n_a|_{\varepsilon_i \to p_i})}{D_a} = 0 \quad \longrightarrow \quad \sum_a \frac{n_a (n_a|_{\varepsilon_i \to p_i})}{D_a} = 0$$

$$c_a = c_b + c_c \qquad\qquad n_a = n_b + n_c$$

For form factors: challenge 2

Another problem: CK-duality can generate spurious poles.

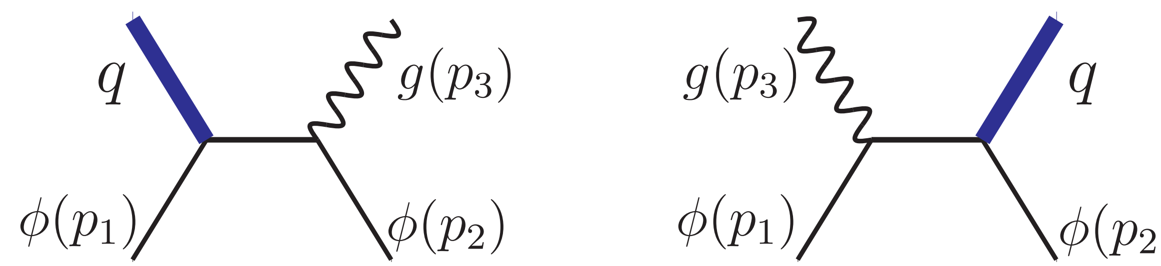
Solution: the spurious poles in gauge theory can become real physical poles in gravity.



2111.12719, 2211.01386, 2306.04672 with Guanda Lin

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^{\phi}, p_2^{\phi}, p_3^{g} \,|\, \mathrm{tr}(\phi^2)(x) \,|\, 0 \rangle$$

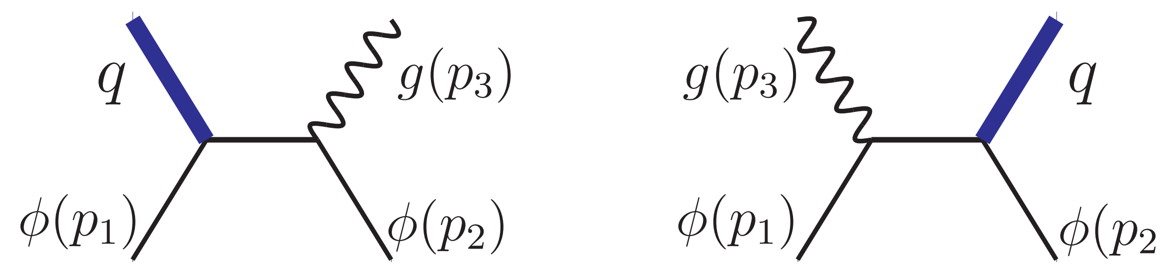


$$\boldsymbol{F}_3(1^{\phi}, 2^{\phi}, 3^{g}) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}}$$

$$C_1 = C_2 = f^{a_1 a_2 a_3}$$

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^{\phi}, p_2^{\phi}, p_3^{g} \,|\, \mathrm{tr}(\phi^2)(x) \,|\, 0\rangle$$



$$\boldsymbol{F}_3(1^{\phi}, 2^{\phi}, 3^{g}) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}} \qquad C_1 = C_2 = f^{a_1 a_2 a_3}$$

A Feynman diagram computation:

$$N_1^{\mathrm{Feyn}} = -\varepsilon_3 \cdot p_2\,, \qquad N_2^{\mathrm{Feyn}} = \varepsilon_3 \cdot p_1\,.$$

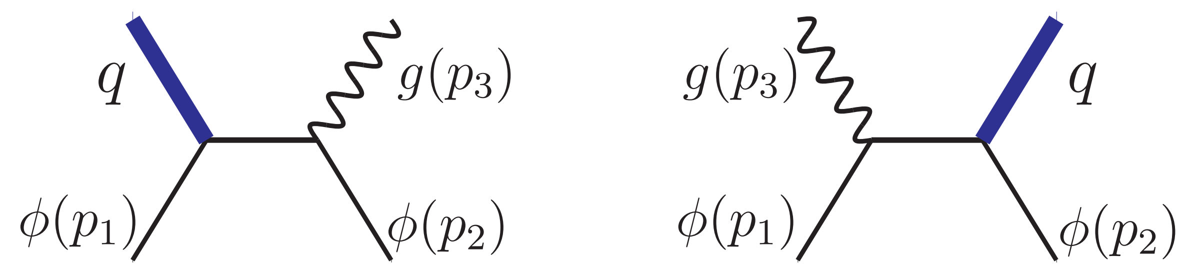
$$\mathcal{G}_3^{\mathrm{naive}} = \frac{(\varepsilon_3 \cdot p_2)^2}{s_{23}} + \frac{(\varepsilon_3 \cdot p_1)^2}{s_{13}}$$

Break diffeomorphism invariance.

$$\varepsilon_i^{\mu\nu} \to \varepsilon_i^{\mu\nu} + p_i^{(\mu} q^{\nu)}$$

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^{\phi}, p_2^{\phi}, p_3^{g} \,|\, \mathrm{tr}(\phi^2)(x) \,|\, 0\rangle$$



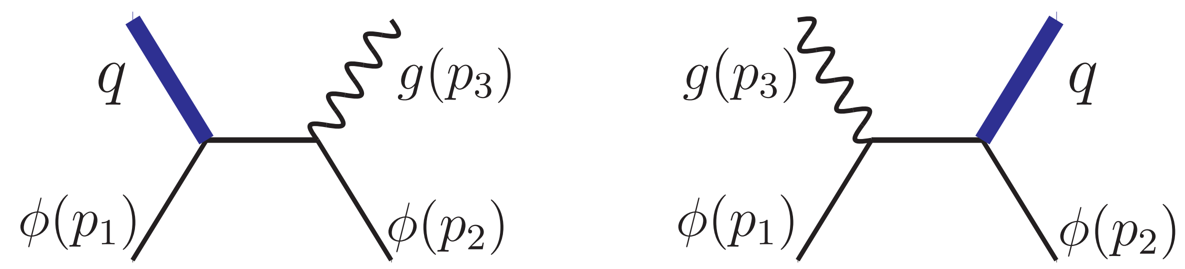
$$\boldsymbol{F}_3(1^{\phi}, 2^{\phi}, 3^{g}) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}}$$

$$C_1 = C_2 = f^{a_1 a_2 a_3} \quad \longrightarrow \quad N_1^{\mathrm{CK}} = N_2^{\mathrm{CK}} = \frac{s_{13} s_{23}}{s_{13} + s_{23}} \mathcal{F}_3(1^{\phi}, 3^{g}, 2^{\phi})$$

Unique solution with a spurious pole

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^{\phi}, p_2^{\phi}, p_3^{g} \,|\, \mathrm{tr}(\phi^2)(x) \,|\, 0 \rangle$$



$$\boldsymbol{F}_3(1^{\phi}, 2^{\phi}, 3^{g}) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}}$$

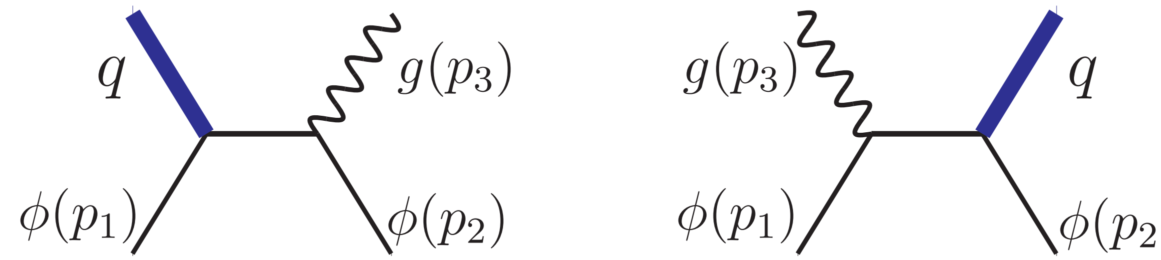
$$C_1 = C_2 = f^{a_1 a_2 a_3} \longrightarrow N_1^{\mathrm{CK}} = N_2^{\mathrm{CK}} = \frac{s_{13} s_{23}}{s_{13} + s_{23}} \mathcal{F}_3(1^{\phi}, 3^{g}, 2^{\phi})$$

Double-copy: $$\mathcal{G}_3 = \frac{(N_1^{\mathrm{CK}})^2}{s_{23}} + \frac{(N_2^{\mathrm{CK}})^2}{s_{13}} = \frac{s_{13} s_{23}}{s_{13} + s_{23}} \Big(\mathcal{F}_3(1^{\phi}, 3^{g}, 2^{\phi})\Big)^2$$

Manifestly diffeomorphism invariant

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^\phi, p_2^\phi, p_3^g \,|\, \mathrm{tr}(\phi^2)(x)\,|\,0\rangle$$



$$\boldsymbol{F}_3(1^\phi, 2^\phi, 3^g) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}}$$

$$C_1 = C_2 = f^{a_1 a_2 a_3} \longrightarrow N_1^{\mathrm{CK}} = N_2^{\mathrm{CK}} = \frac{s_{13} s_{23}}{s_{13} + s_{23}} \mathcal{F}_3(1^\phi, 3^g, 2^\phi)$$

Double-copy: $$\mathcal{G}_3 = \frac{(N_1^{\mathrm{CK}})^2}{s_{23}} + \frac{(N_2^{\mathrm{CK}})^2}{s_{13}} = \frac{s_{13} s_{23}}{s_{13} + s_{23}} \Big(\mathcal{F}_3(1^\phi, 3^g, 2^\phi)\Big)^2$$

However, the spurious pole no longer cancel

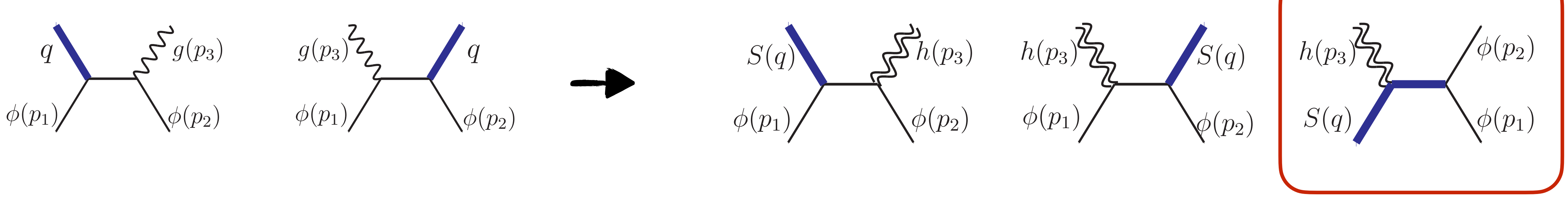
Example: 3-point form factor

$$\mathcal{G}_3 = \frac{(N_1^{\mathrm{CK}})^2}{s_{23}} + \frac{(N_2^{\mathrm{CK}})^2}{s_{13}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}}\Big(\mathcal{F}_3(1^\phi, 3^g, 2^\phi)\Big)^2$$

There is a nice factorization behavior at the new pole:

$$s_{13} + s_{23} = q^2 - s_{12} = 0$$

$$\mathrm{Res}\,[\mathcal{G}_3]_{s_{12}=q^2} = (\epsilon_3 \cdot q)^2 = \Big(\mathcal{F}_2(1^\phi, 2^\phi)\Big)^2 \times \big(\mathcal{A}_3(\mathbf{q}_2^S, 3^g, -q^S)\big)^2$$



A new graph
in gravity

Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- Spectrum and renormalization
- Hidden structure: high-point and high-loop

Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

Spectrum of operators

- 1804.04653, 1904.07260, 1910.09384, with Qingjun Jin
- 2011.02494 with Qingjun Jin, Ke Ren;
- 2202.08285, 2208.08976, 2301.01786 with Qingjun Jin, Ke Ren; Rui Yu

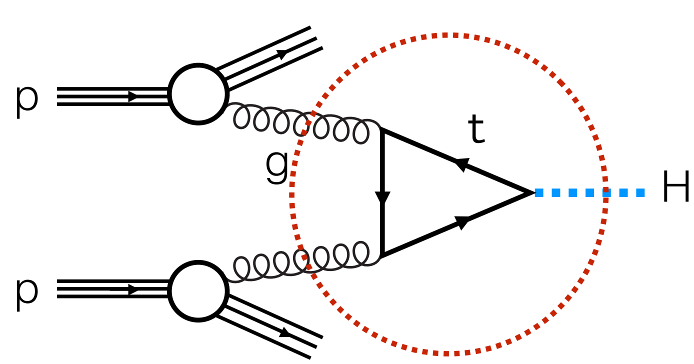
High-dimensional YM operators

We consider Lorentz scalar gauge invariant local operators:

$$\mathcal{O}(x) \sim c(a_1, \ldots, a_n) X(\eta^{\mu\nu}) \left(D_{\mu_{11}} \ldots D_{\mu_{1m_1}} F_{\nu_1\rho_1}\right)^{a_1} \cdots \left(D_{\mu_{n1}} \ldots D_{\mu_{nm_n}} F_{\nu_n\rho_n}\right)^{a_n}(x)$$

$$D_\mu \star = \partial_\mu + ig[A_\mu, \star]\,, \qquad [D_\mu, D_\nu] \star = ig[F_{\mu\nu}, \star] \qquad F_{\mu\nu} = F^a_{\mu\nu} T^a, \qquad [T^a, T^b] = i f^{abc} T^c$$

