Introduction to MadGraph5_aMC@NLO

Physics and Techniques

A Numerical Journey

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QFT Summer School
Shandong University, Jinan
16-17 July 2025

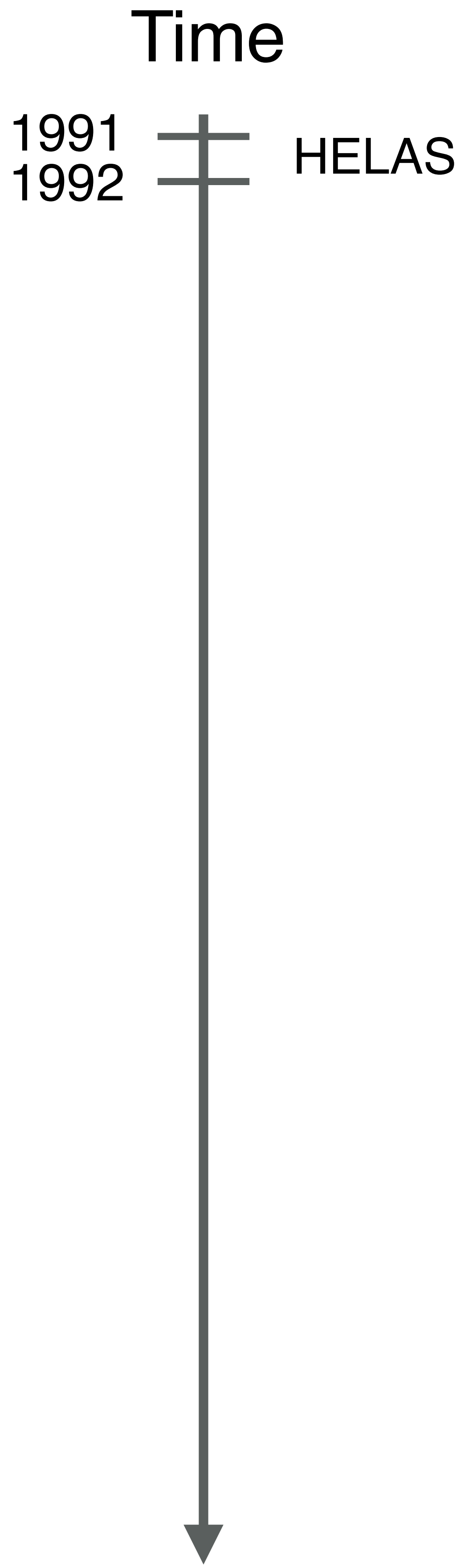
References

- **It is impossible to cover everything in 3 hours.**
- **Many materials can be found online:**
  - Wiki: https://cp3.irmp.ucl.ac.be/projects/madgraph/
  - MadGraph schools
    - Hefei 2018: https://indico.ihep.ac.cn/event/7822/
    - Chennai 2019: https://indico.cern.ch/event/829653/
  - Launchpad (Q&A) : https://launchpad.net/mg5amcnlo
  - aMC@NLO web page : https://amcatnlo.web.cern.ch/ (references)
  - Or simply google it or ask questions to ChatGPT/DeepSeek etc
- **The best way to understand it is by playing it — like playing games in your spare time.**

A Brief Historical Overview

Time

A Brief Historical Overview



HELAS: HELicity Amplitude Subroutines for Feynman Diagram Evaluations

H. MURAYAMA, I. WATANABE and K. HAGIWARA

A Brief Historical Overview

Time

1991
1992 HELAS

1994
1995 MadGraph

University of Wisconsin - Madison

MAD/PH/813
January 1994

Automatic Generation of Tree Level Helicity Amplitudes

T. Stelzer
Physics Department, University of Durham
Durham DH1 3LE, England

and

W. F. Long
Physics Department, University of Wisconsin-Madison
Madison, WI 53706, USA

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MadEvent: automatic event generation with MadGraph*

Fabio Maltoni and Tim Stelzer

Department of Physics, University of Illinois at Urbana-Champaign
1110 West Green Street, Urbana, IL 61801, USA
E-mail: tstelzer@uiuc.edu, maltoni@uiuc.edu

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2008 MadGraph/MadEvent v4

MadGraph/MadEvent v4: the new web generation

Johan Alwall,[a] Pavel Demin,[b] Simon de Visscher,[b] Rikkert Frederix,[b] Michel Herquet,[b] Fabio Maltoni,[b] Tilman Plehn,[c] David L. Rainwater[d] and Tim Stelzer[e]

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2011
2012 — MadGraph 5

MadGraph 5: going beyond

Johan Alwall,[a] Michel Herquet,[b] Fabio Maltoni,[c] Olivier Mattelaer[c] and Tim Stelzer[d]

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2012 MadGraph 5

Higgs discovery at the LHC

2014
2015 MadGraph5_aMC@NLO

(aka MadGraph5 v2)

The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations

J. Alwall,[a] R. Frederix,[b] S. Frixione,[b] V. Hirschi,[c] F. Maltoni,[d] O. Mattelaer,[d] H.-S. Shao,[e] T. Stelzer,[f] P. Torrielli[g] and M. Zaro[h,i]

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2014
2015 MadGraph5_aMC@NLO

(aka MadGraph5 v2)

2018
2019 MadGraph5_aMC@NLO NLO EW automation

(aka MadGraph5 v3)

The automation of next-to-leading order electroweak calculations

R. Frederix,[a] S. Frixione,[b] V. Hirschi,[c] D. Pagani,[a] H.-S. Shao[d] and M. Zaro[e]

What it can bring to me ?

- **A single framework to provide the following types of computations**

  - **fLO**: fixed-order LO (= tree level and/or loop-induced)
  - **fNLO**: fixed-order NLO
  - **LO+PS**: hard LO events and parton shower (external PSMC)
  - **NLO+PS**: hard NLO events and parton shower (MC@NLO+external PSMC)
  - **LO merged**: merging of LO multijet samples (MLM/CKKW-L)
  - **NLO merged**: merging of NLO multijet samples (FxFx/UNLOPS)

- A fully automated chain from the (B)SM Lagrangian to events is:

LO FeynRules $\xrightarrow{\text{UFO}}$ MG5_aMC $\xrightarrow{\text{LHE}}$ PSMC

NLO FeynRules(+NLOCT) $\xrightarrow{\text{UFO}}$ MG5_aMC $\xrightarrow{\text{LHE}}$ PSMC

What it can bring to me ?

