Introduction to Parton Distribution Function

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CTEQ

Reference

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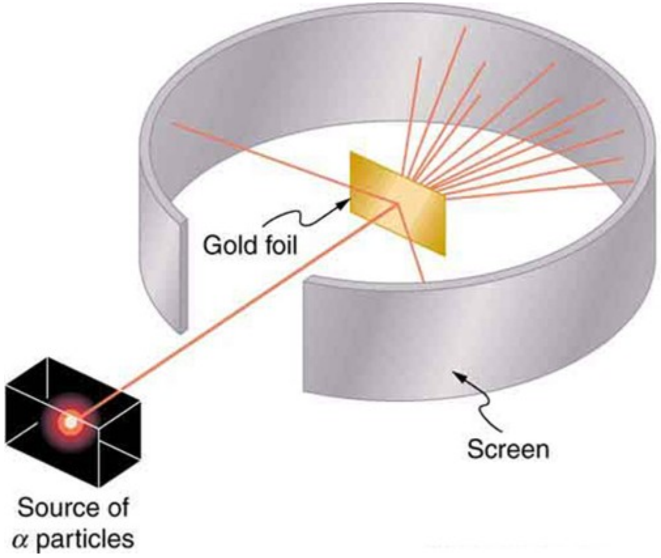
- Marco Stratmann, Lectures on Perturbative QCD or from basic principles to current applications, 2005
- Davison E. Soper, Basics of QCD Perturbation Theory, hep-ph/0011256
- Francis Halzen and Alan D. Martin, Quark and Leptons: An introductory Course in Modern Particle Physics.

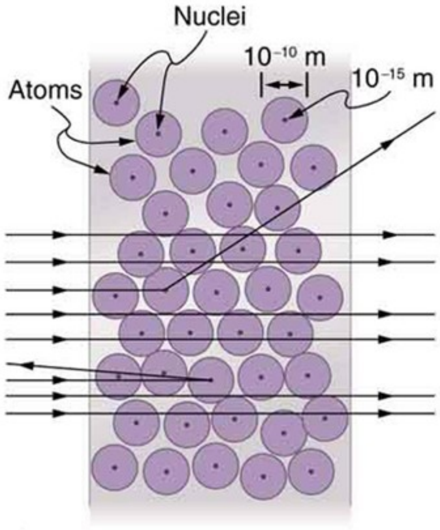
Other Reference

- R. Keith Ellis, W. J. Stirling, B. R. Webber, QCD and Collider Physics
- John Collins, Foundations of Perturbative QCD
- Particle Data Group


Matter
Atom
Electron
Proton
Quarks
Nucleus
Neutron
Matter particles
All ordinary particles belong to this group
These particles existed just after the Big Bang. Now they are found only in cosmic rays and accelerators
LEPTONS
QUARKS
FIRST FAMILY
SECOND FAMILY
THIRD FAMILY
Electron
Responsible for electricity and chemical reactions; it has a charge of -1
Electron neutrino
Particle with no electric charge, and possibly no mass; billions fly through your body every second
Up
Has an electric charge of plus two-thirds; protons contain two, neutrons contain one
Down
Has an electric charge of minus one-third; protons contain one, neutrons contain two
Muon
A heavier relative of the electron; it lives for two-millionths of a second
Muon neutrino
Created along with muons when some particles decay
Charm
A heavier relative of the up; found in 1974
Strange
A heavier relative of the down; found in 1964
Tau
Heavier still; it is extremely unstable. It was discovered in 1975
Tau neutrino
not yet discovered but believed to exist
Top
Heavier still
Bottom
Heavier still; measuring bottom quarks is an important test of electroweak theory
Force particles
These particles transmit the four fundamental forces of nature although gravitons have so far not been discovered
Gluons
Carriers of the strong force between quarks
Felt by: quarks
The explosive release of nuclear energy is the result of the strong force
Photons
Particles that make up light; they carry the electromagnetic force
Felt by: quarks and charged leptons
Electricity, magnetism and chemistry are all the results of electro-magnetic force
Intermediate vector bosons
Carriers of the weak force
W-
W+
Z0
Felt by: quarks and leptons
Some forms of radio-activity are the result of the weak force
Gravitons
Carriers of gravity
Felt by: all particles with mass
All the weight we experience is the result of the gravitational force
GRAPHICS: PETER CROWTHER

Discovery of Strong Interaction



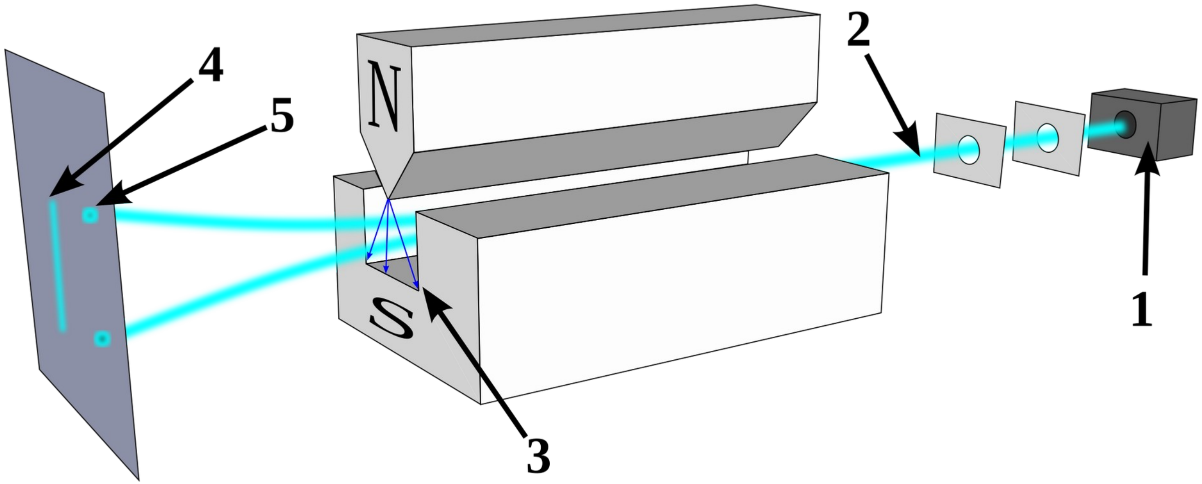


1908-1913: Rutherford gold foil experiments
It shows nucleus is positively charged. Most of the positive charge and mass concentrate at the tiny center of atom, which we call nucleus

1932: Chadwick discovers the neutron

Strong nuclear force, the force binding proton and neutron

Finding of Internal Structure of Proton



Immanuel Estermann and Otto Stern (1933) measured the proton's anomalous magnetic moment

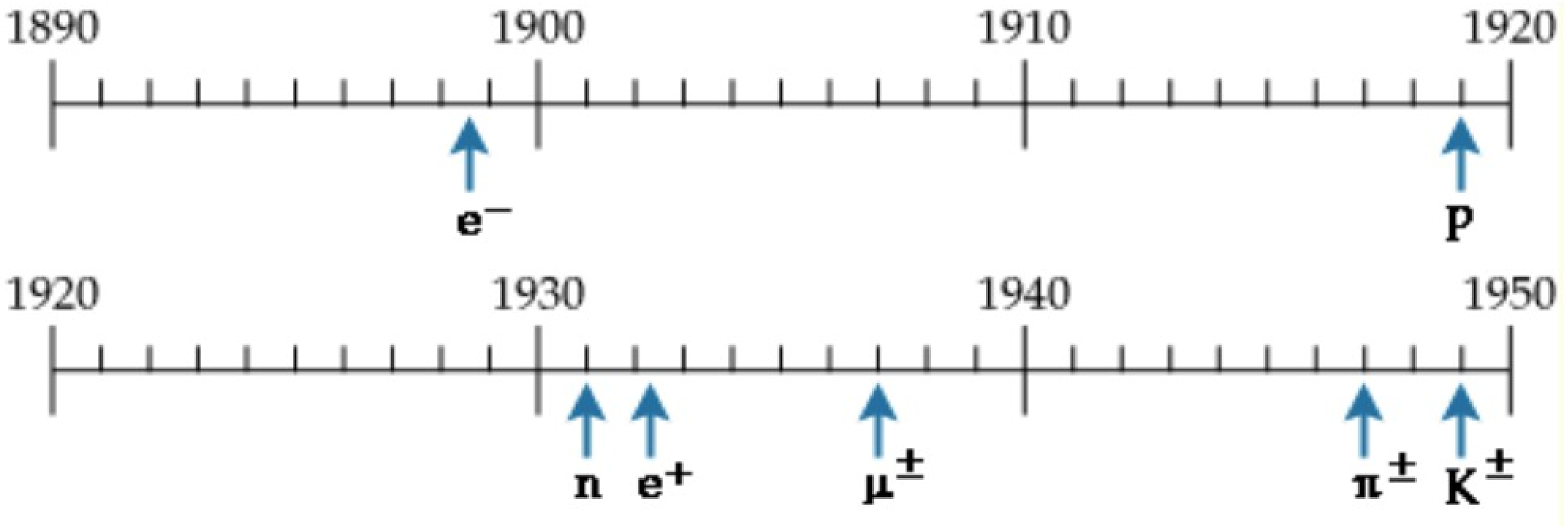
$$\mu_p = g_p \left(\frac{e\hbar}{2m_p} \right)$$

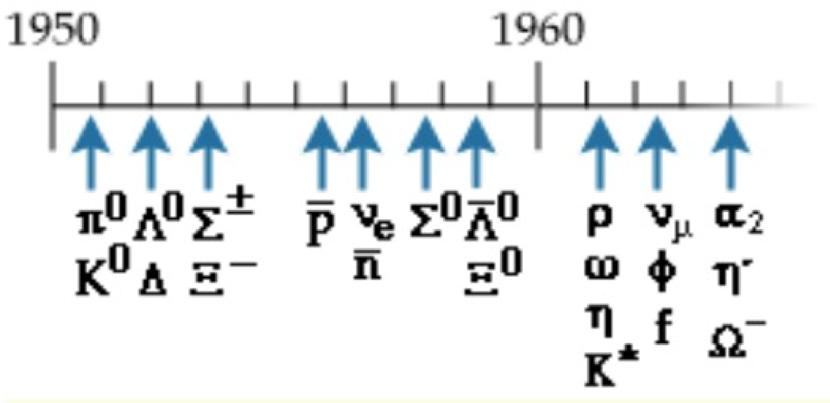
$$g_p = 2.792847356(23) \neq 2!$$

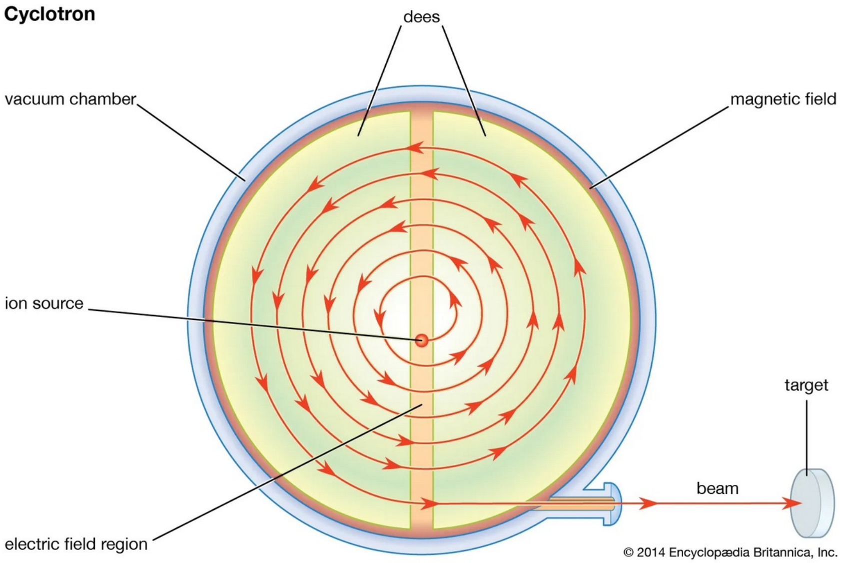
$$\mu_n = -1.913 \left(\frac{e\hbar}{2m_p} \right) \neq 0!$$

Otto Stern
Nobel Prize 1943

Proton is not pointlike and has internal structure!







Isidor I. Rabi asked: who order that?

The Naive quark model

- **Flavor SU(3) – assumption**:
  Physical states for u, d, s, neglecting any mass difference, are represented by 3-eigenstates of the fund'l rep'n of flavor SU(3)

- **Good quantum numbers to label the states**:

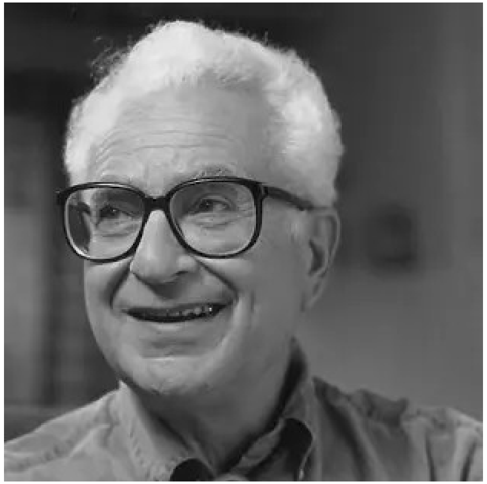
$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

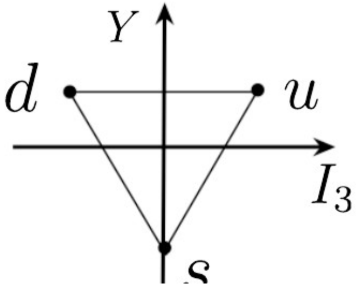
Isospin: $\hat{I}_3 \equiv J_3$ Hypercharge: $\hat{Y} \equiv \frac{2}{\sqrt{3}} J_8$

- **Basis vectors – Eigenstates**:

$$v^1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow u = |\tfrac{1}{2}, \tfrac{1}{3}\rangle \qquad v^2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Longrightarrow d = |\text{-}\tfrac{1}{2}, \tfrac{1}{3}\rangle \qquad v^3 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow s = |0,\text{-}\tfrac{2}{3}\rangle$$

Murray Gell-Mann





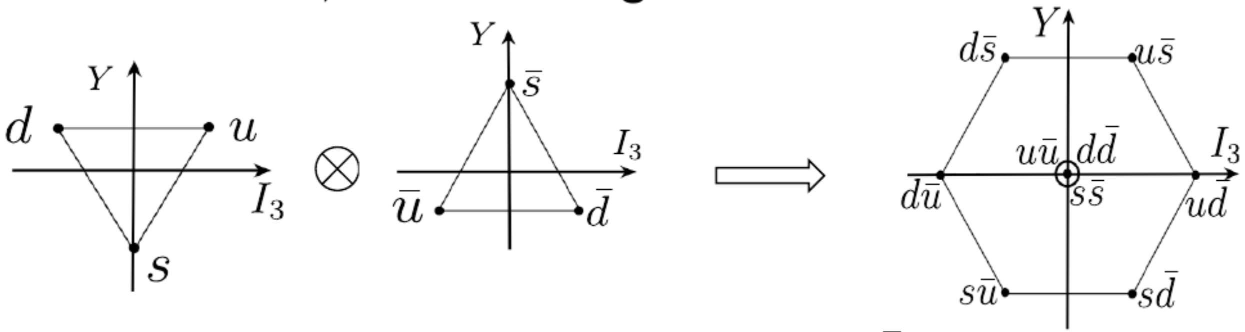
Slide from Jianwei Qiu, 2018

The Naive quark model

❑ Group theory says:

$$q(u,d,s) = \mathbf{3}, \quad \bar{q}(\bar{u},\bar{d},\bar{s}) = \bar{\mathbf{3}}, \quad \text{of flavor SU(3)}$$

$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ $\Longrightarrow$ **1 flavor singlet + 8 flavor octet states**



There are three states with $I_3 = 0, Y = 0$: $u\bar{u}, d\bar{d}, s\bar{s}$

❑ Physical meson states (L=0, S=0):

✧ **Octet states:**

$$A = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \Longrightarrow \pi^0$$

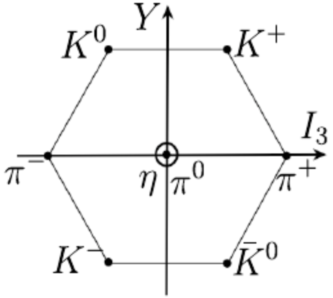
$$B = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \Longrightarrow \eta_8$$

✧ **Singlet states:**

$$C = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \Longrightarrow \eta_1$$

$\eta_8, \eta_1 \;\}\; \eta, \eta'$

$\Longrightarrow$



$\oplus\ \eta'$

Slide from Jianwei Qiu, 2018

The Naive quark model

❑ Group theory says:

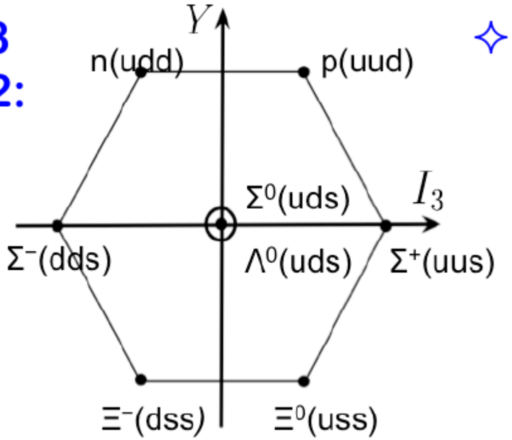
✧ **Flavor:** $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{M_S} \oplus \mathbf{8}_{M_A} \oplus \mathbf{1}_A$

S: symmetric in all 3 q, M_S : symmetric in 1 and 2,
M_A : antisymmetric in 1 and 2, A : antisymmetric in all 3

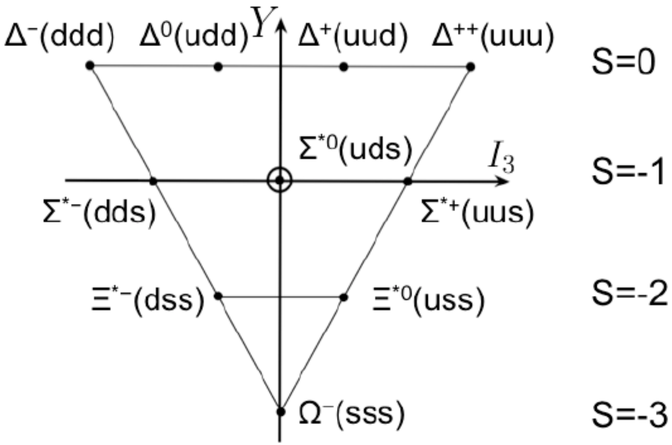
✧ **Spin:** $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_S \oplus \mathbf{2}_{M_s} \oplus \mathbf{2}_{M_A} \Longrightarrow S = \frac{3}{2}, \frac{1}{2}, \frac{1}{2}$

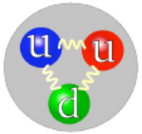
❑ Physical baryon states:

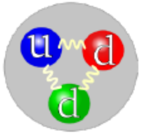
✧ **Flavor-8 Spin-1/2:**



✧ **Flavor-10 Spin-3/2:**



Proton 

Neutron 

Δ^{++}(uuu), …

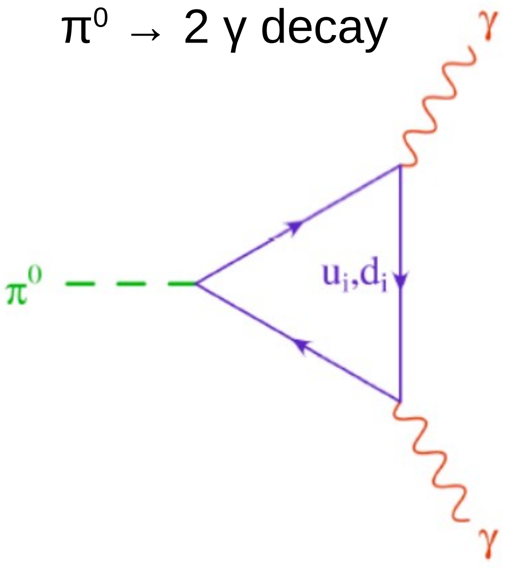
Violation of Pauli exclusive principle

➡ ***Need another quantum number - color!***

Slide from Jianwei Qiu, 2018

How many colors?

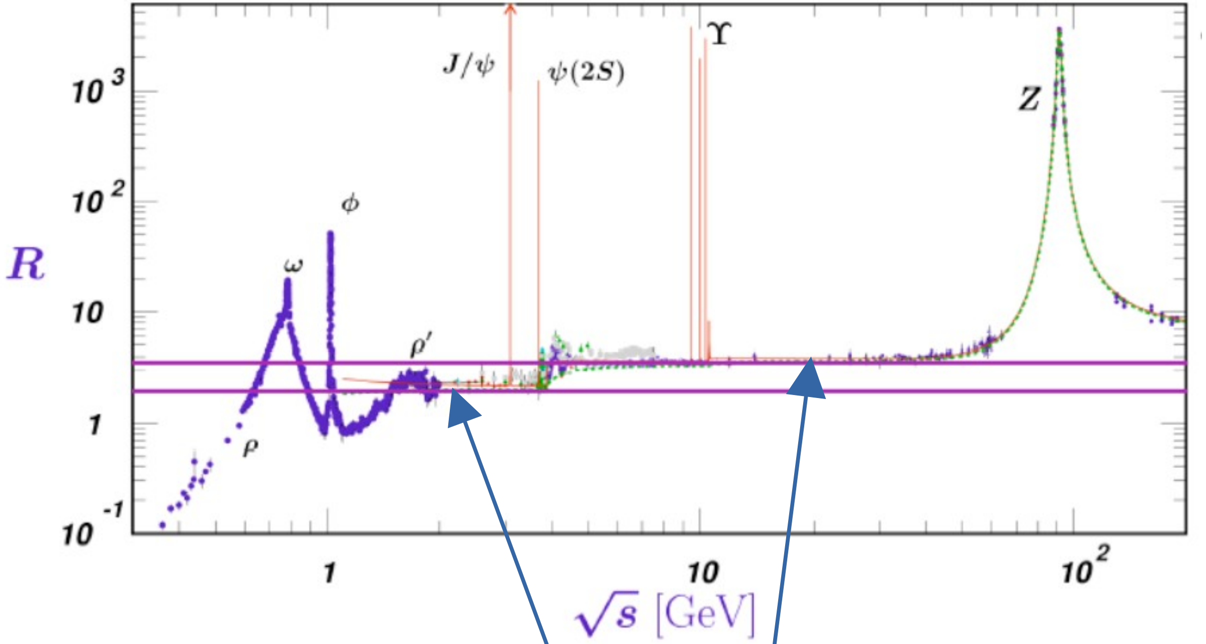
$\pi^0 \to 2\gamma$ decay



$$\Gamma \sim N_c^2 \left[Q_u^2 - Q_d^2\right]^2 \frac{m_\pi^3}{f_\pi^2}$$

$$\Gamma_{TH} = \left(\frac{N_c}{3}\right)^2 7.6\,\text{eV}$$

$$\Gamma_{EXP} = 7.7 \pm 0.6 \quad \text{eV}$$



$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \sim N_c \sum_q e_q^2$$

$$= 2(N_c/3) \quad q = u, d, s$$

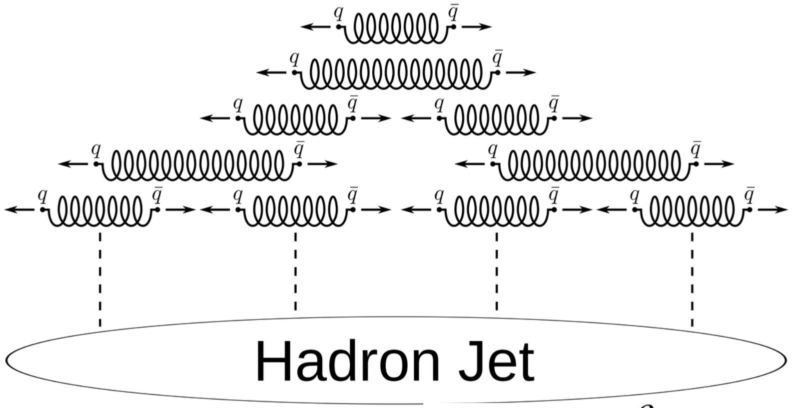
$$= 3.7(N_c/3) \quad q = u, d, s, c, b$$

Slide from Fabio Maltoni, 2020

Experimental Search for Quarks in the Quark Model

- In e^+e^- collisions (e.g., at SLAC), no single quark has ever been observed in the final state—only jets.
- Millikan's oil drop experiment always found integer electric charges; no particles with fractional charge have been detected.
- Rutherford-like scattering off protons: high-energy e-p collisions probe internal structure.
- Jets are composed of mesons and baryons, which are all color-neutral.

Quark confinement → Color confinement.

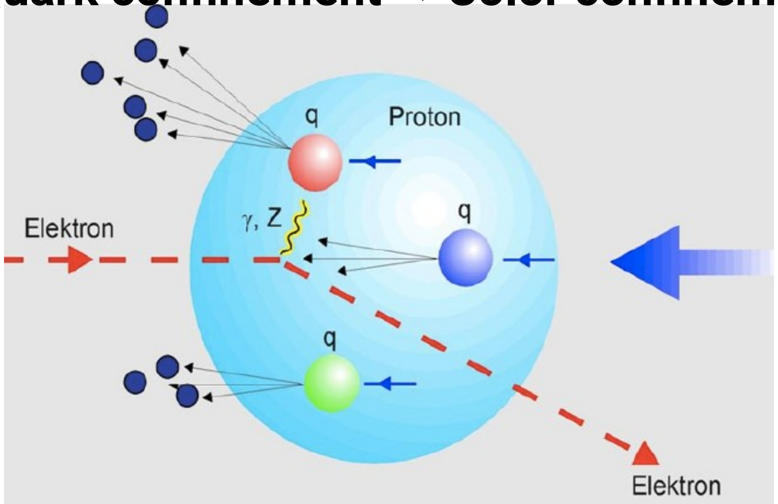


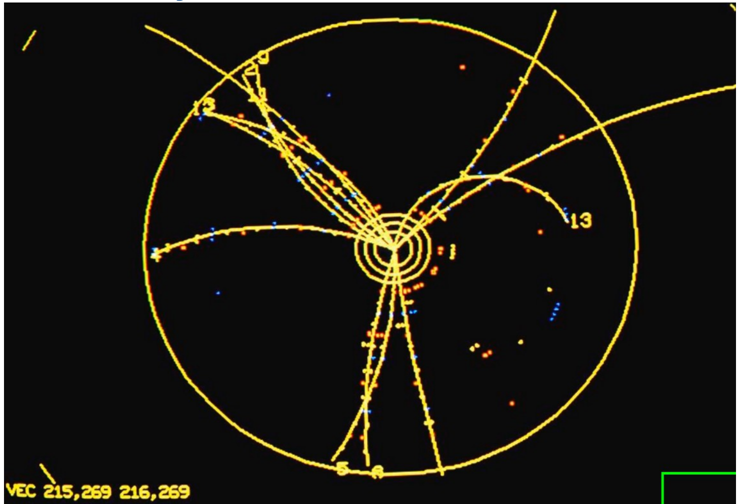
Cornell potential: $V(r) = -\frac{e}{r} + \sigma r$

Phys.Rep.343(2001)1-136

SLAC 3jets event

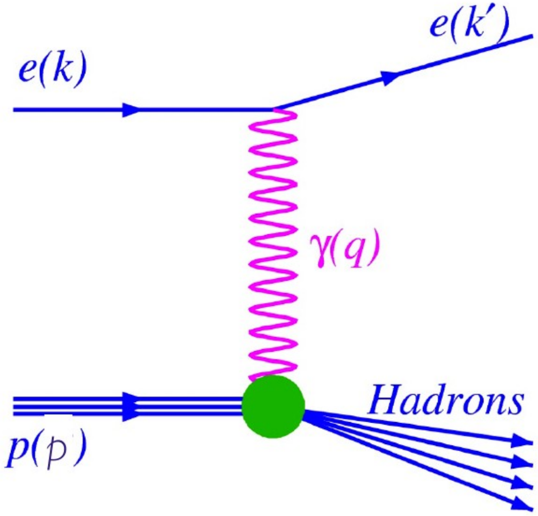
PRL34.369



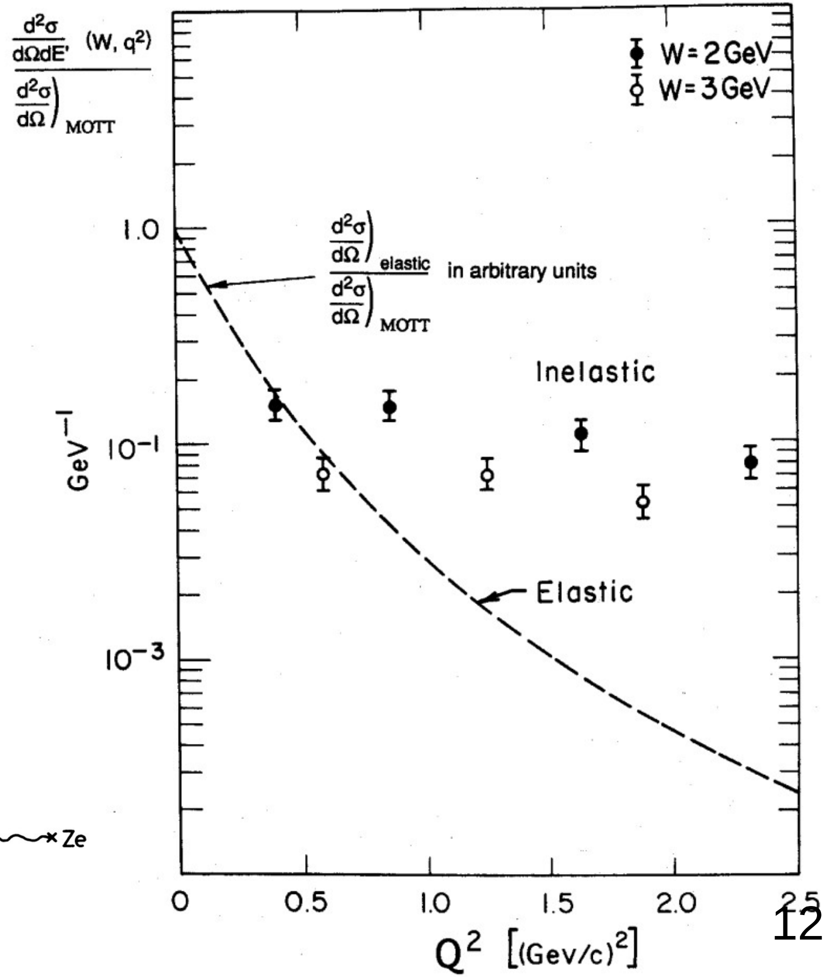


Deep Inelastic Scattering: Evidence of Internal Structure for Proton

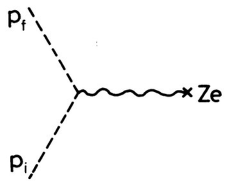
$$e^- + p \rightarrow e^- + X$$



- At low Q region, the data points agree with elastic Mott scattering.
- At high Q region, the data points, corresponding to inelastic scattering, indicating the internal structure of proton.



Mott scattering:
Scattering of Dirac electron on a point charge with infinite mass and charge Ze.