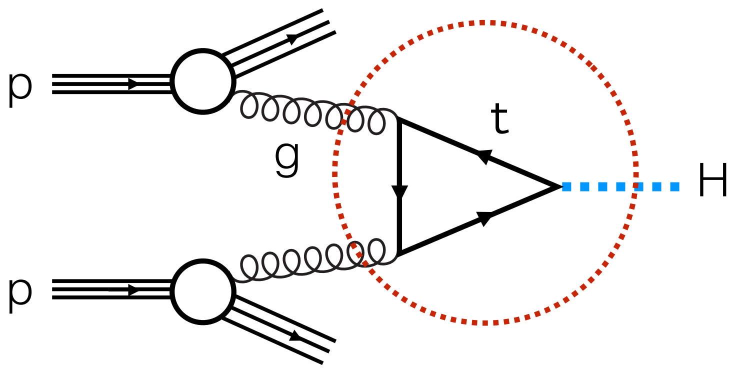
They are color-singlet gluon states and also appear as Higgs-gluon effective interaction vertices in “Higgs” EFT:



$m_t \to \infty$

$$\mathcal{L}_{\rm eff} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum_{k=1}^{\infty} \frac{1}{m_{\rm t}^{2k}} \sum_i \hat{C}_i H \mathcal{O}_{4+2k;i}$$

"Higgs" EFT



$m_t \to \infty$

Effective gluon-Higgs vertex:

$$\mathcal{L}_{\text{eff}} = C_0 O_0 + \frac{1}{m_{\text{t}}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{t}}^4}\right)$$

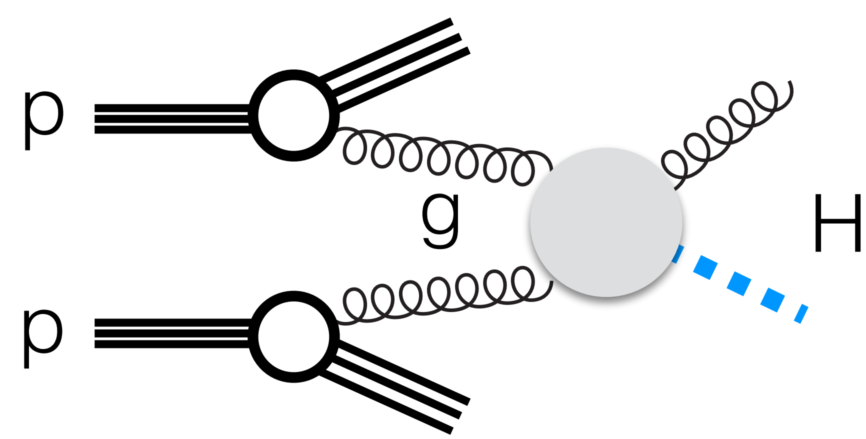
Dimension-5 operator

$$O_0 = H\text{tr}(F_{\mu\nu} F^{\mu\nu})$$

Dimension-7 operators

$$O_1 = H\,\text{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu)\,,$$
$$O_2 = H\,\text{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu})\,,$$
$$O_3 = H\,\text{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu})\,,$$
$$O_4 = H\,\text{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu})\,.$$

Higgs plus jet production



Setup of the problem

Operators:

$$\mathcal{O} \sim \; c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1}\cdots\big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}X(\eta,\epsilon)$$

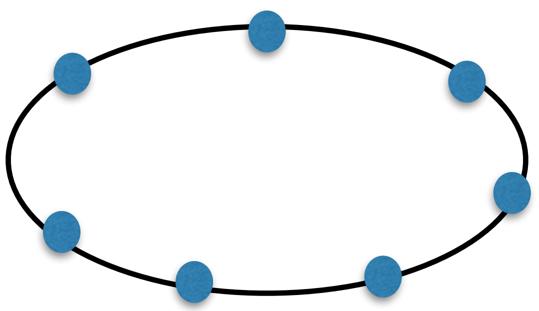
$$D_\mu \star = \partial_\mu + ig[A_\mu, \star]\,, \qquad [D_\mu, D_\nu]\star = ig[F_{\mu\nu}, \star] \qquad\qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a, T^b] = if^{abc}T^c$$

Classical dimension

$$\dim(\mathcal{O}) = \Delta_0(\mathcal{O}) = (\#\text{ of } D\text{'s}) + 2\times(\#\text{ of } F\text{'s})$$

Length of operator

$$\text{len}(\mathcal{O}) = (\#\text{ of } F\text{'s})$$



Lorentz indices

$$F^{\mu_1\mu_2}D_{\mu_1}D_{\mu_5}F^{\mu_3\mu_4}D_{\mu_2}D^{\mu_5}F_{\mu_3\mu_4} \Rightarrow F_{12}D_{15}F_{34}D_{25}F_{34}$$

Setup of the problem

Operators:

$$\mathcal{O} \sim \; c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1}\cdots\big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}X(\eta,\epsilon)$$

$$D_\mu \star = \partial_\mu + ig[A_\mu, \star]\,, \qquad [D_\mu, D_\nu]\star = ig[F_{\mu\nu}, \star] \qquad\qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a, T^b] = if^{abc}T^c$$

Problems to address:

- Independent operator basis (classical)
- Renormalization of operators (quantum UV)
- EFT amplitudes (finite remainder)

High-dimensional YM operators

We consider Lorentz scalar gauge invariant local operators:

$$\mathcal{O}(x) \sim c(a_1, ..., a_n) X(\eta^{\mu\nu}) \left(D_{\mu_{11}} ... D_{\mu_{1m_1}} F_{\nu_1\rho_1}\right)^{a_1} \cdots \left(D_{\mu_{n1}} ... D_{\mu_{nm_n}} F_{\nu_n\rho_n}\right)^{a_n}(x)$$

Classically, operators are generally not independent:

Equation of motion:

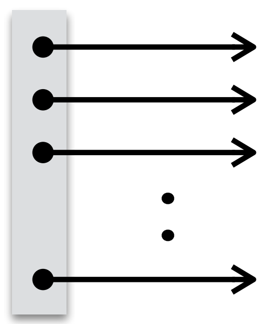
$$D_\mu F^{\mu\nu} = 0$$

Bianchi identities:

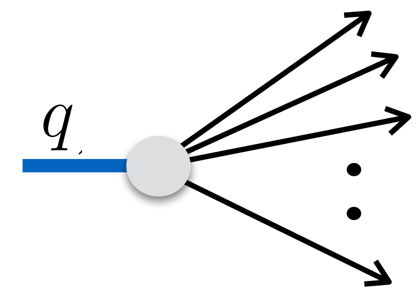
$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$$

At quantum level, different operators can mixing with each other via renormalization.

Minimal tree form factors



$$\mathcal{O}_L \;\Leftrightarrow\; \mathcal{F}_{\mathcal{O}_L,L}(1,\ldots,L)$$



Dictionary for YM operators:

operator	$D_{\dot\alpha\alpha}$	$f_{\alpha\beta}$	$\bar f_{\dot\alpha\dot\beta}$
spinor	$\tilde\lambda_{\dot\alpha}\lambda_\alpha$	$\lambda_\alpha\lambda_\beta$	$-\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}$

4-dim

$$F_{\mu\nu} \to F_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta}$$

$$\mathrm{tr}(\bar F_{\dot\alpha}{}^{\dot\beta}\bar F_{\dot\beta}{}^{\dot\gamma}\bar F_{\dot\gamma}{}^{\dot\alpha}) \to \tilde\lambda_1^{\dot\alpha}\tilde\lambda_{1\dot\beta}\tilde\lambda_2^{\dot\beta}\tilde\lambda_{2\dot\gamma}\tilde\lambda_3^{\dot\gamma}\tilde\lambda_{3\dot\alpha} = [1\,2][2\,3][3\,1]$$

operator	D_μ	$F_{\mu\nu}$
kinematics	p_μ	$p_\mu\varepsilon_\nu - p_\nu\varepsilon_\mu$

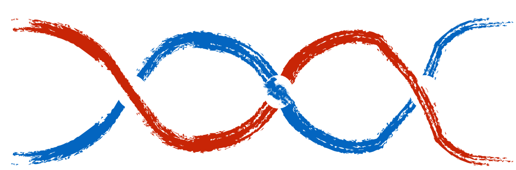
D-dim

Important for capturing
“Evanescent operators”

Unitarity-IBP strategy

$$\mathcal{F}^{(l)}\Big|_{\text{cut}} = \prod(\text{tree blocks}) = \text{cut integrand} \;\; = \sum_i c_i\, M_i|_{\text{cut}}$$

On-shell unitarity



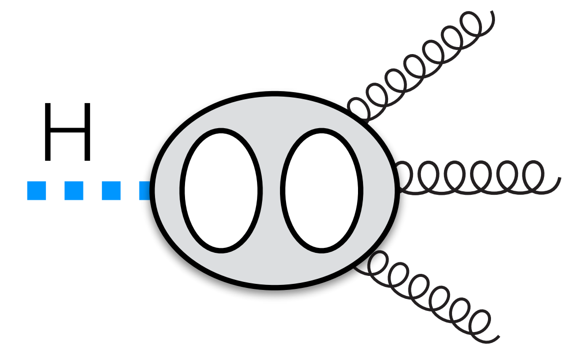
(cut) IBP reduction

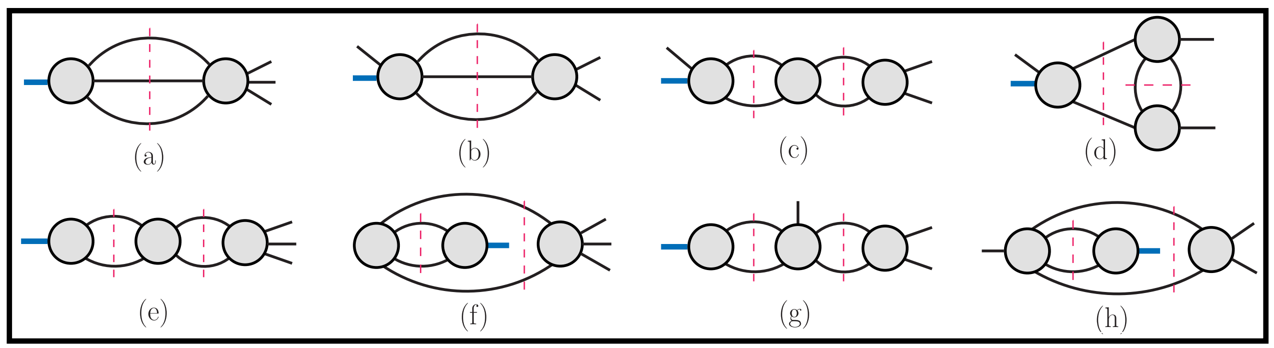
Jin, GY 2018   Boels, Jin, Luo 2018

Numerical unitarity: Abreu, Cordero, Ita, Jaquier, Page, Zeng 2017

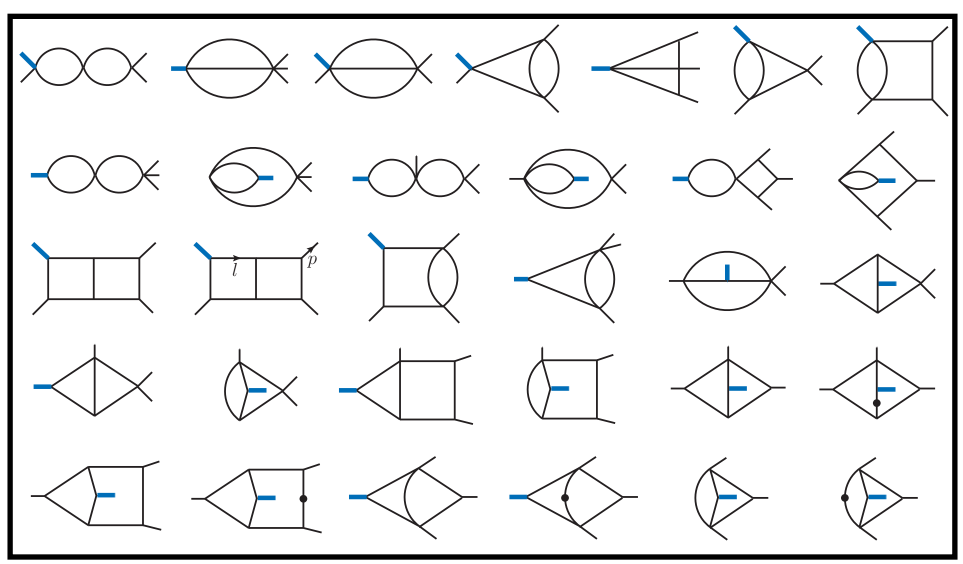
Unitarity cuts and master integrals

All cuts that are needed:





Master integrals are known in terms of 2d Harmonic polylogarithms.