- A framework with many useful tools for phenomenology studies
  - MadWidth: width and branching ratio computations
  - MadWeight: a phase-space generator for matrix-element method
  - MadSpin: spin-entangled decay
  - MadDM: dark-matter observable computations in (in)direct searches
  - MadSTR: simplified treatments of resonances at NLO
  - MadDump: event generation for beam dump experiments
  - e+e- at NLO: (N)LO calculations for e+e- with ISR and beamstrahlung
  - UPC at NLO: (N)LO calculations for gamma-gamma collisions in UPC
  - aMCfast/PineAPPL: fast-interpolation grids of cross sections for PDF fits
  - MadNIS: Neural multiple-channeling importance sampling
  - Reweighting/Systematics/Bias
  - MadAnalysis5: event analysis and reinterpretation of collider searches
- A matrix-element provider
  - e.g. Pythia8 and MatchBox in Herwig7
  - Also your own format with the PLUGIN mode

Plan

- **Lecture 1:**
  - **LO event simulations**
- **Lecture 2:**
  - **NLO calculations and event simulations at NLO QCD**

Lecture 1

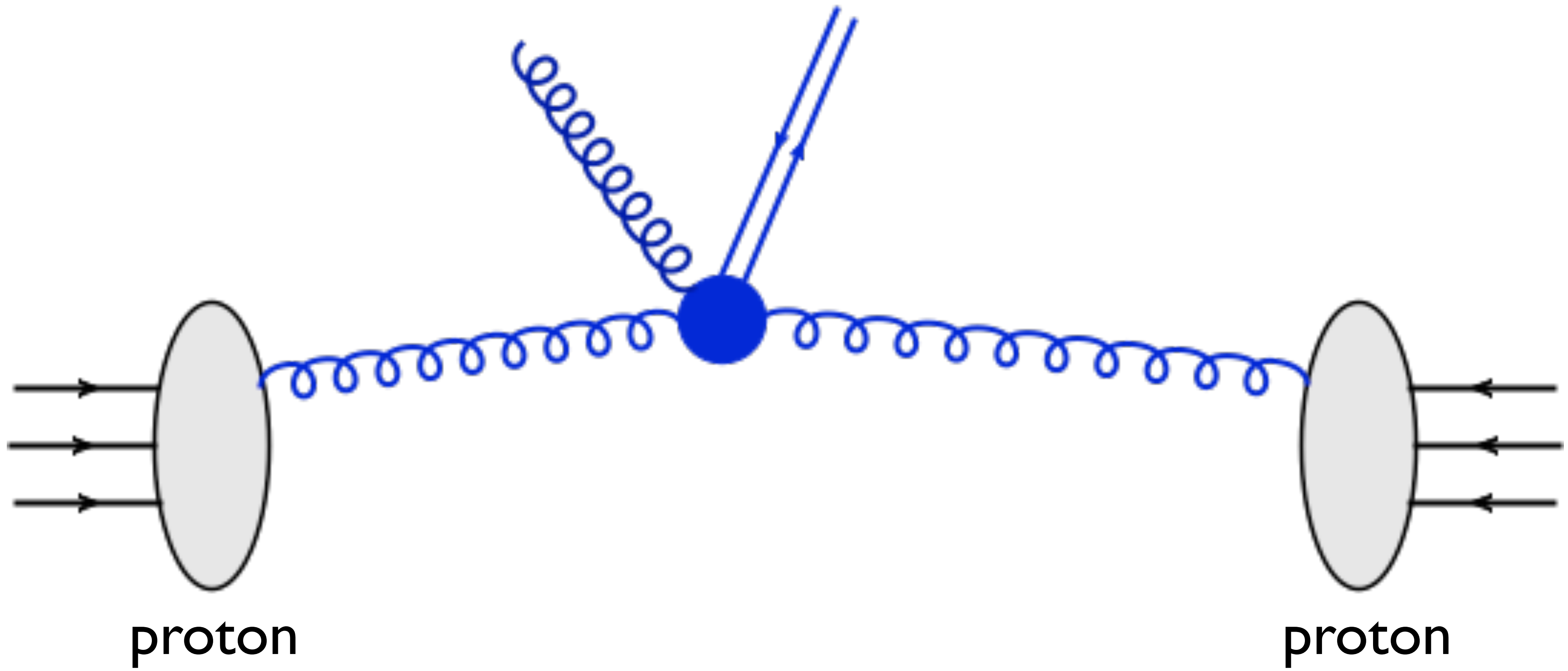
LO Event Simulations

Lecture 1

LO Event Simulations

Introduction

A Typical LHC Event : Monte Carlo Point of View



Hard interaction

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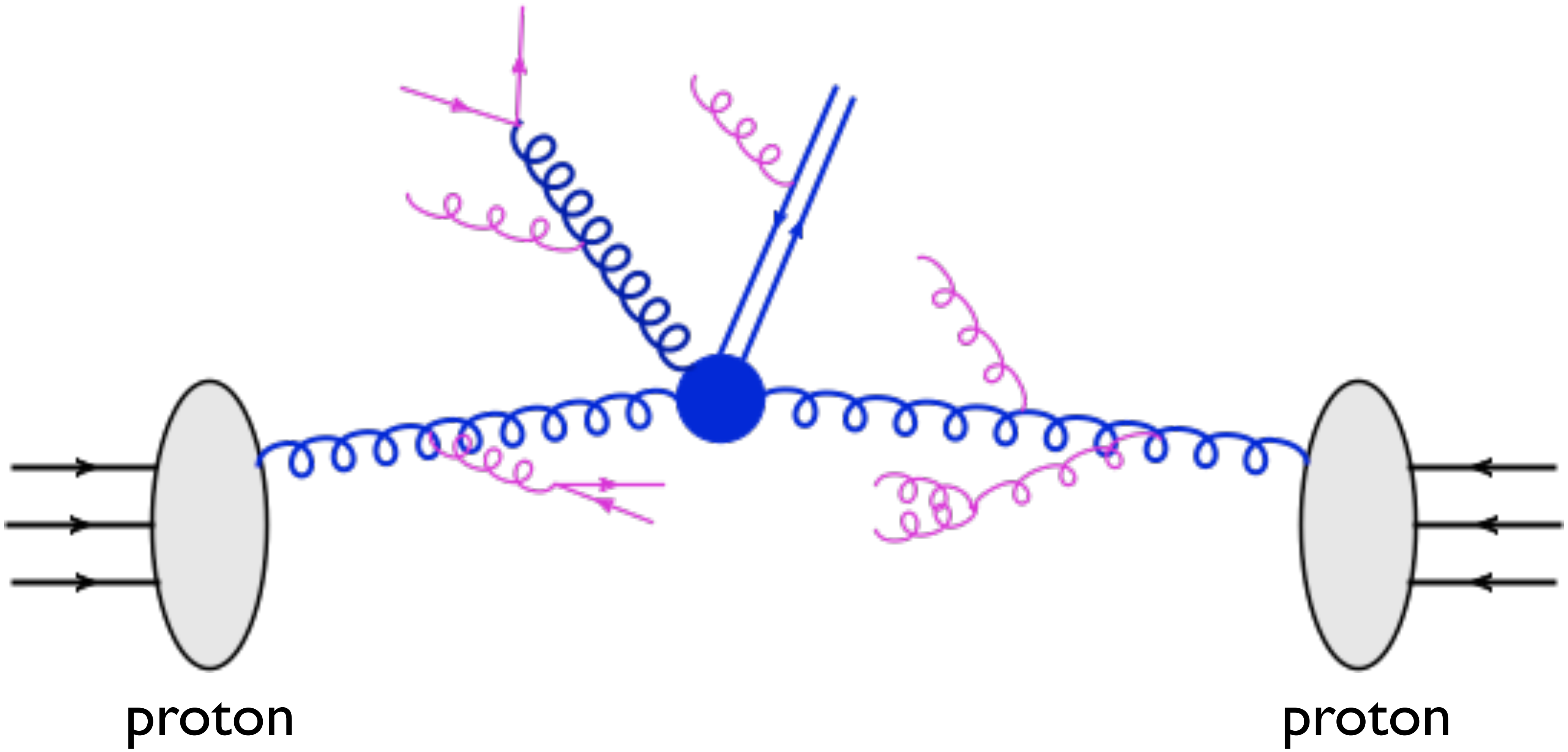