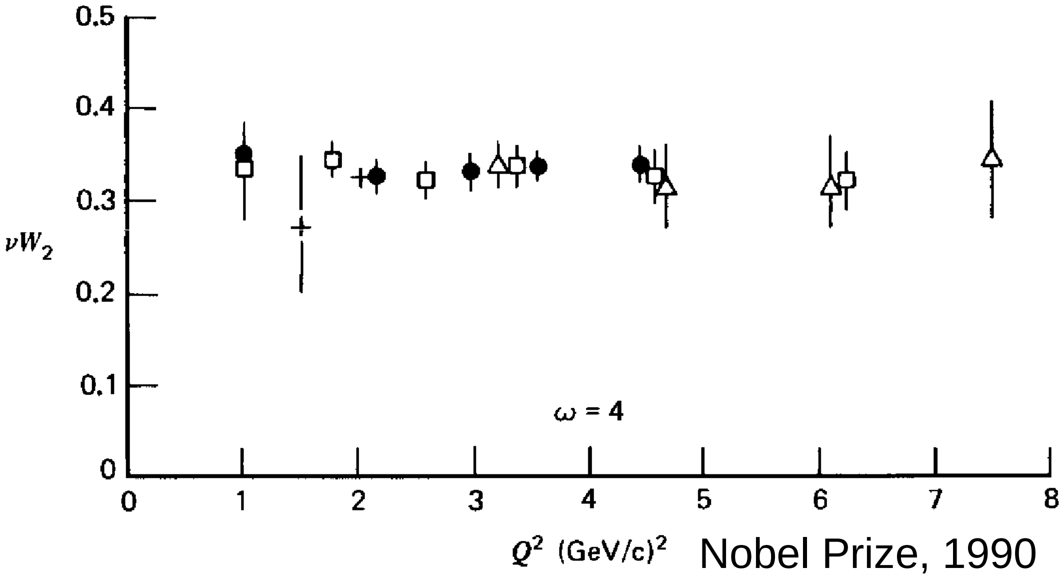


Bjorken scalling

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \left[2W_1(Q^2,\nu)\sin^2(\theta/2) + W_2(Q^2,\nu)\cos^2(\theta/2)\right]$$

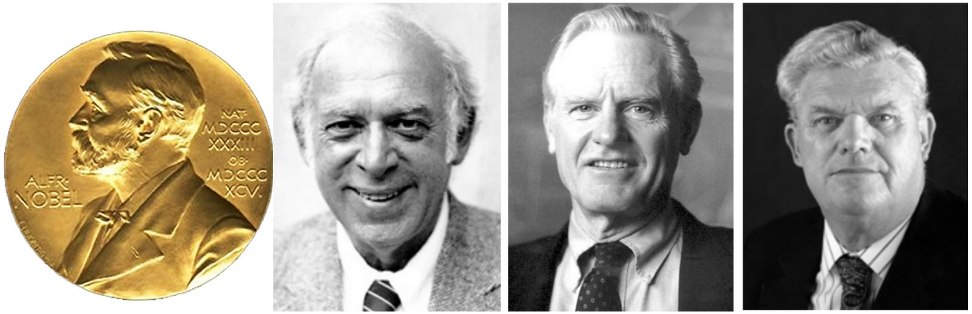
- E: incoming electron energy
- θ: electron scattering angle
- $Q^2 = -q^2$: momentum transfer squared
- $\nu = E - E'$: energy transferred to the proton
- W_1, W_2: structure functions (functions of Q^2 and ν)

$$W_1(Q^2,\nu) \to F_1(x)$$
$$\nu W_2(Q^2,\nu) \to F_2(x)$$

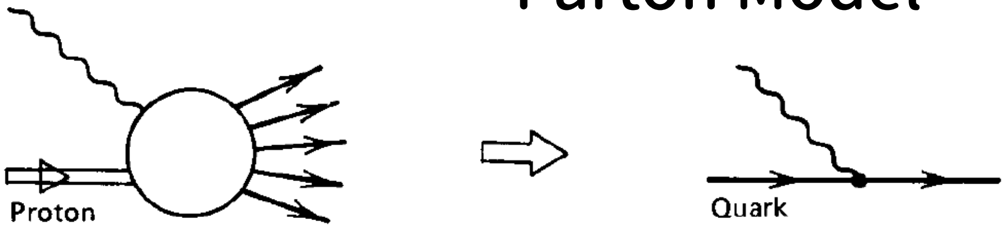


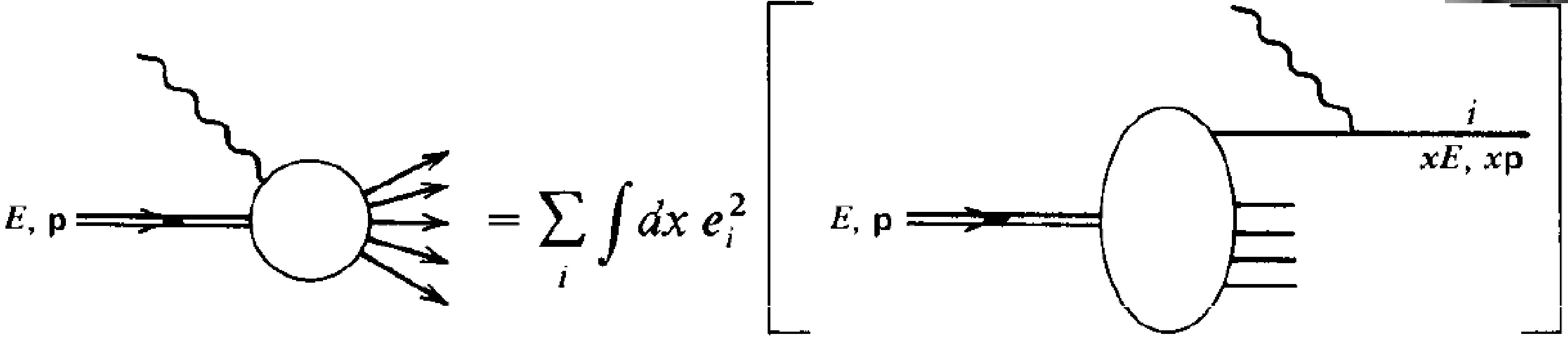
The SLAC data show that, νW_2 is approximately constant over a wide range of Q^2 at fixed x. It suggest the high energy photo found approximate free pointlike particle inside proton.

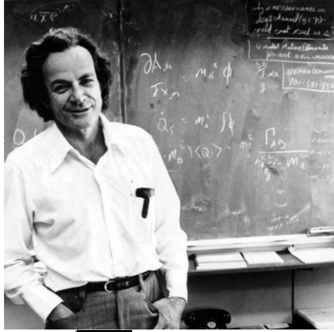
Nobel Prize, 1990



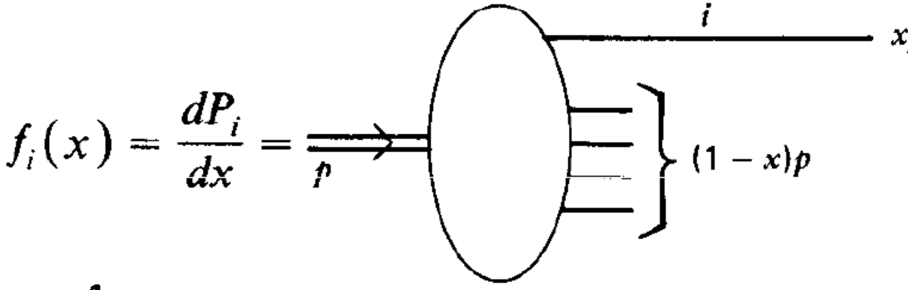
Parton Model







The $f_i(x)$ represent the probability to find a parton i carries a fraction x of the proton's momentum p (leading order perspective).

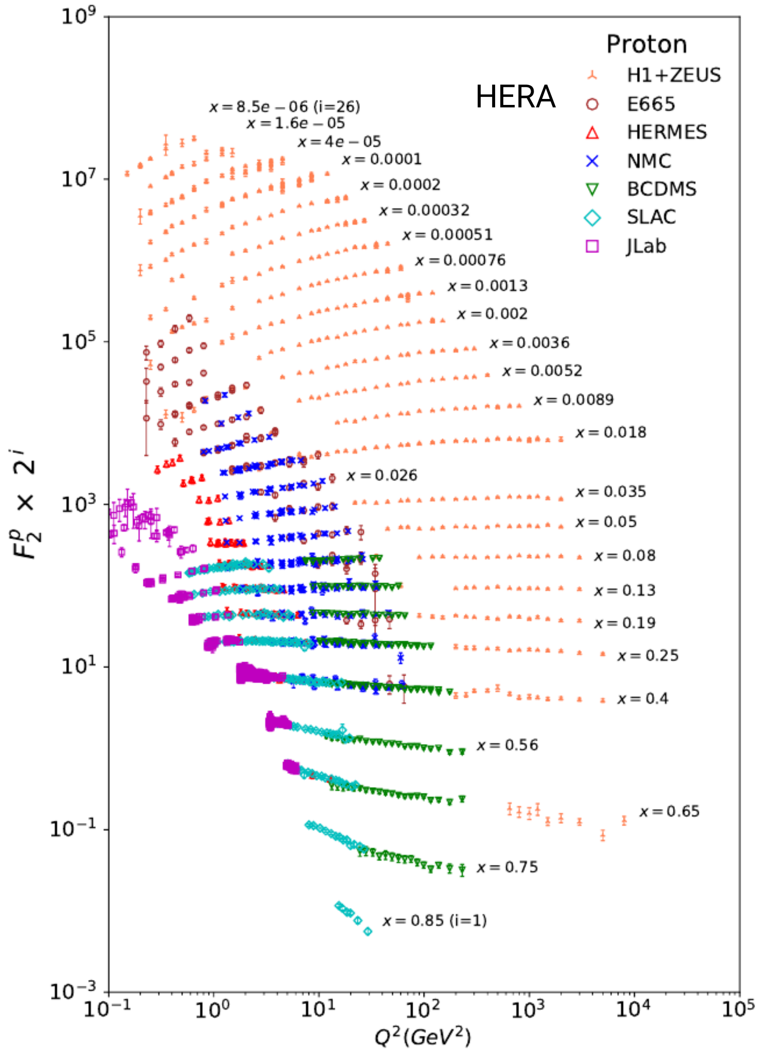


$$\sum_{i'} \int dx\, x f_{i'}(x) = 1$$

What holds the quarks together?
The birth of QCD (1973)
Quark Model + Yang-Mill gauge theory

Figures from Halzem&Martin

DIS experiment



Figures from PDG,2024

- Approximate scaling in middle x region.
- Scale violation in small x region (~ log Q^2)

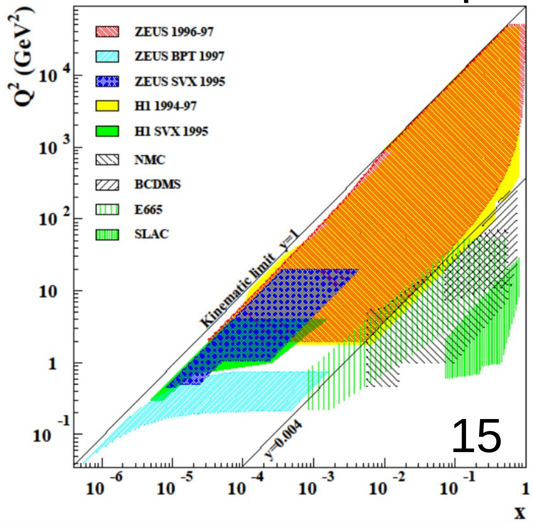
SLAC (Stanford Linear Accelerator Center)

- e^-e^+/e^-p(fixed target)
- Dates of operation: 1966-2006
- Maximum energy: 50 GeV

HERA(Hadron−Electron Ring Accelerator)

- $e^-(e^+)p$
- Dates of operation: 1992-2007
- E_{cm} = 320 GeV

DIS expt.



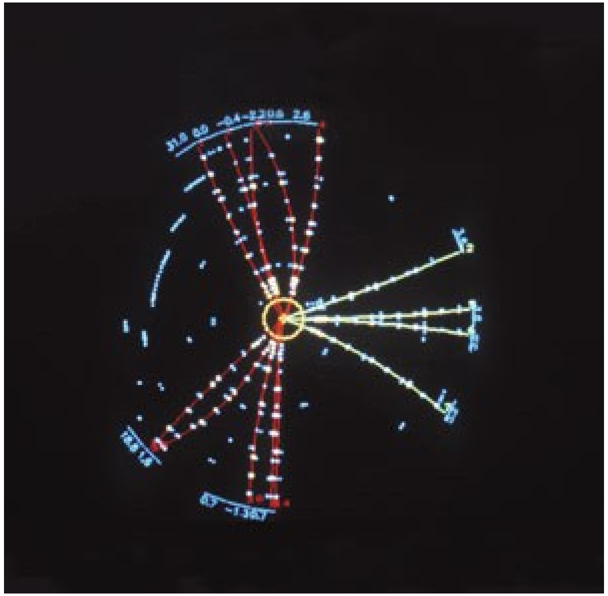
What kind of interaction between quarks?

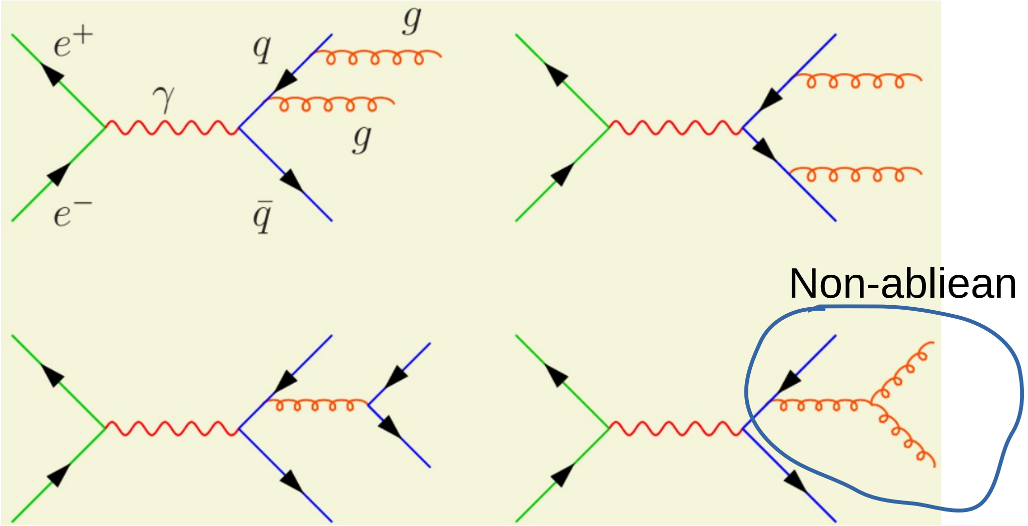
- The $\pi 0 \rightarrow 2\gamma$ decay and the $e^+e^- \rightarrow$ hadrons / $\mu^+\mu^-$ ratio
  - → reveal that quarks carry three color charges.
  - → According to Noether's theorem, conservation laws imply the existence of an internal symmetry.
  - ⇒ This motivates the idea of a gauge symmetry associated with color.
- The discovery of the Δ^{++} (uuu) baryon
  - → requires 3 identical quarks to carry different colors, due to the Pauli exclusion principle.
  - → This means quarks can change color, and color exchange occurs locally in interactions.
  - ⇒ Therefore, the gauge symmetry must be local, not global.
- Bjorken scaling suggests that partons behave as approximately free particles at high energies. But small logarithmic deviations (scaling violation) are observed, especially in small-x region.
  - → These deviations imply a running coupling constant.
  - ⇒ Only non-Abelian gauge theories predict this behavior.

These facts collectively imply that the strong interaction between quarks is best described by a local non-Abelian gauge theory!

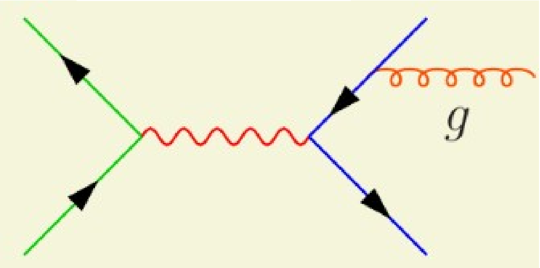
Experimental Evidence for QCD

Three-jet event observed by TASSO at PETRA (1979)





$$e^+e^- \to q\bar{q}g$$



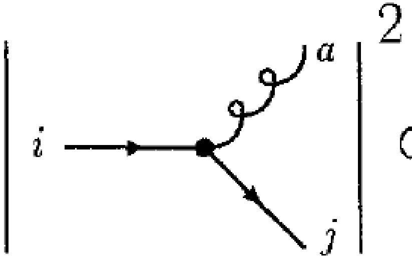
The observation of 3-jet events(PETRA, 1979) confirmed the existence of the gluon and supported the idea that QCD is a local gauge theory.

The angular distributions and cross sections in 4-jet events(PETRA, 1981-1984) revealed the presence of gluon self-interactions, consistent only with a non-Abelian gauge theory.

Evidence of SU(3) as Gauge Group of Strong Interaction

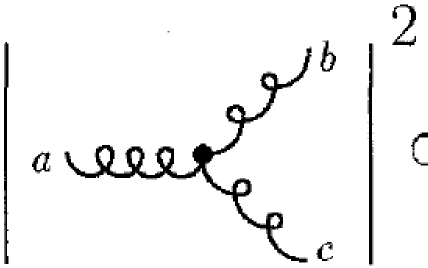
$$[T^i \, , \, T^j] = i \sum f^{ijk} \cdot T^k$$

$$\sum_{k,\eta} T^k_{\alpha\eta} T^k_{\eta\beta} = \delta_{\alpha\beta} C_F$$



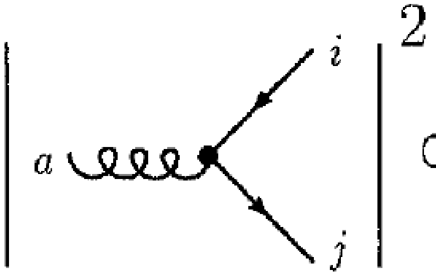
$\propto C_F$

$$\sum_{j,k} f^{jkm} f^{jkn} = \delta^{mn} C_A$$



$\propto C_A$

$$\sum_{\alpha,\beta} T^m_{\alpha\beta} T^n_{\beta\alpha} = \delta^{mn} T_F$$



$\propto T_F$

For example: SU(N_c)

$C_A = N_C$,

$C_F = (N_C^2 - 1)/ 2N_C$

$T_F = 1/2$

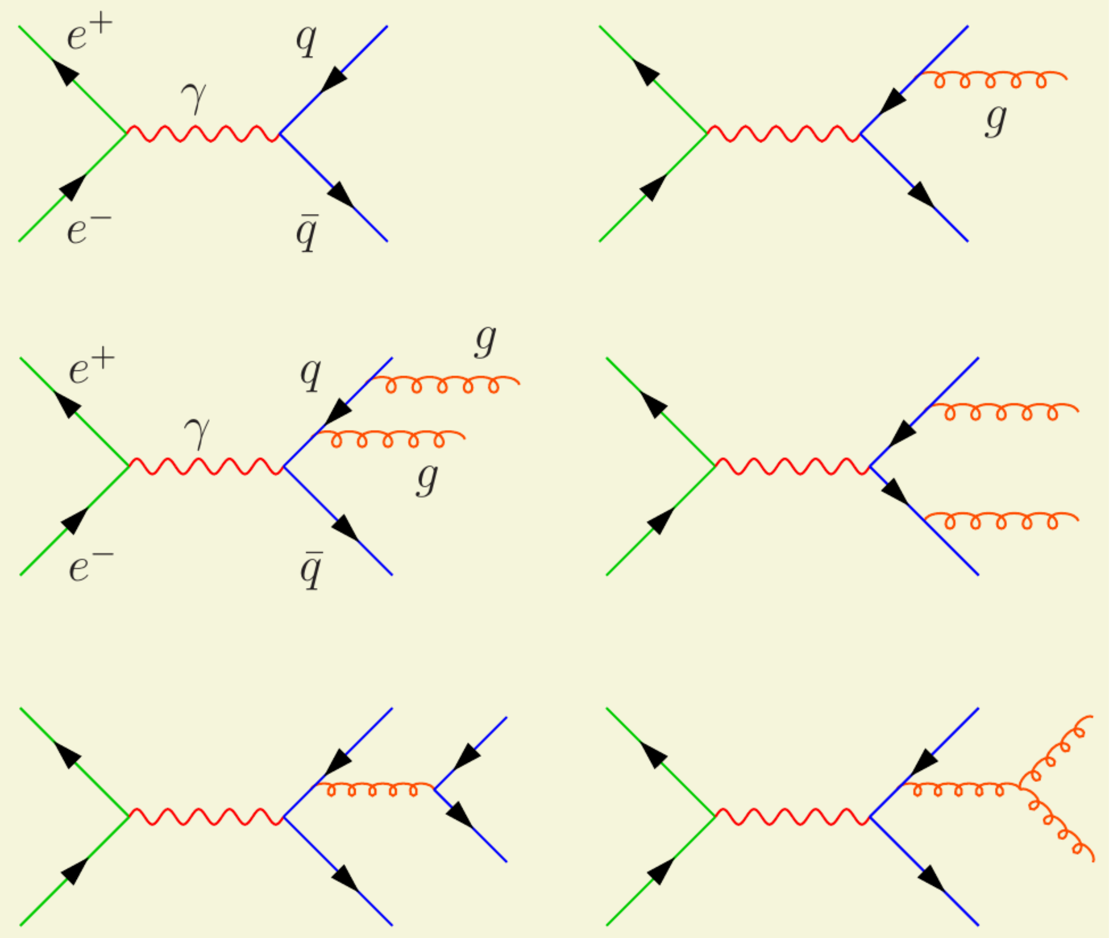
For QCD, SU(3), $N_c = 3$,
hence $C_A = 3$, $C_F = 4/3$.

$$\sigma = f(\alpha_s C_F, \frac{C_A}{C_F}, n_f \frac{T_F}{C_F})$$

Figures from Z.Phys.C59(1993)357-368

Evidence of SU(3) as Gauge Group of Strong Interaction

At e$\bar{e}$ collider like LEP:



The differential cross section can be expressed as a sum of three terms in which the colour factors appear only as coefficients accompanying group-independent kinematic functions

$$\sigma_{q\bar{q}gg} = \sigma_0 \left[F_A(y_{ij}) + \left(1 - \frac{1}{2}\frac{C_A}{\mathbf{C}_F}\right) F_B(y_{ij}) + \frac{\mathbf{C}_A}{\mathbf{C}_F} F_C(y_{ij}) \right],$$

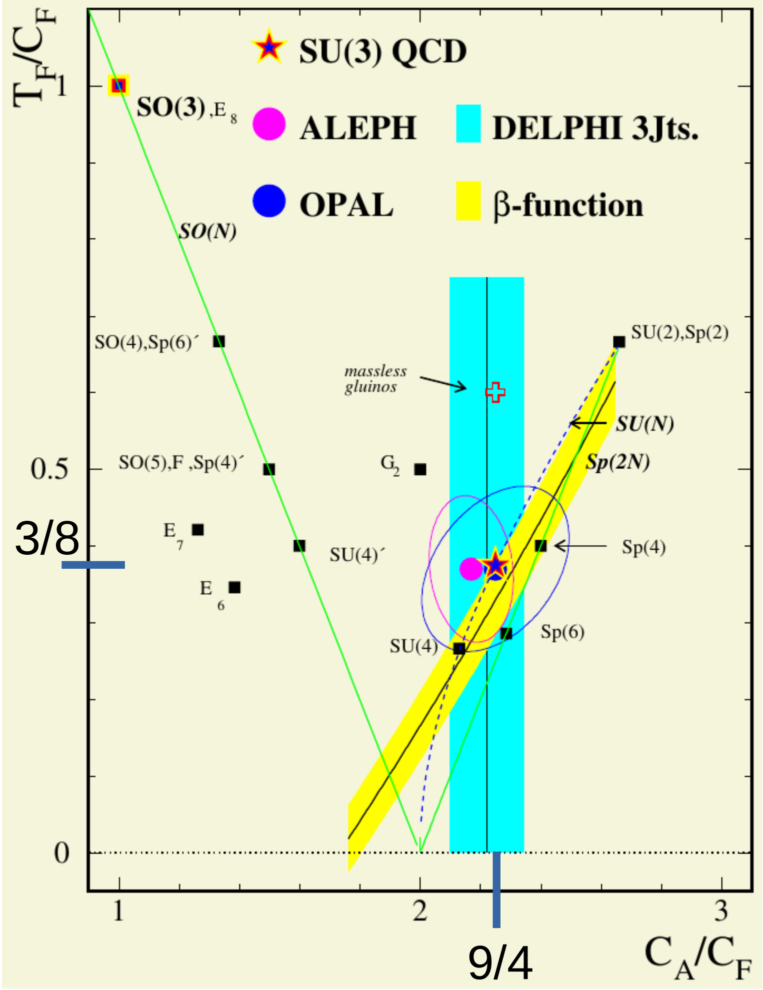
$$\sigma_{q\bar{q}q\bar{q}} = \sigma_0 \left[\frac{\mathbf{T}_R}{\mathbf{C}_F} F_D(y_{ij}) + \left(1 - \frac{1}{2}\frac{\mathbf{C}_A}{\mathbf{C}_F}\right) F_E(y_{ij}) \right]$$

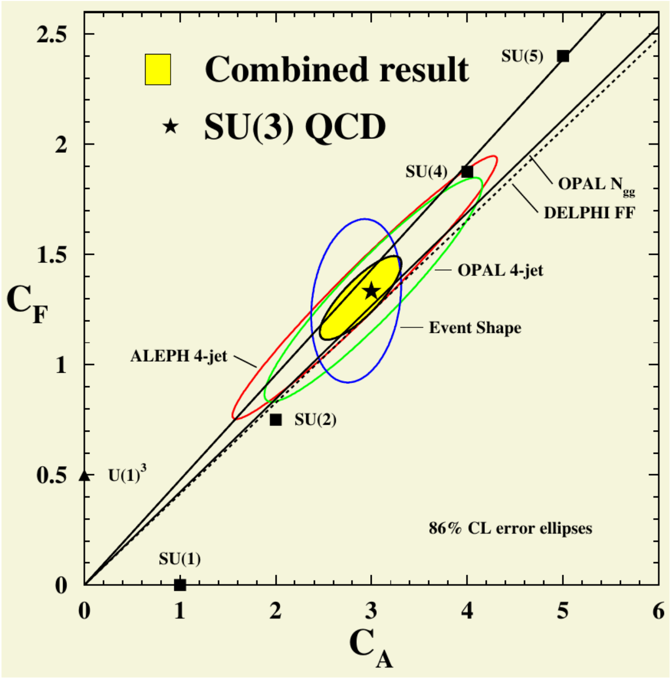
$$\sigma_0 = \frac{4\pi\alpha^2}{3s} N_C \sum_{k=1}^{n_f} e_k^2 \qquad T_R = T_F n_f$$

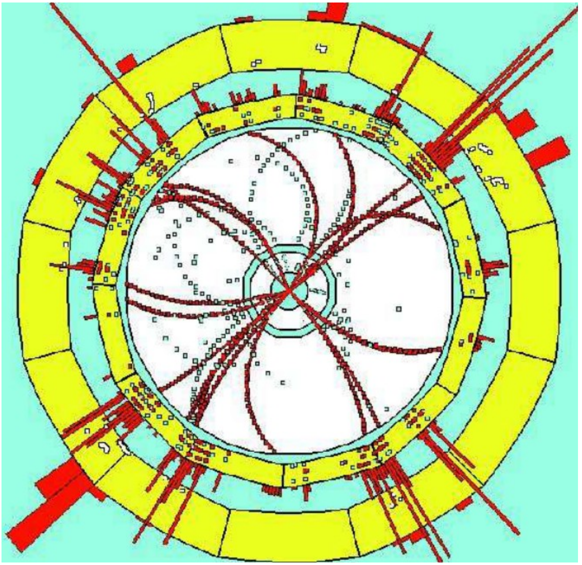
qq̄ (2jets), qq̄g (3jets) and
qq̄,q ′ q̄ ′ (4jets), qq̄gg (4jets)

Figure from F. JEGERLEHNER, 2008

Evidence of SU(3) as Gauge Group of Strong Interaction







LEP 4 jets event

$$\frac{C_A}{C_F} = 2.20 \pm 0.09\,(stat) \pm 0.13\,(syst)$$

$$\frac{T_F}{C_F} = 0.29 \pm 0.05\,(stat) \pm 0.06\,(syst)$$

Figures from F. JEGERLEHNER, 2008

CERN-OPEN-97-015(1997)

Quantum Chromo-dynamics (QCD)

= A quantum field theory of quarks and gluons =

- **Fields:**

$\psi_i^f(x)$ — **Quark fields: spin-½ Dirac fermion (like electron)**

Color triplet: $i = 1, 2, 3 = N_c$

Flavor: $f = u, d, s, c, b, t$

$A_{\mu,a}(x)$ — **Gluon fields: spin-1 vector field (like photon)**

Color octet: $a = 1, 2, ..., 8 = N_c^2 - 1$

- **QCD Lagrangian density:**

$$\mathcal{L}_{QCD}(\psi, A) = \sum_f \overline{\psi}_i^f \left[(i\partial_\mu \delta_{ij} - gA_{\mu,a}(t_a)_{ij})\gamma^\mu - m_f \delta_{ij} \right] \psi_j^f - \frac{1}{4}\left[\partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - gC_{abc}A_{\mu,b}A_{\nu,c} \right]^2 + \text{gauge fixing} + \text{ghost terms}$$

- **QED – force to hold atoms together:**

$$\mathcal{L}_{QED}(\phi, A) = \sum_f \overline{\psi}^f \left[(i\partial_\mu - eA_\mu)\gamma^\mu - m_f \right] \psi^f - \frac{1}{4}\left[\partial_\mu A_\nu - \partial_\nu A_\mu \right]^2$$

QCD is much richer in dynamics than QED

Slide from Jianwei Qiu, 2018

Gluons are dark, but, interact with themselves, NO free quarks and gluons

Gauge property of QCD

❑ Gauge Invariance:

$$\psi_i(x) \to \psi'_j(x) = U(x)_{ji}\,\psi_i(x)$$

$$A_\mu(x) \to A'_\mu(x) = U(x)\,A_\mu(x)\,U^{-1}(x) + \frac{i}{g}\left[\partial_\mu U(x)\right]U^{-1}(x)$$

where $A_\mu(x)_{ij} \equiv A_{\mu,a}(x)(t_a)_{ij}$

$$U(x)_{ij} = \left[e^{i\,\alpha_a(x)\,t_a}\right]_{ij} \qquad \text{Unitary } [\det{=}1,\ \text{SU}(3)]$$

❑ Color matrices:

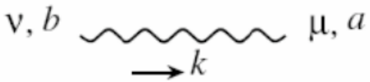
$$[t_a, t_b] = i\,C_{abc}\,t_c$$

Generators for the fundamental representation of SU3 color

❑ Gauge Fixing:

$$\mathcal{L}_{gauge} = -\frac{\lambda}{2}(\partial_\mu A^\mu_a)(\partial_\nu A^\nu_a)$$

Allow us to define the gauge field propagator:



$$G_{\mu\nu}(k)_{ab} = \frac{\delta_{ab}}{k^2}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\left(1 - \frac{1}{\lambda}\right)\right]$$

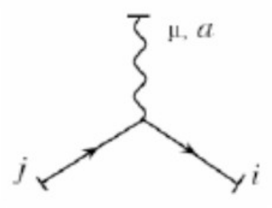
with $\lambda = 1$ **the Feynman gauge**

Slide from Jianwei Qiu, 2018

Quantum Chromo-dynamics (QCD)

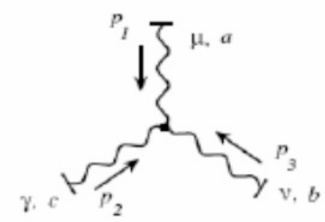
❑ Interactions:

$$-g\bar{\psi}\gamma^{\mu}A_{\mu,a}t_a\psi$$



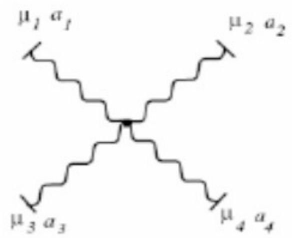
$$-ig(t_a)_{ij}\gamma_{\mu}$$

$$\frac{1}{2}gC_{abc}(\partial_{\mu}A_{\nu,a} - \partial_{\nu}A_{\mu,a})A_b^{\mu}A_c^{\nu}$$



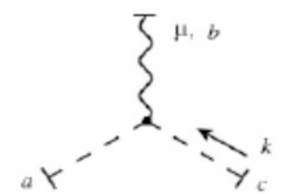
$$-gC_{abc}\left[g_{\mu\nu}(p_1 - p_2)_{\gamma} + g_{\nu\gamma}(p_2 - p_3)_{\mu} + g_{\gamma\mu}(p_3 - p_1)_{\nu}\right]$$

$$-\frac{g^2}{4}C_{abc}C_{ab'c'} * A_b^{\mu}A_c^{\nu}A_{\mu,b'}A_{\nu,c'}$$



$$-ig^2\left[C_{ea_1a_2}C_{ea_3a_4} * (g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}) + \ldots\right]$$

$$\partial_{\mu}\bar{\eta}_a\left(gC_{abc}A_b^{\mu}\right)\eta_c$$



$$gC_{abc}k_{\mu}$$

Slide from Jianwei Qiu, 2018

Renormalization Group

- Physical quantity should not depend on renormalization scale μ → renormalization group equation:

$$\mu^2 \frac{d}{d\mu^2} \sigma_{\rm Phy}\left(\frac{Q^2}{\mu^2}, g(\mu), \mu\right) = 0 \quad \Longrightarrow \quad \sigma_{\rm Phy}(Q^2) = \sum_n \hat{\sigma}^{(n)}(Q^2, \mu^2) \left(\frac{\alpha_s(\mu)}{2\pi}\right)^n$$

- Running coupling constant:

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g) \qquad \alpha_s(\mu) = \frac{g^2(\mu)}{4\pi}$$

- QCD β function:

$$\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu} = +g^3 \frac{\beta_1}{16\pi^2} + \mathcal{O}(g^5) \qquad \beta_1 = -\frac{11}{3} N_c + \frac{4}{3}\frac{n_f}{2} < 0 \quad \text{for } n_f \le 6$$

- QCD running coupling constant:

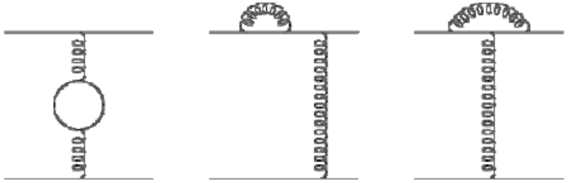
$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 - \frac{\beta_1}{4\pi}\alpha_s(\mu_1) \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \Rightarrow 0 \quad \text{as } \mu_2 \to \infty \quad \text{for } \beta_1 < 0$$

Asymptotic freedom!