[Gehrmann, Remiddi 2001]

Loop structure of form factors

General structure of (bare) amplitudes/form factors:

Loop correction = IR + UV + finite remainder + $\mathcal{O}(\epsilon)$

Mixed in dim-reg

Loop structure of form factors

General structure of (bare) amplitudes/form factors:

Loop correction = IR + UV + finite remainder + $\mathcal{O}(\epsilon)$

IR structure is “universal”: [Catani 1998]

$$
\begin{aligned}
\mathcal{F}^{(1)}_{\mathcal{O},R} &= I^{(1)}(\epsilon)\mathcal{F}^{(0)}_{\mathcal{O}} + \mathcal{F}^{(1)}_{\mathcal{O},\text{fin}} + \mathcal{O}(\epsilon)\,, \\
\mathcal{F}^{(2)}_{\mathcal{O},R} &= I^{(2)}(\epsilon)\mathcal{F}^{(0)}_{\mathcal{O}} + I^{(1)}(\epsilon)\mathcal{F}^{(1)}_{\mathcal{O},R} + \mathcal{F}^{(2)}_{\mathcal{O},\text{fin}} + \mathcal{O}(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
I^{(1)}(\epsilon) &= -\frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\Big(\frac{N_c}{\epsilon^2} + \frac{\beta_0}{2\epsilon}\Big)\sum_{i=1}^{E}(-s_{i,i+1})^{-\epsilon}\,, \\
I^{(2)}(\epsilon) &= -\frac{1}{2}\big(I^{(1)}(\epsilon)\big)^2 - \frac{\beta_0}{\epsilon}I^{(1)}(\epsilon) + \frac{e^{-\gamma_E\epsilon}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\Big(\frac{\beta_0}{\epsilon} + \frac{67}{9} - \frac{\pi^2}{3}\Big)I^{(1)}(2\epsilon) \\
&\quad + E\frac{e^{\gamma_E\epsilon}}{\epsilon\Gamma(1-\epsilon)}\Big(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11\pi^2}{144}\Big)\,.
\end{aligned}
$$

UV renormalization: operator mixing

By subtracting the universal IR, one can obtain the UV renormalization matrix.

- Operators (of same classical dimension) can mix with each other at quantum level via renormalization:

$$\mathcal{O}_{R,i} = Z_i^{\ j} \mathcal{O}_{B,j}$$

- From the renormalization matrix, one can obtain the dilatation operator:

$$\mathscr{D} = -\frac{d \log Z}{d \log \mu}$$

- The anomalous dimensions are are given by the eigenvalues of dilatation operator:

$$\mathscr{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}$$

Example for UV mixing

Jin, Ren, GY 2020

$$\mathcal{F}^{(2)}_{\mathcal{O}_{8;\alpha;f;1}}(1^-,2^-,3^+)\Big|_{\frac{1}{\epsilon}\text{ UV-div.}} = \mathcal{F}^{(0)}_{\mathcal{O}_{8;\alpha;f;1}}(1^-,2^-,3^+)\times\frac{N_c^2}{\epsilon}\Big(-\frac{1}{3vw}+\frac{269}{72}\Big),$$

$$\mathcal{F}^{(2),\alpha}_{\mathcal{O}_{8;\beta;f;1}}(1^-,2^-,3^+)\Big|_{\frac{1}{\epsilon}\text{ UV-div.}} = \mathcal{F}^{(0)}_{\mathcal{O}_{8;\alpha;f;1}}(1^-,2^-,3^+)\times\frac{N_c^2}{\epsilon}\Big(-\frac{1}{vw}\Big).$$

$$\rightarrow\quad (Z^{(2)})_{\mathcal{O}_{8;\alpha;f;1}}^{\ \mathcal{O}_{8;0}} = -\frac{N_c^2}{3\epsilon},\quad (Z^{(2)})_{\mathcal{O}_{8;\alpha;f;1}}^{\ \mathcal{O}_{8;\alpha;f;1}} = \frac{269N_c^2}{72\epsilon},\quad (Z^{(2)})_{\mathcal{O}_{8;\beta;f;1}}^{\ \mathcal{O}_{8;0}} = -\frac{N_c^2}{\epsilon}.$$

$$\mathcal{F}^{(2)}_{\mathcal{O}_{8;\alpha;f;1}}(1^-,2^-,3^-)\Big|_{\frac{1}{\epsilon}\text{ UV-div.}} = \mathcal{F}^{(0)}_{\mathcal{O}_{8;\beta;f;1}}(1^-,2^-,3^-)\times\frac{N_c^2}{\epsilon}\Big(-\frac{1}{3uvw}+\frac{5}{2}\Big),$$

$$\mathcal{F}^{(2)}_{\mathcal{O}_{8;\beta;f;1}}(1^-,2^-,3^-)\Big|_{\frac{1}{\epsilon}\text{ UV-div.}} = \mathcal{F}^{(0)}_{\mathcal{O}_{8;\beta;f;1}}(1^-,2^-,3^-)\times\frac{N_c^2}{\epsilon}\Big(-\frac{1}{uvw}+\frac{25}{12}\Big).$$

$$\rightarrow\quad (Z^{(2)})_{\mathcal{O}_{8;\alpha;f;1}}^{\ \mathcal{O}_{8;0}} = -\frac{N_c^2}{3\epsilon},\qquad (Z^{(2)})_{\mathcal{O}_{8;\alpha;f;1}}^{\ \mathcal{O}_{8;\beta;f;1}} = \frac{5N_c^2}{2\epsilon},$$

$$(Z^{(2)})_{\mathcal{O}_{8;\beta;f;1}}^{\ \mathcal{O}_{8;0}} = -\frac{N_c^2}{\epsilon},\qquad (Z^{(2)})_{\mathcal{O}_{8;\beta;f;1}}^{\ \mathcal{O}_{8;\beta;f;1}} = \frac{25N_c^2}{12\epsilon}.$$

$$\{\mathcal{O}_{8;0},\mathcal{O}_{8;\alpha;f;1},\mathcal{O}_{8;\beta;f;1}\}$$

$$Z^{(2)}_{\mathcal{O}_8}\Big|_{\frac{1}{\epsilon}\text{-part.}} = \frac{N_c^2}{\epsilon}\begin{pmatrix} -\frac{34}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{269}{72} & \frac{5}{2} \\ -1 & 0 & \frac{25}{12}\end{pmatrix}$$

$$\mathbb{D}_{\mathcal{O}_8} = \begin{pmatrix} -\frac{22}{3}\hat\lambda - \frac{136}{3}\hat\lambda^2 & 0 & 0 \\ -\frac{\hat\lambda^2}{\hat g} & \frac{7}{3}\hat\lambda + \frac{269}{18}\hat\lambda^2 & 10\hat\lambda^2 \\ -3\frac{\hat\lambda^2}{\hat g} & 0 & \hat\lambda + \frac{25}{3}\hat\lambda^2 \end{pmatrix} \quad\rightarrow\quad \hat\gamma^{(1)}_{\mathcal{O}_8} = \Big\{-\frac{22}{3};1;\frac{7}{3}\Big\},\qquad \hat\gamma^{(2)}_{\mathcal{O}_8} = \Big\{-\frac{136}{3};\frac{25}{3};\frac{269}{18}\Big\}.$$

Mixing matrix and spectrum

Results were known previously at one-loop up to dimension-8.

See e.g.: Gracey 2002; Dawson, Lewis, Zeng 2014

We obtain new one- and two-loop results up to dimension16.

$$\mathbb{D}_{\mathcal{O}_8} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix} \qquad \hat{\gamma}^{(1)}_{\mathcal{O}_8} = \left\{-\frac{22}{3}; 1; \frac{7}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_8} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}\right\}$$

$$\mathbb{D}_{\mathcal{O}_{10,f}} = \begin{pmatrix} -\frac{22\hat{\lambda}}{3} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7\hat{\lambda}}{3} + \frac{269}{18}\hat{\lambda}^2 & 0 & 10\hat{\lambda}^2 & 0 \\ -\frac{209}{300}\frac{\hat{\lambda}^2}{\hat{g}} & -\frac{6\hat{\lambda}}{5} - \frac{5579\hat{\lambda}^2}{4500} & \frac{71\hat{\lambda}}{15} + \frac{2848}{125}\hat{\lambda}^2 & \frac{1493}{300}\hat{\lambda}^2 & \frac{5}{9}\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 & 0 \\ -\frac{19}{12}\frac{\hat{\lambda}^2}{\hat{g}} & \frac{139}{600}\hat{\lambda}^2 & \frac{499}{200}\hat{\lambda}^2 & -2\hat{\lambda} - \frac{143}{72}\hat{\lambda}^2 & \frac{17\hat{\lambda}}{3} + \frac{2195}{72}\hat{\lambda}^2 \end{pmatrix}$$

$$\hat{\gamma}^{(1)}_{\mathcal{O}_{10,f}} = \left\{-\frac{22}{3}; 1; \frac{7}{3}; \frac{71}{15}, \frac{17}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_{10,f}} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}; \frac{2848}{125}, \frac{2195}{72}\right\}$$

Mixing matrix and spectrum

Jin, Ren, GY 2020

Dim-16 at 1-loop:

$$Z^{(1)}_{\mathcal{O}_{16,f}} = \frac{N_c}{\epsilon}\left(\begin{array}{c|ccccccccc|ccccc}
-\frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \frac{7}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{5} & \frac{71}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{4} & \frac{221}{60} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & \frac{1}{10} & -\frac{19}{30} & \frac{37}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{84} & -\frac{17}{28} & -\frac{47}{70} & -\frac{17}{28} & \frac{337}{84} & \frac{5}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{20} & \frac{9}{20} & -1 & -\frac{31}{20} & -\frac{1}{4} & \frac{31}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{13}{30} & -\frac{13}{15} & \frac{13}{10} & -\frac{13}{10} & -\frac{5}{2} & \frac{13}{15} & \frac{961}{210} & \frac{8}{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{71}{105} & -\frac{212}{105} & \frac{141}{35} & -\frac{71}{35} & -\frac{141}{35} & \frac{79}{105} & -\frac{38}{35} & \frac{223}{35} & \frac{5}{14} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{70} & \frac{19}{105} & -\frac{19}{70} & -\frac{121}{70} & -\frac{11}{42} & \frac{16}{105} & -\frac{6}{5} & \frac{127}{210} & \frac{559}{105} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{17}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{9}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & -2 & \frac{1}{3} & \frac{43}{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & \frac{5}{2} & -\frac{11}{4} & \frac{67}{12}
\end{array}\right)$$

$$Z^{(1)}_{\mathcal{O}_{16,d}} = \frac{N_c}{\epsilon}\left(\begin{array}{cccccc|cc}
\frac{13}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{41}{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & -2 & \frac{301}{60} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\
-1 & 1 & -\frac{3}{10} & \frac{25}{6} & 0 & 0 & 0 & 0 \\
-\frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5} & \frac{307}{60} & \frac{7}{20} & 0 & 0 \\
\frac{1}{3} & -1 & \frac{1}{2} & -\frac{7}{3} & \frac{13}{12} & \frac{67}{12} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{12} & \frac{67}{12}
\end{array}\right)$$

Mixing matrix and spectrum

Jin, Ren, GY 2020

Dim-16 at 2-loop:

$$Z^{(2)}_{\mathcal{O}_{16,f}}\Big|_{\frac{1}{\epsilon}-\text{part.}} = \frac{N_c^2}{\epsilon}\left(\begin{array}{c|cccccccccccccc}
-\frac{34}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\frac{1}{3} & \frac{269}{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \\
-\frac{209}{900} & -\frac{5579}{18000} & \frac{712}{125} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1493}{1200} & \frac{5}{36} & 0 & 0 & 0 \\
-\frac{31}{180} & \frac{53}{3600} & -\frac{36227}{28800} & \frac{3575983}{432000} & \frac{9793}{21600} & 0 & 0 & 0 & 0 & 0 & \frac{13}{16} & \frac{16877}{14400} & -\frac{7319}{14400} & 0 & 0 \\
-\frac{181}{900} & -\frac{60979}{36000} & \frac{78487}{72000} & -\frac{2177}{2000} & \frac{704167}{72000} & 0 & 0 & 0 & 0 & 0 & \frac{1229}{1200} & \frac{115501}{43200} & -\frac{9803}{43200} & 0 & 0 \\
-\frac{523}{3920} & -\frac{2201287}{29635200} & \frac{605939}{1975680} & -\frac{64128769}{24696000} & \frac{3303367}{9878400} & \frac{332422343}{29635200} & \frac{6699071}{14817600} & 0 & 0 & 0 & \frac{37547}{78400} & \frac{75071}{39200} & -\frac{497}{576} & \frac{103}{1440} & 0 \\
-\frac{809}{5600} & -\frac{12166789}{21168000} & \frac{11202299}{7056000} & -\frac{73487}{36750} & -\frac{9182209}{7056000} & \frac{37249}{156800} & \frac{26302879}{2116800} & 0 & 0 & 0 & \frac{1613}{3360} & \frac{17401}{6720} & \frac{19}{225} & \frac{1187}{2880} & 0 \\
-\frac{269}{2520} & \frac{125599}{10584000} & \frac{50369}{1323000} & -\frac{98317}{1176000} & \frac{73489}{392000} & -\frac{8625329}{3528000} & -\frac{97913}{756000} & \frac{90760559}{7408800} & \frac{25354501}{21168000} & \frac{40519}{56448} & \frac{184259}{1058400} & \frac{65297}{23520} & -\frac{420373}{211680} & \frac{248791}{235200} & -\frac{2747}{9408} \\
-\frac{19717}{176400} & \frac{3374557}{7408800} & -\frac{102465523}{74088000} & \frac{5260289}{1764000} & -\frac{6201763}{4939200} & -\frac{115070197}{24696000} & \frac{10687837}{9261000} & \frac{6498287}{9261000} & \frac{1025255701}{74088000} & -\frac{25511}{493920} & \frac{347437}{1764000} & \frac{863371}{302400} & -\frac{230747}{105840} & \frac{938797}{705600} & -\frac{78243}{196000} \\
-\frac{19717}{176400} & -\frac{2733089}{9261000} & \frac{88146899}{74088000} & -\frac{5678651}{3528000} & -\frac{1966229}{12348000} & \frac{17842339}{18522000} & -\frac{6878309}{4630500} & -\frac{58976629}{37044000} & \frac{8569667}{9261000} & \frac{179275483}{12348000} & \frac{28489}{661500} & \frac{54403}{14700} & -\frac{228689}{88200} & \frac{687461}{264600} & -\frac{485507}{5292000} \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{12} & 0 & 0 & 0 & 0 \\
-\frac{19}{36} & \frac{139}{2400} & \frac{499}{800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{143}{288} & \frac{2195}{288} & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{4}{15} & \frac{121}{400} & \frac{637}{800} & -\frac{211}{800} & 0 & 0 & 0 & 0 & 0 & \frac{119}{120} & -\frac{15643}{7200} & \frac{79313}{7200} & 0 & 0 \\
-\frac{209}{900} & \frac{6299}{21168} & \frac{6767}{35280} & \frac{71063}{88200} & -\frac{34723}{176400} & \frac{25841}{58800} & -\frac{36091}{264600} & 0 & 0 & 0 & \frac{22723}{21600} & -\frac{35}{48} & -\frac{2861}{5400} & \frac{443801}{36000} & 0 \\
-\frac{31}{180} & \frac{13843}{105840} & \frac{8317}{15120} & -\frac{797}{35280} & \frac{5477}{35280} & \frac{2417}{3528} & \frac{611}{105840} & \frac{13975}{14112} & -\frac{5377}{10584} & -\frac{3581}{10080} & \frac{114221}{151200} & \frac{6017}{15120} & \frac{121}{216} & -\frac{3661627}{1411200} & \frac{63879443}{4233600}
\end{array}\right)$$

$$Z^{(2)}_{\mathcal{O}_{16,d}}\Big|_{\frac{1}{\epsilon}-\text{part.}} = \frac{N_c^2}{\epsilon}\begin{pmatrix}
\frac{575}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{23347}{14400} & \frac{46517}{5760} & 0 & 0 & 0 & 0 & \frac{487}{1800} & 0 \\
\frac{3883}{4032} & -\frac{171823}{37800} & \frac{36597791}{3024000} & -\frac{29581}{16800} & 0 & 0 & -\frac{1789}{4800} & 0 \\
-\frac{9271}{11200} & -\frac{35239}{50400} & \frac{74209}{168000} & \frac{188599}{18900} & 0 & 0 & \frac{2101}{4800} & 0 \\
\frac{3287}{84000} & -\frac{2048479}{1176000} & \frac{422283}{392000} & -\frac{2501309}{1764000} & \frac{49211483}{3528000} & \frac{293221}{392000} & \frac{2764807}{2116800} & -\frac{61}{20160} \\
\frac{947587}{1058400} & -\frac{1555357}{705600} & \frac{16831}{29400} & -\frac{239641}{75600} & -\frac{381527}{2116800} & \frac{5839021}{423360} & -\frac{5807}{201600} & \frac{118933}{1411200} \\
\frac{3349}{7200} & -\frac{2591}{2400} & 0 & 0 & 0 & 0 & \frac{150391}{14400} & 0 \\
-\frac{45083}{44100} & \frac{16564}{11025} & \frac{5447}{117600} & \frac{380791}{176400} & \frac{1063}{29400} & -\frac{545189}{352800} & \frac{1176541}{1058400} & \frac{174229}{12600}
\end{pmatrix}$$

Mixing matrices and spectrum

Two-loop anomalous dimensions for length-3 operators up to dimension 16:

Jin, Ren, GY 2020

dim	4	6	8	10	12	14	16
$\gamma^{(1)}_{f,\alpha}$	$-\frac{22}{3}$	/	$\frac{7}{3}$	$\frac{71}{15}$	$\frac{241}{30}, \frac{101}{15}$	$\frac{61}{6}, \frac{172}{21}$	$\frac{331}{35}, \frac{1212\pm\sqrt{3865}}{105}$
$\gamma^{(2)}_{f,\alpha}$	$-\frac{136}{3}$	/	$\frac{269}{18}$	$\frac{2848}{125}$	$\frac{49901119}{1404000}, \frac{8585281}{234000}$	$\frac{4392073141}{87847200}, \frac{685262197}{15373260}$	$\frac{231568398949}{4253886000}, \frac{355106171452034\pm95588158951\sqrt{3865}}{6576507756000}$
$\gamma^{(1)}_{f,\beta}$	$-\frac{22}{3}$	1	/	$\frac{17}{3}$	9	$\frac{43}{5}$	$\frac{67}{6}$
$\gamma^{(2)}_{f,\beta}$	$-\frac{136}{3}$	$\frac{25}{3}$	/	$\frac{2195}{72}$	$\frac{79313}{1800}$	$\frac{443801}{9000}$	$\frac{63879443}{1058400}$
$\gamma^{(1)}_{d,\alpha}$	/	/	/	$\frac{13}{3}$	$\frac{41}{6}$	$\frac{551\pm3\sqrt{609}}{60}$	$\frac{321\pm\sqrt{1561}}{30}$
$\gamma^{(2)}_{d,\alpha}$	/	/	/	$\frac{575}{36}$	$\frac{46517}{1440}$	$\frac{5809305897\pm19635401\sqrt{609}}{131544000}$	$\frac{229162584707\pm225658792\sqrt{1561}}{4130406000}$
$\gamma^{(1)}_{d,\beta}$	/	/	/	/	9	/	$\frac{67}{6}$
$\gamma^{(2)}_{d,\beta}$	/	/	/	/	$\frac{150391}{3600}$	/	$\frac{174229}{3150}$

Two-loop renormalization for higher length operators. Jin, Ren, GY, Yu 2022

Evanescent operators

Evanescent operator ("倏逝算符")：
Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

Evanescent operators

Evanescent operator (“倏逝算符”)：
Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\mathbf{F}^{(0)}_{\mathcal{O}^{\mathrm{e}}_L, n \geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\mathrm{e}}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

Four-fermion dimension-6 operators:

$$\mathcal{O}^{(n)}_{\text{4-ferm}} = \bar{\psi}\gamma^{[\mu_1}\ldots\gamma^{\mu_n]}\psi\bar{\psi}\gamma_{[\mu_1}\ldots\gamma_{\mu_n]}\psi\,, \qquad n \geq 5\,.$$

Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich and U. Nierste 1994

Evanescent operators

Evanescent operator ("倏逝算符"):
Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

Gluonic evanescent operators (start to appear at dimension 10):

$$\mathcal{O}_{\rm e} = \frac{1}{16}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{\nu_1\nu_2\nu_3\nu_4\nu_5}\mathrm{tr}(D_{\nu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}D_{\mu_5}F_{\nu_1\nu_2}F_{\nu_3\nu_4})$$

$$\delta^{\mu_1..\mu_n}_{\nu_1...\nu_n} = \det(\delta^{\mu}_{\nu}) = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_1}_{\nu_n} \\ \vdots & & \vdots \\ \delta^{\mu_n}_{\nu_1} & \ldots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}$$

Length-4 basis counting

Δ_0	$N^{\rm p}_+ = N^{\rm p}$	$N^{\rm e}_+$	$N^{\rm e}_-$
8	4	0	0
10	20	4	0
12	82	24	1
14	232	88	4
16	550	246	13

Jin, Ren, GY, Yu, 2202.08285, 2208.08976

Evanescent operators

Evanescent operators are important for renormalization beyond one-loop order.

$$\begin{pmatrix} Z_{\mathrm{pp}}^{(1)} & Z_{\mathrm{pe}}^{(1)} \\ 0 & Z_{\mathrm{ee}}^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} Z_{\mathrm{pp}}^{(l)} & Z_{\mathrm{pe}}^{(l)} \\ Z_{\mathrm{ep}}^{(l)} & Z_{\mathrm{ee}}^{(l)} \end{pmatrix}, \quad l \geq 2$$

One can use finite renormalization scheme such that

$$\begin{pmatrix} \hat{\mathcal{D}}_{\mathrm{pp}}^{(l)} & \hat{\mathcal{D}}_{\mathrm{pe}}^{(l)} \\ 0 & \hat{\mathcal{D}}_{\mathrm{ee}}^{(l)} \end{pmatrix}$$

but the lower-loop evanescent operator result are needed.

For example, $\hat{\mathcal{D}}_{\mathrm{pp}}^{(2)}$ contains $(-2\epsilon \hat{Z}_{\mathrm{pe}}^{(1)} \hat{Z}_{\mathrm{ep}}^{(1)})$

Jin, Ren, GY, Yu, 2202.08285, 2208.08976

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

  The answer is NO.

  YM theory is non-unitary in non-integer spacetime dimensions, due to the existence of evanescent operators.

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

The answer is NO.

YM theory is non-unitary in non-integer spacetime dimensions, due to the existence of evanescent operators.

$$
\begin{aligned}
&\partial_\nu\partial_\rho\Big[\delta^{12456\nu}_{3789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big],\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2357\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2367\nu}_{589\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2378\nu}_{569\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{5}\delta^{2347\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{2}_{4}\delta^{1567\nu}_{389\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]
\end{aligned}
$$

Dim-12 evanescent operators

$$
\begin{pmatrix}
-\frac{38}{3\epsilon} & \frac{2}{\epsilon} & -\frac{13}{12\epsilon} & 0 & \frac{14}{3\epsilon} & 0 & \frac{14}{3\epsilon} & \frac{28}{3\epsilon}\\
-\frac{1}{2\epsilon} & -\frac{85}{6\epsilon} & \frac{2}{\epsilon} & \frac{5}{6\epsilon} & -\frac{2}{3\epsilon} & -\frac{5}{12\epsilon} & -\frac{7}{3\epsilon} & -\frac{16}{3\epsilon}\\
0 & -\frac{4}{\epsilon} & -\frac{22}{3\epsilon} & \frac{16}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & \frac{16}{3\epsilon}\\
0 & -\frac{4}{3\epsilon} & \frac{7}{3\epsilon} & -\frac{34}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & 0\\
\frac{1}{12\epsilon} & -\frac{1}{12\epsilon} & -\frac{3}{8\epsilon} & \frac{1}{12\epsilon} & -\frac{44}{3\epsilon} & \frac{5}{8\epsilon} & \frac{1}{2\epsilon} & \frac{2}{\epsilon}\\
0 & \frac{4}{3\epsilon} & \frac{2}{3\epsilon} & 0 & 0 & -\frac{18}{\epsilon} & 0 & -\frac{16}{3\epsilon}\\
\frac{1}{6\epsilon} & \frac{3}{2\epsilon} & \frac{9}{16\epsilon} & -\frac{1}{2\epsilon} & \frac{29}{6\epsilon} & -\frac{5}{12\epsilon} & -\frac{49}{6\epsilon} & \frac{13}{3\epsilon}\\
-\frac{5}{6\epsilon} & -\frac{1}{3\epsilon} & \frac{13}{32\epsilon} & -\frac{5}{6\epsilon} & \frac{3}{4\epsilon} & \frac{1}{4\epsilon} & \frac{5}{12\epsilon} & -\frac{91}{6\epsilon}
\end{pmatrix}
$$

One-loop mixing matrix

A pair of complex eigenvalues:

$$\boxed{1.90386 \pm 0.181142\,\mathrm{i}\,.}$$

Jin, Ren, GY, Yu, 2301.01786

Similar complex AD was observed in phi^4 theory starting at dim-23 operators.