Hard interaction

Parton shower

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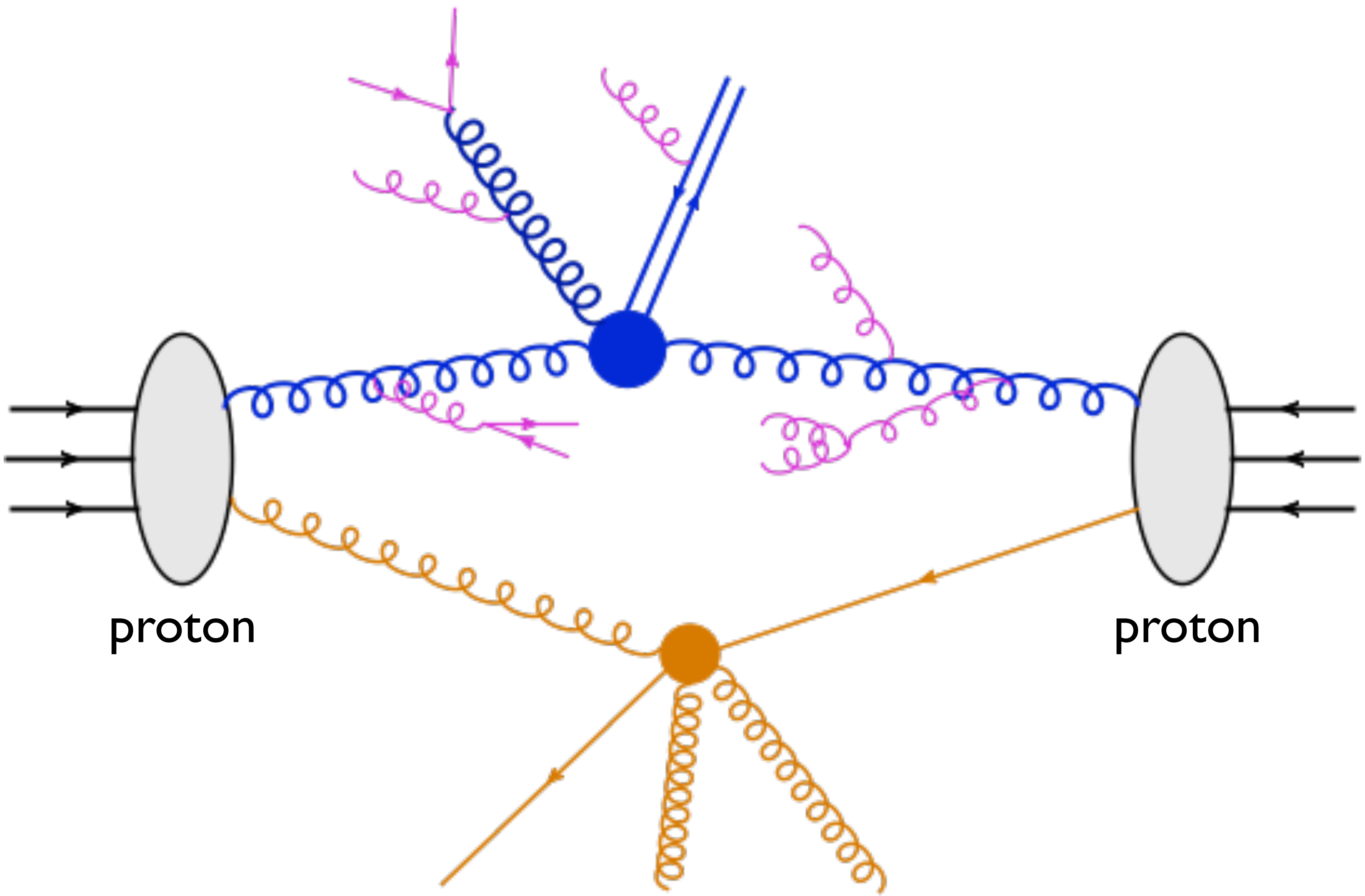

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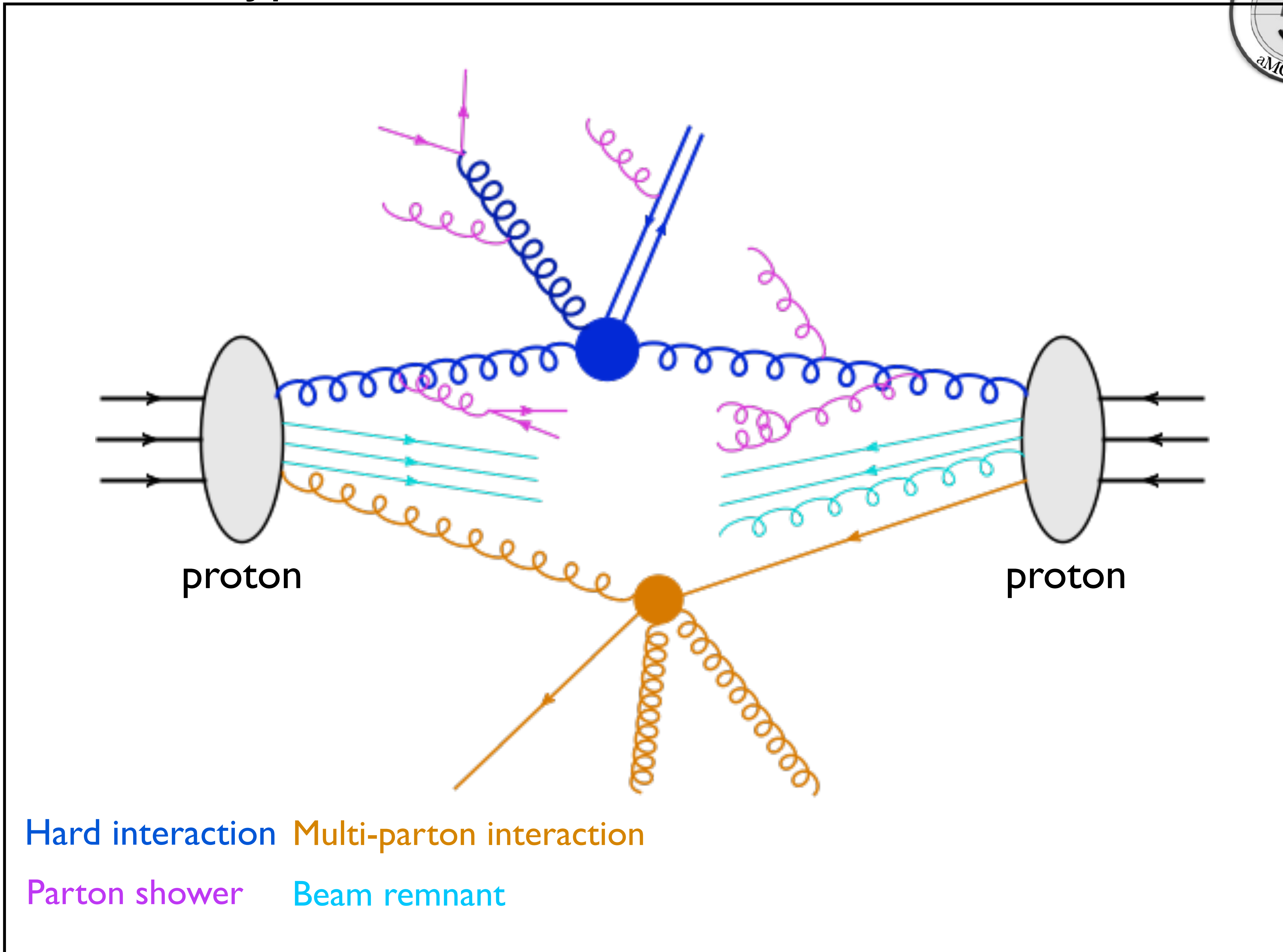