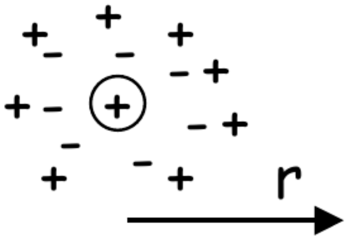
Slide from Jianwei Qiu, 2018

Renormalization Group

value of strong coupling α_s depends on distance (i.e. energy)

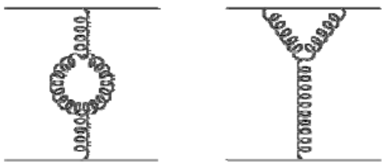


"screening" of the charge

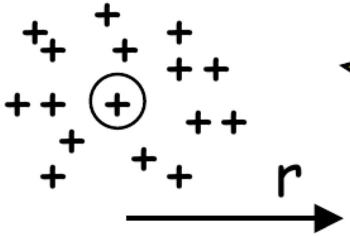


$\alpha_s(r) \uparrow$ if $r \downarrow$

like QED



"anti-screening"



$\alpha_s(r) \downarrow$ if $r \downarrow$

non abelian

Who wins? $$\alpha_s(Q^2) = \frac{g^2}{4\pi} \approx \frac{4\pi}{(11 - \frac{2}{3}N_f)\ln(Q^2/\Lambda^2)}$$ $Q \sim 1/r$

coupling at some reference scale Q_0

Slide from Marco Stratmann, 2005

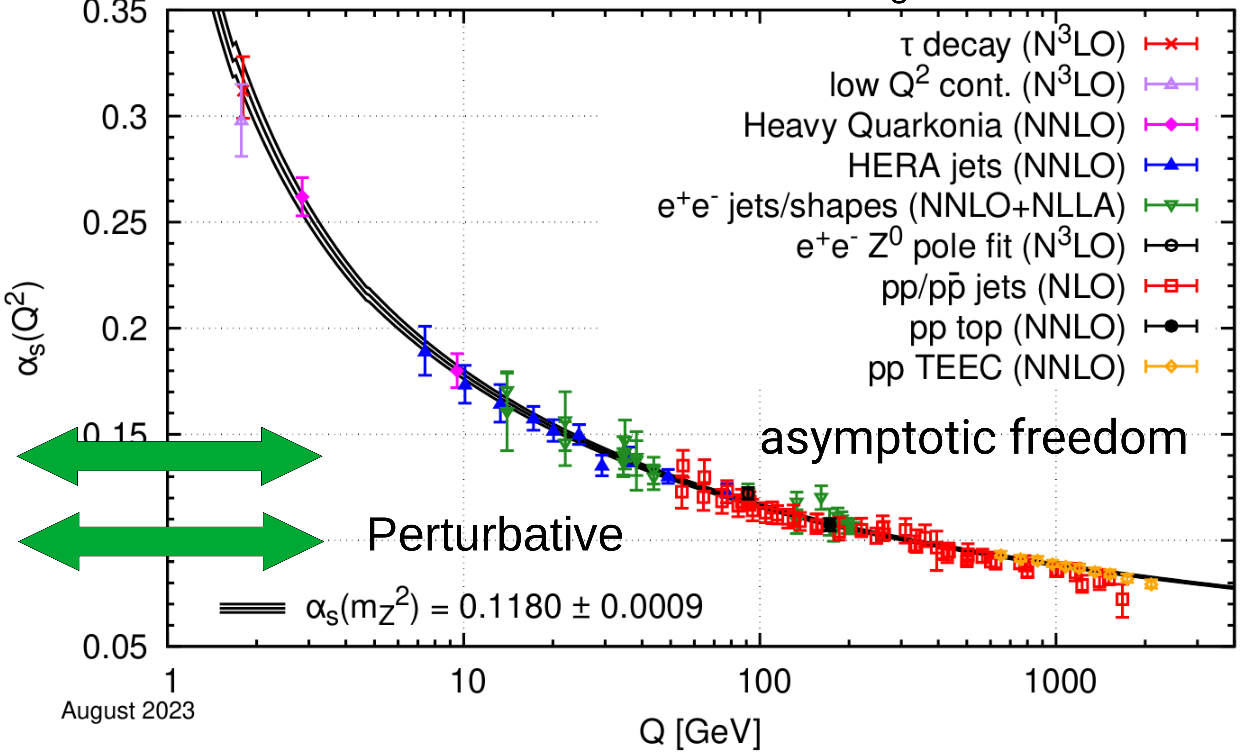
Asymptotic Freedom

Nobel Prize 2024

confinement

Non-Perturbative

Figure from PDG 2024



$$\alpha_s(Q) = \frac{2\pi}{\beta_0 \ln(Q/\Lambda_{\rm QCD})}$$

Landau pole

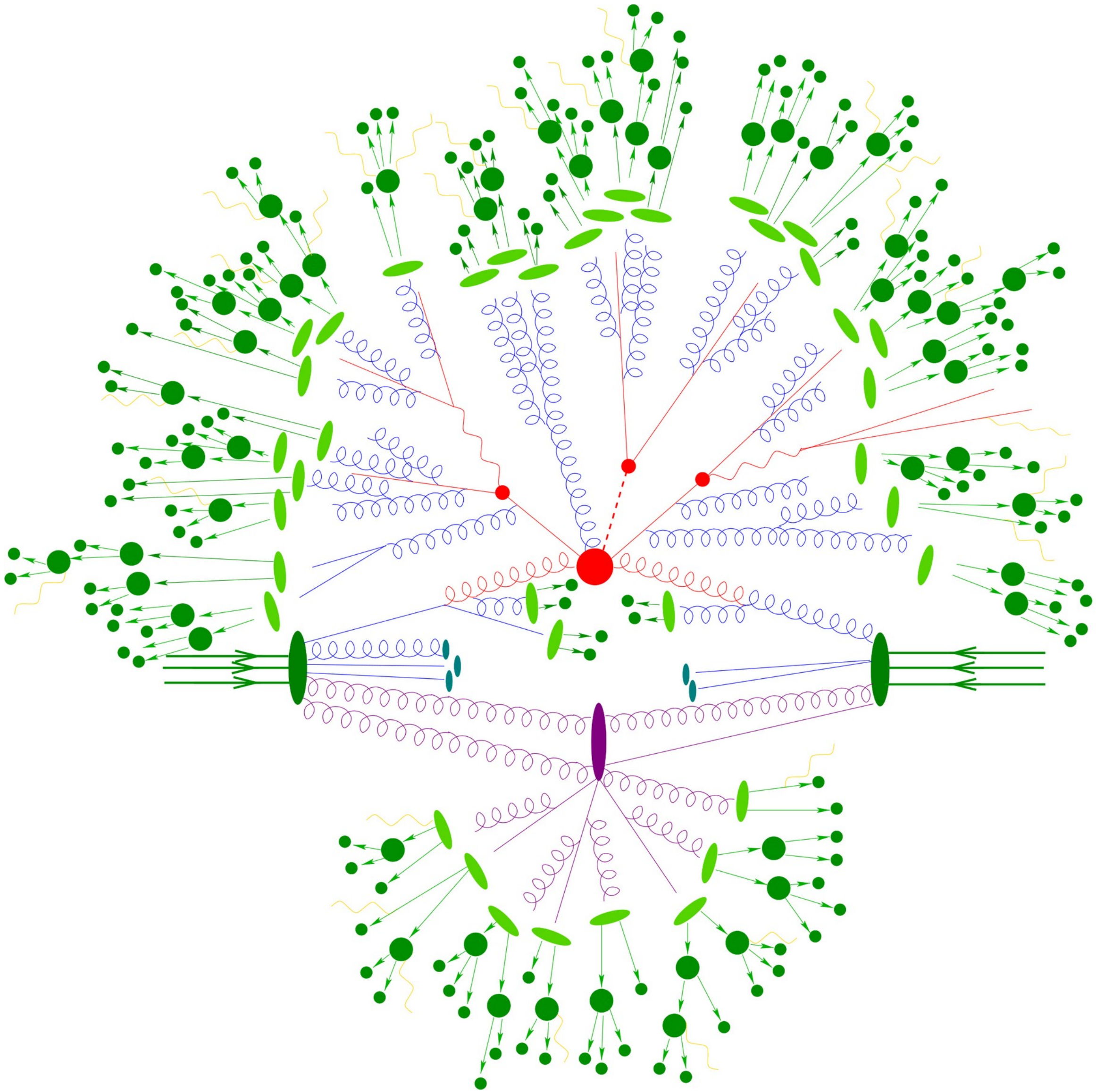
$\Lambda_{\rm QCD}^{\rm N4LO}(N_f)$ =

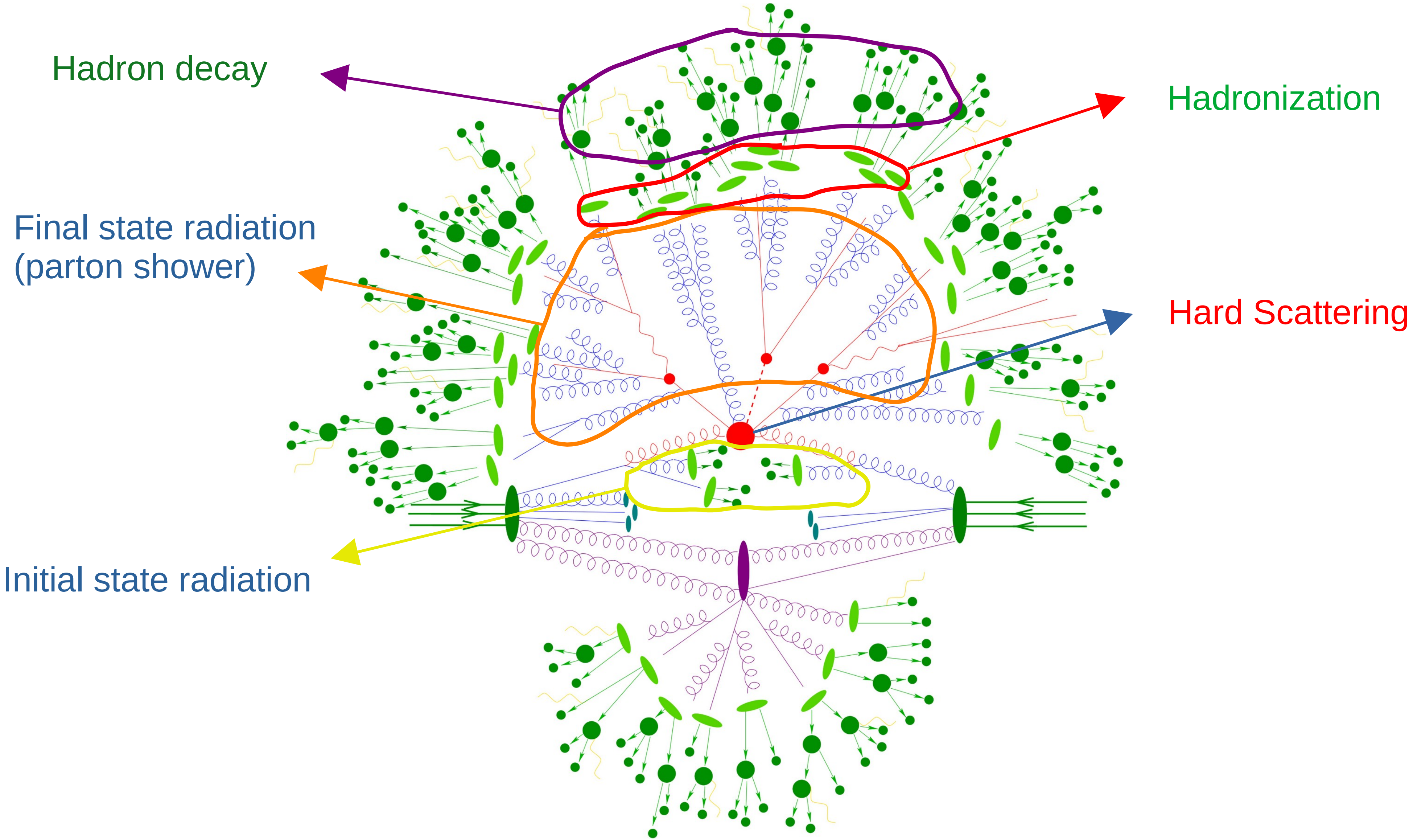
$$\begin{aligned}\Lambda_{(3)} &\approx 359.526\ \text{MeV}\\ \Lambda_{(4)} &\approx 309.560\ \text{MeV}\\ \Lambda_{(5)} &\approx 220.263\ \text{MeV}\\ \Lambda_{(6)} &\approx 93.270\ \text{MeV}\end{aligned}$$

~ 1 fm

Perturbative calculation is valid in high energy region (≳ 1 GeV)

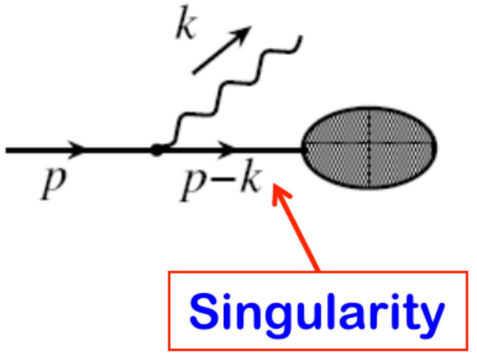
Everything you wanted to know about Lambda-QCD but were afraid to ask
Randall J. Scalise and Fredrick I. Olness




Hadron decay
Hadronization
Final state radiation
(parton shower)
Hard Scattering
Initial state radiation

Infrared and collinear divergences

❑ **Consider a general diagram:**



$p^2 = 0, \quad k^2 = 0$ **for a massless theory**

✧ $k^\mu \to 0 \;\Rightarrow\; (p-k)^2 \to p^2 = 0$

⟹ **Infrared (IR) divergence**

✧ $k^\mu \,||\, p^\mu \;\Rightarrow\; k^\mu = \lambda p^\mu \quad \text{with} \quad 0 < \lambda < 1$

$\Rightarrow\; (p-k)^2 \to (1-\lambda)^2 p^2 = 0$

⟹ **Collinear (CO) divergence**

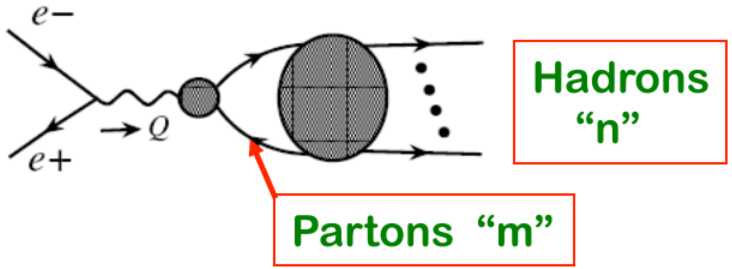
IR and CO divergences are generic problems of a massless perturbation theory

Slide from Jianwei Qiu, 2018

IR Safe in the Final State – Inclusive cross section

- ❑ e^+e^- ➔ hadron **total** cross section – not a specific hadron!

$$\sigma^{\text{tot}}_{e^+e^-\to\text{hadrons}} \propto \left| \quad \right|^2$$



If there is no quantum interference between partons and hadrons,

$$\sigma^{\text{tot}}_{e^+e^-\to\text{hadrons}} \propto \sum_n P_{e^+e^-\to n} = \sum_n \sum_m P_{e^+e^-\to m} P_{m\to n} = \sum_m P_{e^+e^-\to m} \sum_n P_{m\to n} = 1$$

Unitarity

$$\sigma^{\text{tot}}_{e^+e^-\to\text{partons}} \propto \sum_m P_{e^+e^-\to m}$$

$$\Rightarrow \quad \sigma^{\text{tot}}_{e^+e^-\to\text{hadrons}} = \sigma^{\text{tot}}_{e^+e^-\to\text{partons}}$$

Finite in perturbation theory – KLN theorem

- ❑ e^+e^- ➔ parton total cross section:

$$\sigma^{\text{tot}}_{e^+e^-\to\text{partons}}(s=Q^2) = \sum_n \sigma^{(n)}(Q^2,\mu^2)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

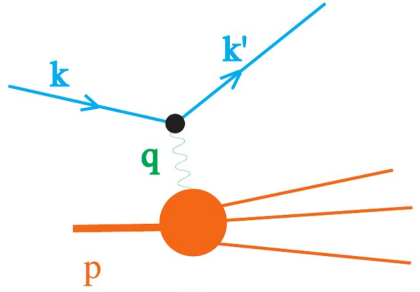
Calculable in pQCD

Slide from Jianwei Qiu, 2018

IR Divergence in Initial State – DIS revisit

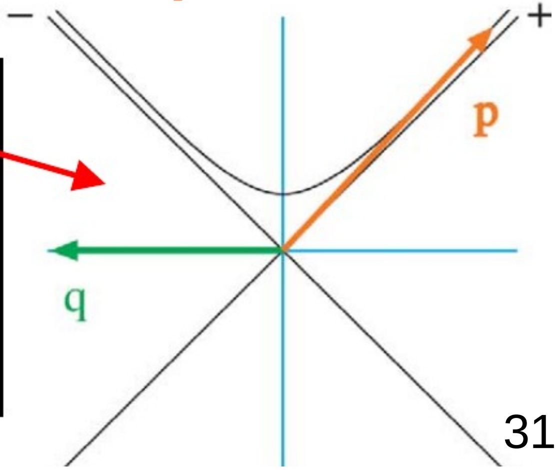
$x^\mu = (t, x^1, x^2, x^3)$ —Light cone cor.→ $x^\mu = (x^+, x^-, \vec{x}_T)$

$x^+ = \frac{1}{\sqrt{2}}(t+z)$ light-front time $x^- = \frac{1}{\sqrt{2}}(t-z)$ conjugate light-front time $\vec{x}_\perp = (x^1, x^2)$



In a reference frame where the proton moves very fast and $Q \gg m_h$ is big

4-vector	hadron rest frame	Breit frame
$(p^+, p^-, \vec{p}_T)$	$\frac{1}{\sqrt{2}}(m_h, m_h, \vec{0})$	$\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{x m_h^2}{Q}, \vec{0})$
$(q^+, q^-, \vec{q}_T)$	$\frac{1}{\sqrt{2}}(-m_h x, \frac{Q^2}{m_h x}, \vec{0})$	$\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$



Slide from Marco Stratmann, 2005

IR Divergence in Initial State – DIS revisit

simple estimate for typical time-scale of interactions among the partons inside a fast-moving hadron:

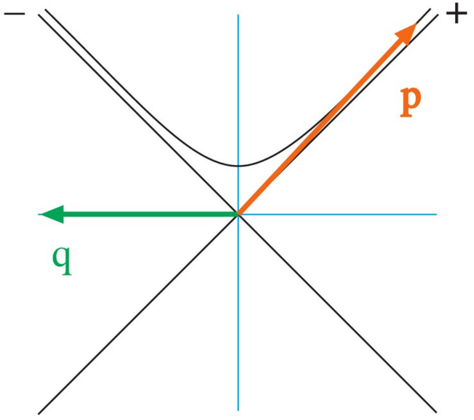
$$\textbf{rest frame:}\quad \Delta x^+ \sim \Delta x^- \sim \frac{1}{m}$$

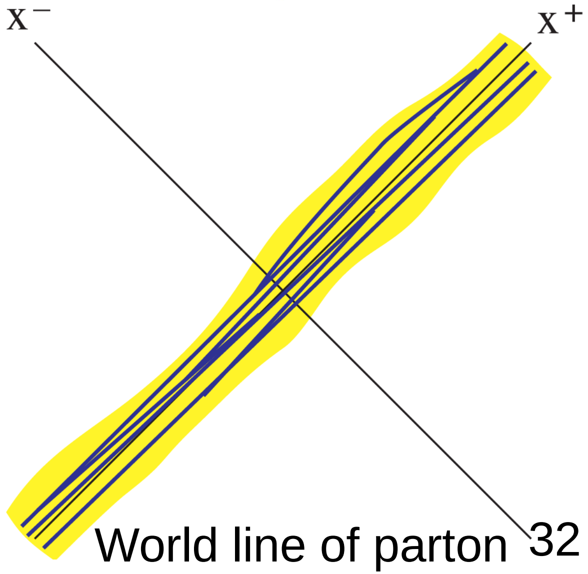
$$\textbf{Breit frame:}\quad \Delta x^+ \sim \frac{1}{m}\frac{Q}{m} = \frac{Q}{m^2} \quad \text{large}$$

$$\Delta x^- \sim \frac{1}{m}\frac{m}{Q} = \frac{1}{Q} \quad \text{small}$$

Interactions between partons are spread out inside a fast moving hadron.

4-vector	hadron rest frame	Breit frame
$(p^+, p^-, \vec{p}_T)$	$\frac{1}{\sqrt{2}}(m_h, m_h, \vec{0})$	$\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{x m_h^2}{Q}, \vec{0})$
$(q^+, q^-, \vec{q}_T)$	$\frac{1}{\sqrt{2}}(-m_h x, \frac{Q^2}{m_h x}, \vec{0})$	$\frac{1}{\sqrt{2}}(-Q, Q, \vec{0})$





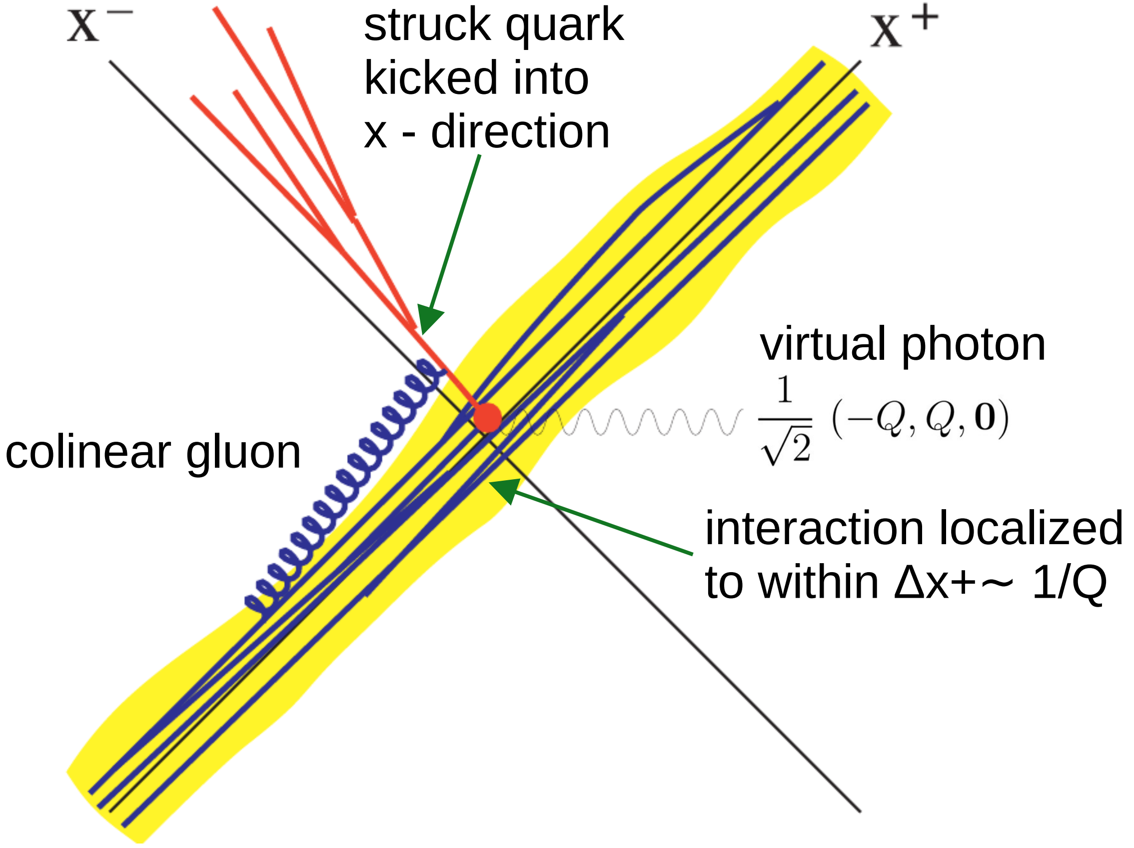
World line of parton

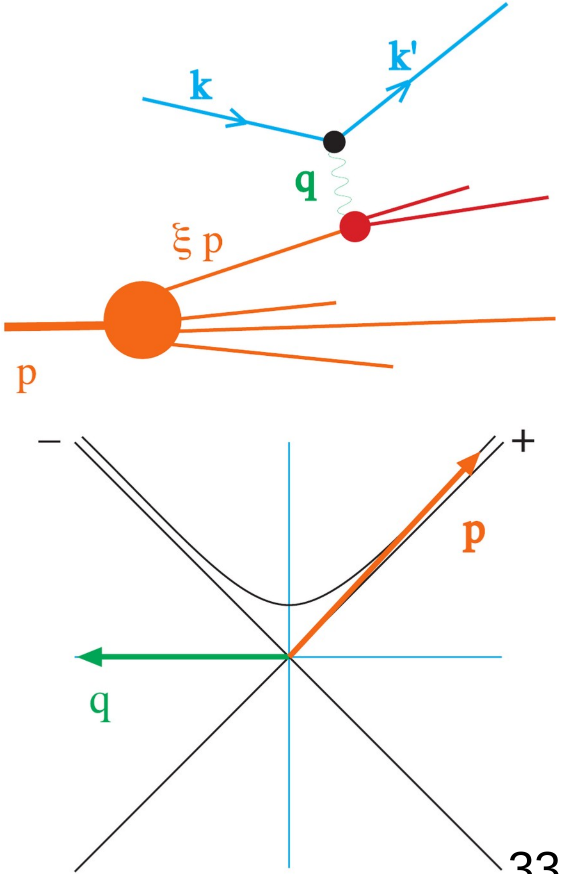
Slide from Marco Stratmann, 2005

Figures from Davison E. Soper, 2000

IR Divergence in Initial State – DIS revisit

The space-time picture suggests the possibility of separating short and long-distance physics

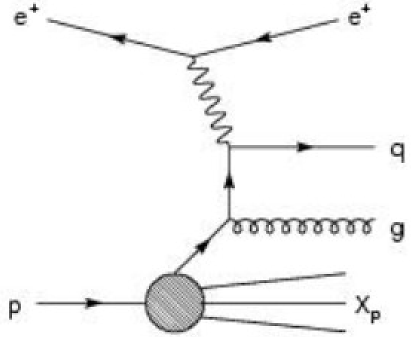




Slide from Marco Stratmann, 2005

Figures from Davison E. Soper, 2000

Factorization



► Let's consider leading order QCD effect to DIS

$\kappa^2 \to 0$

Splitting function (Probability of q→qg) × **Collinear divergence**

$$F_2(x,Q^2) = x\sum_q e_q{}^2\left[q_o{}^2(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0(\xi)\{P(\frac{x}{\xi})\ln(\frac{Q^2}{\kappa^2}) + C(\frac{x}{\xi})\}\right]$$

Naïve Quark-Parton Model

Everything else that can be calculable

► Factorize at scale μ_F

$$\ln(\frac{Q^2}{\kappa^2}) = \ln(\frac{Q^2}{\mu_F{}^2}) + \ln(\frac{\mu_F{}^2}{\kappa^2})$$

$$F_2(x,Q^2) = x\sum_q e_q{}^2\int_x^1 \frac{d\xi}{\xi} q(\xi,\mu_F{}^2)\times\{\delta(1-\frac{x}{\xi}) + \frac{\alpha_S}{2\pi}P(\frac{x}{\xi})\ln\frac{Q^2}{\mu_F{}^2} + C'\}$$

$\kappa^2 \to 0$

$$q(x,\mu_F{}^2) = q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 q_0(\xi)\{P(\frac{x}{\xi})\ln\frac{\mu_F{}^2}{\kappa^2} + C''\}$$

Arbitrary choice to split C btw F_2 and PDF

Factorization

- Arbitrary choice on "C" → Factorization scheme
  - -- $\overline{\text{MS}}$, DIS schemes, etc.

- PDF absorbs collinear divergence
  - -- Cannot be fully calculated
  - -- However, its variation with μ_F is given by

DGLAP evolution equation

$$\frac{\partial q_i(x,\mu_F^2)}{\partial \ln \mu_F^2} = \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_i(\xi,\mu_F^2) P\left(\frac{x}{\xi}\right)$$

Take derivative with $\ln \mu_F^2$

$$F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi,\mu_F^2)\times\left\{\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_S}{2\pi} P\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{\mu_F^2} + C'\right\}$$

$\kappa^2 \to 0$

PDF is universal!