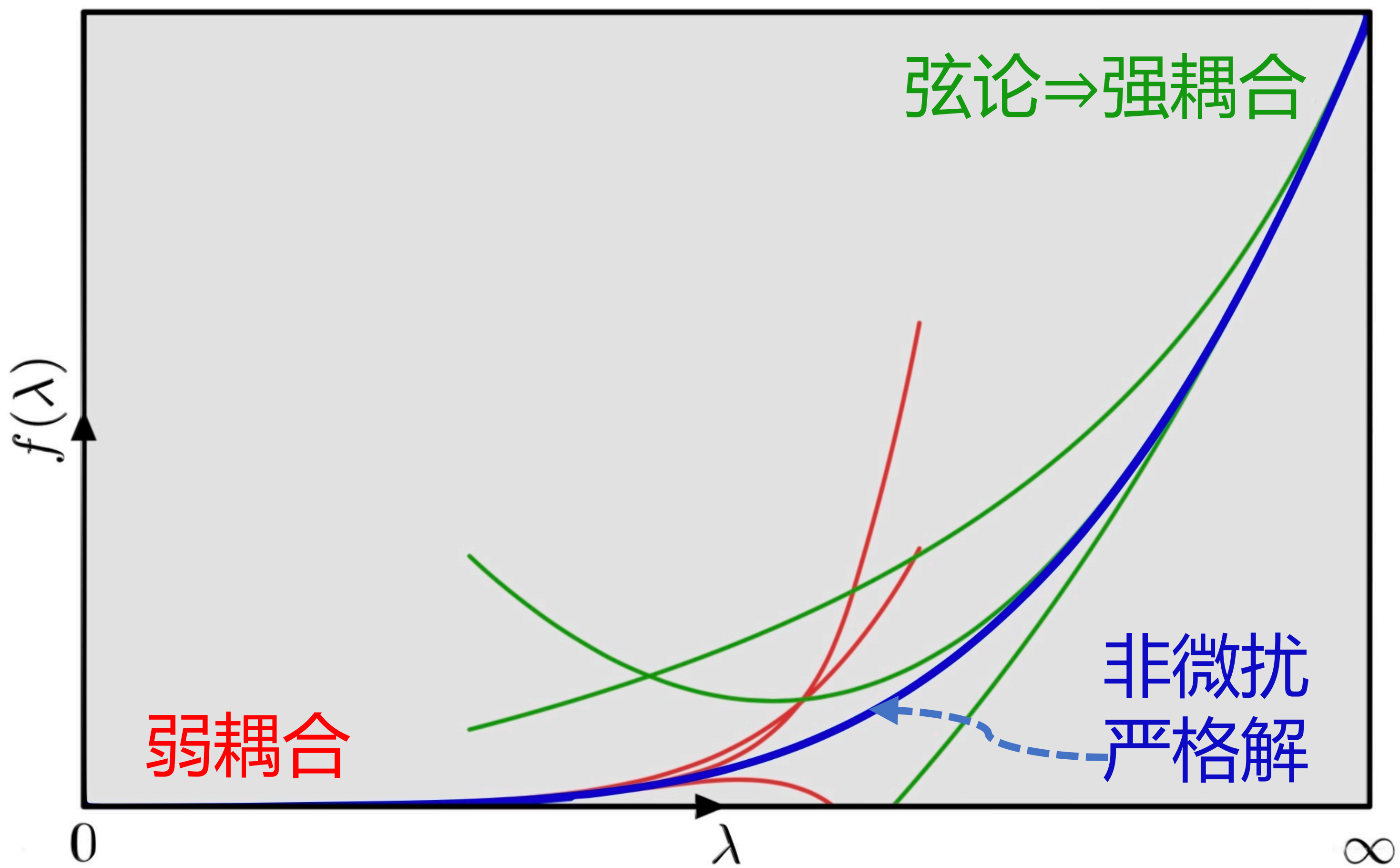
Hogervorst, Rychkov, van Rees 2015

Outline

- Introduction and background
- On-shell methods
- Tree-level form factors
- Sudakov FF and IR divergences
- CK duality and double copy
- Operator classification and renormalization
- Form factor / Wilson line duality

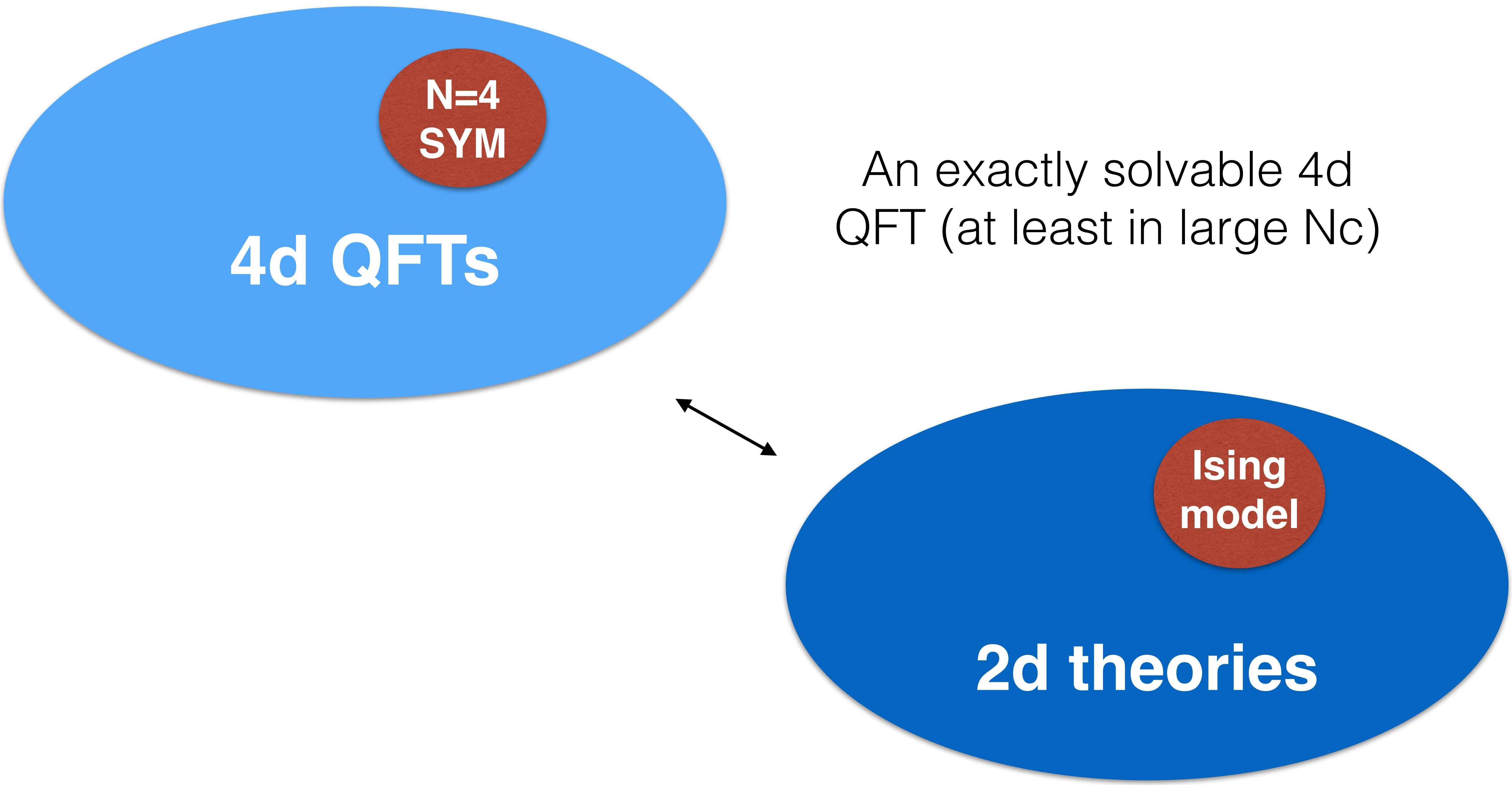
Exact solutions

Planar N=4 SYM is exactly solvable:



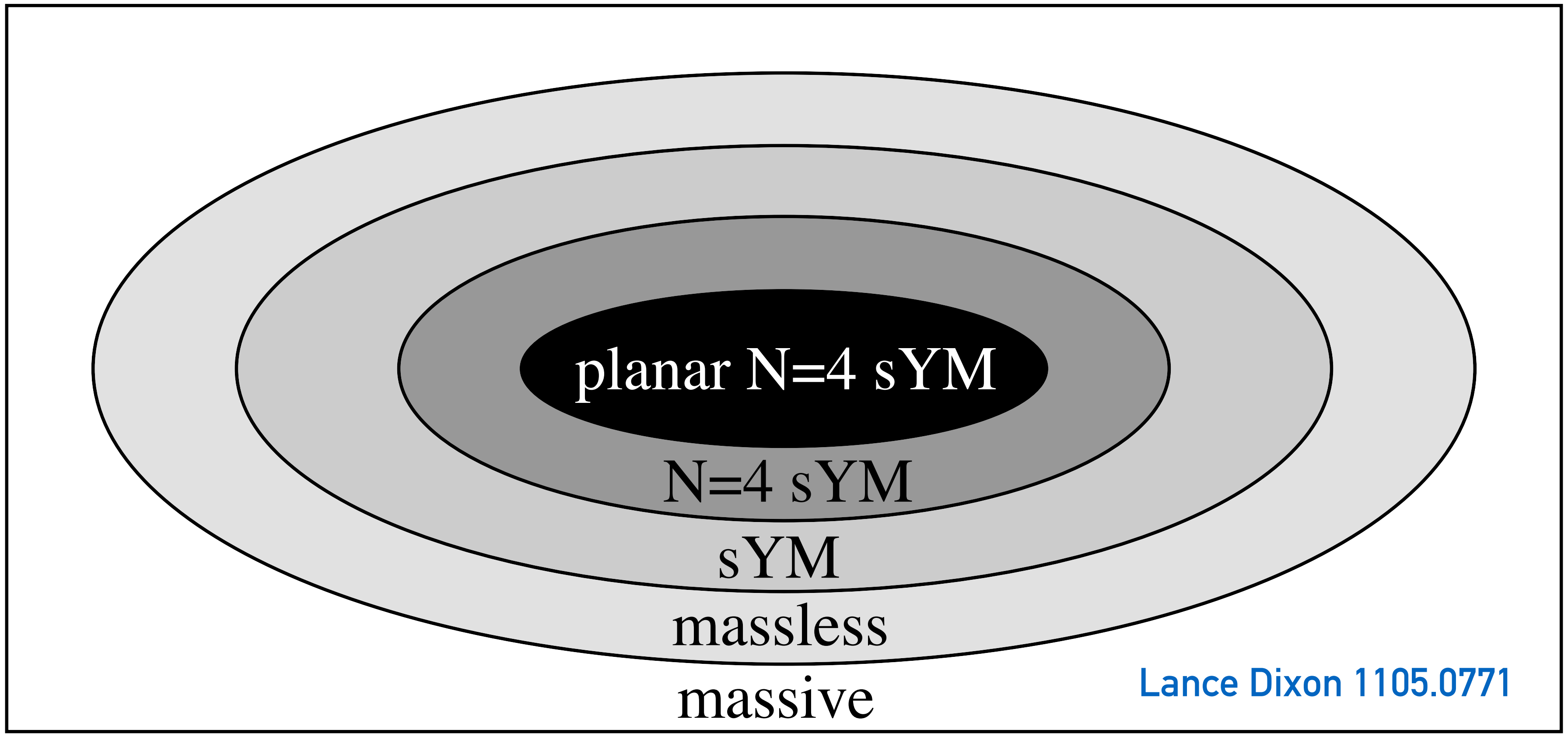
N. Beisert et al., Lett. Math. Phys. 99 (2012) 3

N=4 SYM



An exactly solvable 4d QFT (at least in large Nc)

N=4 SYM and amplitudes



N=4 SYM has also been the main source for modern amplitudes development.

Bern, Dixon, Durban, Kosower 1994; Witten 2003; Britto, Cachazo, Feng, Witten, 2004; …

"BCFW recursion relation" and "unitarity methods" were all first developed by studying N=4 SYM.

N=4 SYM v.s. QCD

N=4 SYM theory : -> QCD's maximally supersymmetric cousin

$$\mathscr{L}_{N=4} = -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \text{fermions} + \text{scalars}$$

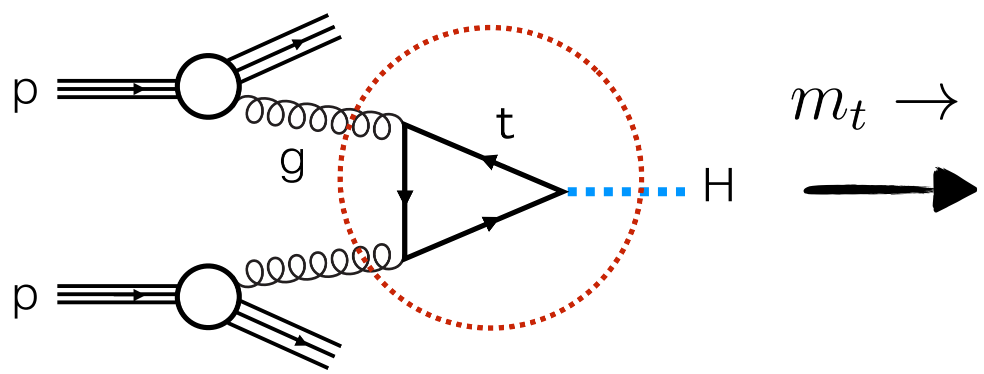
where all fields are the in the adjoint representation of the gauge group SU(Nc).

A four-dimensional theory with non-trivial interaction and also many special symmetries.