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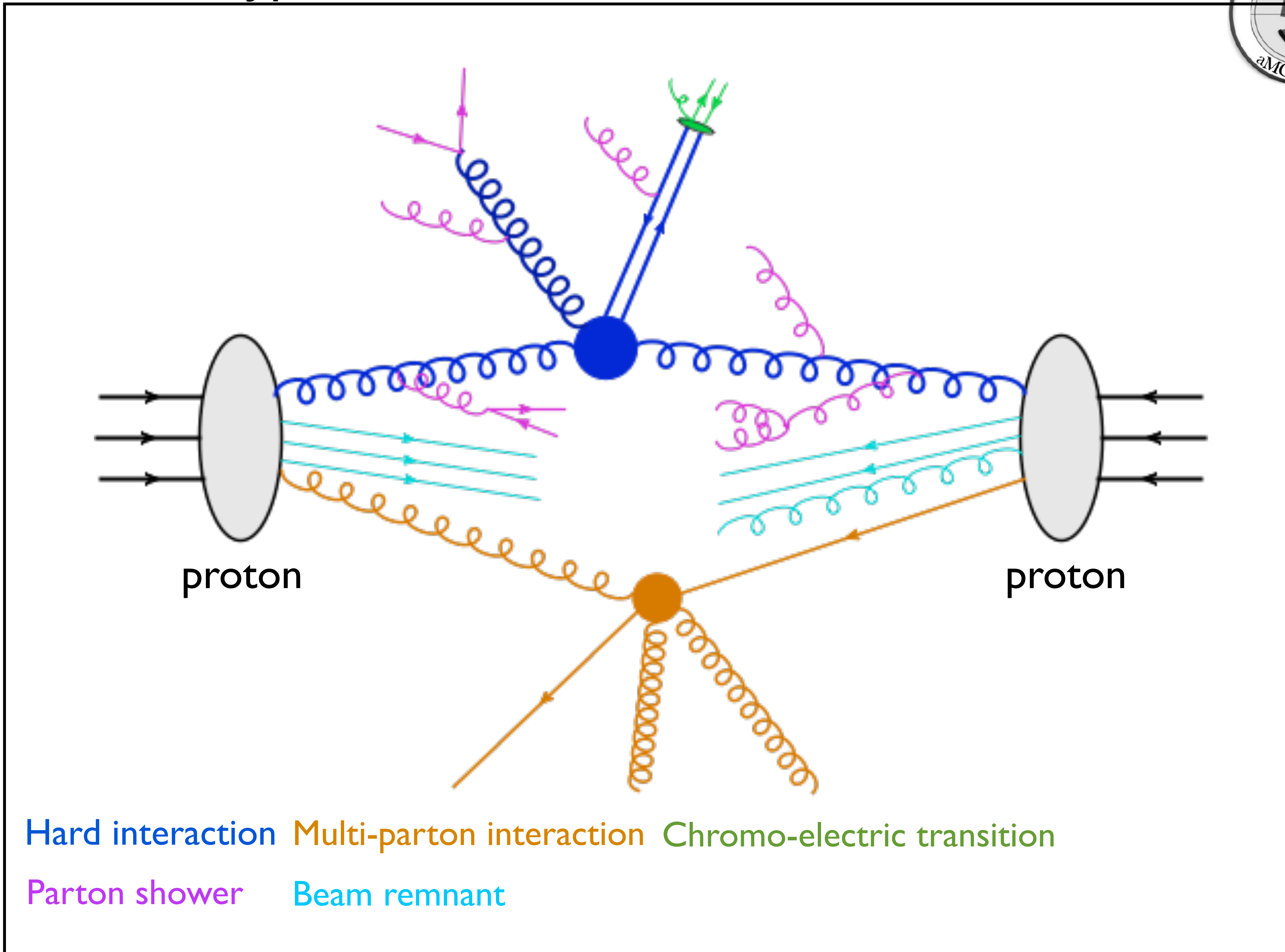

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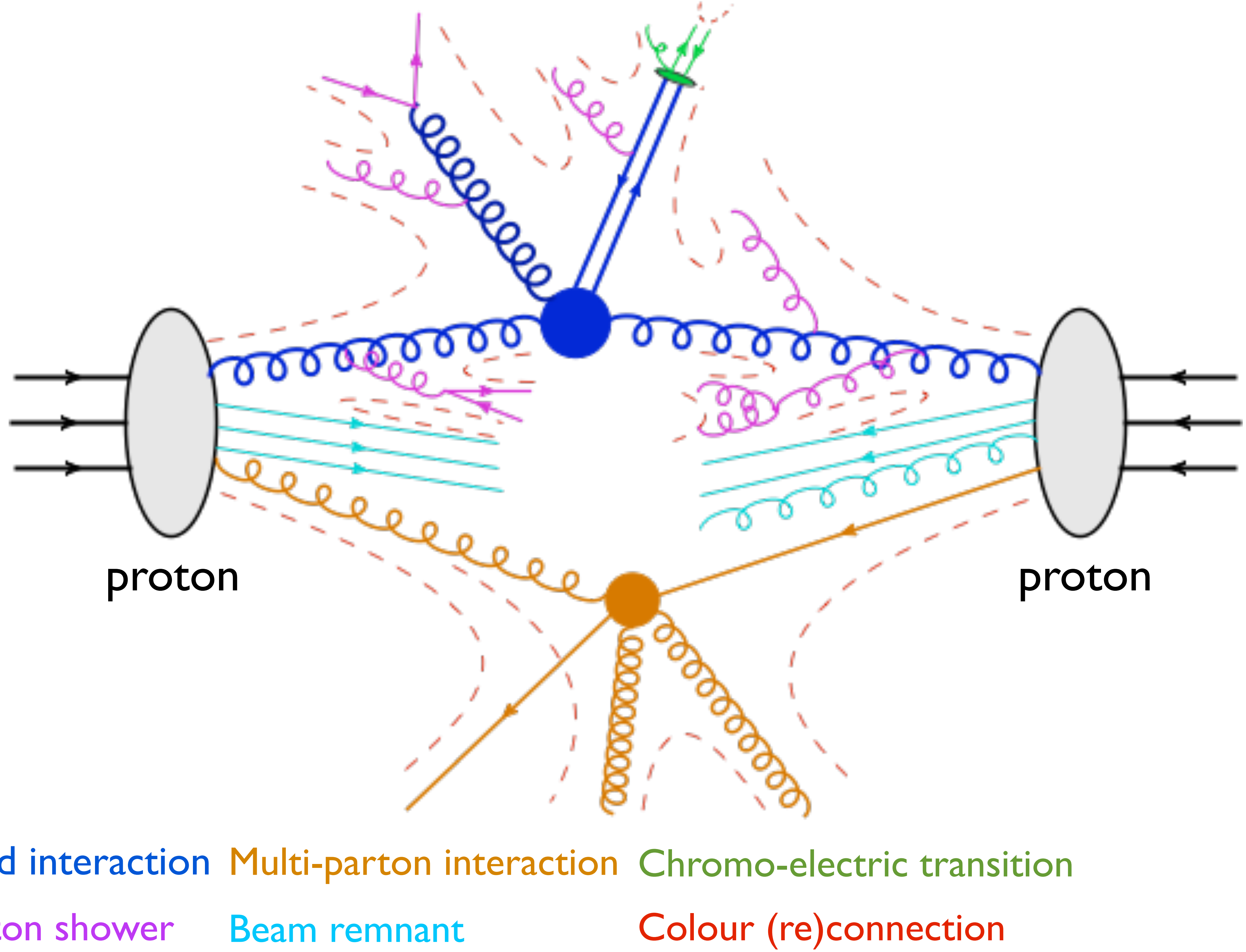