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 q_0(\xi)\left\{P\left(\frac{x}{\xi}\right)\ln\frac{\mu_F^2}{\kappa^2} + C''\right\}$$

Arbitrary choice to split C btw F_2 and PDF

The DGLAP evolution equations

The Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q \\ g \end{pmatrix} = \begin{pmatrix} P_{q\leftarrow q} & P_{q\leftarrow g} \\ P_{g\leftarrow q} & P_{g\leftarrow g} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

$$P_{qg}(z) = T_R\left[z^2 + (1-z)^2\right], \qquad P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right],$$

$$P_{gg}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \delta(1-z)\frac{(11C_A - 4n_f T_R)}{6}$$

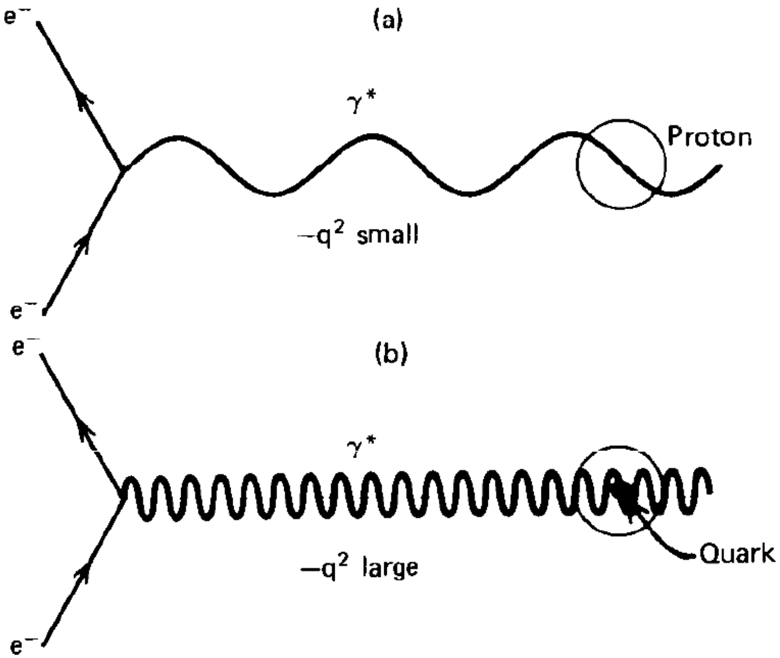
P_{qg}, P_{gg}: *symmetric* $z \leftrightarrow 1-z$ (except virtuals)

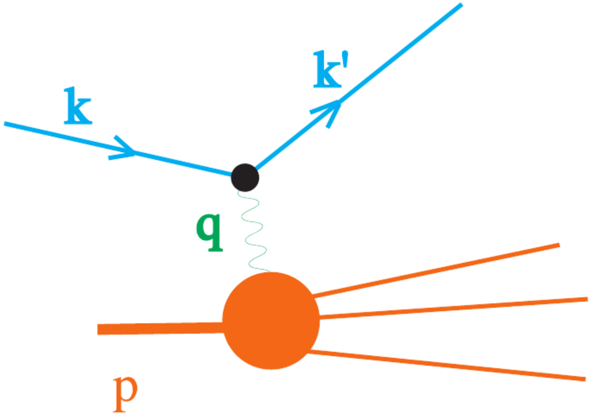
P_{qq}, P_{gg}: *diverge for* $z \to 1$ soft gluon emission

P_{gg}, P_{gq}: *diverge for* $z \to 0$ Implies PDFs grow for $x \to 0$

The QCD-improved parton model

Q^2





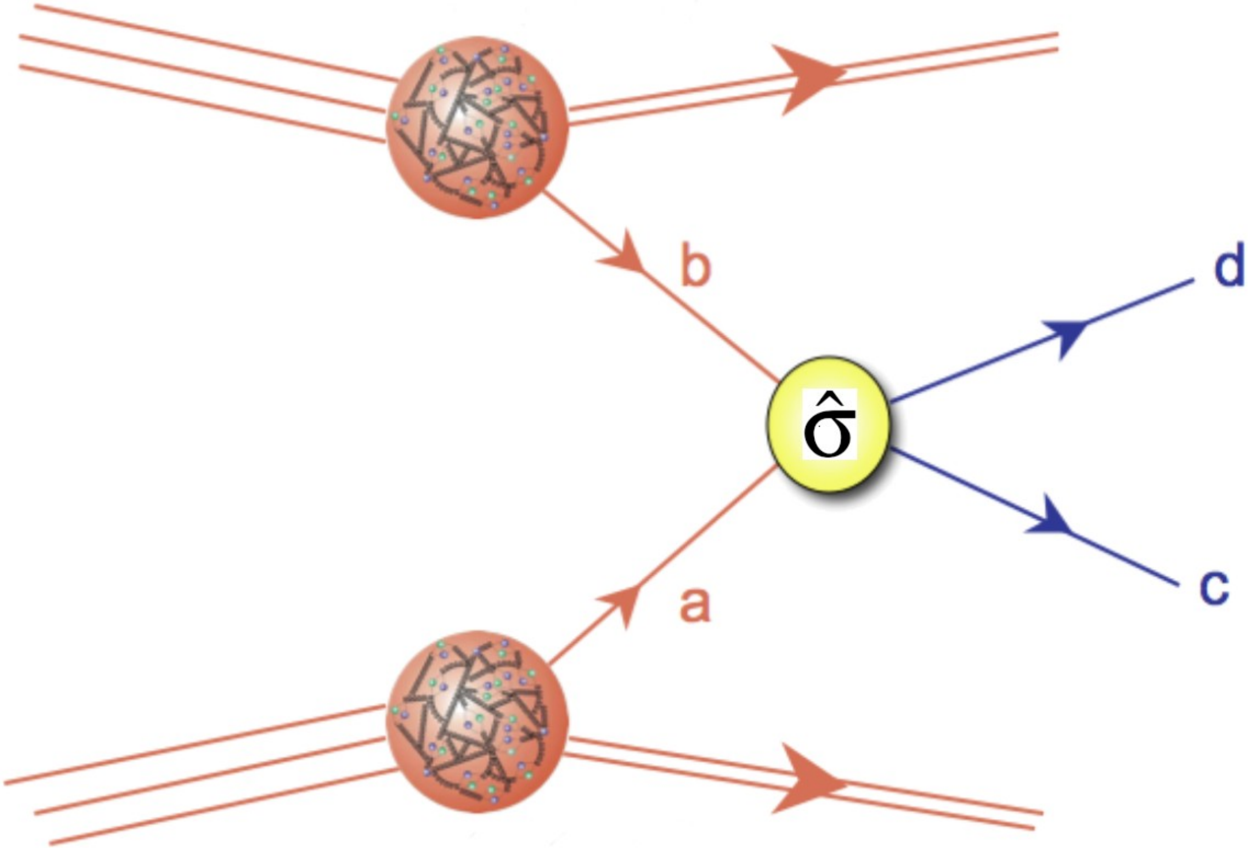
As the probe energy increases, more details of quantum fluctuations, such as sea quarks and gluons, become visible.

valence quark

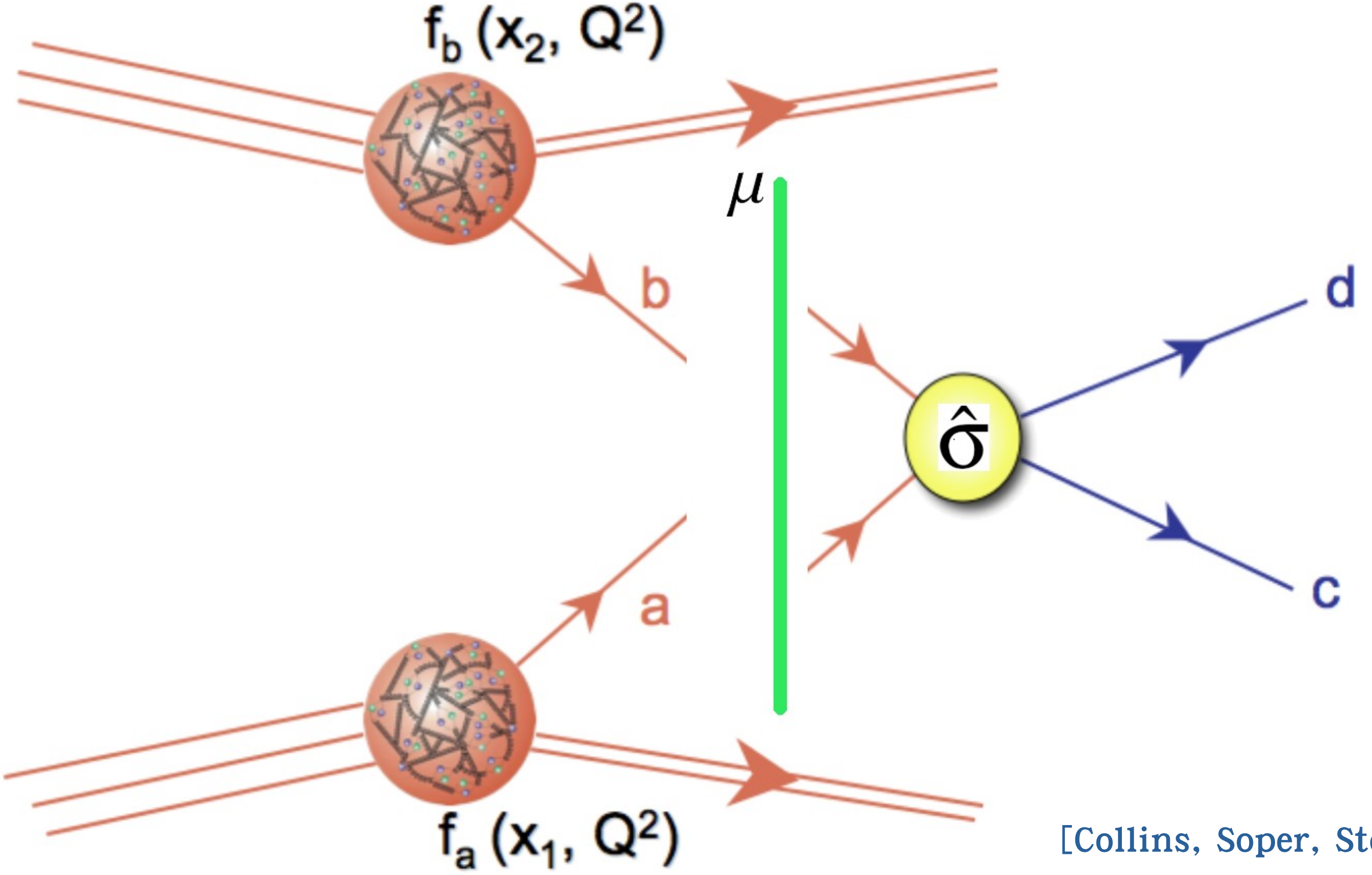
What do we know by far?

- Hadrons, like baryons and mesons, are composed of quarks and gluons.
- The strong interaction between quarks is described by QCD, a SU(3) local gauge theory.
- Physical observables must be IR safe, such as inclusive cross sections and jet observables.
- Initial-state infrared singularities are absorbed into the Parton Distribution Functions (PDFs).
- The PDFs evolve with energy scale, governed by the DGLAP equation.
- The PDFs have a dual role:
  - They describe the internal structure of hadrons (quarks and gluons).
  - They are an essential, universal input for calculating any hadron collision process.

How can we extract such non-perturbative, yet universal and essential functions for hadron collisions?



The collision is characterized by
low-energy **Parton Distribution Functions (PDFs)** and
high-energy **hard-core interactions**.



[Collins, Soper, Sterman, 1989]

$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

$$\frac{d\sigma}{d\ln\mu^2} = 0$$

$$\frac{\partial f_a(x,\mu^2)}{\partial \ln\mu^2} = \sum_{j=g,q,\bar{q}} P_{a/j}(x,\mu^2) \otimes f_{j/p}(x,\mu^2)$$

$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

$$\frac{d\sigma}{d\ln\mu^2} = 0$$

$$\frac{\partial f_a(x,\mu^2)}{\partial \ln\mu^2} = \sum_{j=g,q,\bar{q}} P_{a/j}(x,\mu^2) \otimes f_{j/p}(x,\mu^2)$$

$$x f_a(x, Q_0, \{a_1, a_2, \ldots\}) = x^{a_1}(1-x)^{a_2} P_a(x)$$

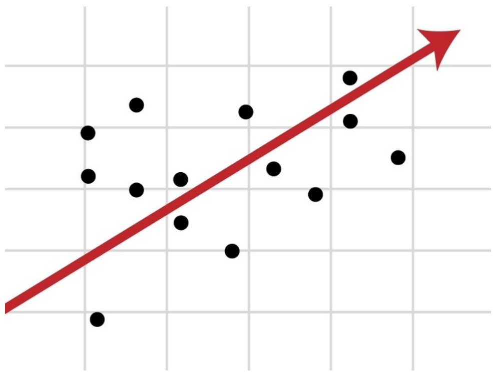
- PDFs evolve with energy scale.
- With a given parametrized PDFs at specific energy scale Q_0 scale, the PDFs $f(x,\mu^2)$ is determined

$$\sigma = f_a(x_1, \mu^2) \otimes f_b(x_2, \mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

How can we extract such non-perturbative, yet universal and essential functions for hadron collisions?

$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

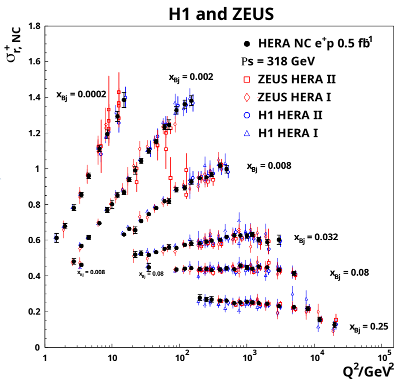
Experiments

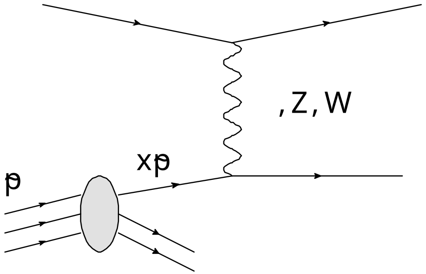


theoretical prediction

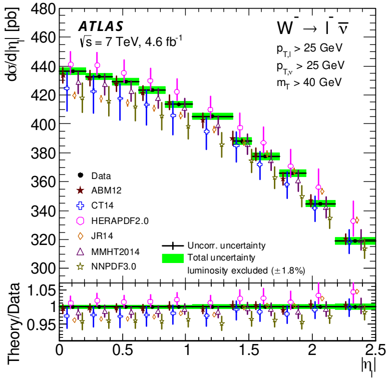
$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$

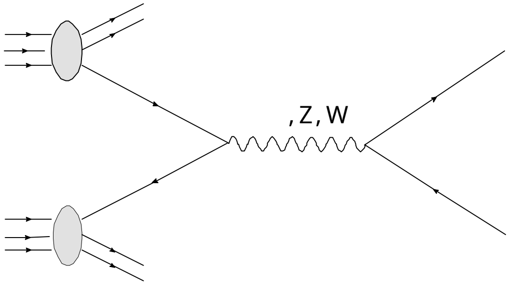
DIS



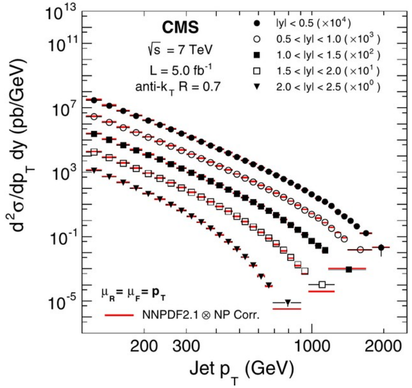


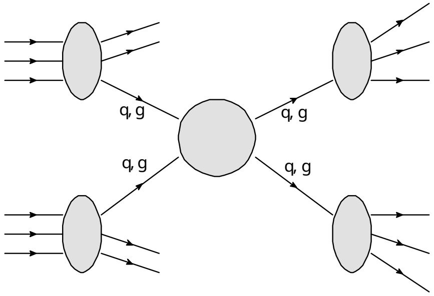
Drell-Yan



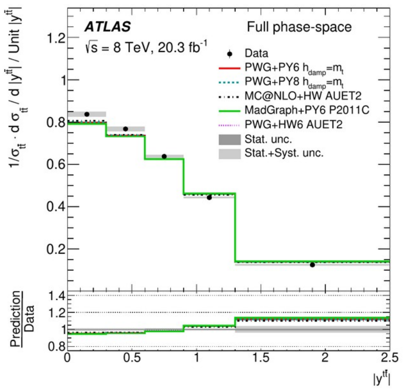


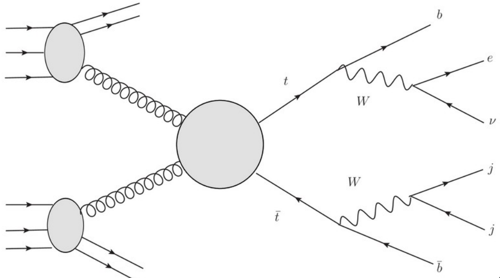
Jet



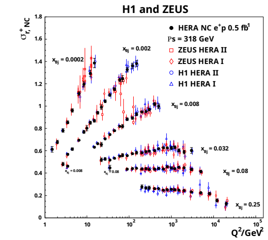


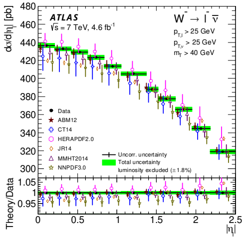
$t\bar{t}$

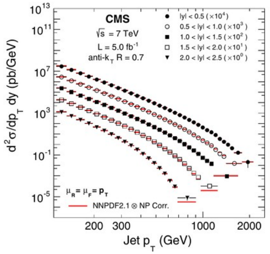


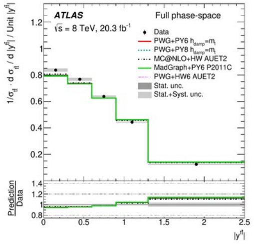


$$\sigma = f_a(x_1,\mu^2) \otimes f_b(x_2,\mu^2) \otimes \hat{\sigma}_{ab}(\mu^2)$$





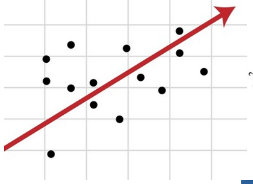


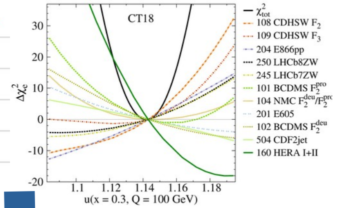


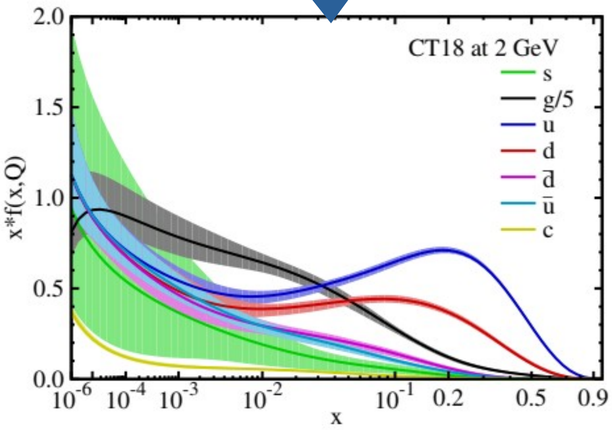
Statistic

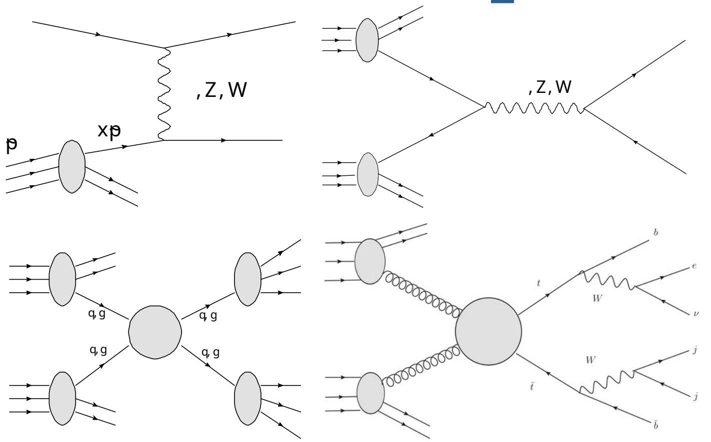
Regression analysis

Uncertainty Estimation









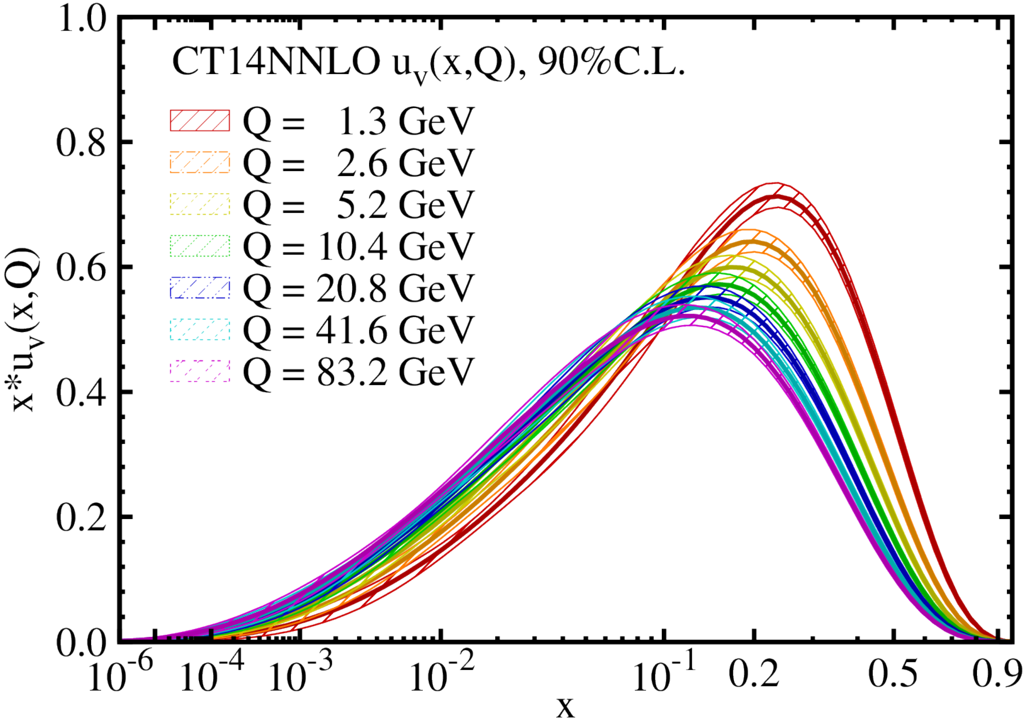
$$\frac{\partial f_a(x,\mu^2)}{\partial \ln \mu^2} = \sum_{j=g,q,\bar{q}} P_{a/j}(x,\mu^2) \otimes f_{j/p}(x,\mu^2)$$

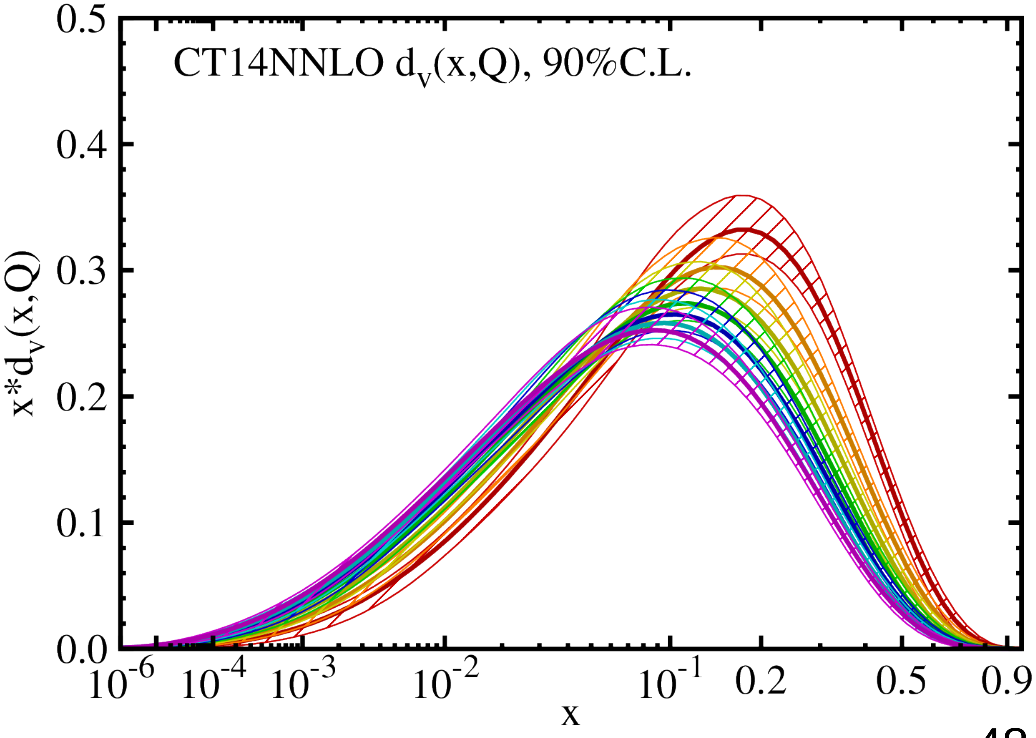
$$x f_a(x, Q_0, \{a_1, a_2, \ldots\}) = x^{a_1}(1-x)^{a_2} P_a(x)$$

- With a given parametrized PDFs at specific energy scale Q_0 scale, the PDFs $f(x,\mu^2)$ is determined

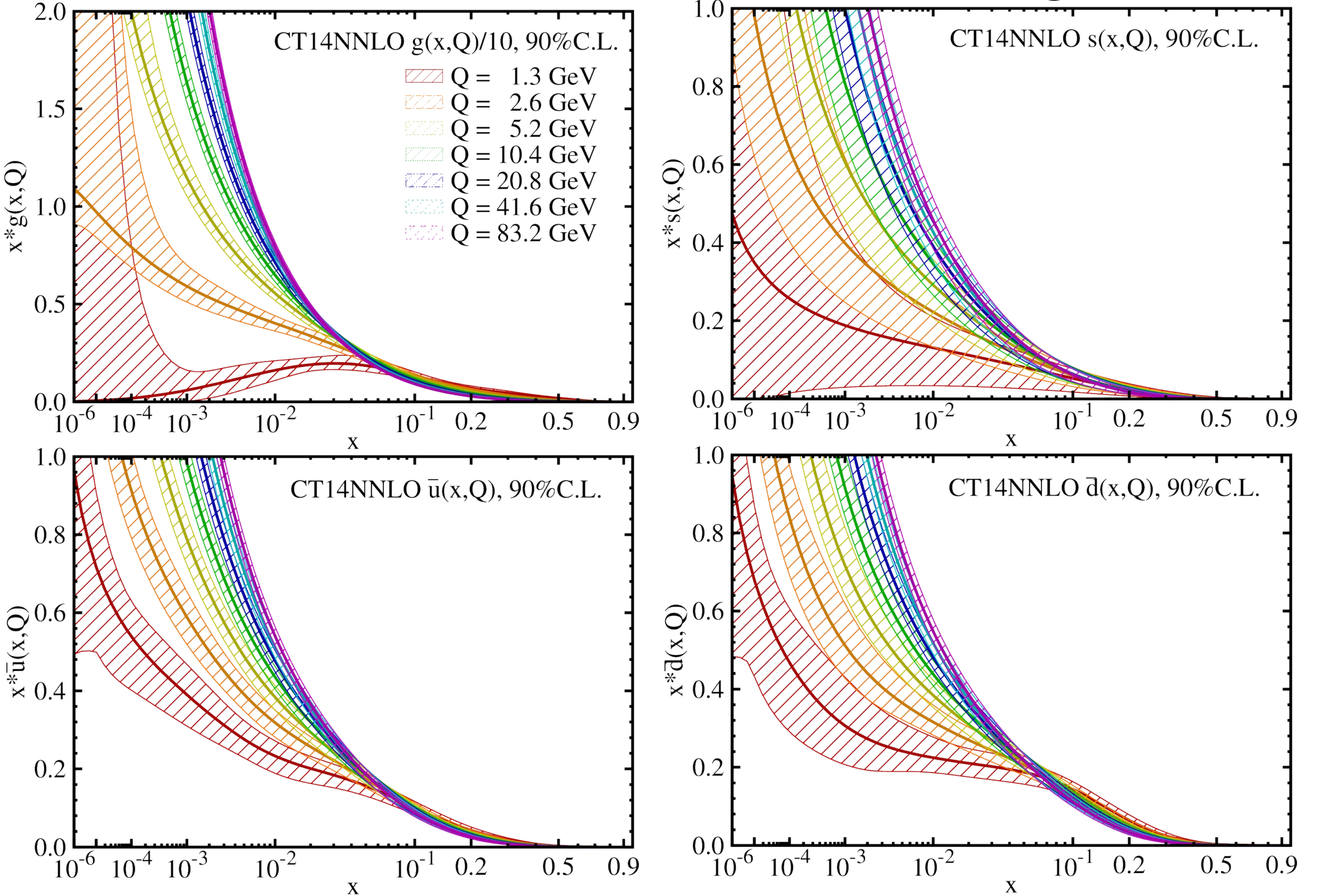
But….where? Top down or Bottom up?

Evolution of PDFs: valence quark



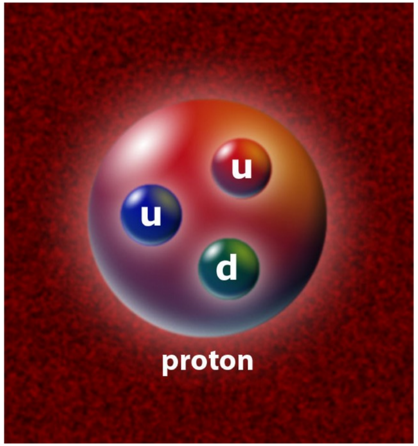


Evolution of PDFs: sea quark and gluon

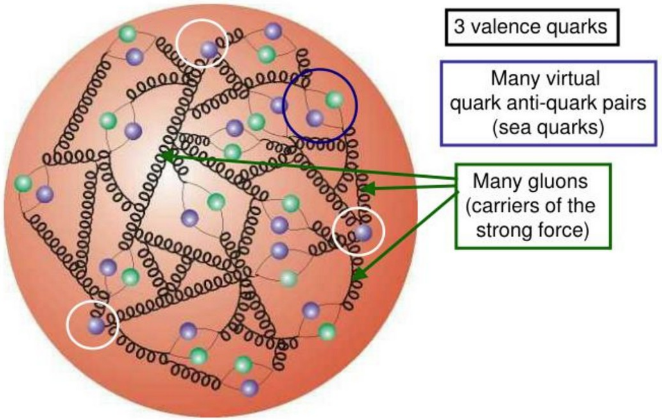


Content of Proton

What PDF flavors to fit?



Naive quark model



QCD-improved quark model

$$u_{valence}, d_{valence}, \text{gluon, sea quarks....}$$

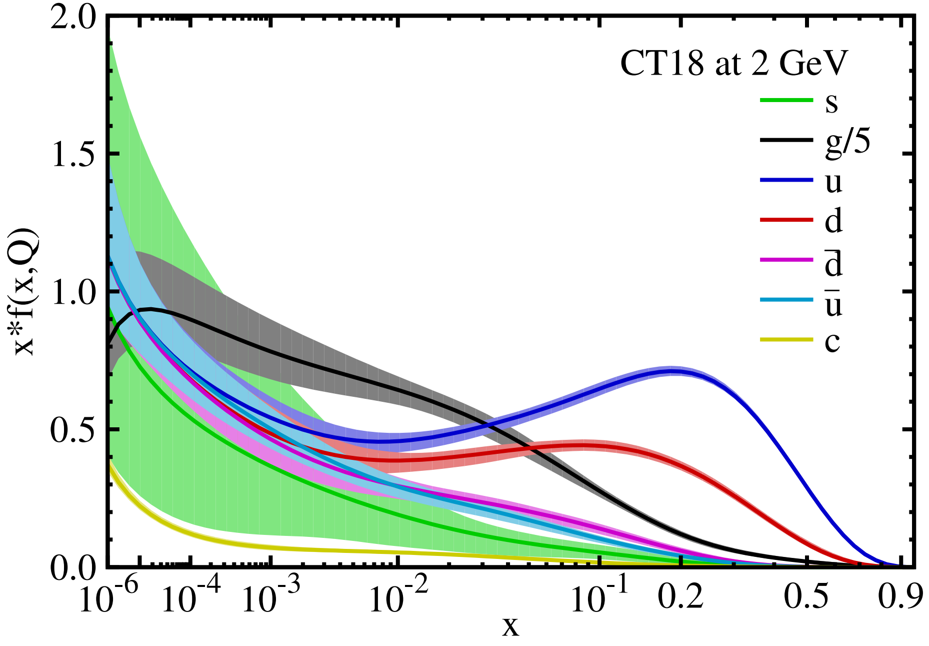
If we consider flavor SU(3) symmetry, i.e., $m_u = m_d = m_s$ are massless.