QCD $$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \text{quarks}$$

Higgs amplitudes

Operators also appear as interaction vertices in effective field theories (EFT)

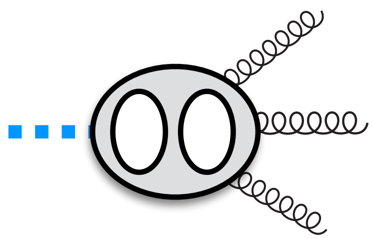


$m_t \to \infty$

Effective gluon-Higgs vertex:

$$\mathscr{L}_{\text{HEFT}} = c_0 H \mathrm{tr}(F^2) + \mathcal{O}(1/m_t^2)$$

Higgs + multi-gluon scattering is a form factor



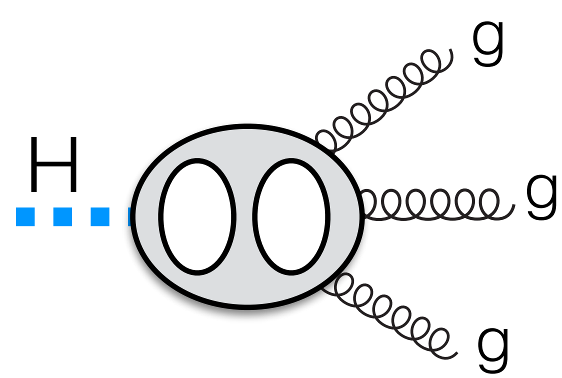
$$A(q^H, 1^g, 2^g, \ldots, n^g) = F_{\mathcal{O}=\mathrm{tr}(F^2)}(1^g, 2^g, \ldots, n^g)$$

Maximal Transcendentality Principle

N=4 SYM 3-point form factor of stress-tensor supermultiplet ⟷ Higgs plus 3-gluon amplitudes $m_t \to \infty$

Brandhuber, Travaglini, GY 2012

Gehrmann, Jaquier, Glover, Koukoutsakis 2011



Maximally transcendental parts are equal between two theories!

N=4 SYM ⟷ **QCD**

"maximal transcendentality principle" Kotikov, Lipatov, Onishchenko, Velizhanin 2004

Maximal Transcendentality Principle

Conjecture for certain quantities

N=4 SYM ⟷ **QCD**

The maximally transcendental parts are equal in two theories.

- Such a relation was first observed for anomalous dimension of twist-2 operators

$$\gamma^{\mathcal{N}=4}(j) = \gamma^{\text{QCD}}(j)|_{\text{max. trans}}$$

Kotikov, Lipatov 2001; Kotikov, Lipatov, Onishchenko, Velizhanin 2004

Lev Lipatov
1940-2017

- Also for certain Wilson lines [Li, Manteuffel, Schabinger, Zhu 2014]

Maximal Transcendentality Principle

Maximal transcendental part of
Higgs amplitudes:

Gehrmann, Jaquier, Glover, Koukoutsakis 2011

$$
\begin{aligned}
&-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
&+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
&-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
&-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
&-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
&+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
&-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
&-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
&-\cdots-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
&+\cdots v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
&+\cdots)-4G(1-v,-v,-v,u)H(1,v)+2G(-v,0,1-v,u)H(1,v)+2G(-v,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
&-4\cdots,v)+4G(-v,-v,-v,u)H(1,v)+G(0,0,u)H(0,0,v)+G(0,1-v,u)H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
&-G(0,0,u)H(0,1,v)+G(0,-v,u)H(0,1,v)-G(1,0,u)H(0,1,v)+2G(1-v,0,u)H(0,1,v)+2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
&-G(-v,0,u)H(0,1,v)-2G(-v,1-v,u)H(0,1,v)+4G(-v,-v,u)H(0,1,v)-G(0,0,u)H(1,0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
&+2G(1-v,0,u)H(1,0,v)-2G(1-v,1-v,u)H(1,0,v)+G(1-v,-v,u)H(1,0,v)-G(-v,0,u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
&-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
&+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
&-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
&+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
&+4G(-v,1-v,-v,1-v,u)
\end{aligned}
$$

QCD

$$G(1-v,1-v,v,1-v;u)$$

Multiple polyLogarithm

$\updownarrow$

Brandhuber, Travaglini, GY 2012

N=4

$$
-2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-\frac{1}{u_i}\right)+\frac{\log^4 u_i}{4!}\right]-2\left[\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right]^2
$$

$$
+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2+2(J_2^2-\zeta_2 J_2)-\frac{\log^4(uvw)}{4!}-\zeta_3\log(uvw)-\frac{123}{8}\zeta_4
$$

$$
J_4(x)=\mathrm{Li}_4(x)-\log(-x)\mathrm{Li}_3(x)+\frac{\log^2(-x)}{2!}\mathrm{Li}_2(x)-\frac{\log^3(-x)}{3!}\mathrm{Li}_1(x)-\frac{\log^4(-x)}{48},\quad J_2=\sum_{i=1}^{3}\left(\mathrm{Li}_2(1-u_i)+\frac{1}{2}\log(u_i)\log(u_{i+1})\right),\quad u=\frac{s_{12}}{q^2},\ v=\frac{s_{23}}{q^2},\ w=\frac{s_{13}}{q^2},\ \text{where}\ q^2=s_{123}
$$

Transcendental numbers and functions in QFT

Riemann zeta value:

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}\,, \qquad k \geq 2$$

Polylogarithm:

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t} dt$$

$$\mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathrm{Li}_k(1) = \zeta_k$$

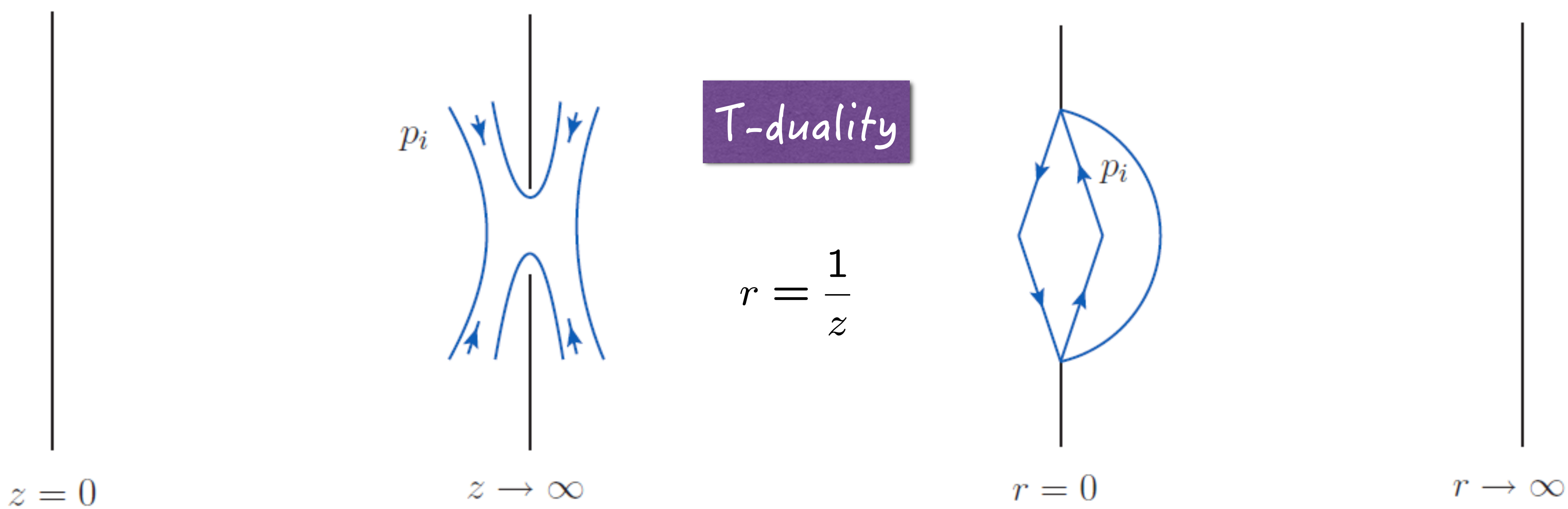
transcendental degree k

Ref: C. Duhr “Mathematical aspects of scattering amplitudes” arXiv:1411.7538

Form factor / Wilson line and dual conformal symmetry

Amplitudes / Wilson loop duality

N=4 SYM ⟷ (AdS/CFT) Type IIB string theory in $AdS_5 \times S^5$

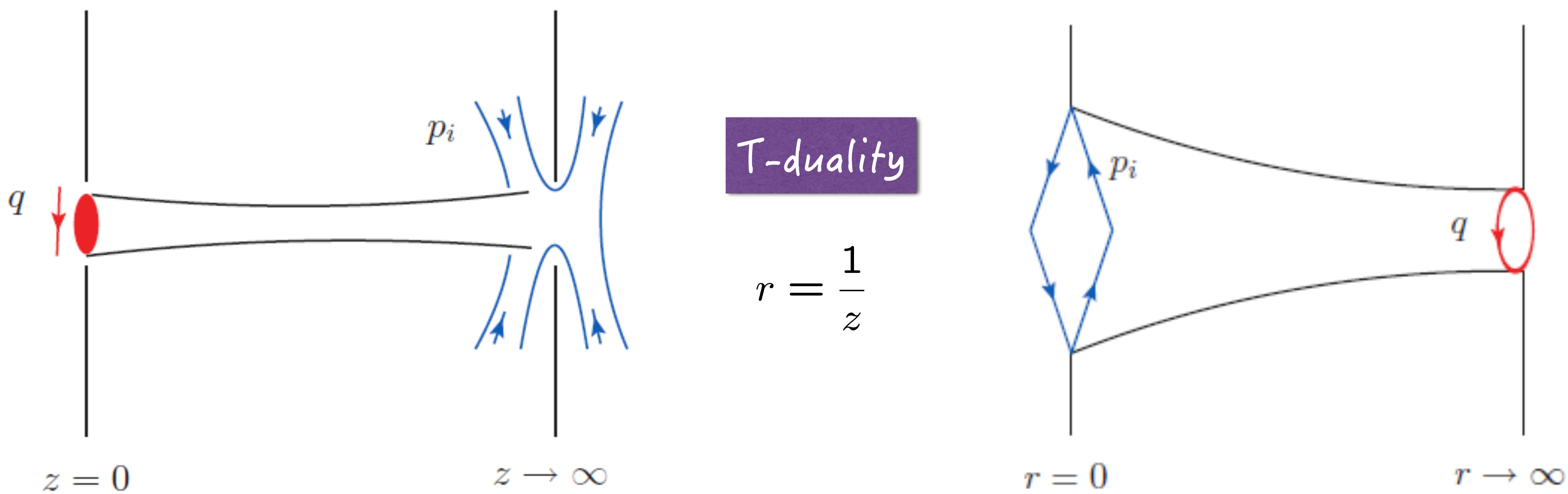


Amplitudes ⟷ minimal surface of Light-like Wilson loops

[Alday and Maldacena 2007]

Form factors picture

N=4 SYM ⟷ (AdS/CFT) Type IIB string theory in $AdS_5 \times S^5$

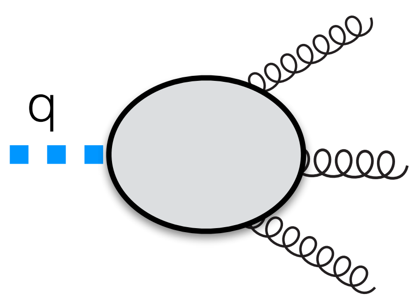


Form factors as minimal surfaces in one period [Alday and Maldacena 2007]

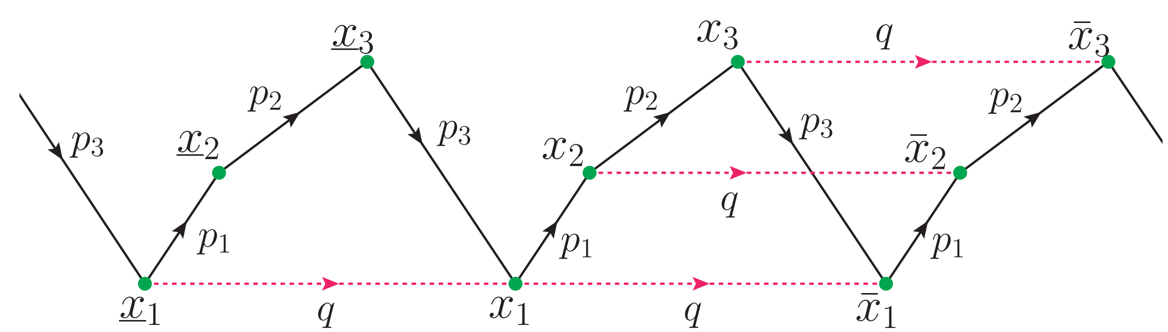
Y-system formulation (Maldacena and Zhiboedov in AdS3; Gao and GY in AdS5)

Form factor / Wilson loop duality

Strong coupling T-duality implies the duality:



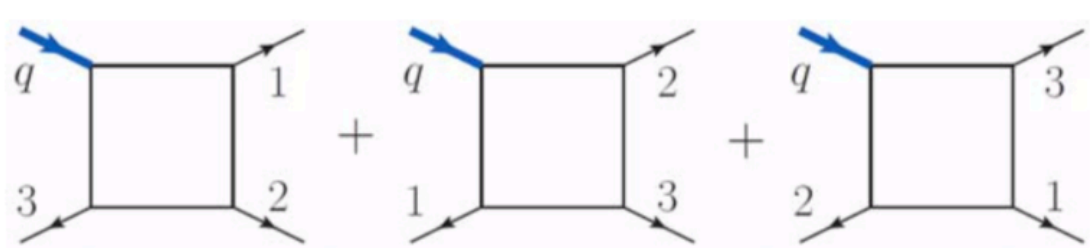
$$x_i - x_{i+1} = p_i \quad \longleftrightarrow$$