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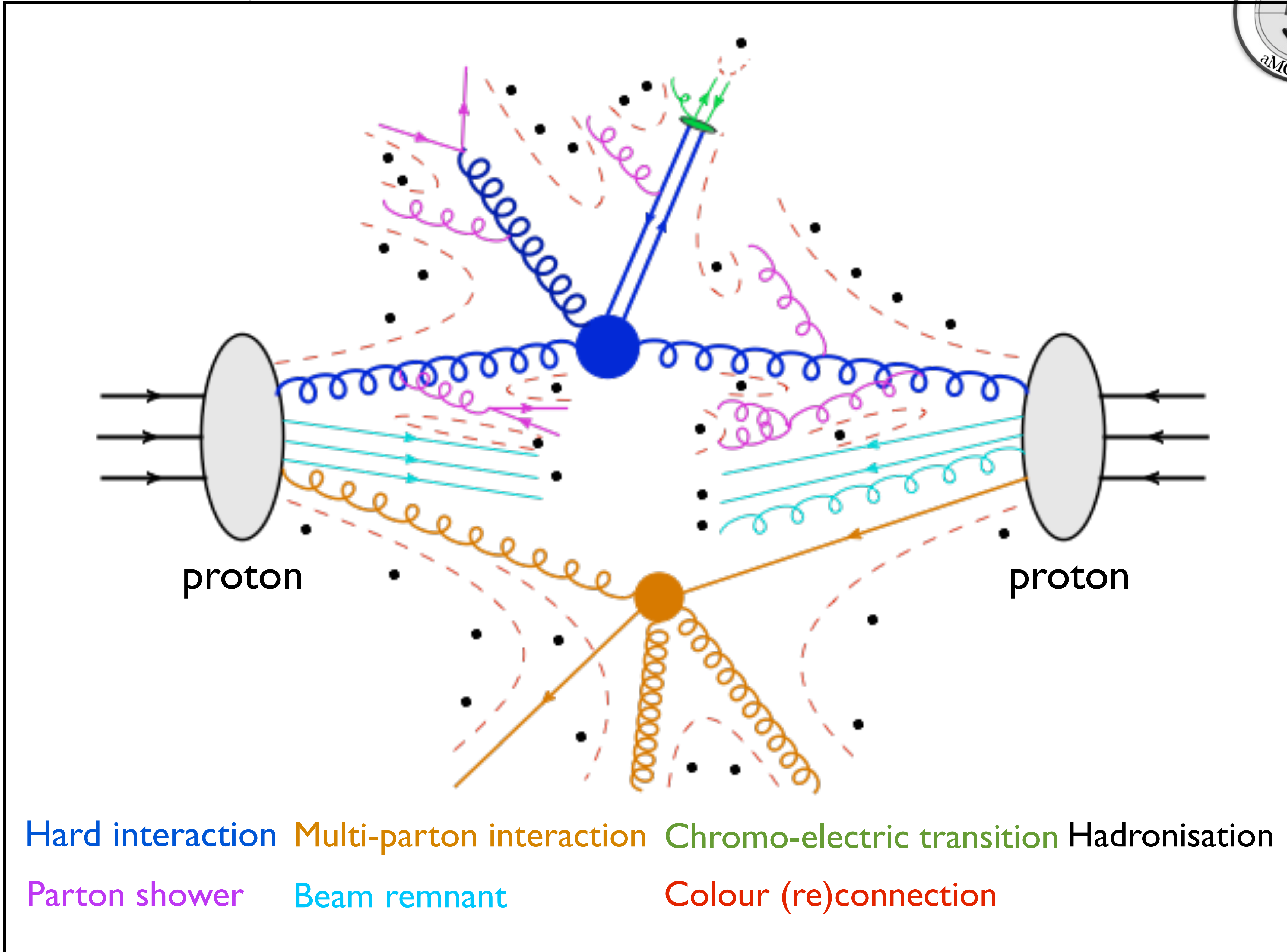

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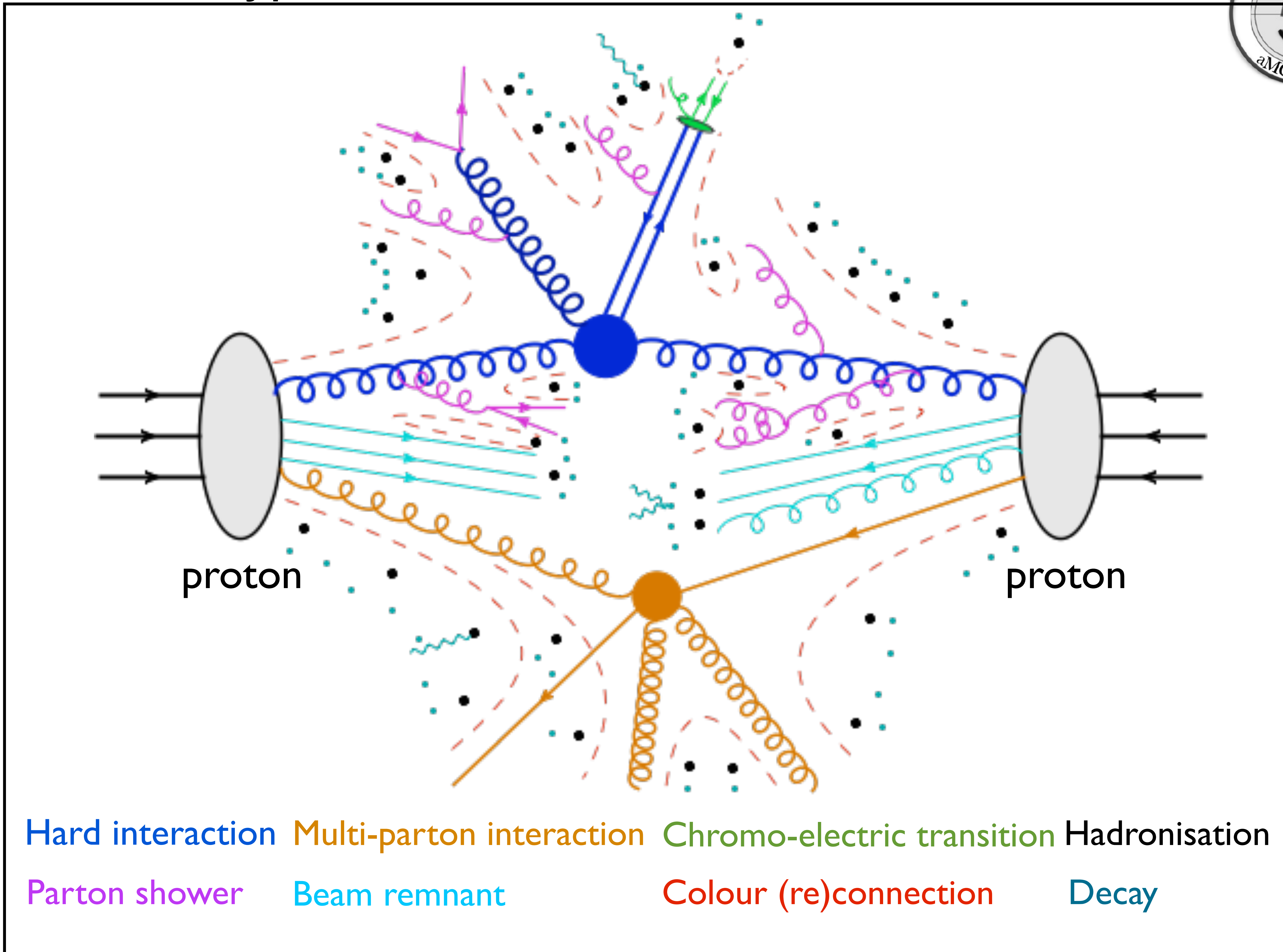