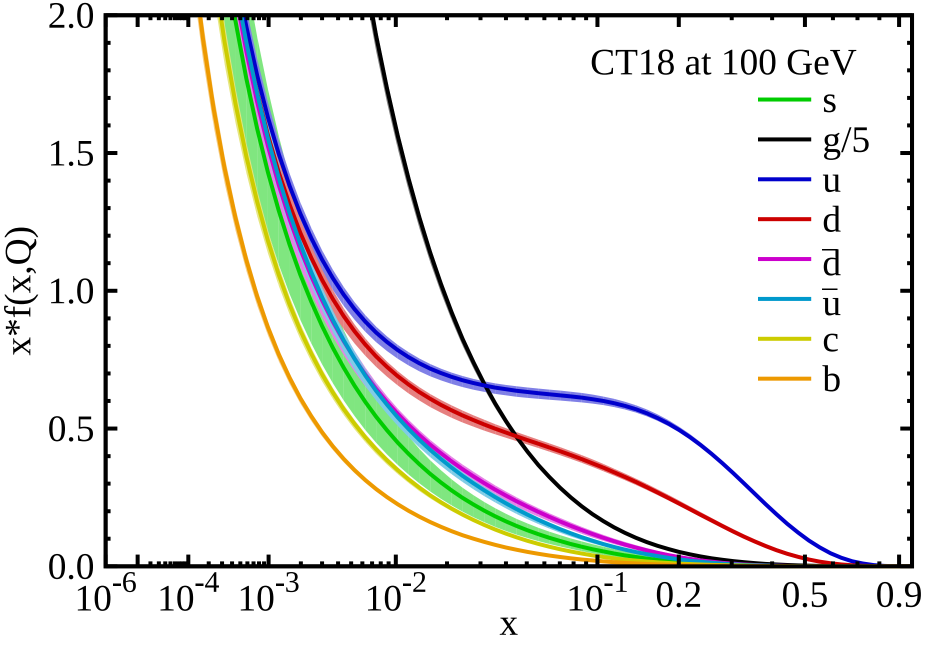
$$u_{sea} = \bar{u}_{sea} = d_{sea} = \bar{d}_{sea} = s_{sea} = \bar{s}_{sea}$$

But further experiement shows that,

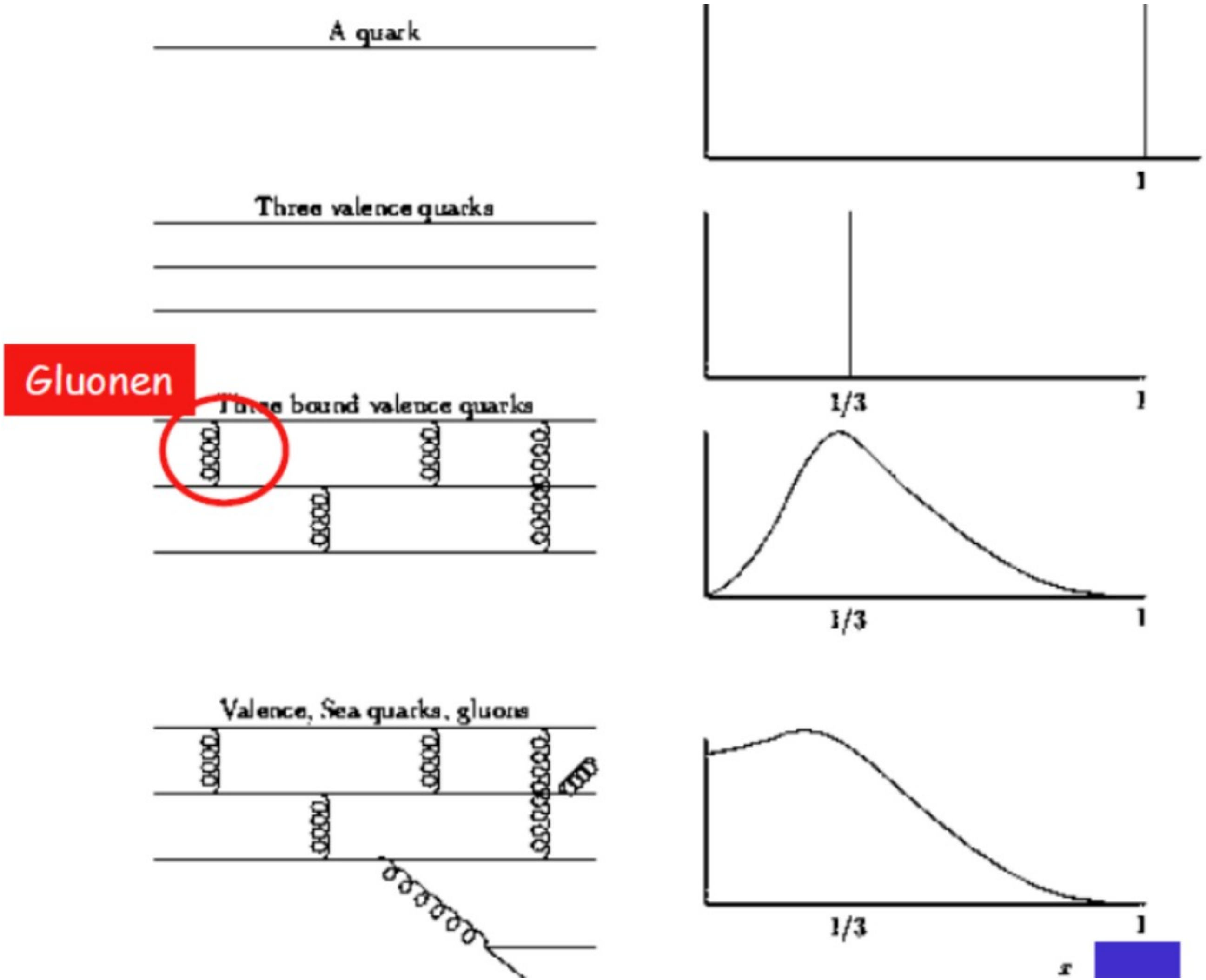
$$\bar{u}_{sea} \neq \bar{d}_{sea} \neq \bar{s}_{sea}$$

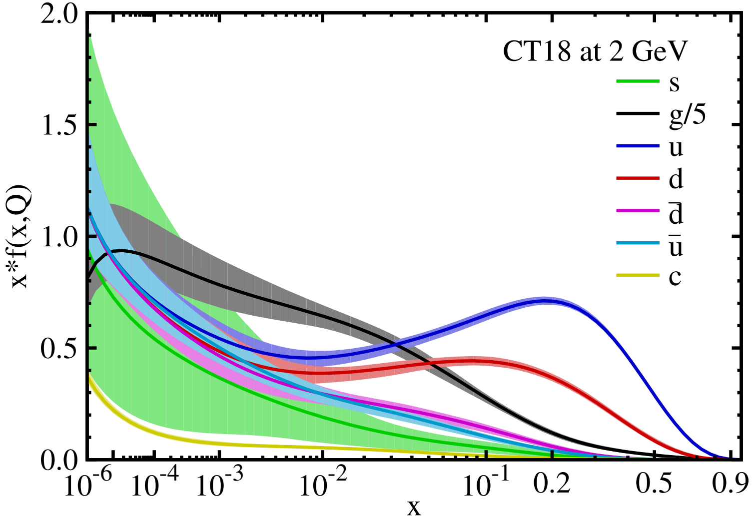
CT18

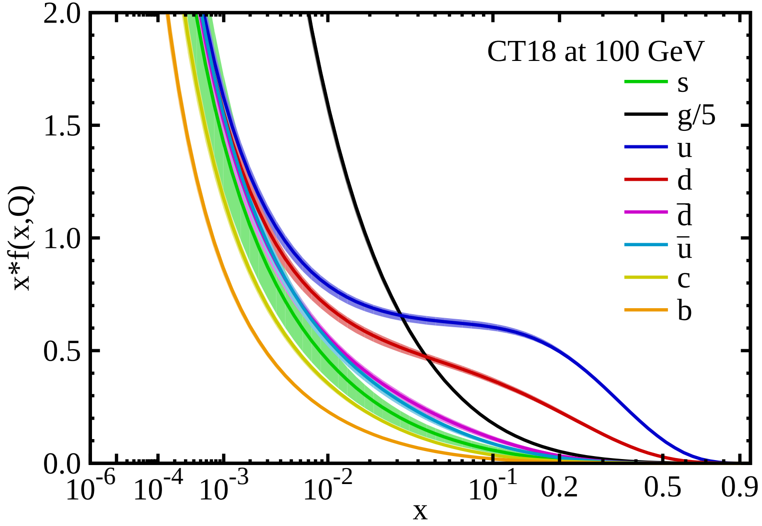

CT18 at 2 GeV
s
g/5
u
d
d̄
ū
c
x*f(x,Q)
2.0
1.5
1.0
0.5
0.0
10^-6 10^-4 10^-3 10^-2 10^-1 0.2 0.5 0.9
x


CT18 at 100 GeV
s
g/5
u
d
d̄
ū
c
b
x*f(x,Q)
2.0
1.5
1.0
0.5
0.0
10^-6 10^-4 10^-3 10^-2 10^-1 0.2 0.5 0.9
x

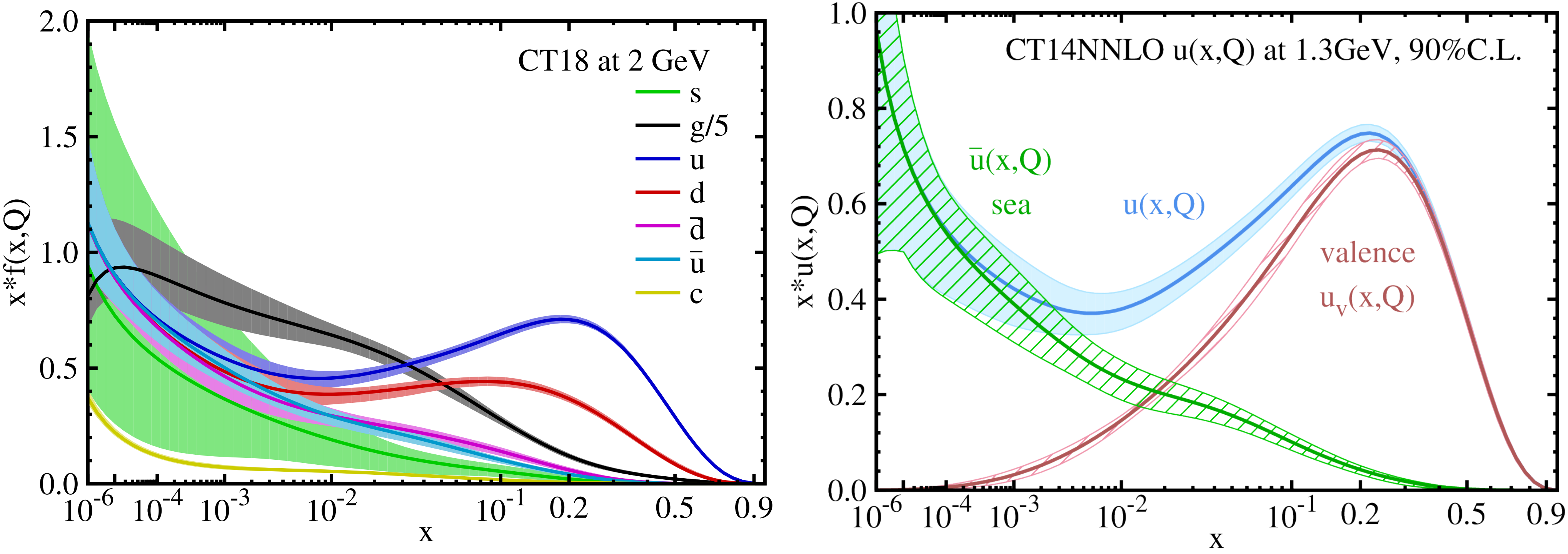
Valence quark

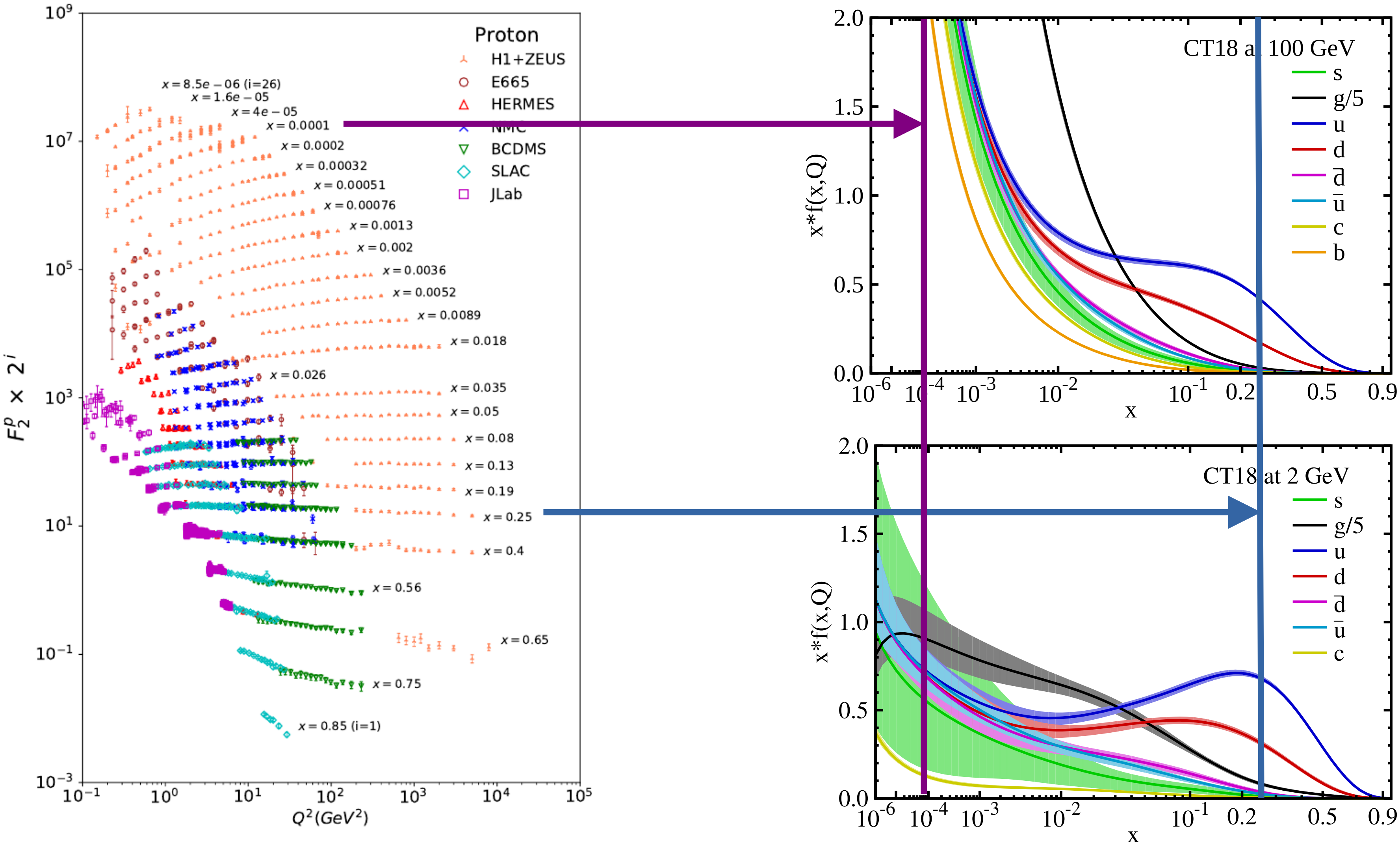






Valence quark and Sea Quark





Proton
H1+ZEUS
E665
HERMES
NMC
BCDMS
SLAC
JLab
$F_2^p \times 2^i$
$Q^2 (GeV^2)$
CT18 at 100 GeV
CT18 at 2 GeV
s
g/5
u
d
$\bar{d}$
$\bar{u}$
c
b
x*f(x,Q)
x

Parametrization of PDFs

The choice of input Q_0 scale is tricky:

- Q_0 need to be large enough to away from Λ_{QCD} ,
- Q_0 need to be smaller enough to reduce input PDF flavors

CTEQ	1.3	GeV
MSHT	1.0	GeV
NNPDF	1.65	GeV

$$g, \quad u_v, \quad d_v, \quad s, \quad \bar{u}_s = u_s, \quad \bar{d}_s = d_s$$

Where $u = u_v + u_s$, and $d = d_v + d_s$.

Notice that, $\bar{u}$, $\bar{d}$ and s are considered to have different PDF momentum fraction.

We will back to this later….

u $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

$m_u = 2.2^{+0.6}_{-0.4}$ MeV Charge $= \frac{2}{3}\ e$ $I_z = +\frac{1}{2}$
$m_u/m_d = 0.38$–0.58

d $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

$m_d = 4.7^{+0.5}_{-0.4}$ MeV Charge $= -\frac{1}{3}\ e$ $I_z = -\frac{1}{2}$
$m_s/m_d = 17$–22
$\overline{m} = (m_u + m_d)/2 = 3.5^{+0.7}_{-0.3}$ MeV

s $I(J^P) = 0(\frac{1}{2}^+)$

$m_s = 96^{+8}_{-4}$ MeV Charge $= -\frac{1}{3}\ e$ Strangeness $= -1$
$m_s\ /\ ((m_u + m_d)/2) = 27.3 \pm 0.7$

c $I(J^P) = 0(\frac{1}{2}^+)$

$m_c = 1.27 \pm 0.03$ GeV Charge $= \frac{2}{3}\ e$ Charm $= +1$
$m_c/m_s = 11.72 \pm 0.25$
$m_b/m_c = 4.53 \pm 0.05$
$m_b - m_c = 3.45 \pm 0.05$ GeV

b $I(J^P) = 0(\frac{1}{2}^+)$

Charge $= -\frac{1}{3}\ e$ Bottom $= -1$

$m_b(\overline{MS}) = 4.18^{+0.04}_{-0.03}$ GeV
$m_b(1S) = 4.66^{+0.04}_{-0.03}$ GeV

t $I(J^P) = 0(\frac{1}{2}^+)$

Charge $= \frac{2}{3}\ e$ Top $= +1$

Mass (direct measurements) $m = 173.21 \pm 0.51 \pm 0.71$ GeV [a,b]
Mass ($\overline{MS}$ from cross-section measurements) $m = 160^{+5}_{-4}$ GeV [a]
Mass (Pole from cross-section measurements) $m = 174.2 \pm 1.4$ GeV

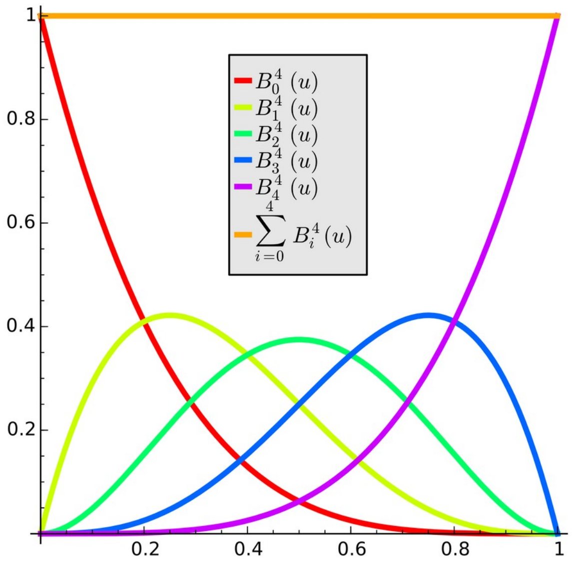
Parametrization of PDFs

$$xf(x, \{a_1, a_2, ...\}) = x^{a_1}(1-x)^{a_2}P(x)$$

- $x \to 0$: $f \propto x^{a_1}$, Regge-like behavior
- $x \to 1$: $f \propto (1-x)^{a_2}$, quark counting rules
- $P(x; a_3, a_4, ...)$: smooth function for intermediate x range, ex: Bernstein polynomial.

$$b_{\nu,n}(x) \equiv \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$$

$$\nu = 0, \ldots, n,$$



Requirements for PDF parametrization

- Valence quark number sum rule

$$\int_0^1 [u(x) - \bar{u}(x)]dx = 2, \int_0^1 [d(x) - \bar{d}(x)]dx = 1$$

$$\int_0^1 [s(x) - \bar{s}(x)]dx = 0$$

Where $u = u_v + \bar{u}$, $d = d_v + \bar{d}$. $(s(x) - \bar{s}(x))$ can be non-zero.

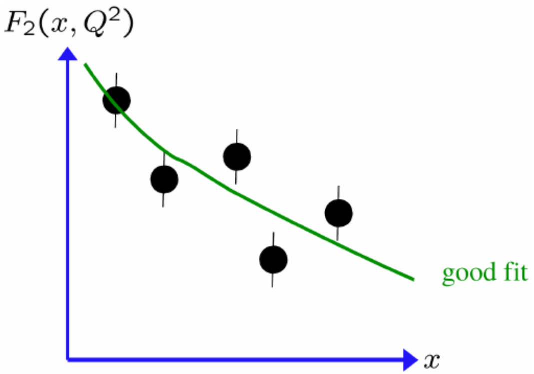
- Momentum sum rule

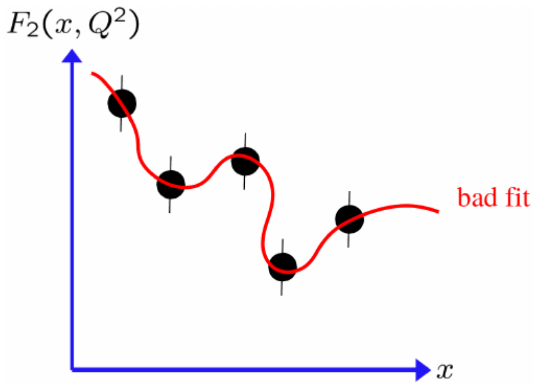
$$\sum_{a=q,\bar{q},g} \int_0^1 x f_{a/p}(x, Q)dx = 1$$

CT18 NNLO	gluon	u_v	d_v	dbar+ubar
momentum fraction(1.3GeV, %)	38.4	32.5	13.4	12.9

More Requirements for Fitting

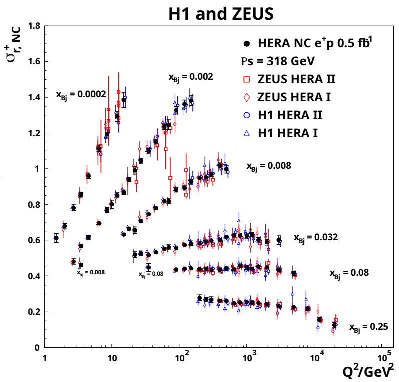
- A valid PDF set must not produce unphysical predictions for observables
  - Any concievable hadron cross section σ must be non-negative: $\sigma > 0$. This is typically realized by requiring $f_{a/p}(x, Q) > 0$.
  - Any cross section asymmetry A must lie in the range $-1 \leq A \leq 1$. This constrains the range of allowed PDF parametrizations.
- PDF parametrization for $f_{i/p}(x, Q)$ must be ”flexible just enough” to reach agreement with the data, without reproducing random fluctuation.



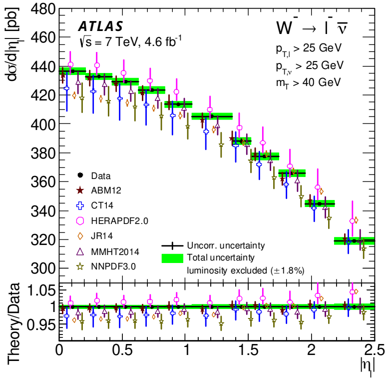


$$\sigma = f_a(x_1,\mu^2)\otimes f_b(x_2,\mu^2)\otimes \hat{\sigma}_{ab}(\mu^2)$$

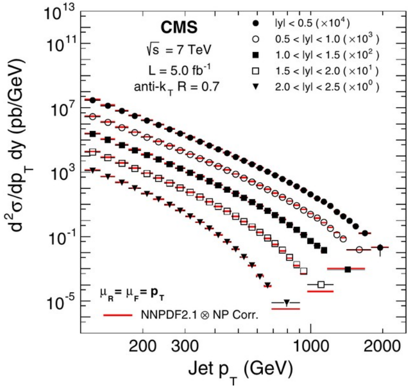
DIS



Drell-Yan



Jet



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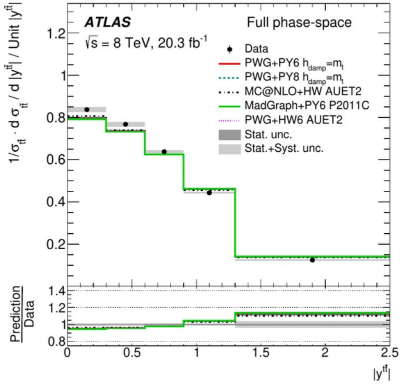
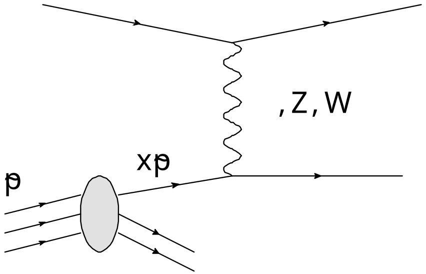
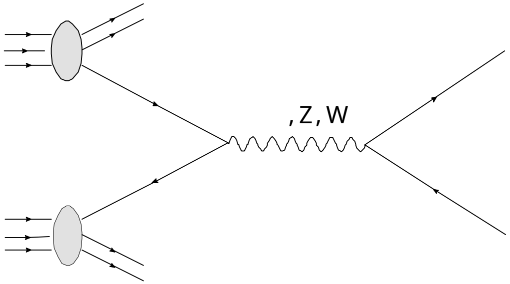
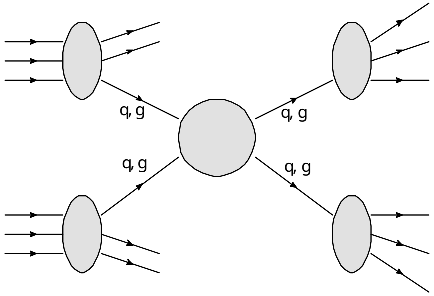
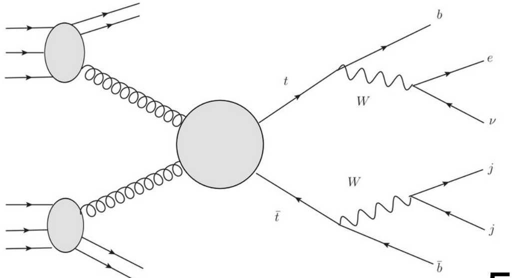


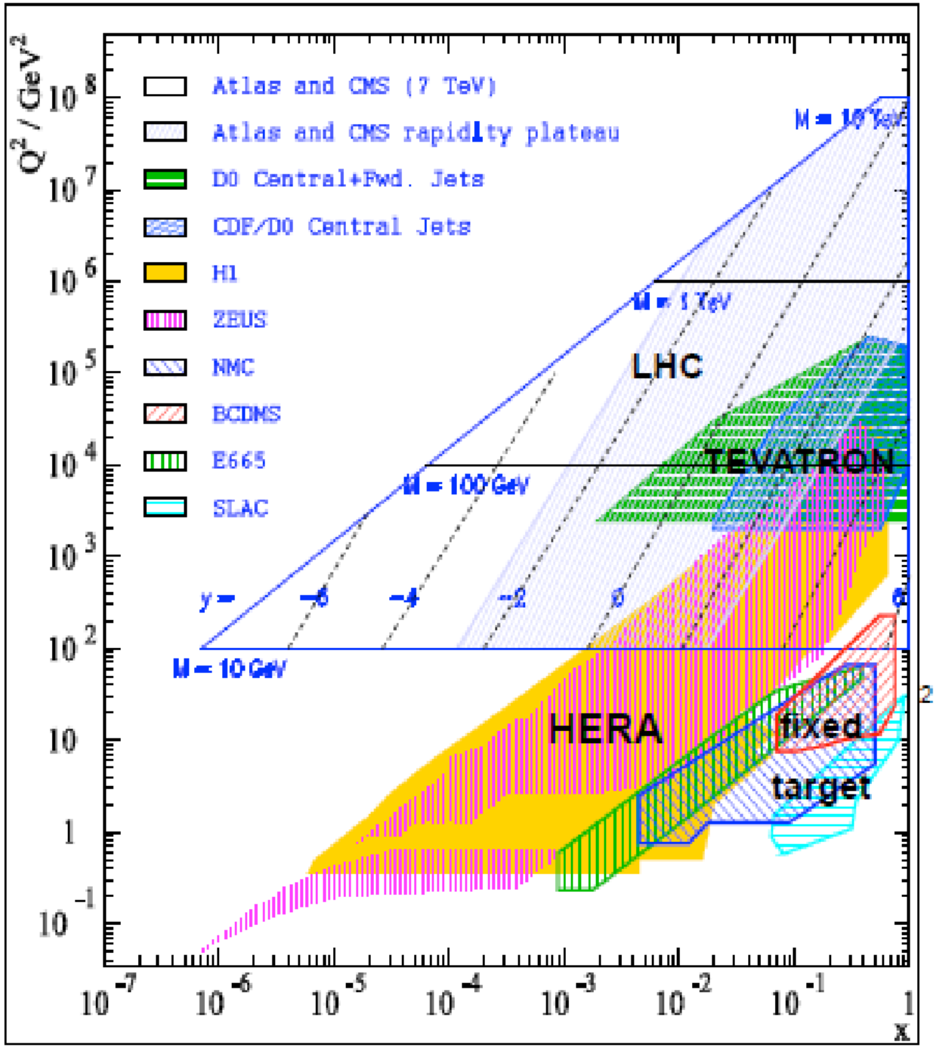




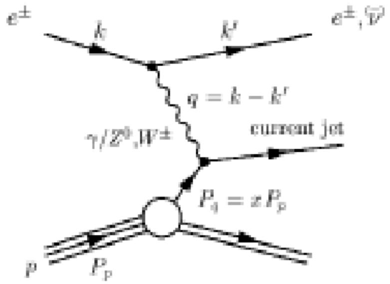




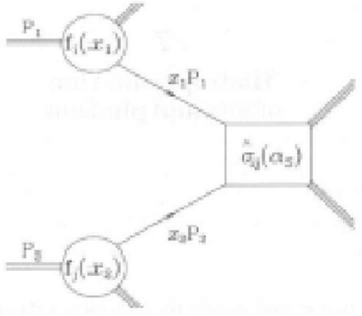
Experimental Data Access to Proton Structure



HERA: low and medium x



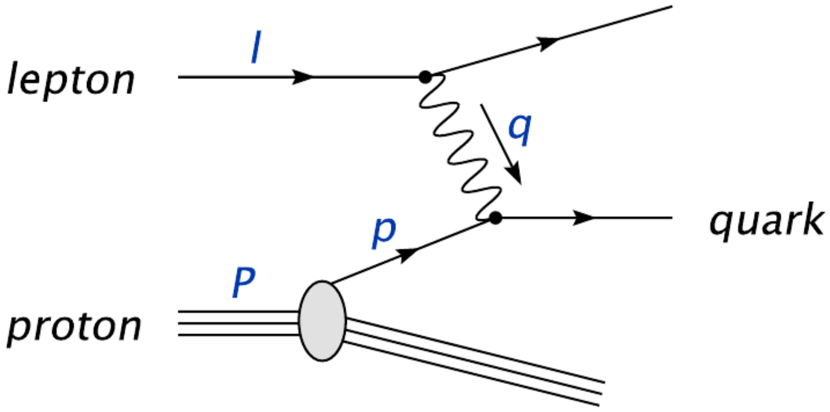
LHC: important constraints on g(x), flavour separation



Fixed Target: high x, nuclear PDFs

DIS data

- Deep-inelastic lepton-hadron scattering ($e^{\pm}p$, $e^{\pm}n$, νp, $\bar{\nu}p$, ...)