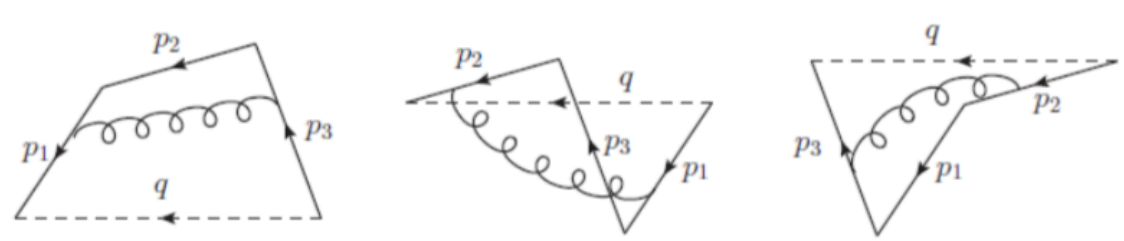


Weak coupling picture at one loop:

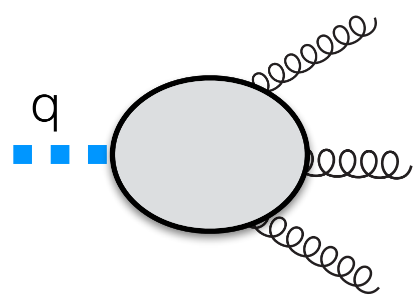
Brandhuber, Spence, Travaglini, GY 2010

Fin

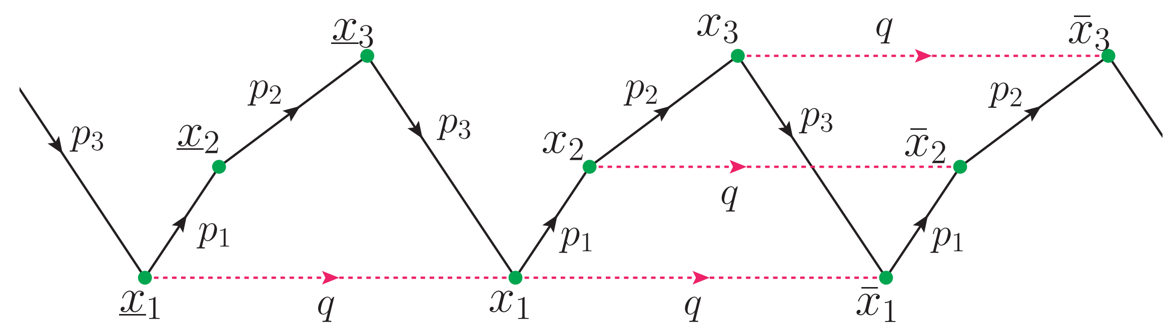




Form factor / Wilson loop duality

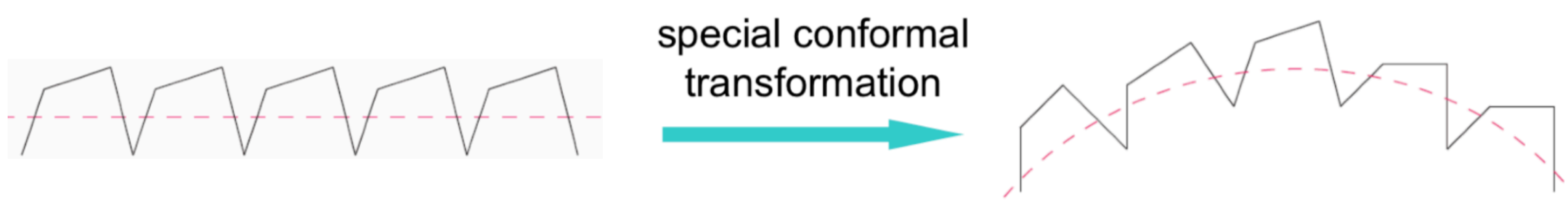


$$x_i - x_{i+1} = p_i$$

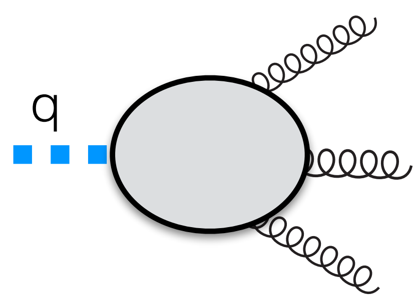


Dual periodic WL picture

No exact dual conformal symmetry for general q.

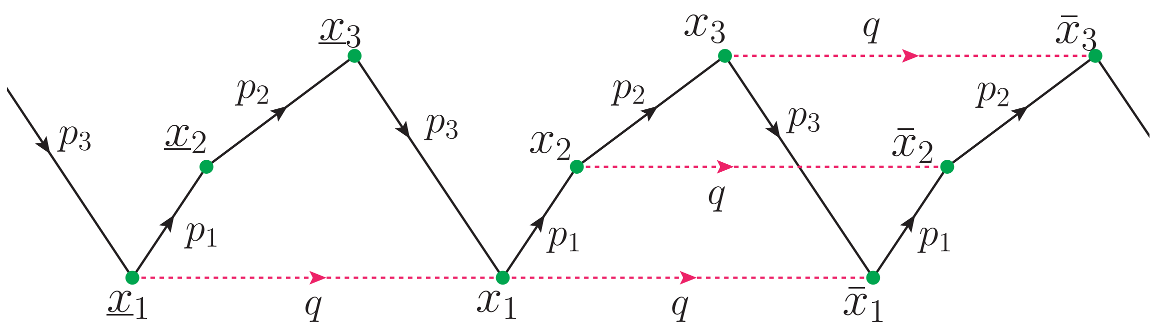


Form factor / Wilson loop duality



$x_i - x_{i+1} = p_i$

$\longleftrightarrow$

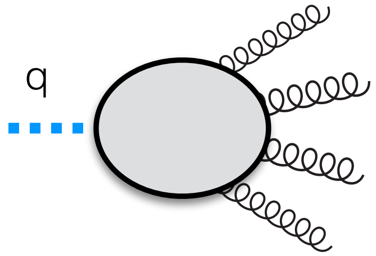


Dual periodic WL picture

For lightlike q, one expects an exact dual conformal symmetry:

$$\delta_q x_i^\mu \equiv \frac{1}{2} x_i^2 q^\mu - (x_i \cdot q)\, x_i^\mu \qquad \boxed{q^2 = 0}$$

The first non-trivial lightlike FF is the 4-point FF.



Number of independent variables

Counting the degree of freedom:

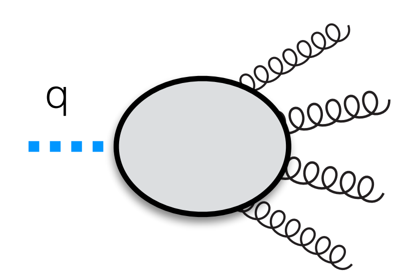
Conformal group

Amplitudes: 3n - 15

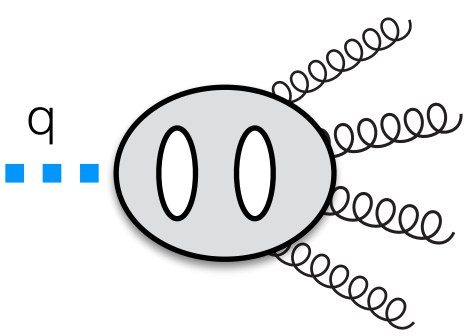
FF ($q^2 \neq 0$): 3n + 4 - 10 - 1 = 3n - 7

FF ($q^2 = 0$): 3n - 7 -1 -1 = 3n - 9

The first non-trivial lightlike FF is the 4-point FF.



A bootstrap computation

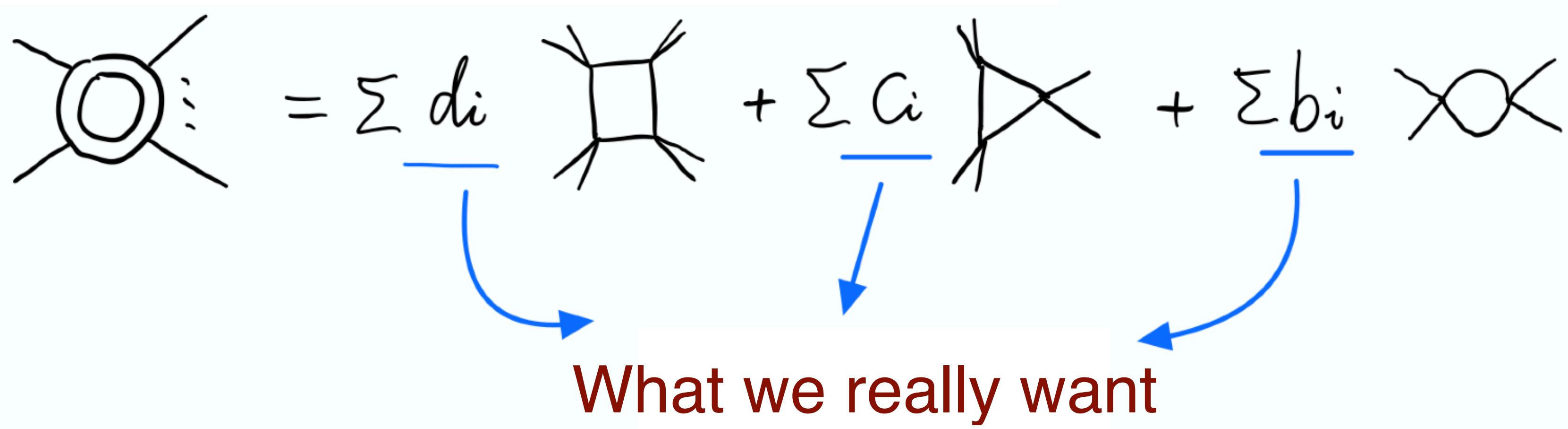


Master-integral bootstrap

Based on the fact:

any amplitude or form factor can be expanded in a set of integral basis

Consider one-loop amplitudes:



Ansatz in master integrals → **Physical constraints** → Solution of coefficients

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

$$\mathcal{F}^{(l)} = \sum_i C_i I_i^{(l)}$$

Master-integral bootstrap

A bootstrap strategy to compute amplitudes or form factors: Guo, Wang, GY 2021

Ansatz → **Physical constraints** → **Solution**

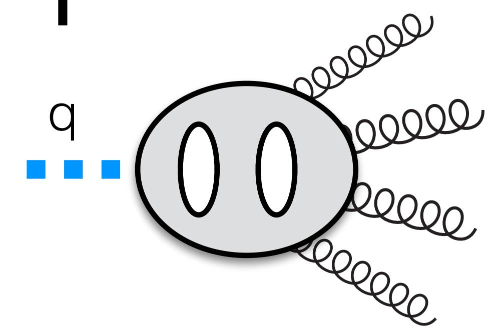
$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR divergences,
Collinear limits,
Spurious pole cancellation,
Unitarity cuts,
etc

$$\mathcal{F}^{(l)} = \sum_i C_i I_i^{(l)}$$

Remarks:
the method does not rely on special symmetries of the theory and can be applied to general theories.