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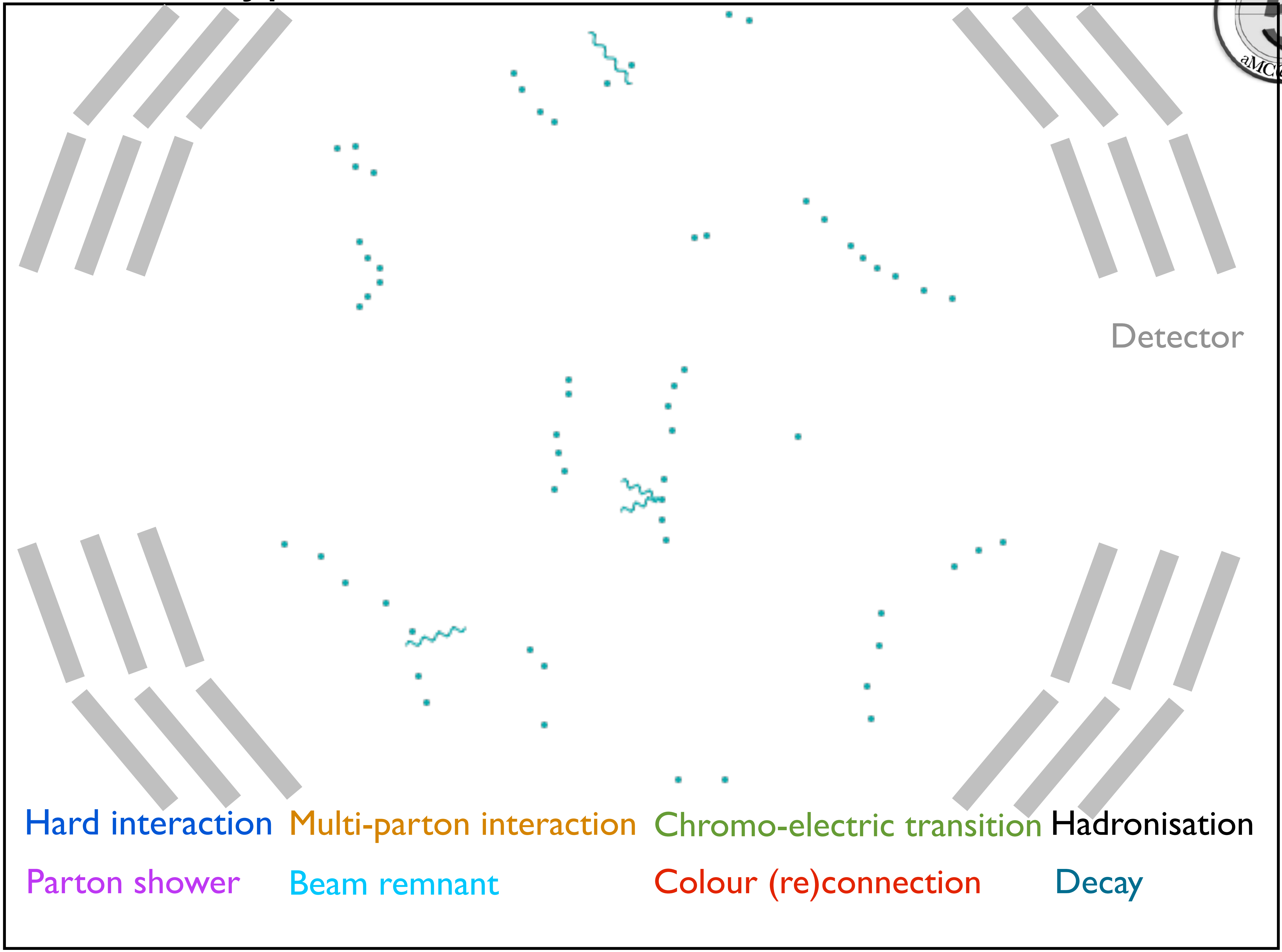
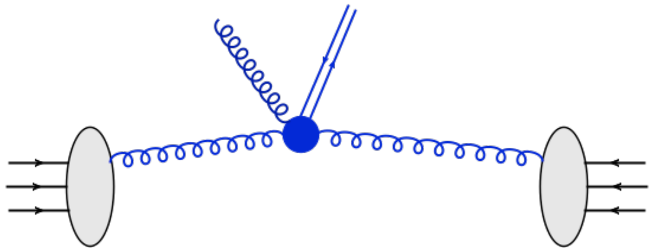


How To Improve Theoretical Predictions ?

Hard interaction 

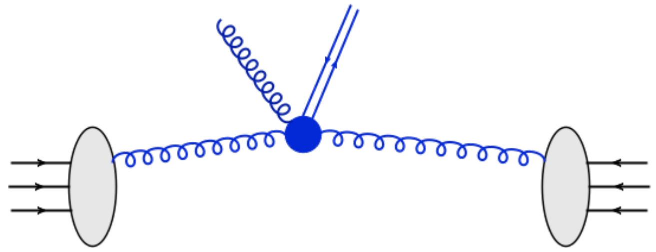
$$\sigma = \mathrm{PDF}_1 \otimes \mathrm{PDF}_2 \otimes \hat{\sigma}$$

LO $\rightarrow$ NLO $\rightarrow$ NNLO $\rightarrow$ N^{3}LO

$$\hat{\sigma} = \hat{\sigma}_0 \left[1 + \frac{\alpha_s}{2\pi}\Delta_1 + \left(\frac{\alpha_s}{2\pi}\right)^2 \Delta_2 + \left(\frac{\alpha_s}{2\pi}\right)^3 \Delta_3 + \ldots \right]$$

How To Improve Theoretical Predictions ?

Hard interaction

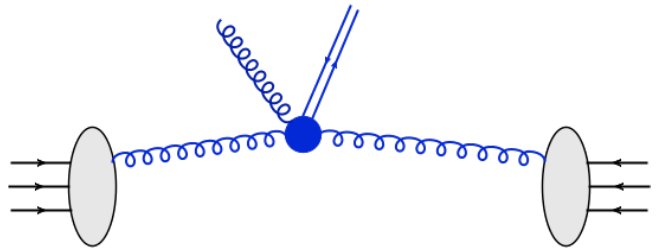


Parton shower

- Formally leading log and leading colour accuracy
- Beyond depends on some details (evolution variables, recoil schemes, …)
- Active theoretical developments (NLO kernels, NLL, NLC,…)

How To Improve Theoretical Predictions ?