- **Kinematic variables**
  - momentum transfer $Q^2 = -q^2$
  - Bjorken variable $x = Q^2/(2p \cdot q)$
- **Gauge boson exchange**
  - neutral current: γ, Z
  - charged current: $W^{\pm}$

- Cross section $\sigma \simeq L^{\mu\nu} W_{\mu\nu}$
  - leptonic tensor $L_{\mu\nu}$ for neutral/charged current
  - hadronic tensor parametrized through structure functions

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

Slide from Sven-Olaf Moch, 2007

DIS data

NC (neutral-current)

- NC $e^{\pm}p$ cross section

$$\frac{d^2\sigma^{NC}(e^{\pm}p)}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}[(1+(1-y)^2)F_2 - y^2 F_L \mp (1-(1-y)^2)xF_3]$$

  - valence/sea parton distributions $q \pm \bar{q}$

$$F_2 = \sum_q A_q\, x\,(q+\bar{q}) \text{ with } A_q = e_q^2 + \mathcal{O}(\frac{Q^2}{Q^2+M_Z^2})$$

$$F_3 = \sum_q B_q\, x\,(q-\bar{q}) \text{ with } B_q = \mathcal{O}(\frac{Q^2}{Q^2+M_Z^2})$$

Slide from Sven-Olaf Moch, 2007

DIS data

NC (neutral-current)

- NC $e^{\pm}p$ cross section

$$\frac{d^2\sigma^{NC}(e^{\pm}p)}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}[(1+(1-y)^2)F_2 - y^2 F_L \mp (1-(1-y)^2)xF_3]$$

  - valence/sea parton distributions $q \pm \bar{q}$

$$F_2 = \sum_q A_q\, x\,(q+\bar{q}) \text{ with } A_q = e_q^2 + \mathcal{O}(\frac{Q^2}{Q^2+M_Z^2})$$

$$F_3 = \sum_q B_q\, x\,(q-\bar{q}) \text{ with } B_q = \mathcal{O}(\frac{Q^2}{Q^2+M_Z^2})$$

CC (charged-current)

- CC $e^{\pm}p$ cross section $\longrightarrow$ flavour separation:

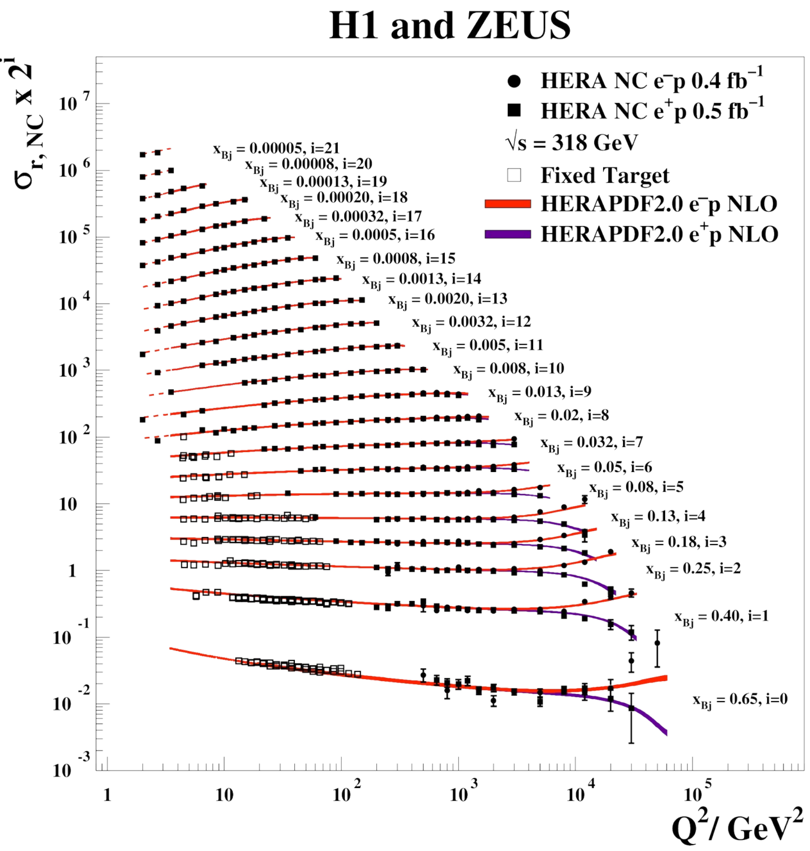
$$\frac{d^2\sigma^{CC}(e^{+}p)}{dxdQ^2} = \frac{G_F^2}{2\pi}(\frac{M_W^2}{M_W^2+Q^2})^2\left[\bar{u}+\bar{c}+(1-y)^2(d+s)\right]$$

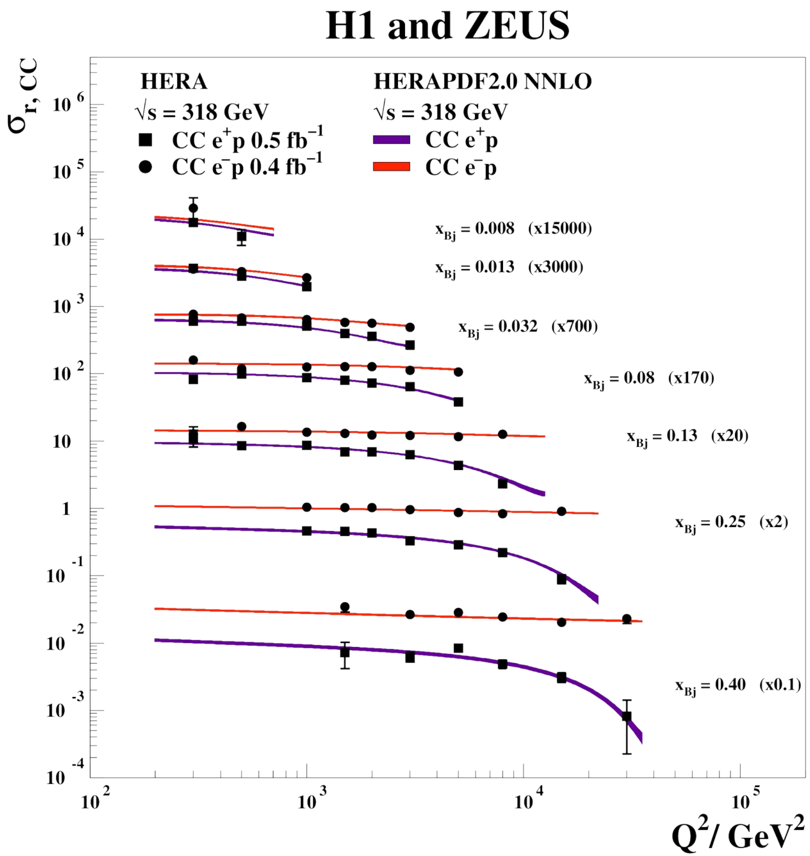
$$\frac{d^2\sigma^{CC}(e^{-}p)}{dxdQ^2} = \frac{G_F^2}{2\pi}(\frac{M_W^2}{M_W^2+Q^2})^2\left[u+c+(1-y)^2(\bar{d}+\bar{s})\right]$$

Slide from Sven-Olaf Moch, 2007

Final combined DIS cross sections at HERA

41 data sets on NC and CC DIS from H1 and ZEUS are combined into 1 set. 2927 data points are combined into 1307 data points. 165 correlated systematic errors are reanalyzed and calibrated.





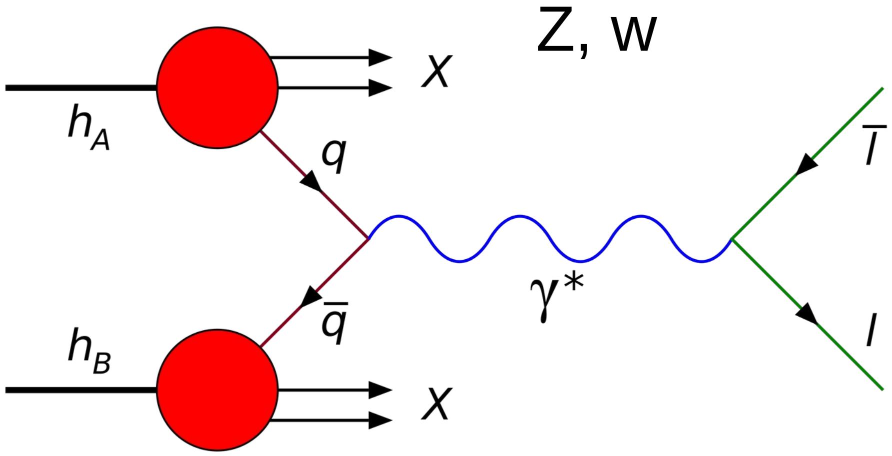
Drell-Yan Process

The Sidney Drell and Tung-Mow Yan(颜东茂) proposed the process at 1970.
Drell-Yan process is “clean” in hadron collider.

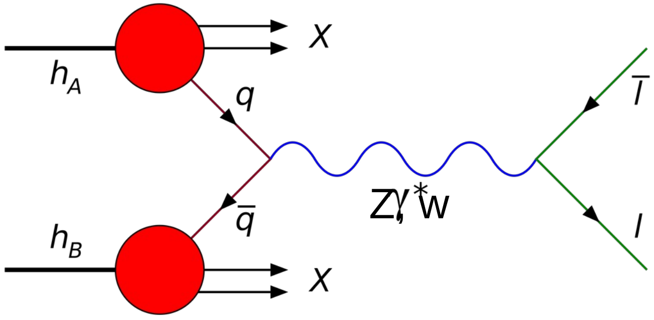
$$\frac{d\sigma^Z}{dy} \propto \frac{4}{9}[u_A\bar{u}_B + \bar{u}_A u_B] + \frac{1}{9}[d_A\bar{d}_B + \bar{d}_A d_B] + \ldots$$

$$\frac{d\sigma^{W^+}}{dy} \propto u_A\bar{d}_B + \bar{d}_A u_B + \ldots$$

$$\frac{d\sigma^{W^-}}{dy} \propto \bar{u}_A d_B + d_A\bar{u}_B + \ldots$$



Drell-Yan Process

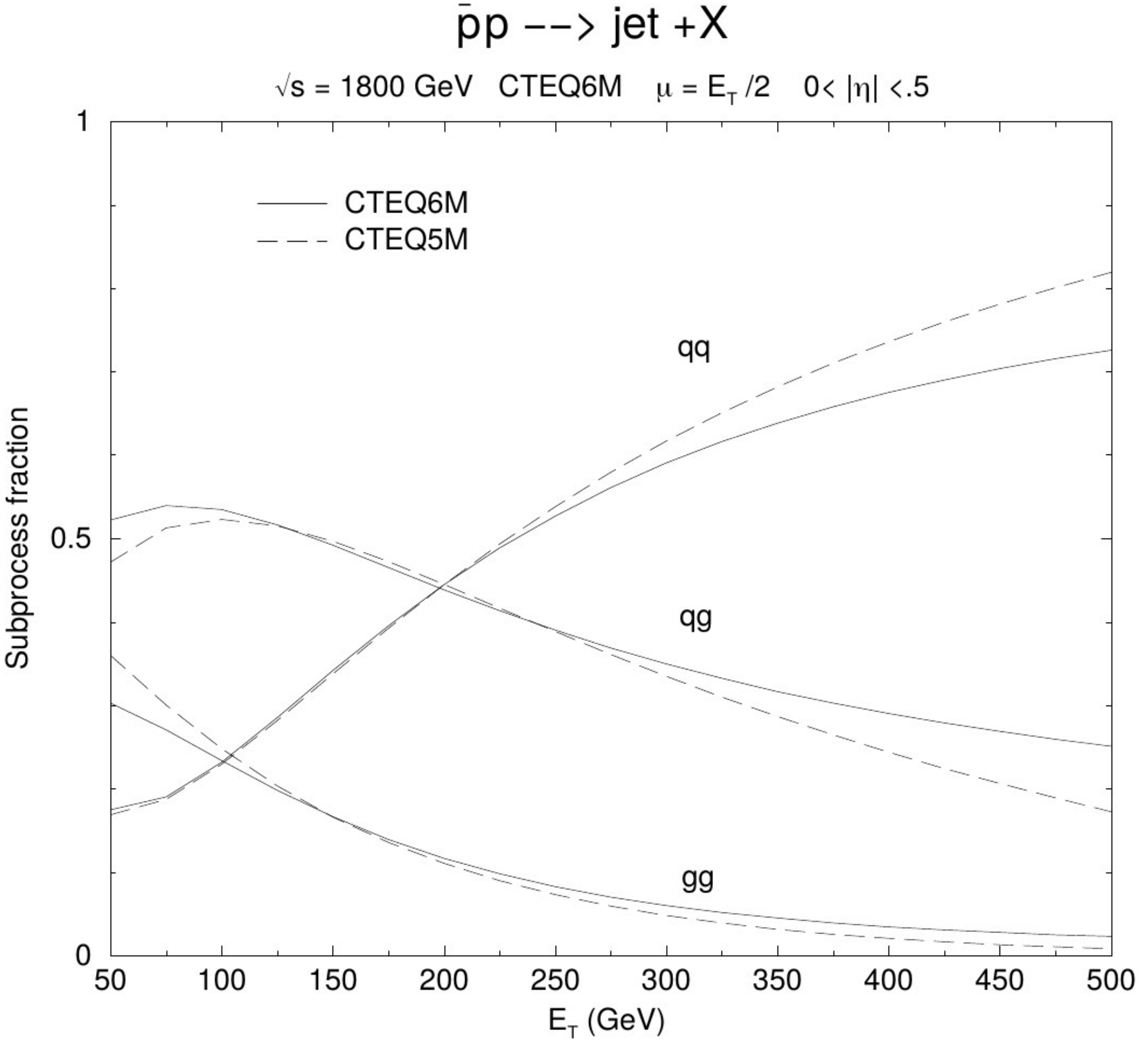


$$A_{ch}(y) \equiv \frac{\frac{d\sigma^{W^+}}{dy} - \frac{d\sigma^{W^-}}{dy}}{\frac{d\sigma^{W^+}}{dy} + \frac{d\sigma^{W^-}}{dy}}$$

$A_{ch}(y)$ constrains PDF ratio at $Q \approx m_W$:

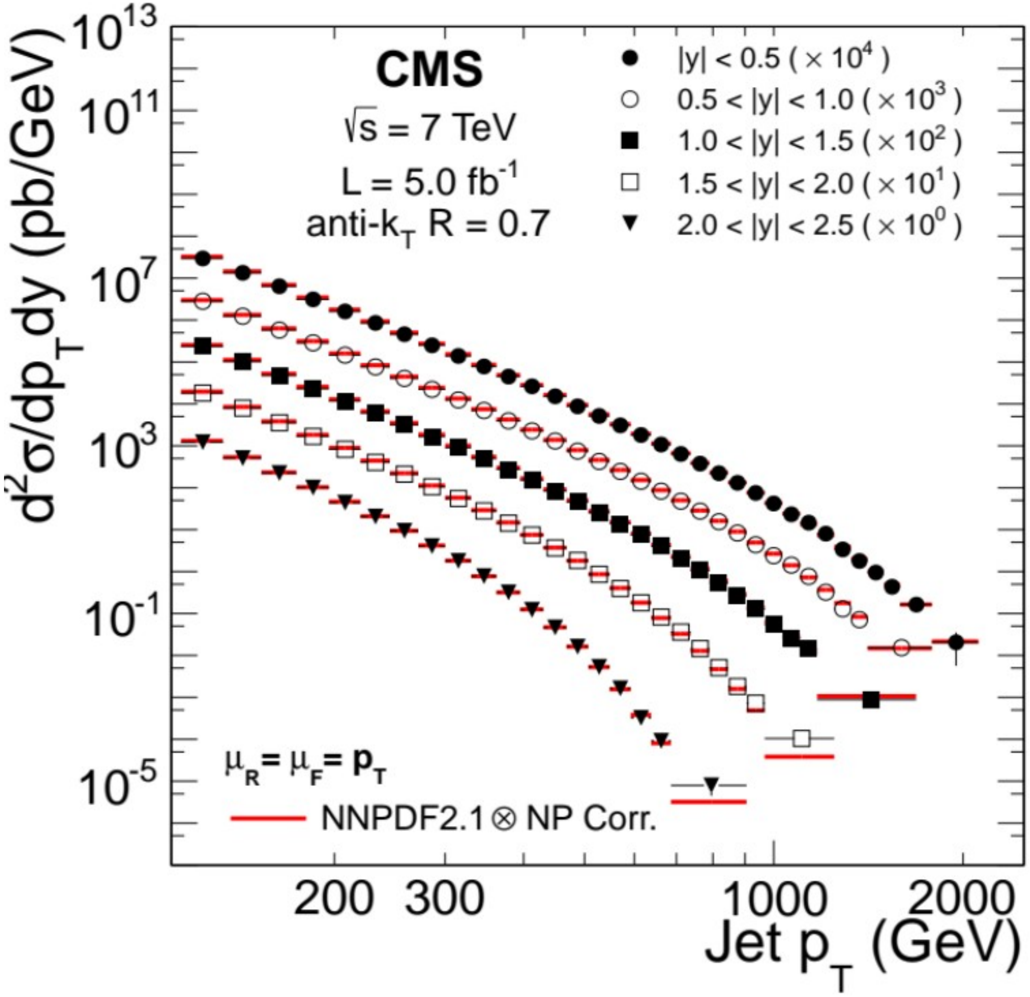
- d/u at $x \to 1$ at Tevatron 1.96 TeV ($p\bar{p}$)
- d/u at $x > 0.1$ and $\bar{d}/\bar{u}$ at $x \sim 0.01$ at the LHC 7TeV(pp)

Inclusive jet production

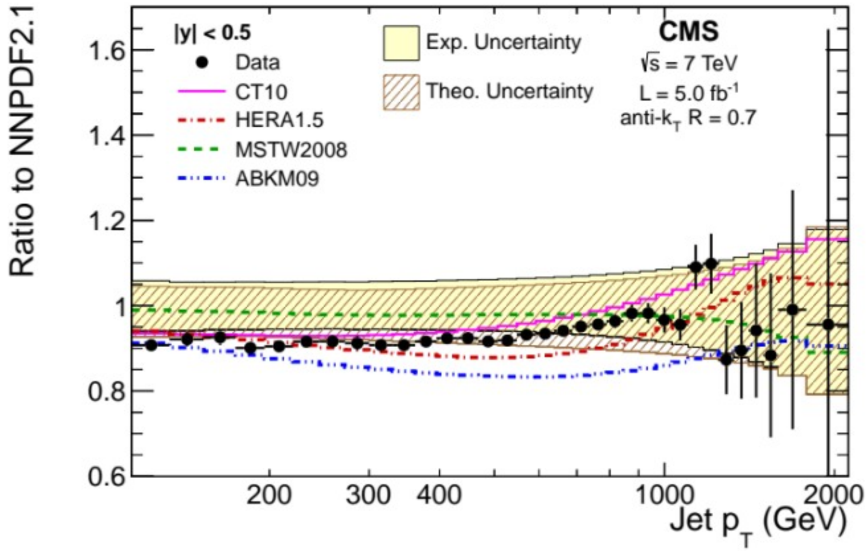


- Jet process is IR safe by including all final state hadron in a jet.
- High-E_T jets are mostly produced in qq scattering; yet most of the PDF uncertainty arises from qg and gg contributions.
- Typical x is of order $2E_T / \sqrt{s} \gtrsim 0.1$; e.g. $x \approx 0.2$ for $E_T = 200$ GeV, $\sqrt{s} = 1.8$ TeV. At such x, u quark and d quark are known very well, uncertainty arise mostly from gluon.

Inclusive jet production in pp → jet + X



- The cross sections span 12 orders of magnitude
- Almost negligible statistical error
- Systematic uncertainties dominate, both from the experiment (up to 90 correlated sources of uncertainty) and NLO theoretical cross section (QCD scale dependence)



Criteria for determining PDFs

$$\chi^2_{global} = \sum_i \frac{[D_i - \sum_k \lambda_k \beta_{ki} - T_i(\{a\})]^2}{\sigma_i^2} + \sum_k \lambda_k^2.$$

Where
D_i is the central value of data,
$T_i(\{a\})$ is the theoretical prediction of the data,
σ_i^2 is the qudratic sum of the statistical error and uncorrelated error,
β_{ki} is the matrix for correlated error,
and λ_k are the nuisance parameters.

The PDF is obtained by minimizing the global χ^2 function respect to shape parameters $\{a\}$ and nuisance parameters $\{\lambda\}$.

Source of PDF uncertainty

Factorization Theorem:

Data = PDFs $\otimes$ Hard part cross sections (Wilson coeff.)

Experimental errors:

- Statistical
- Systematic
- ▪ uncorrelated
- ▪ correlated
- χ^2definition (experimental or t_0)
- Possible tensions among data sets

Extracted with errors, dependent of methodology of analysis

- Non-perturbative parametrization forms of PDFs
- Additional theory prior
- Choice of Tolerance (T^2) value

Theoretical errors:

- Which order: (NLO, NNLO, …, resummation – BFKL, qT, threshold)
- Which scale: (μ_F, μ_R)
- Which code: (antenna subtraction, sector decomposition,…, qT, N-jettiness,,…)
- Monte Carlo error: (most efficient implementation,…)

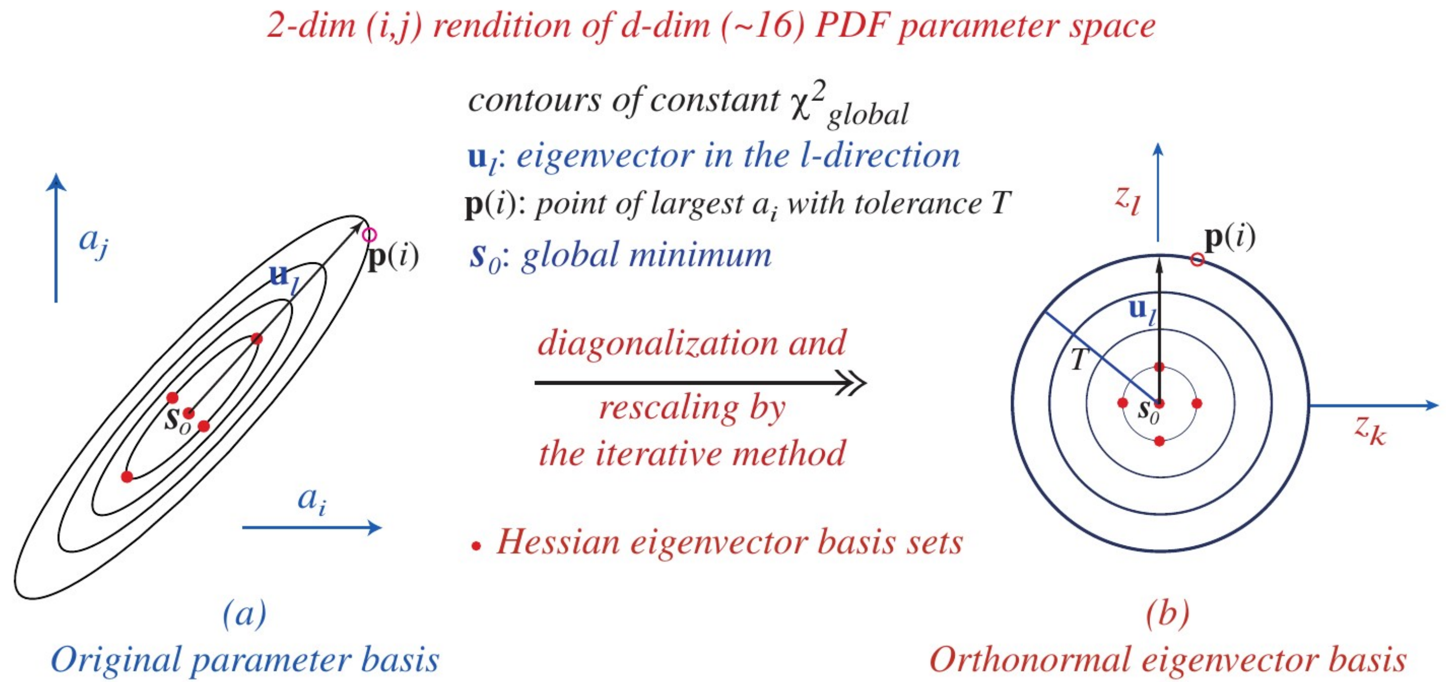
Slide from C.-P. Yuan, 2023

PDFs Uncertainty: Hessian Method

One of the main method to estimate the uncertainty of PDFs is the Hessian method.

$$\chi^2 = \chi_0^2 + \sum_{i,j} H_{ij} y_i y_j , \quad H_{ij} = \frac{1}{2}\left(\frac{\partial^2 \chi^2}{\partial y_i \partial y_j}\right)_0 ,$$

Where $y_i = a_i - a_i^0$ with a_i^0 to be the parameters at minimal χ_0^2.



PDFs Uncertainty: Hessian Method

Let $X = X(\{a_i\})$ to be the observable as a function of fitting parameter.
Using the linear approximation of parameter $\{z_i\}$, the symmetry uncertainty of X is,

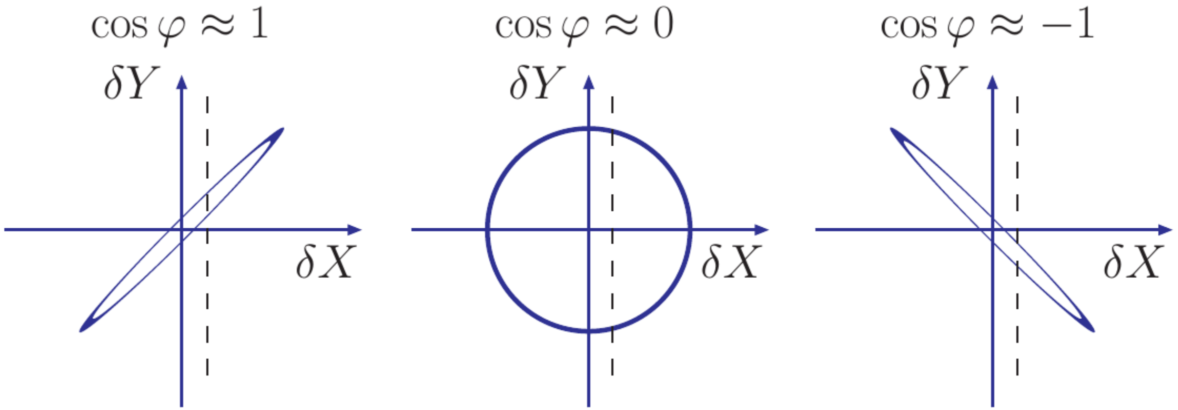
$$\Delta X = \frac{1}{2}\left(\sum_{i=1}^{N_p}\left[X(\{z_i^+\}) - X(\{z_i^-\})\right]^2\right)^{1/2},$$

Where $\{z_1^\pm\} = \{\pm T, 0\ldots.\}$, $\{z_2^\pm\} = \{0, \pm T, 0\ldots.\}$ and so on.
The asymmetry uncertainty of X is,

$$\delta^+ X = \sqrt{\sum_{i=1}^{N_a}\left[\max\left(X_i^{(+)} - X_0, X_i^{(-)} - X_0, 0\right)\right]^2},$$

$$\delta^- X = \sqrt{\sum_{i=1}^{N_a}\left[\max\left(X_0 - X_i^{(+)}, X_0 - X_i^{(-)}, 0\right)\right]^2},$$

Application of Hessian method : Correlation



In the framework of the Hessian, the correlation between two variables X and Y can be worked out as.

$$\cos\varphi = \frac{\vec{\nabla}X \cdot \vec{\nabla}Y}{\Delta X \Delta Y} = \frac{1}{4\Delta X\,\Delta Y}\sum_{\alpha=1}^{N}\left(X_\alpha^{(+)} - X_\alpha^{(-)}\right)\left(Y_\alpha^{(+)} - Y_\alpha^{(-)}\right)$$

where the ΔX and ΔY are their symmetric uncertainties. By this correlation angle φ, the tolerance ellipse is defined by

$$X = X_0 + \Delta X\cos\theta,\; Y = Y_0 + \Delta Y\cos(\theta+\varphi),$$

Application of Hessian method : Sensitivity

- The correlation cosine between PDF $f(x,\mu)$ and theoretical prediction T_i contains no information of the experimental uncertainty.
- The correlation cosine $C_f(x_i,\mu_i)$ between PDF $f(x,\mu)$ and residual r_i contains no information of the experimental uncertainty in practice

$$C_f(x_i,\mu_i) = \frac{\vec{\nabla} f(x_i,\mu_i) \cdot \vec{\nabla} r_i}{\Delta f(x_i,\mu_i)\Delta r_i}, \quad \text{where}$$

$$\chi^2 = \sum_i^N r_i^2 + \sum_k \lambda_k^2, \quad r_i(\vec{a}) = \frac{D_i - \sum_k \lambda_k \beta_{ki} - T_i(\{a\})}{\sigma_i}$$

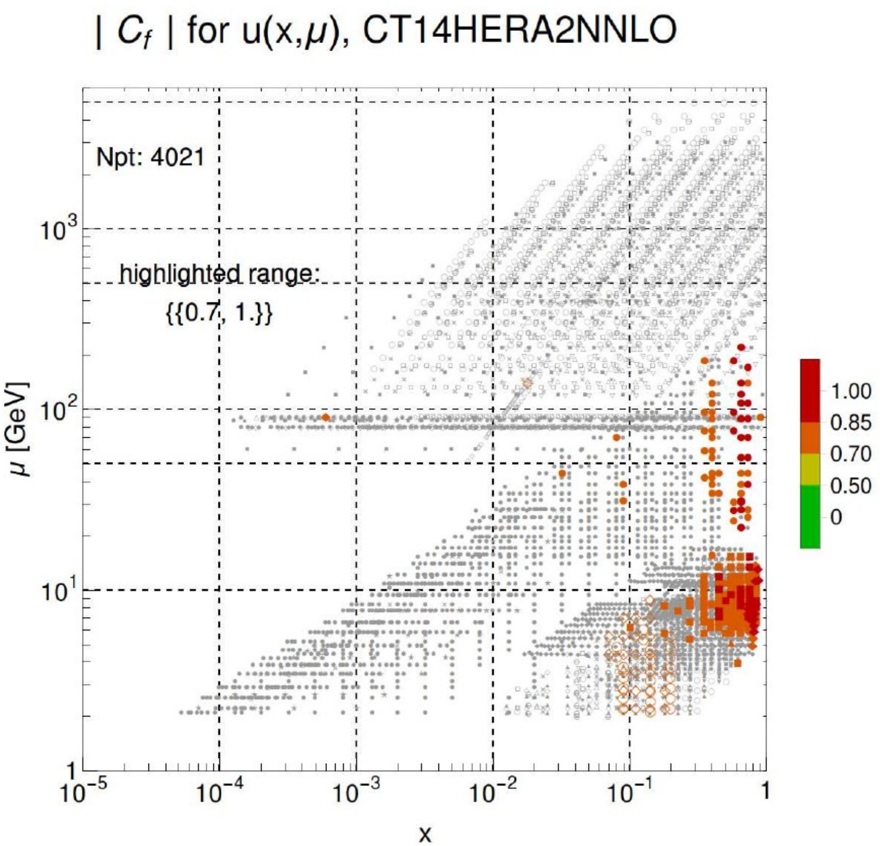
- Instead, we concern the "sensitivity" $S_f(x_i,\mu_i)$

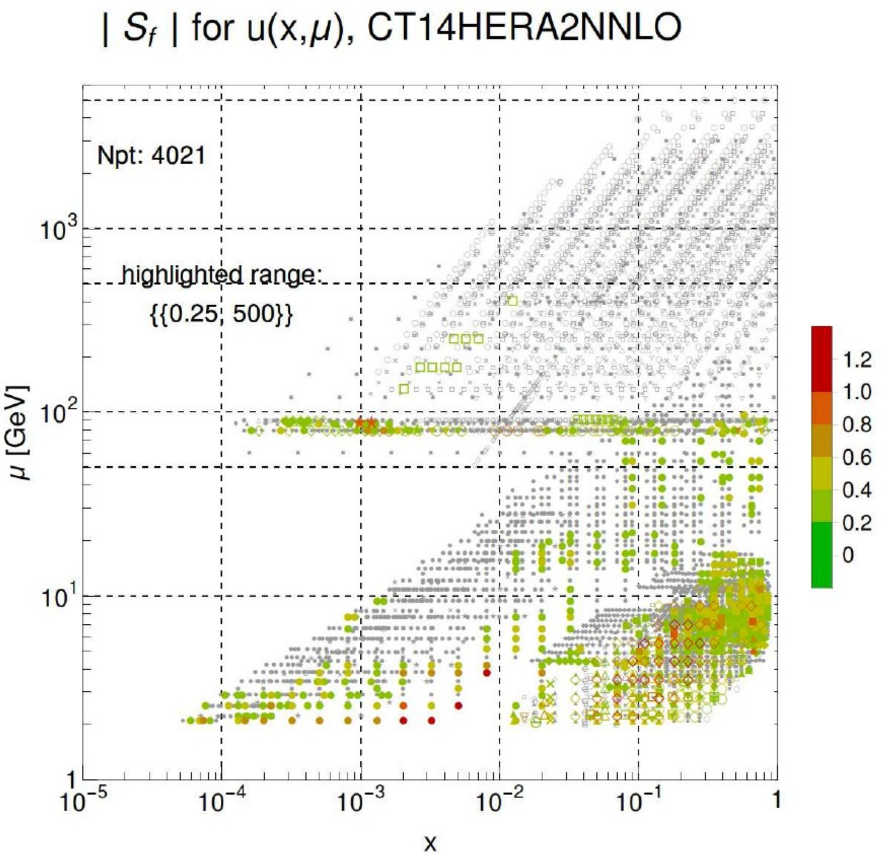
$$S_f(x_i,\mu_i) = C_f(x_i,\mu_i) \frac{\Delta r_i}{\sqrt{\frac{\sum_i^N r_i^2}{N}}}$$

Application of Hessian method : Sensitivity

The sensitivity $S_f(x_i, \mu_i)$ help us to visualize the potential impact on PDF in x − Q plane.