Bootstrapping the two-loop FF



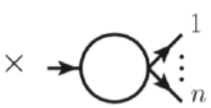
Based on the one-loop results:

$$\mathcal{F}_4^{\mathrm{LL},(1)} = \mathcal{F}_4^{\mathrm{LL},(0)} \left(\mathcal{G}_1^{(1)} + B\, \mathcal{G}_2^{(1)} \right)$$

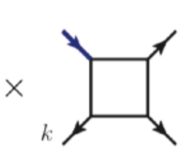
Pure functions

$$B \equiv \frac{s_{12}s_{34} + s_{23}s_{14} - s_{13}s_{24}}{4i\varepsilon(1234)}$$

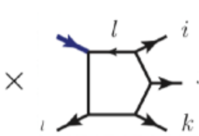
$$I^{(1)}_{\mathrm{Bub}}(1,\ldots,n) = \frac{1-2\epsilon}{\epsilon} \times$$



$$I^{(1)}_{\mathrm{Box}}(i,j,k) = (s_{ij}s_{jk} - p_j^2 q^2) \times$$



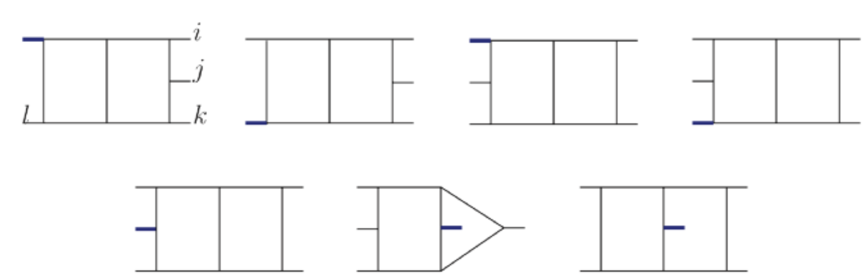
$$I^{(1)}_{\mathrm{Pen}}(i,j,k,l) = 4i\varepsilon(1234) \times \mu \times$$



We propose the ansatz at two loops:

$$\mathcal{F}_4^{\mathrm{LL},(2)} = \mathcal{F}_4^{\mathrm{LL},(0)} \left(\mathcal{G}_1^{(2)} + B\, \mathcal{G}_2^{(2)} \right)$$

$$\mathcal{G}_a^{(2)} = \sum_{i=1}^{590} c_{a,i} I_i^{(2)}$$



UT master integrals: Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, Lo Presti, Mitev, Page, Papadopoulos, Tommasini, Sotnikov, Wasser, Wever, Zeng, Zhang, Zoia

Bootstrapping the two-loop FF

$$\mathcal{F}_4^{\mathrm{LL},(2)} = \mathcal{F}_4^{\mathrm{LL},(0)} \Big(\mathcal{G}_1^{(2)} + B\, \mathcal{G}_2^{(2)} \Big) \qquad \mathcal{G}_a^{(2)} = \sum_{i=1}^{590} c_{a,i} I_i^{(2)}$$

$$\hat{\mathcal{F}}_4^{(2)} = \frac{1}{2}\big(\hat{\mathcal{F}}_4^{(1)}(\epsilon)\big)^2 + f^{(2)}(\epsilon)\hat{\mathcal{F}}_4^{(1)}(2\epsilon) + \mathcal{R}_4^{(2)} + \mathcal{O}(\epsilon)$$

$$\mathcal{R}_4^{\mathrm{LL},(2)} \xrightarrow{p_i \parallel p_{i+1}} \mathcal{R}_3^{\mathrm{LL},(2)} = -6\zeta_4$$

Constraints	Parameters left
Starting ansatz	590×2
Symmetries of external legs	168
IR (Symbol)	109
Collinear limit (Symbol)	43
IR (Function)	39
Collinear limit (Function)	21
Keeping up to ϵ^0 order (or via unitarity)	0

Guo, Wang, GY 2209.06816

Bootstrapping the two-loop FF

$$\mathcal{F}_4^{\mathrm{LL},(2)} = \mathcal{F}_4^{\mathrm{LL},(0)}\left(\mathcal{G}_1^{(2)} + B\,\mathcal{G}_2^{(2)}\right) \qquad \mathcal{G}_a^{(2)} = \sum_{i=1}^{590} c_{a,i} I_i^{(2)}$$

$$\hat{\mathcal{F}}_4^{(2)} = \frac{1}{2}\left(\hat{\mathcal{F}}_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\hat{\mathcal{F}}_4^{(1)}(2\epsilon) + \mathcal{R}_4^{(2)} + \mathcal{O}(\epsilon)$$

$$\mathcal{R}_4^{\mathrm{LL},(2)} \xrightarrow{p_i \parallel p_{i+1}} \mathcal{R}_3^{\mathrm{LL},(2)} = -6\zeta_4$$

Constraints	Parameters left
Starting ansatz	590×2
Symmetries of external legs	168
IR (Symbol)	109
Collinear limit (Symbol)	43
IR (Function)	39
Collinear limit (Function)	21
Keeping up to ϵ^0 order (or via unitarity)	0

IR and collinear properties are sufficient to determine the two-loop lightlike FF up to finite order!

Guo, Wang, GY 2209.06816

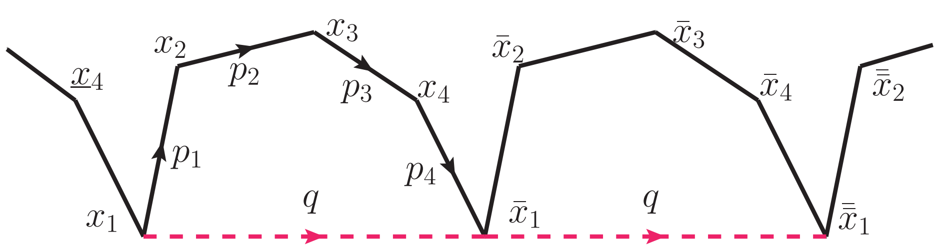
Dual conformal symmetry

The finite remainder depends only on three ratios:

$$u_1 \equiv \frac{s_{12}}{s_{34}} = \frac{x_{13}^2}{x_{3\bar{1}}^2}\,, \quad u_2 \equiv \frac{s_{23}}{s_{14}} = \frac{x_{24}^2}{x_{4\bar{2}}^2}\,, \quad u_3 \equiv \frac{s_{123}s_{134}}{s_{234}s_{124}} = \frac{x_{14}^2 x_{3\bar{2}}^2}{x_{2\bar{1}}^2 x_{4\bar{3}}^2}$$

which satisfies precisely the dual conformal symmetry

$$\delta_q x_i^\mu \equiv \frac{1}{2} x_i^2 q^\mu - (x_i \cdot q)\, x_i^\mu$$



$$\delta_q R_4^{LL,(2)} = 0$$

Guo, Wang, GY 2209.06816

Other 4-point form factors

Ansatz → **Physical constraints** → **Solution**

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits,
unitarity-cut, etc

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

The same strategy has been used to compute four-point form factors of length-3 operators:

$$F^{(2)}_{\text{tr}(\phi^3)}(1^\phi, 2^\phi, 3^\phi, 4^g)$$

Guo, Wang, GY 2021

$$F^{(2)}_{\text{tr}(F^3)}(1^g, 2^g, 3^g, 4^g)$$

Guo, Jin, Wang, GY 2022

Proof of MTP for form factors

Ansatz → **Physical constraints** → **Solution**

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

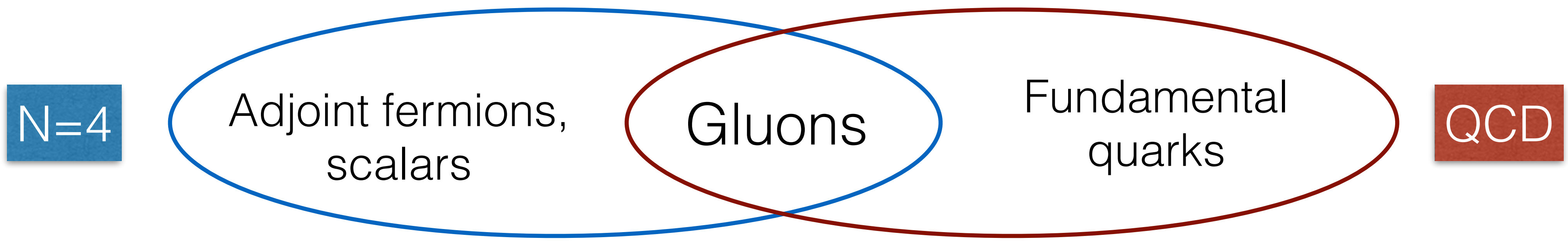
IR, collinear limits,
unitarity-cut, etc

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

Theory-independent constraints → **Theory-independent results**

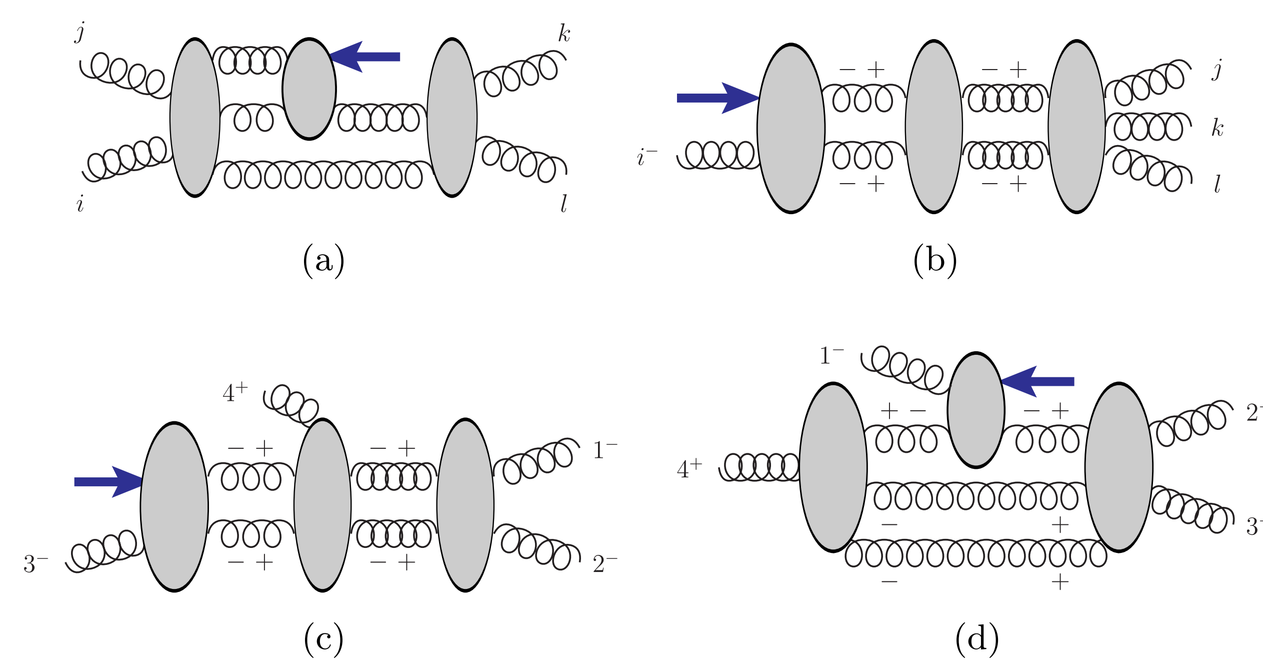
IR and collinear are universal at MT level,
and some unitarity cuts are also universal.

Physical constraints



There are universal cuts that involve only gluon states and thus are also universal for general gauge theories.

Unitarity cut



Proof of MTP for form factors

Ansatz → **Physical constraints** → **Solution**

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

IR, collinear limits,
unitarity-cut, etc

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

Theory-independent constraints → **Theory-independent results**

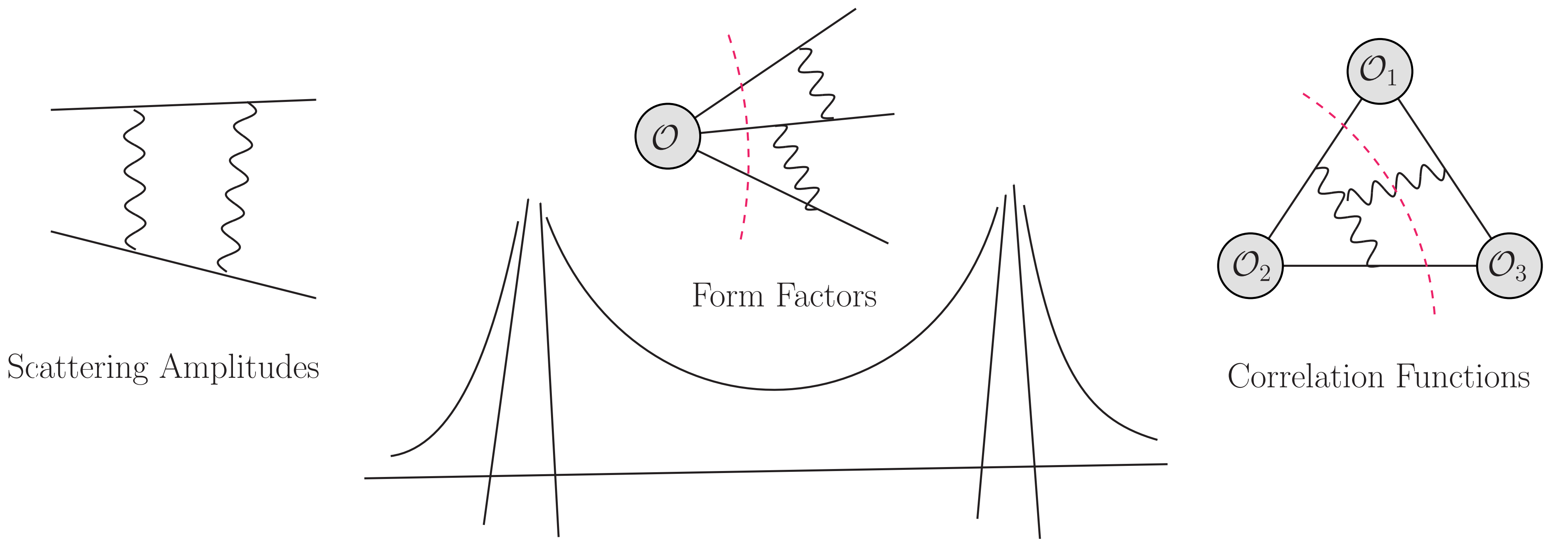
IR and collinear are universal at MT level,
and some unitarity cuts are also universal.

The master-integral bootstrap provides a proof of MTP for various form factors.

Guo, Jin, Wang, GY 2205.12969

Summary

- Form factors provide a framework to study many interesting physical quantities using powerful on-shell amplitude methods:



- IR divergences
- UV renormalization
- Finite remainder

- New hidden structure of form factor.