Hard interaction



Parton shower

Pure perturbative

How To Improve Theoretical Predictions ?

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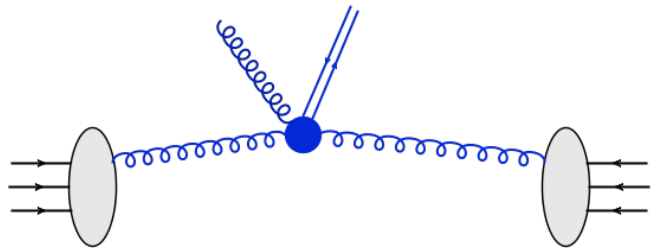


Parton shower

Pure perturbative

Mostly non-perturbative

Multi-parton interaction

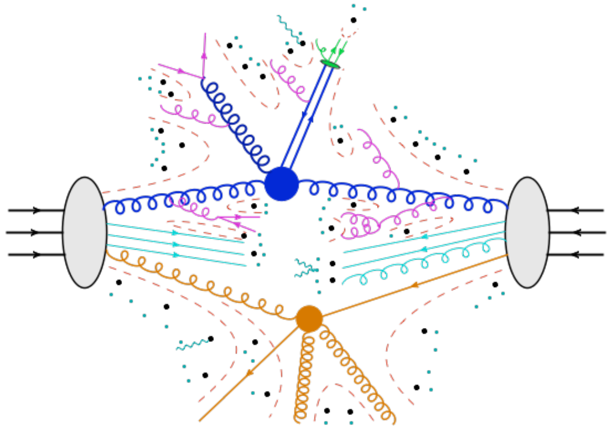
Beam remnant

Chromo-electric transition

Colour (re)connection

Hadronisation

Decay

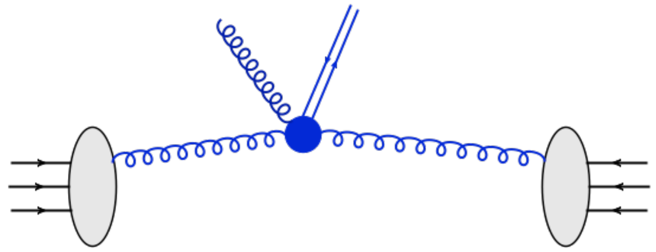


Comments:

- Tough !
- Most are model dependent
- Tuning to data
- Very flexible
- Needs the efforts of experimentalists

How To Improve Theoretical Predictions ?

Hard interaction



More rigorous

Parton shower

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Mostly non-perturbative

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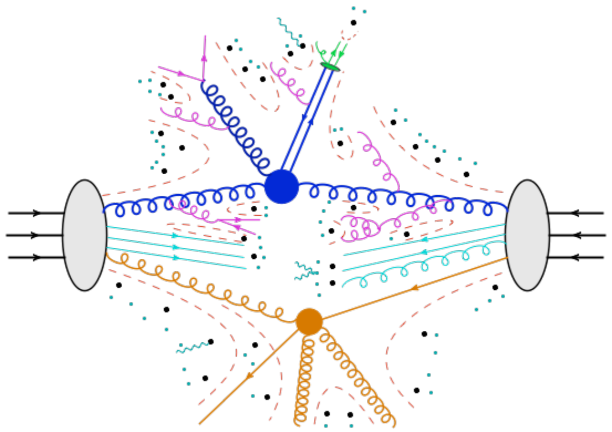
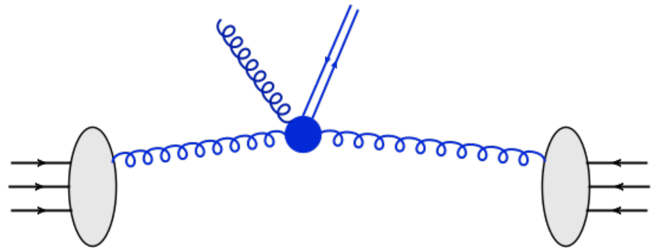


Less rigorous

How To Improve Theoretical Predictions ?

Less details

Hard interaction



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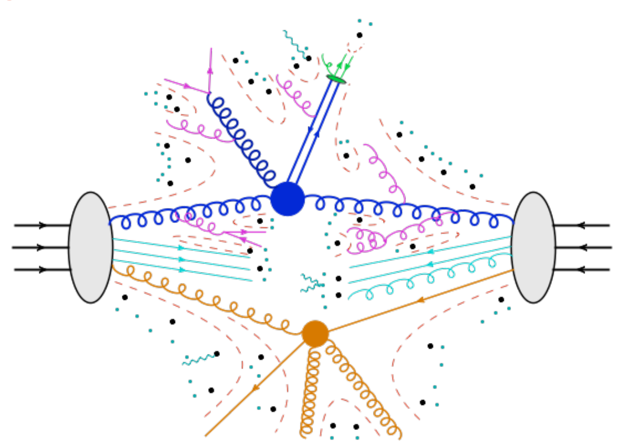


More details

How To Improve Theoretical Predictions ?

- **Common Principle:**
  - Adopt the advantages of different aspects

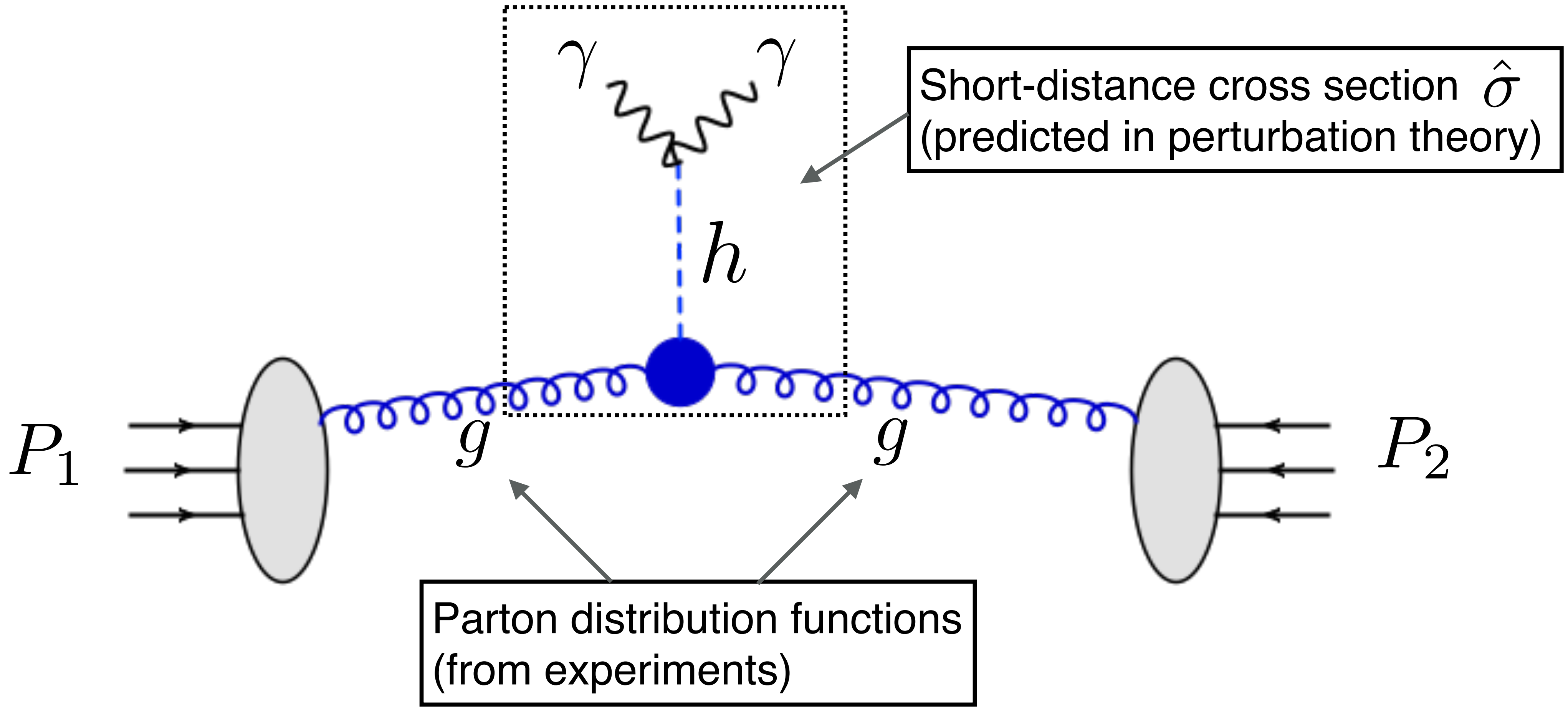
Hard interaction	Parton shower
Hard and Resolved	Soft and/or Collinear