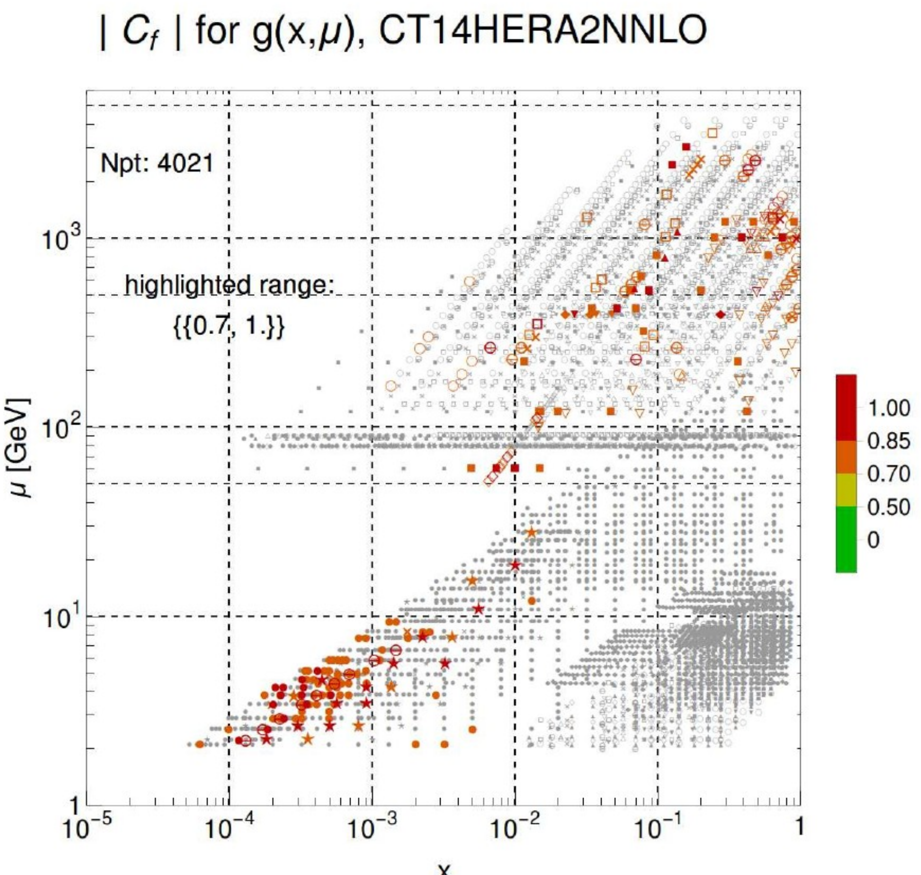
$$u(x,\mu)$$

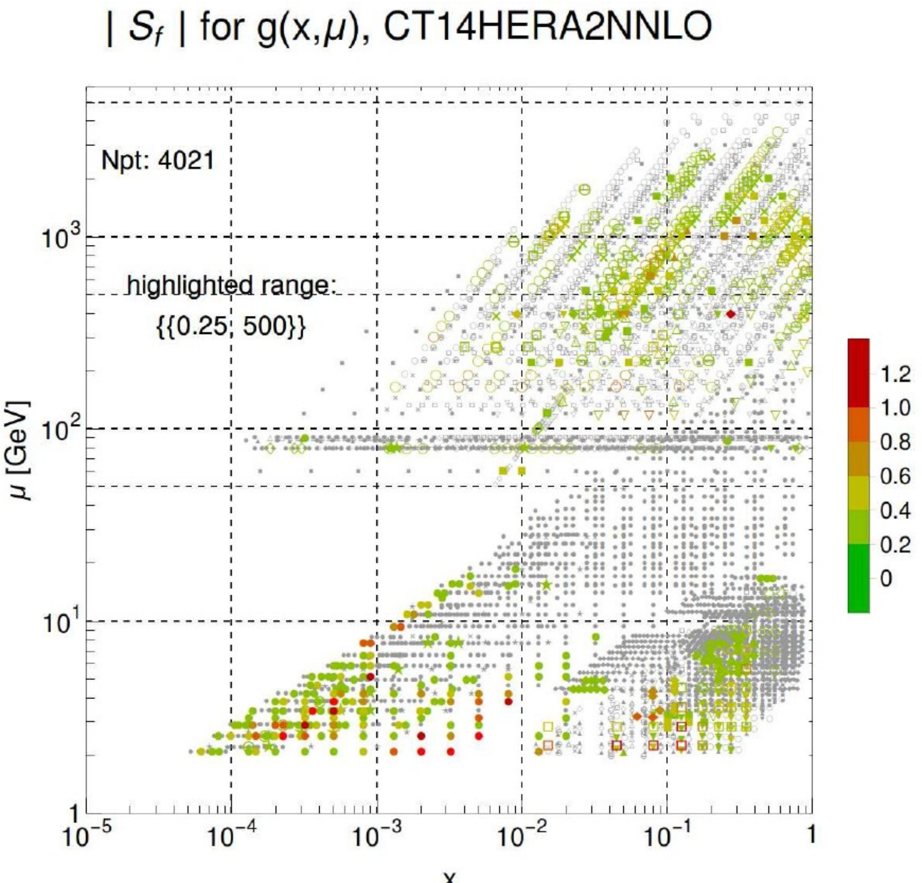




Application of Hessian method : Sensitivity

$$g(x,\mu)$$



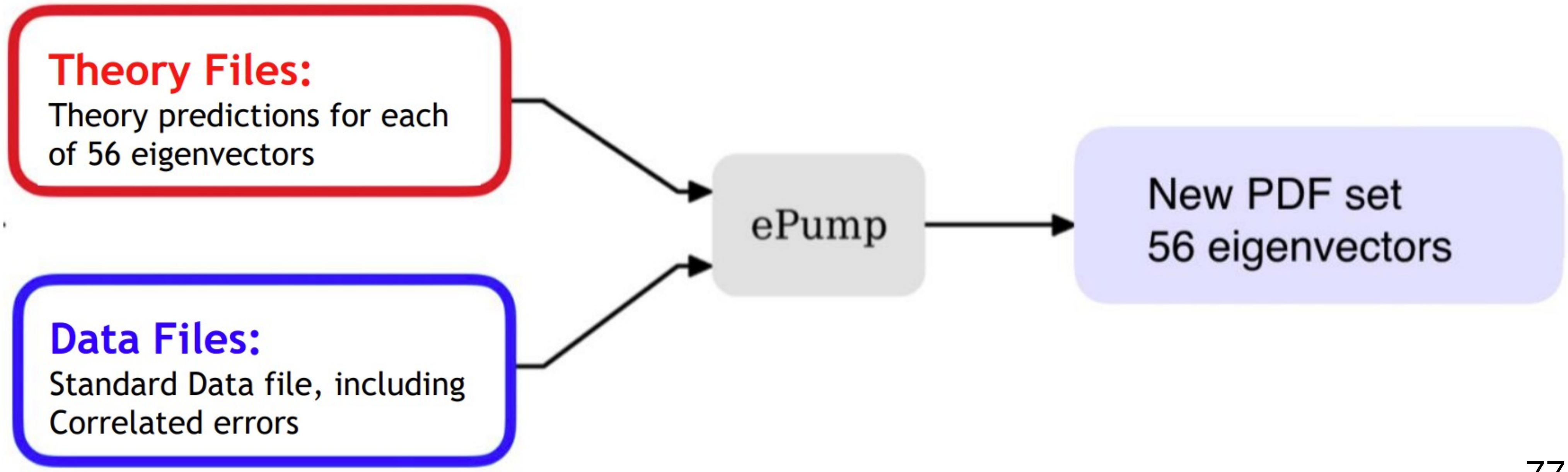


Application of Hessian method : Hessian Updating

ePump
error PDF Update Method Package

https://epump.hepforge.org/

Update PDF by yourself!



Application of Hessian method : Hessian Updating

- New best-fit PDF : $f^0_{\text{new}} = f^0 + \boldsymbol{\Delta f} \cdot \mathbf{z}$
- New error PDFs : $f^{\pm(r)} = f^0_{\text{new}} \pm \frac{1}{\sqrt{1+\lambda^{(r)}}} \boldsymbol{\Delta f} \cdot \mathbf{U}^{(r)}$

  where $/^{(r)}$ and $\mathbf{U}^{(r)}$ are the eigenvalues and eigenvectors of matrix $\mathbf{M}$

- Extensions :
  - Best choices for $\boldsymbol{\Delta f}$ within the linear approximation
  - Dynamical tolerances : $\pm\mathbf{e}^i \rhd \pm\left(T^{\pm i}/T\right)\mathbf{e}^i$
  - Inclusion of diagonal quadratic terms in expansion of $X_a\left(\mathbf{z}\right)$
  - Direct update of other observables :

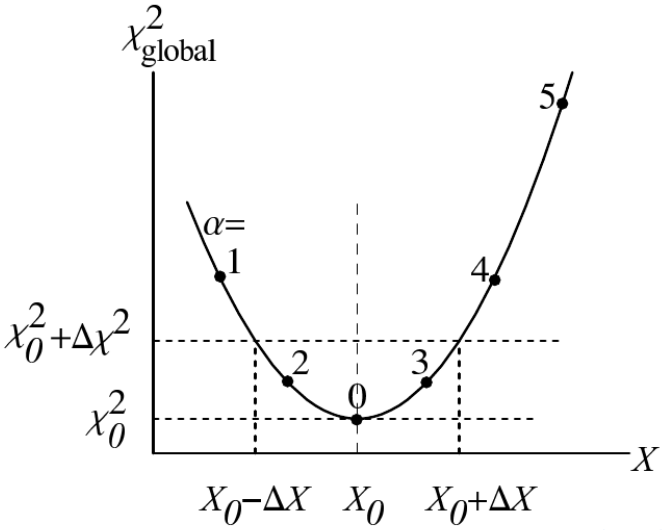
$$Y^0_{\text{new}} = Y^0 + \boldsymbol{\Delta Y} \cdot \mathbf{z} \quad , \quad \left|\Delta Y\right| = \boldsymbol{\Delta Y} \cdot \left(\mathbf{1} + \mathbf{M}\right)^{-1} \cdot \boldsymbol{\Delta Y}$$

Lagrange Multiplier Method

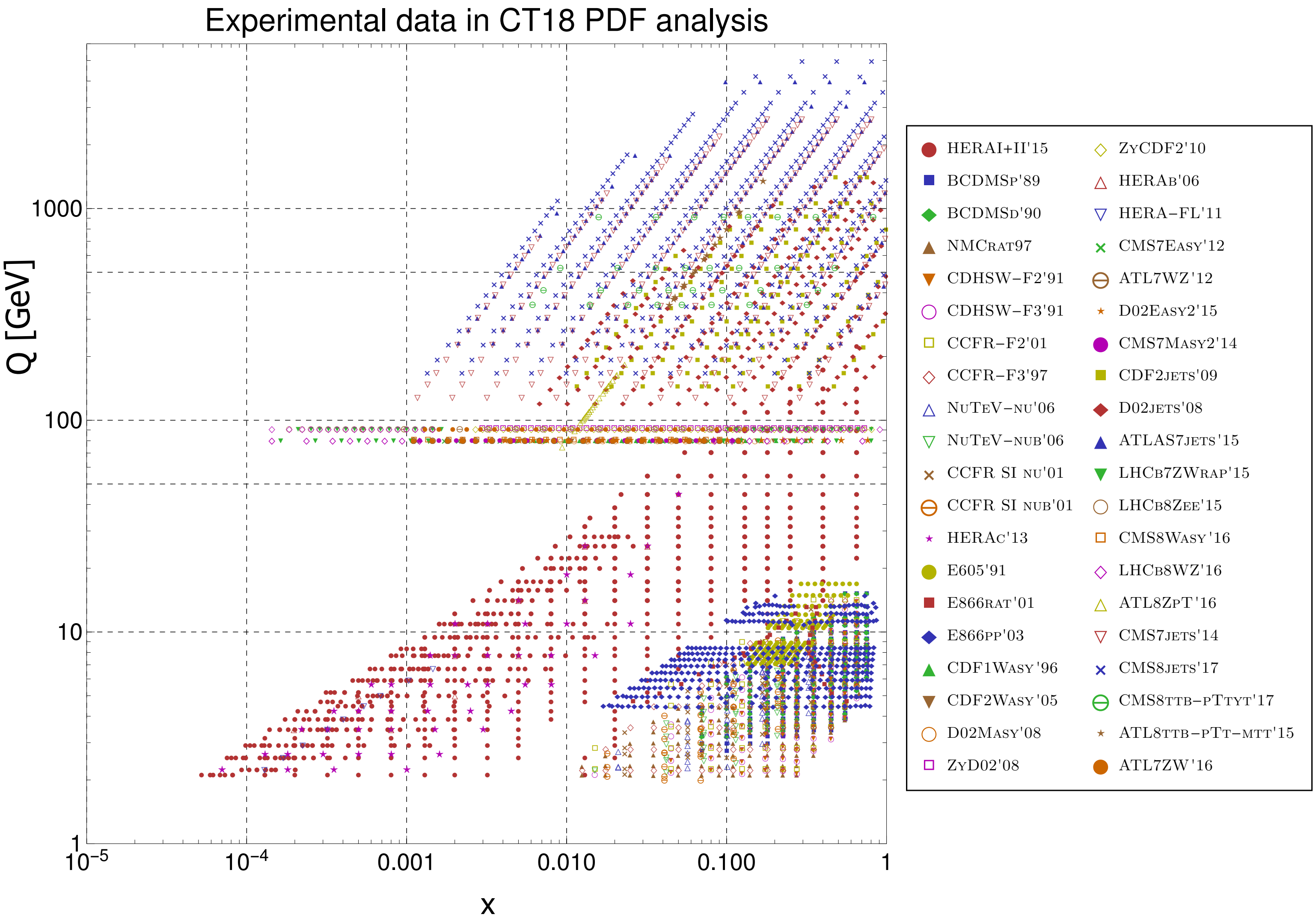
Conaider a particular physical quantity, say $X(\{a_i\})$, which is a function of PDFs.

$$F(\lambda, \{a_i\}) = \chi^2(\{a_i\}) + \lambda(X(\{a_i\}) - X(\{a_i^{(0)}\}))$$

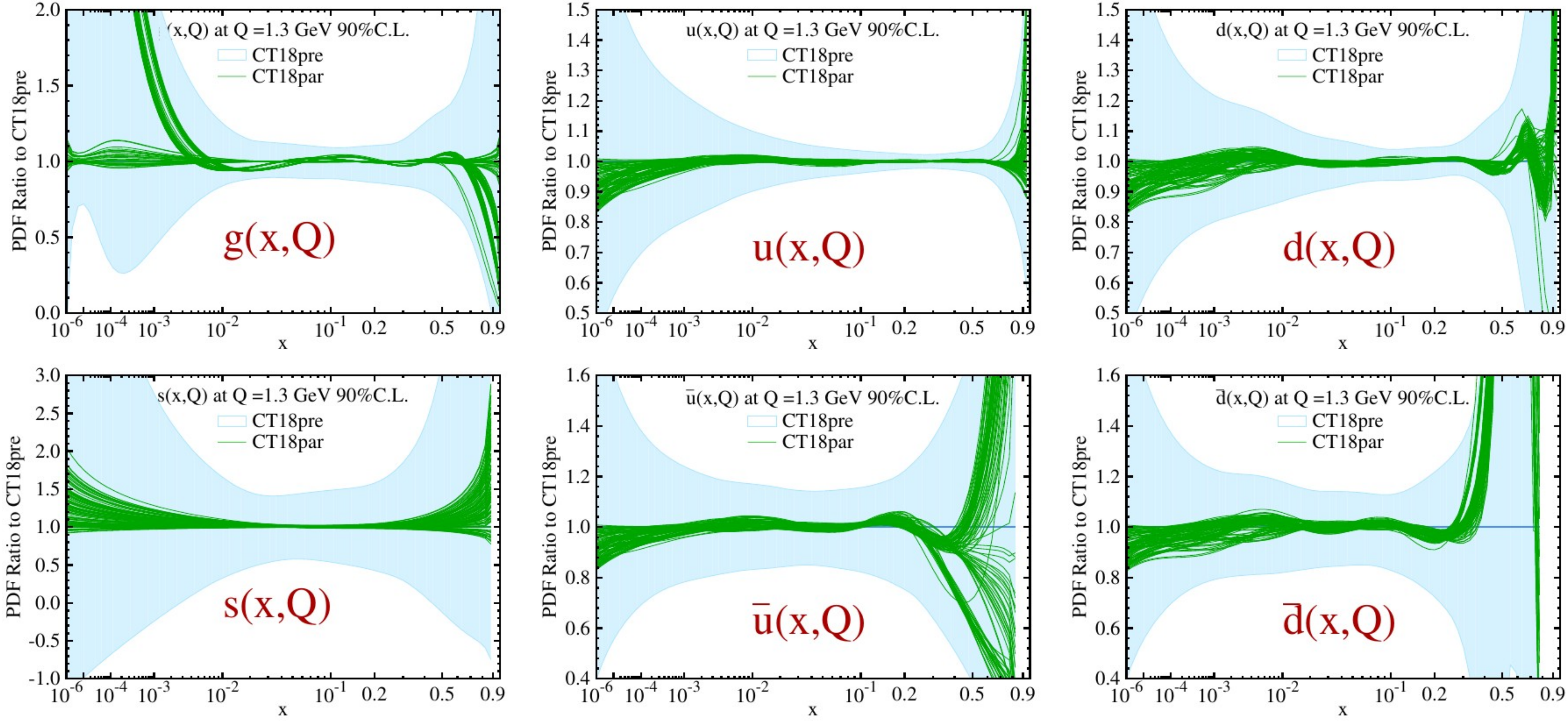
By minimizing this function with various fixed λ value, say $\lambda_1, \ldots\lambda_j\ldots, \lambda_n$, we will obtain n parameter sets $\{a_i(\lambda_j)\}$ and corresponding $X(\{a_i(\lambda_j)\})$ and $\chi^2(\{a_i(\lambda_j)\})$. With suitable choice of $\Delta\chi^2$, we obtain the uncertainty of the physical quantity $X(\{a_i\})$.



Taking CT18 as Example

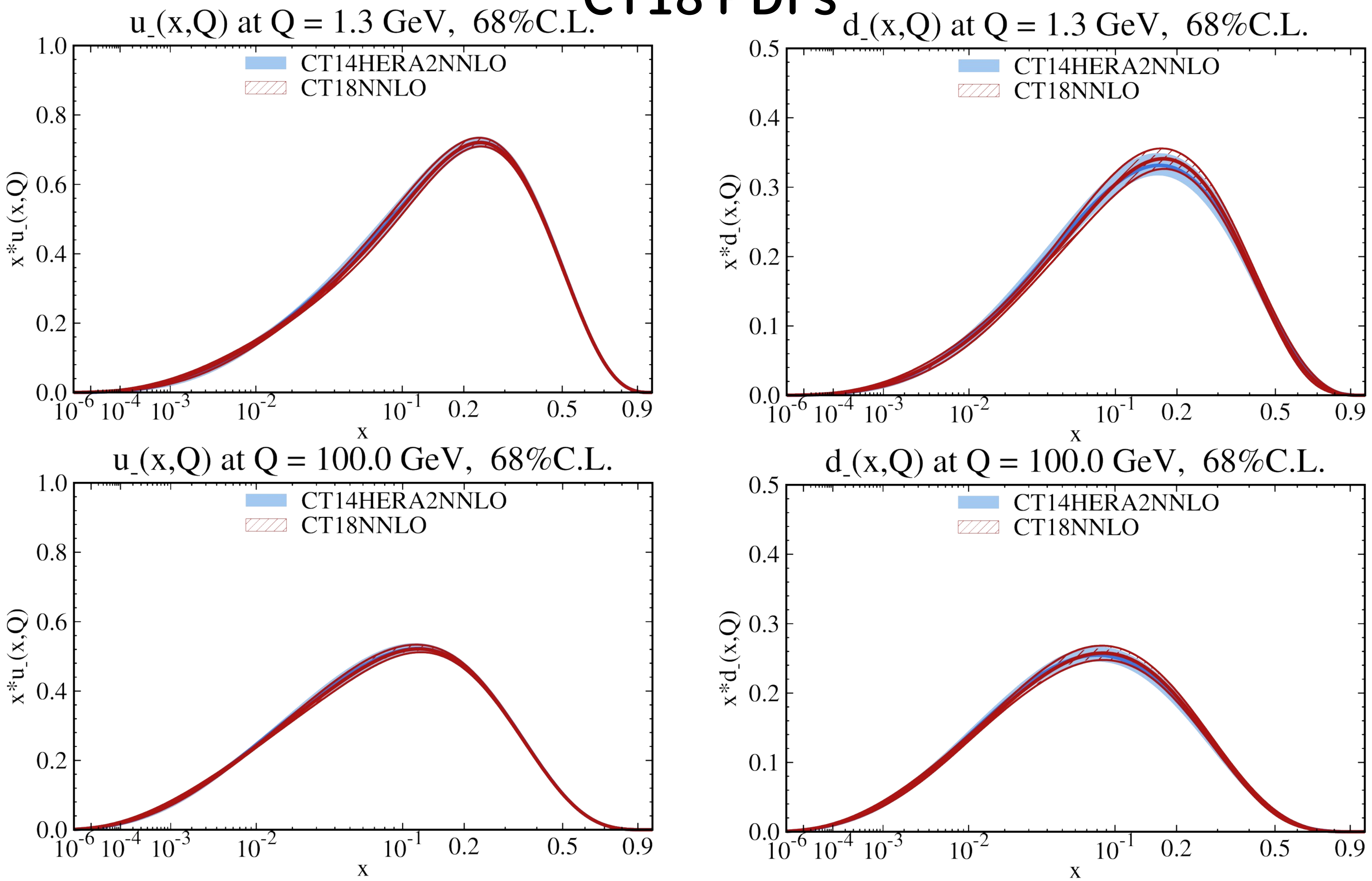


CT18 parametrization

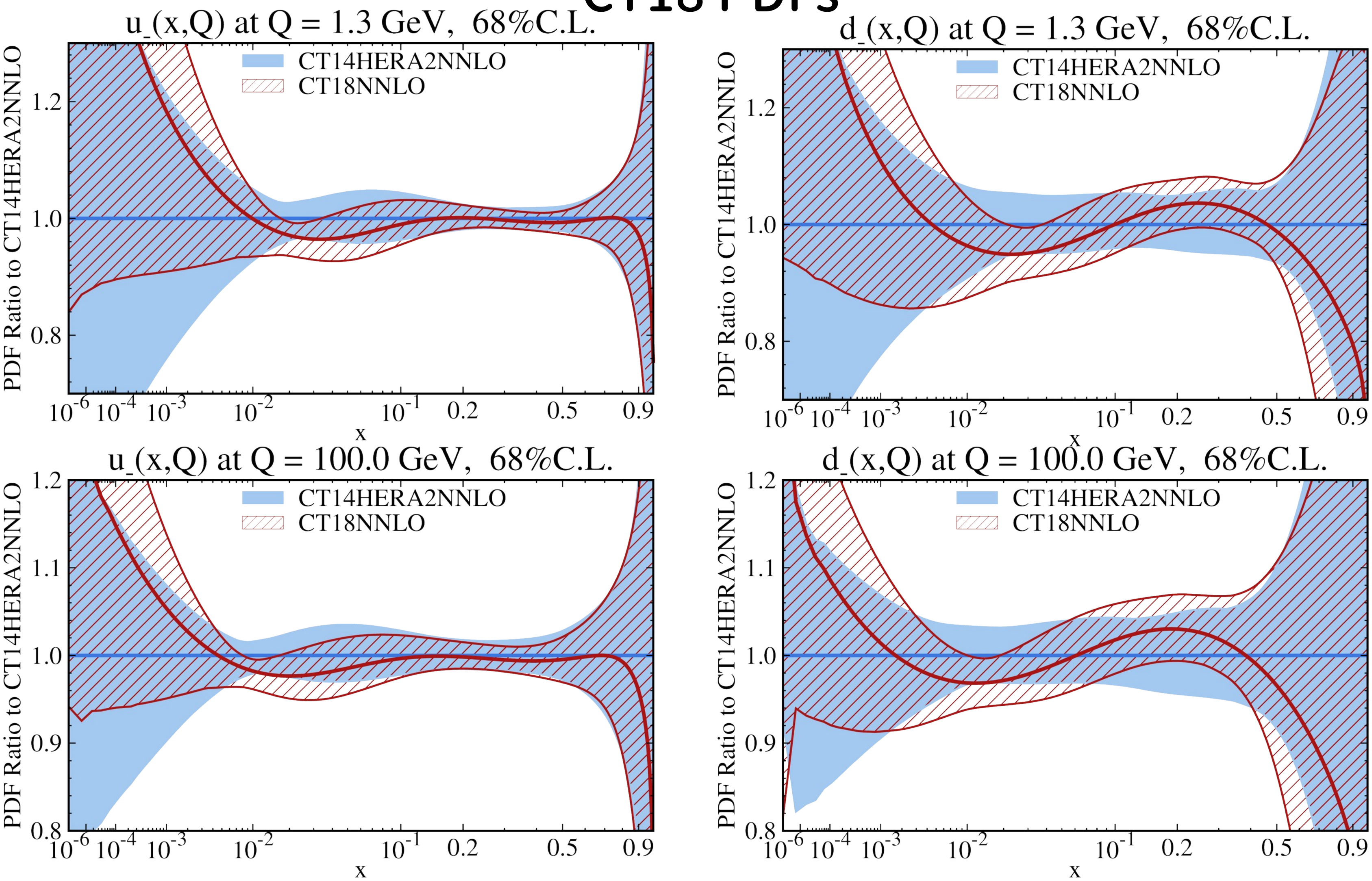


- CT18 – sample result of exploring various non-perturbative parametrization forms.
- There is no data to constrain very large or very small x region.

CT18 PDFs

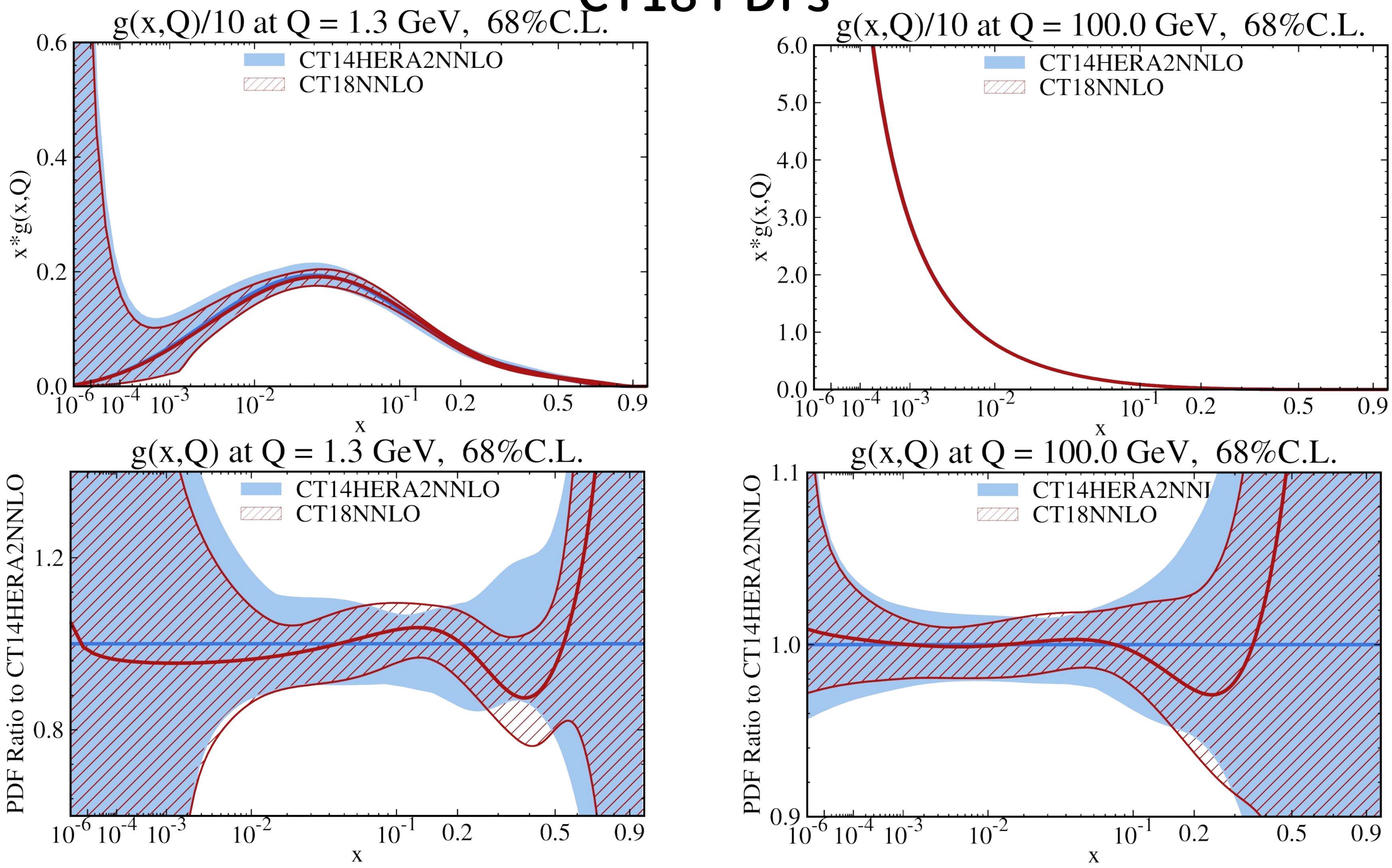

u_(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*u_(x,Q)
x
d_(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*d_(x,Q)
x
u_(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*u_(x,Q)
x
d_(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*d_(x,Q)
x

CT18 PDFs

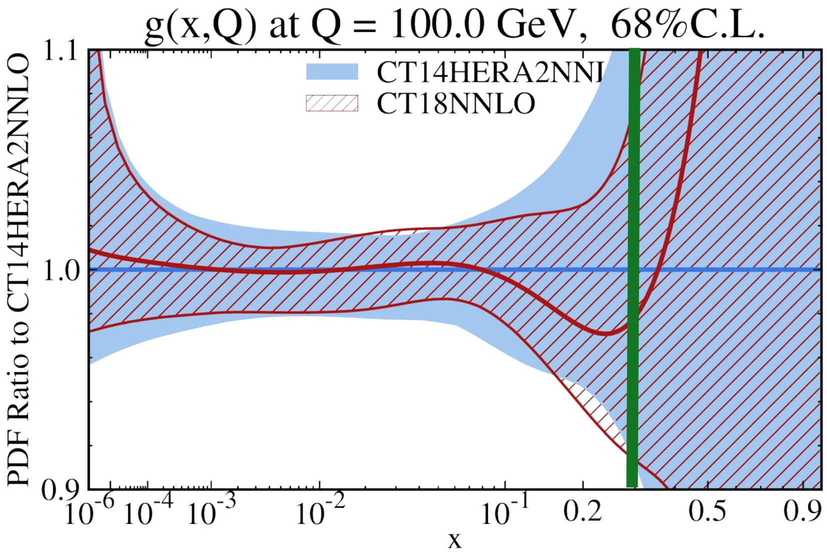


u_(x,Q) at Q = 1.3 GeV, 68%C.L.
d_(x,Q) at Q = 1.3 GeV, 68%C.L.
u_(x,Q) at Q = 100.0 GeV, 68%C.L.
d_(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x

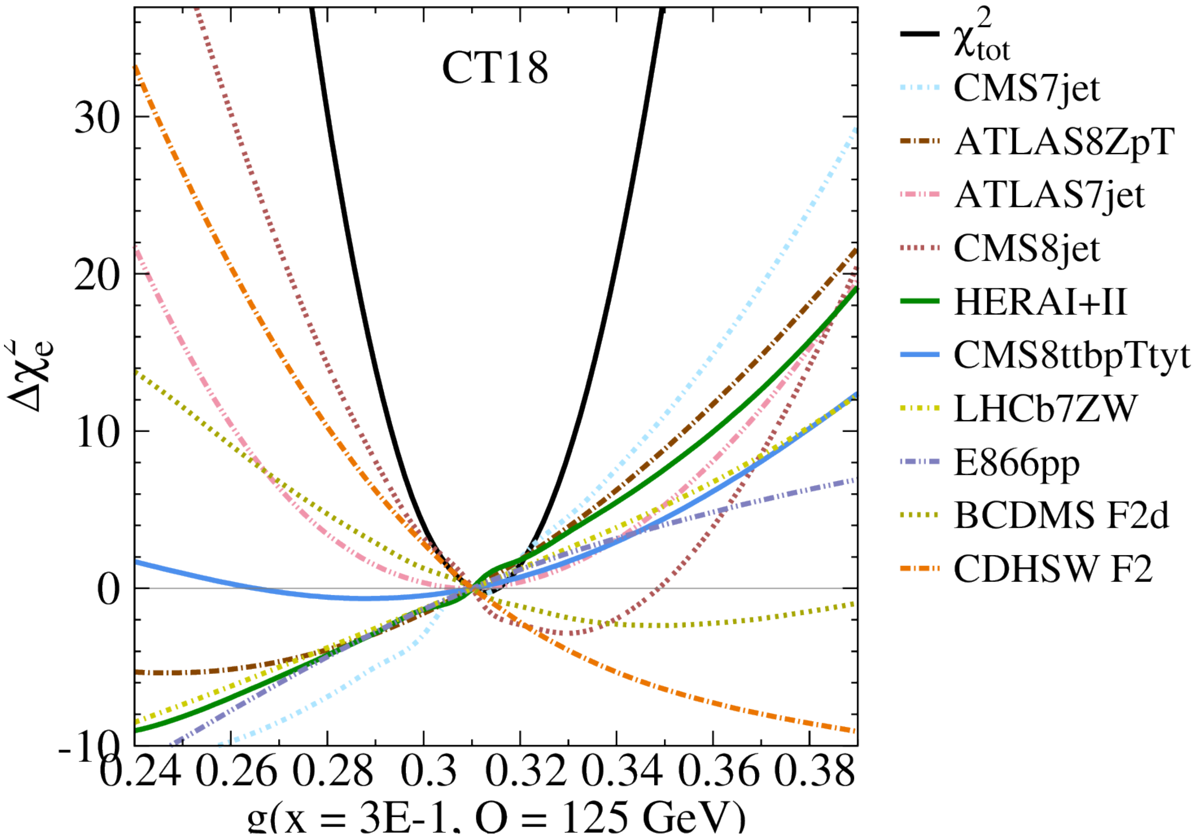
CT18 PDFs


g(x,Q)/10 at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*g(x,Q)
x
g(x,Q)/10 at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
x*g(x,Q)
x
g(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x
g(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x

CT18 PDFs

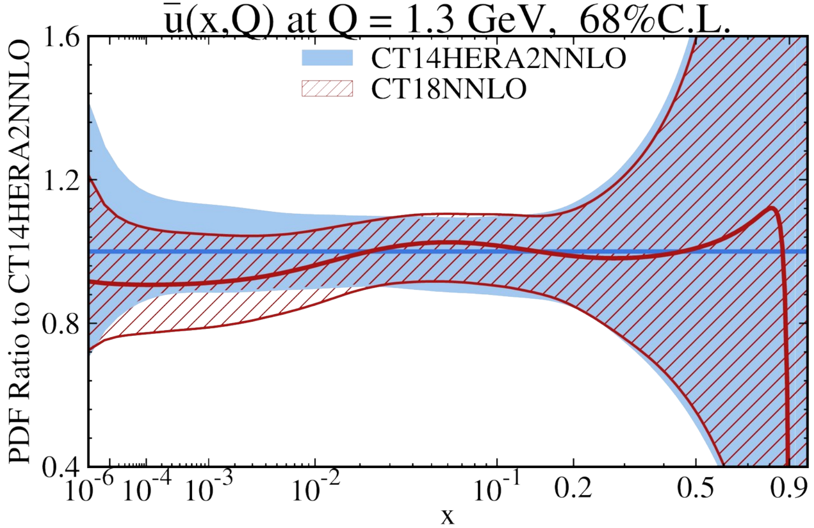


g(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
1.1
1.0
0.9
10^-6 10^-4 10^-3 10^-2 10^-1 0.2 0.5 0.9
x

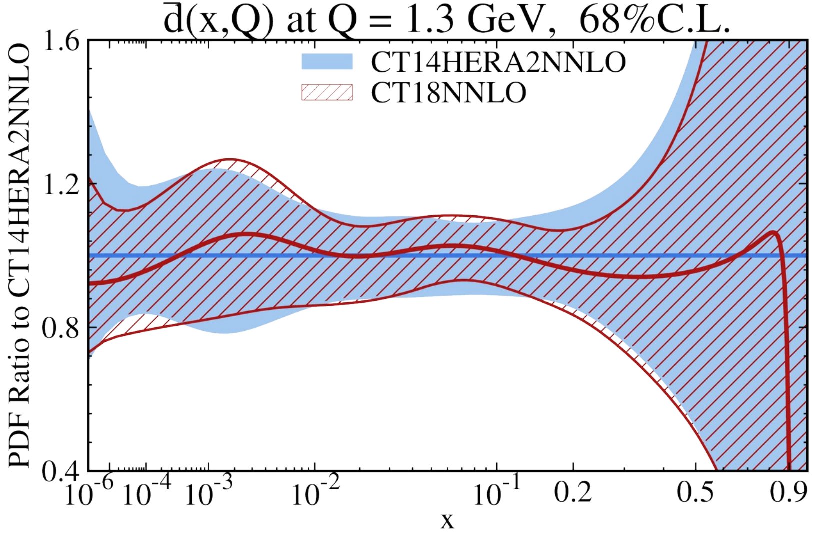


CT18
Δχ²
30
20
10
0
-10
0.24 0.26 0.28 0.3 0.32 0.34 0.36 0.38
g(x = 3E-1, Q = 125 GeV)
χ²tot
CMS7jet
ATLAS8ZpT
ATLAS7jet
CMS8jet
HERAI+II
CMS8ttbpTtyt
LHCb7ZW
E866pp
BCDMS F2d
CDHSW F2

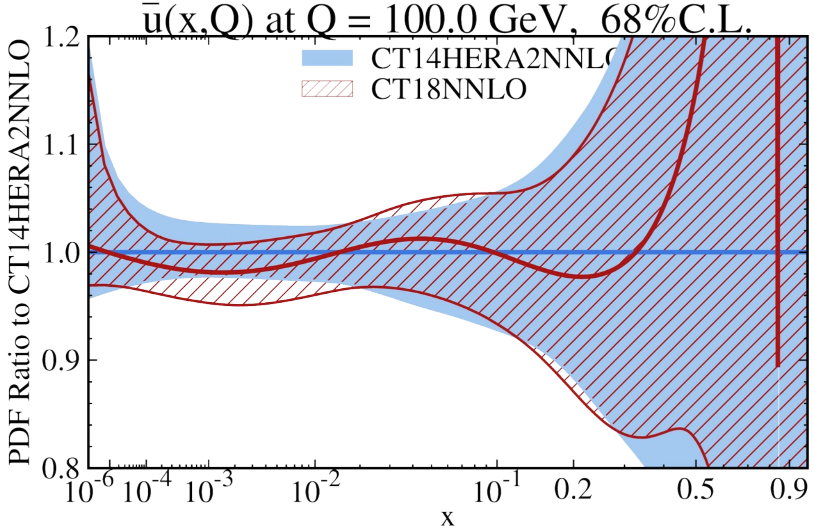
CT18 PDFs



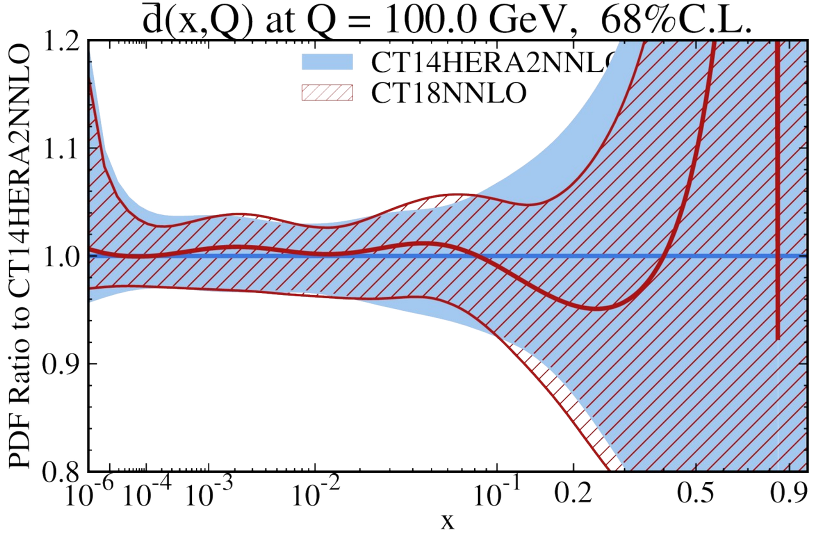
ū(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x



d̄(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x

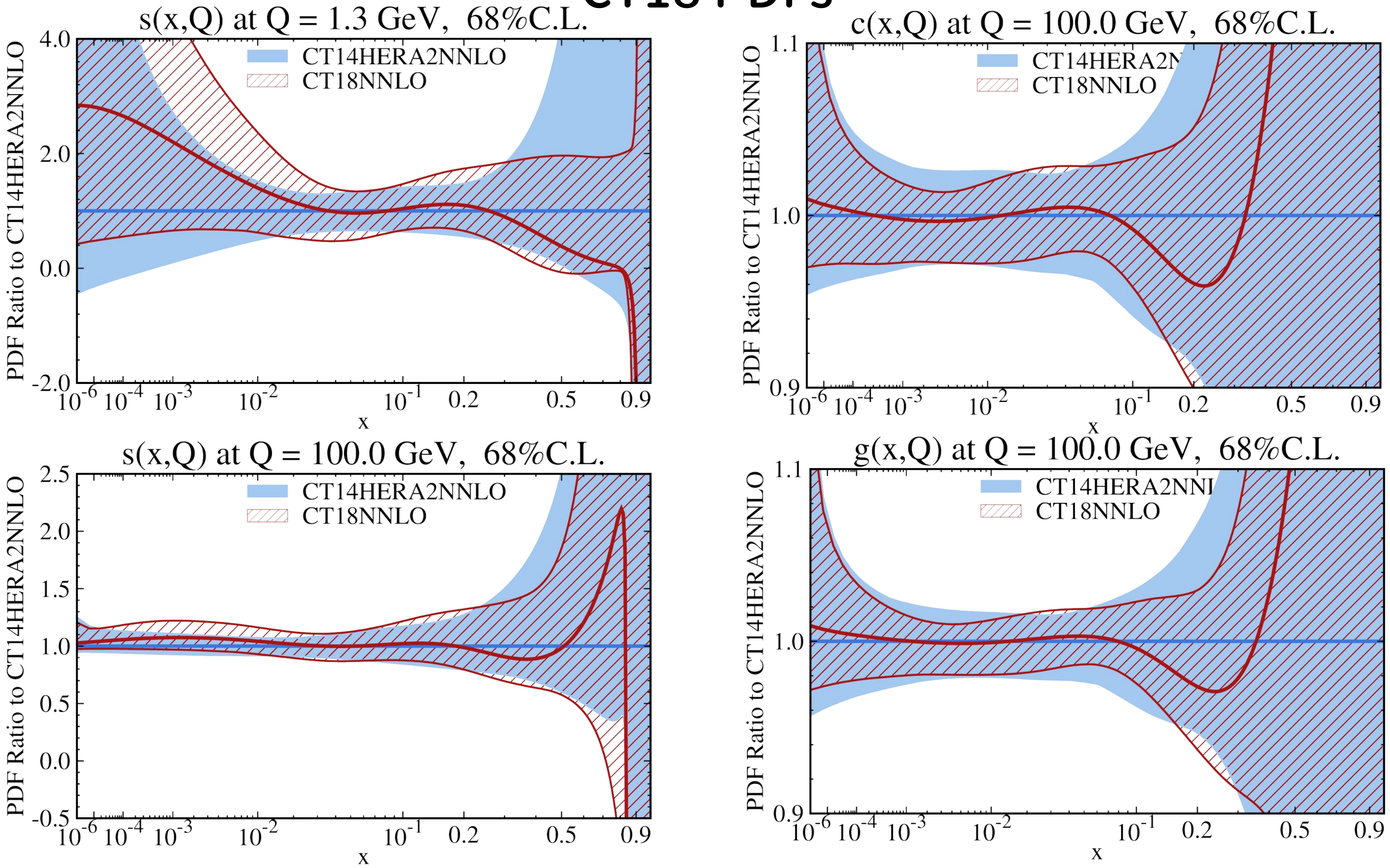


ū(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x



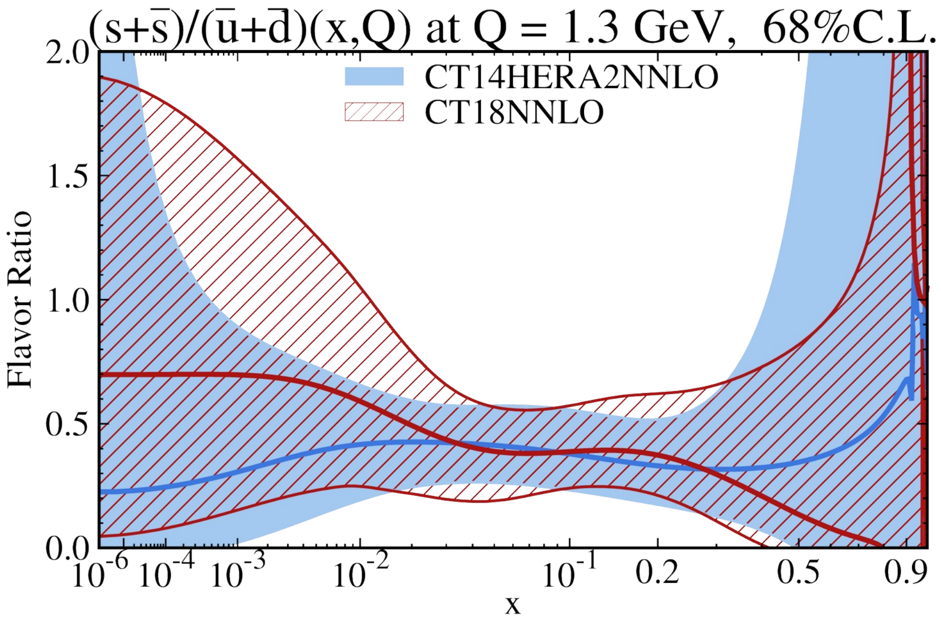
d̄(x,Q) at Q = 100.0 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x

CT18 PDFs

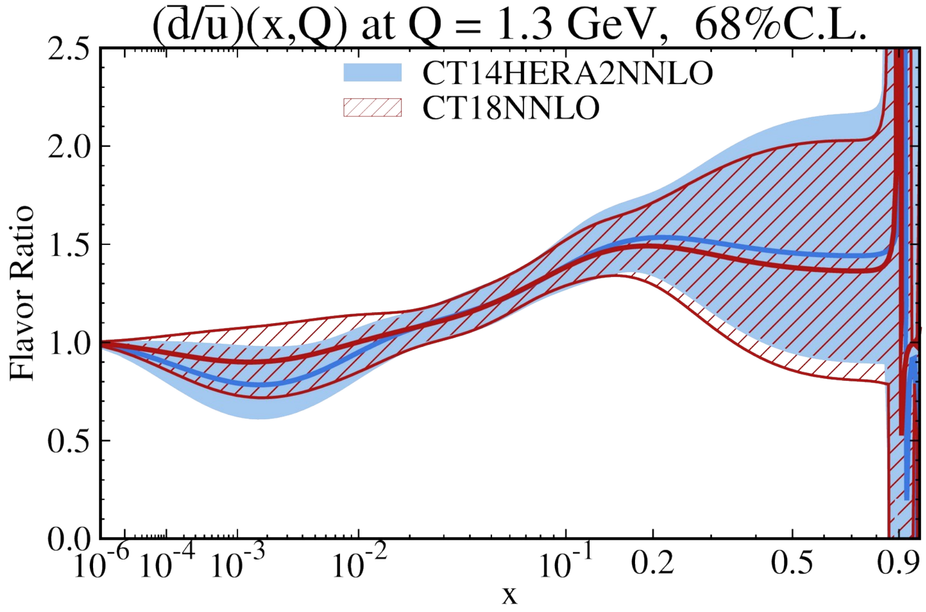


s(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
PDF Ratio to CT14HERA2NNLO
x
c(x,Q) at Q = 100.0 GeV, 68%C.L.
s(x,Q) at Q = 100.0 GeV, 68%C.L.
g(x,Q) at Q = 100.0 GeV, 68%C.L.

CT18 PDFs

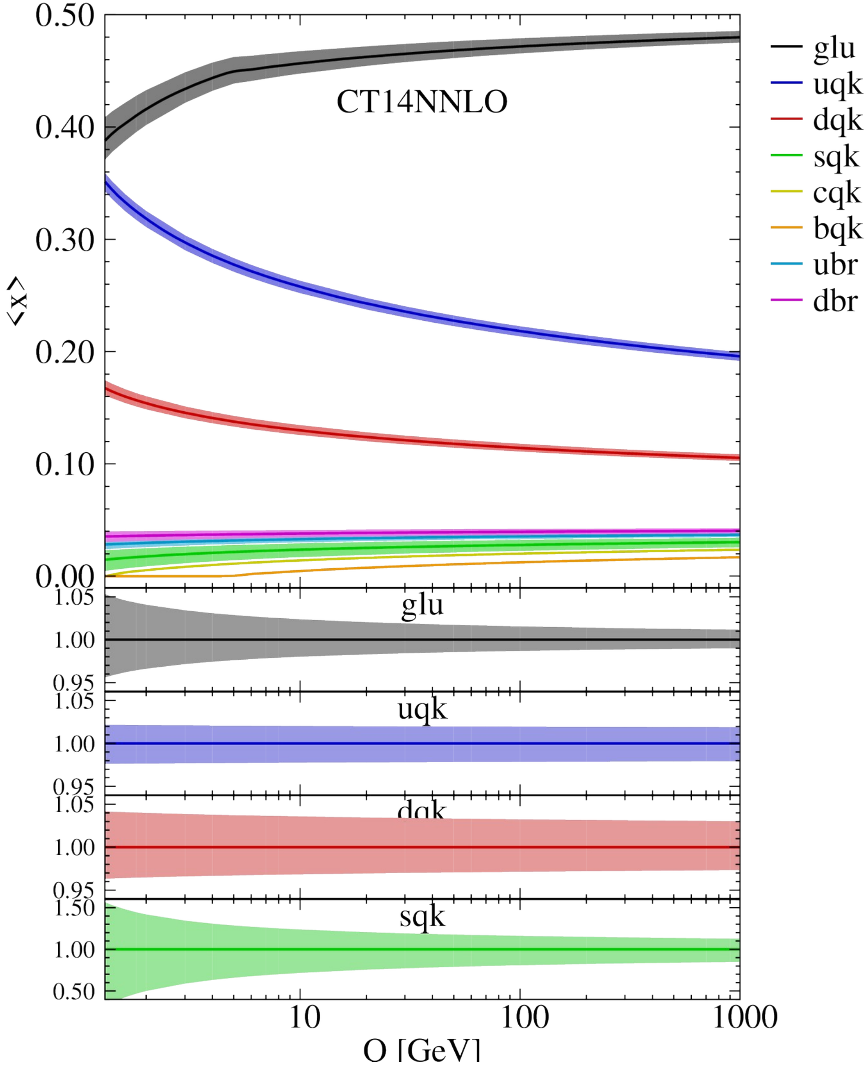


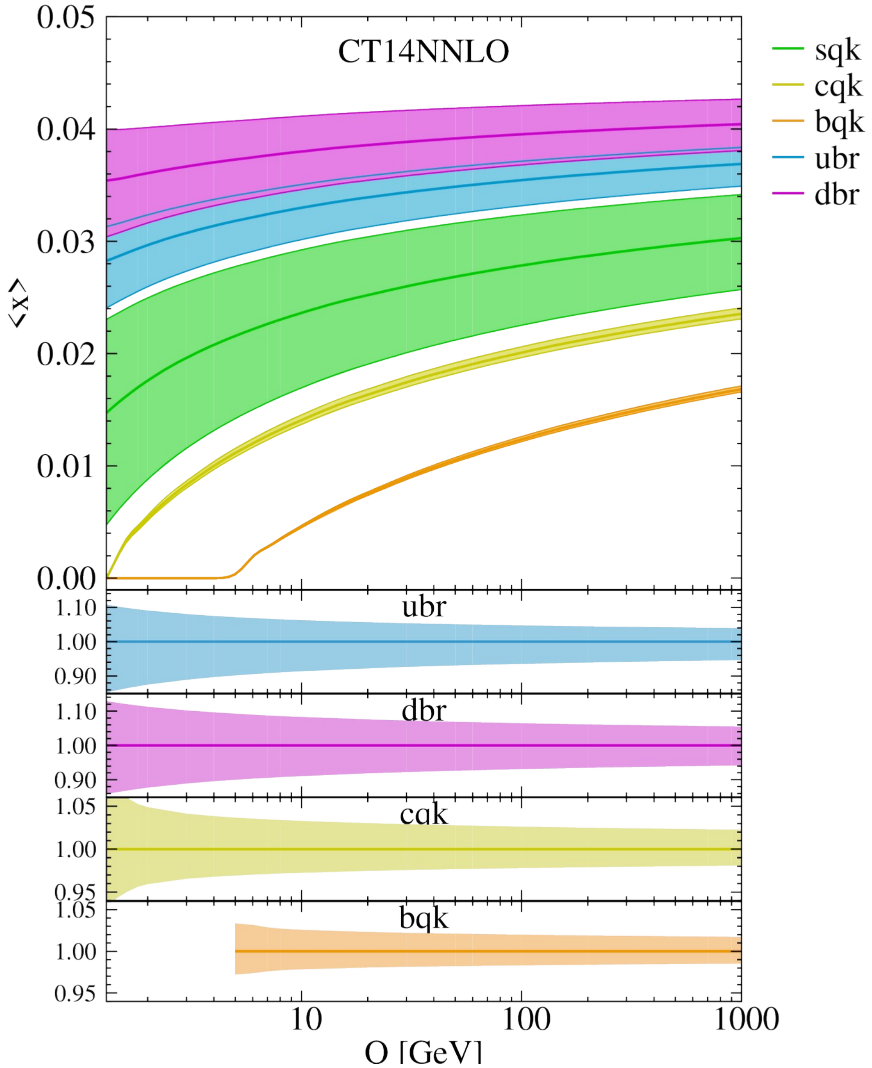
(s+s̄)/(ū+d̄)(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
Flavor Ratio
x



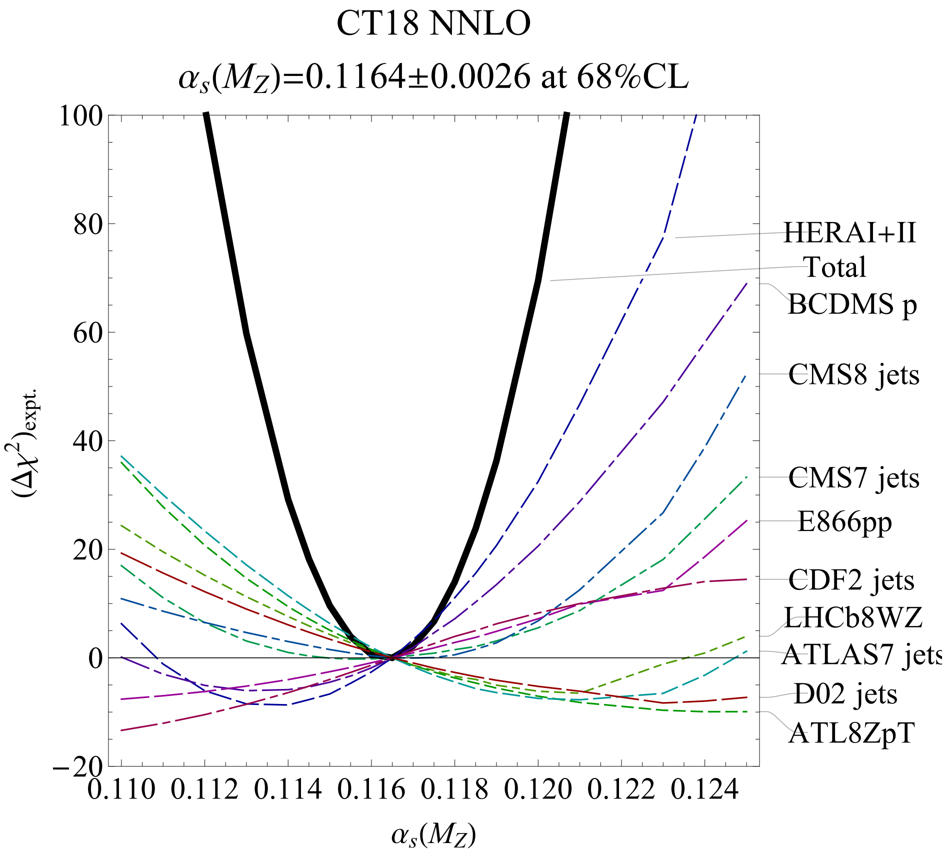
(d̄/ū)(x,Q) at Q = 1.3 GeV, 68%C.L.
CT14HERA2NNLO
CT18NNLO
Flavor Ratio
x

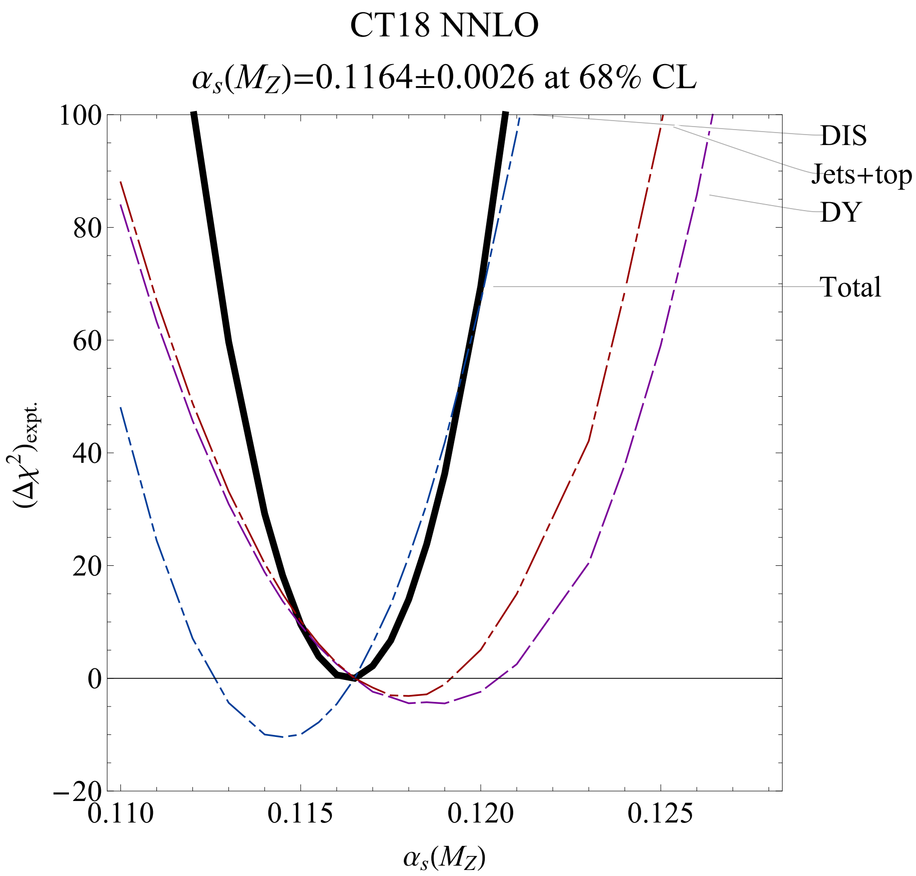
Momentum Fraction evolution with Q





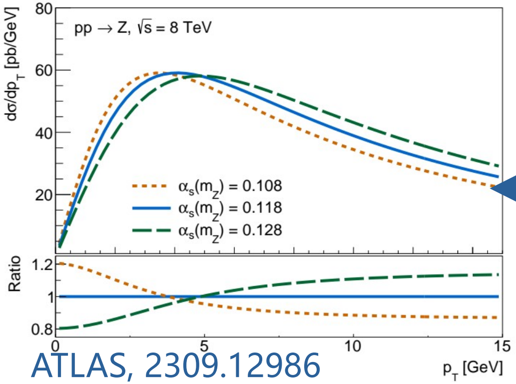
α_s Extraction from LM Method



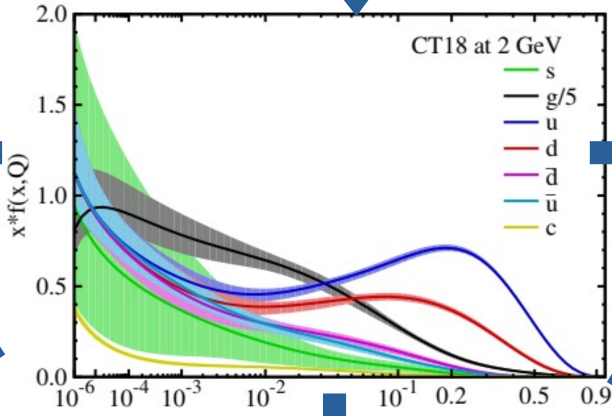


$$\sigma = f_a(x_1,\mu^2)\otimes f_b(x_2,\mu^2)\otimes \hat{\sigma}_{ab}(\mu^2)$$

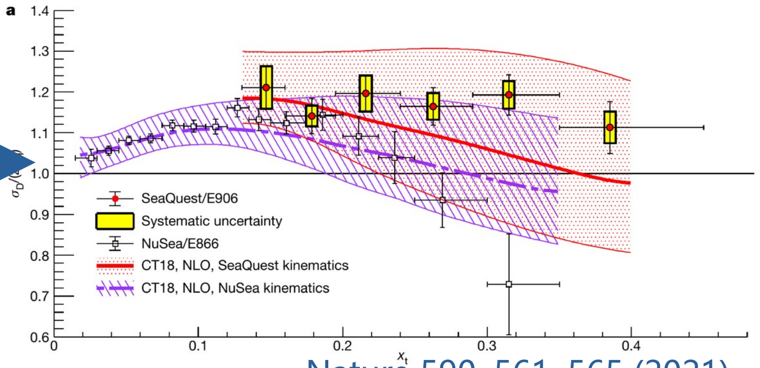
α_s



ATLAS, 2309.12986

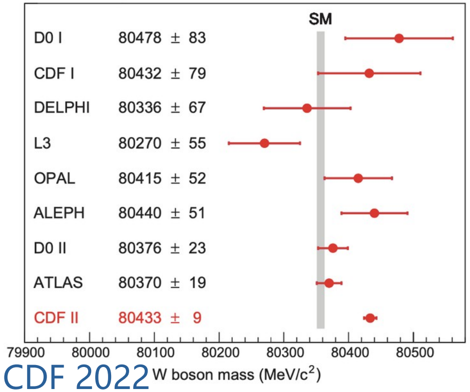


$\sigma_D/(2\sigma_H)$



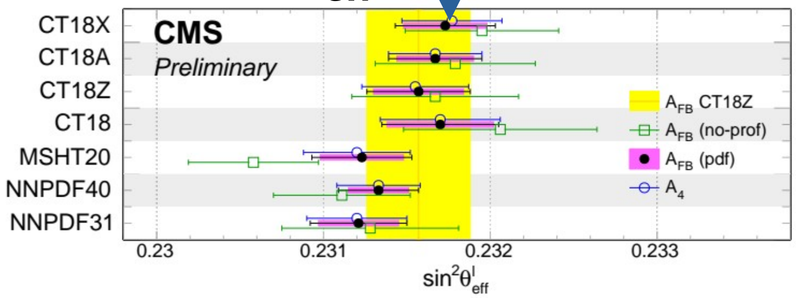
Nature 590, 561–565 (2021)

W Boson mass



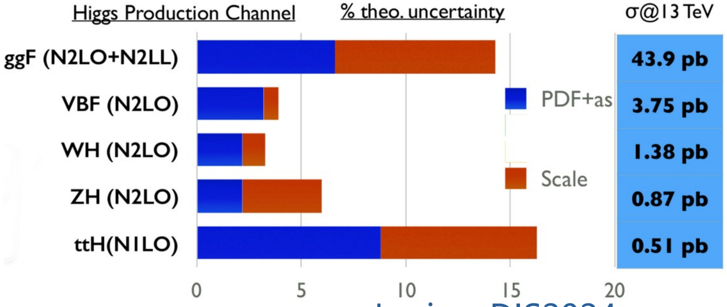
CDF 2022

$\sin^2\theta^l_{eff}$



CMS PAS SMP-22-010

Higgs Production



Lucian, DIS2024

Summary

- The parton distribution functions (PDFs) play an essential role in hadron collisions — they are both a description of the proton's internal structure and a necessary input for predicting cross sections in QCD processes.
- One of the primary methods for determining PDFs is global analysis, which uses regression techniques to fit a wide range of infrared-safe experimental data.
- The uncertainty of the PDFs is quantified using statistical frameworks, among which the Hessian method remains widely used due to its efficiency and interpretability.