  - Balance the accuracy over the steps in the simulation chain
  - Improve not only the single steps but also their merging

- **Two main directions:**
  - Matching
    - Avoid double counting between N^kLO ($k>0$) MEs and PSs
  - Merging
    - Include more real radiation MEs (formally higher order, improve kin.)
    - Subtract double counting of real radiation from MEs and PSs

The LHC master formula

- The initial condition of the MC simulation



$$\sigma(P_1P_2 \to h+X) = \int dx_1 dx_2 f_g^{(P_1)}(x_1,\mu_F^2) f_g^{(P_2)}(x_2,\mu_F^2)\hat{\sigma}$$

$$\hat{\sigma} = \frac{1}{2s}\int d\Phi_n |\mathcal{M}_n|^2$$

Matrix element: n! diags

Phase space: (3n-4) dims

Lecture 1

LO Event Simulations

Matrix element

Helicity Amplitudes

ChatGPT

Can you write a code to calculate the cross section of $e^+e^-\to e^+e^-$?

✅ Python Code (Symbolic Bhabha scattering using SymPy)

python

Copy Edit

```
import sympy as sp

# Constants and symbols
alpha, s, t, u, m = sp.symbols('alpha s t u m', real=True, positive=True)

# Define the differential cross section (tree-level QED, unpolarized)
# Taken from standard QED textbooks (e.g. Peskin & Schroeder, Halzen & Martin)
dsigma_dt = (2 * sp.pi * alpha**2 / s**2) * (
    (s**2 + u**2) / t**2 +
    (t**2 + u**2) / s**2 +
    2 * u**2 / (s * t)
)

# Display the symbolic result
sp.pprint(dsigma_dt, use_unicode=True)
```

- The general idea:
  - Evaluate $\mathcal{M}_n$ for a given helicity of external particles
    - Multiply with its complex conjugate to get amplitude squared
    - Loop over helicities, diagrams, and color configurations

Helicity Amplitudes

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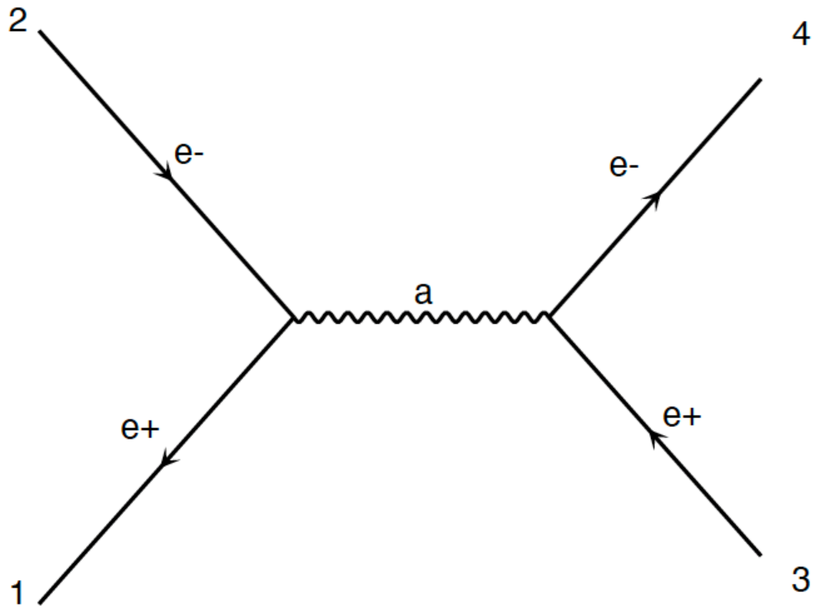

$$\mathcal{M}_n =$$

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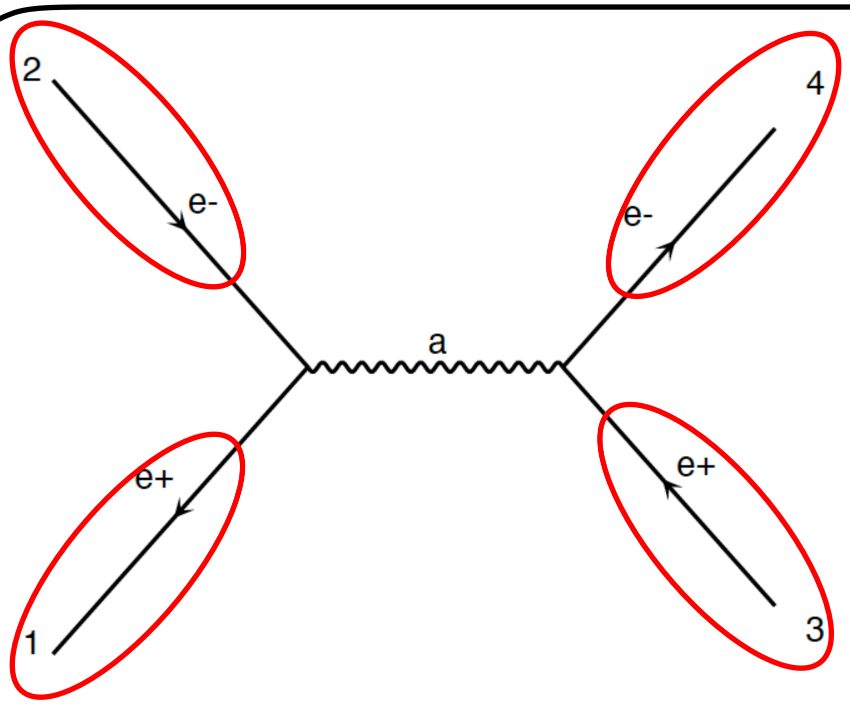

$$\mathcal{M}_n = \bar{u}_4 \quad v_3 \quad \bar{v}_1 \quad u_2$$

Matrices for given helicity and momenta of external legs

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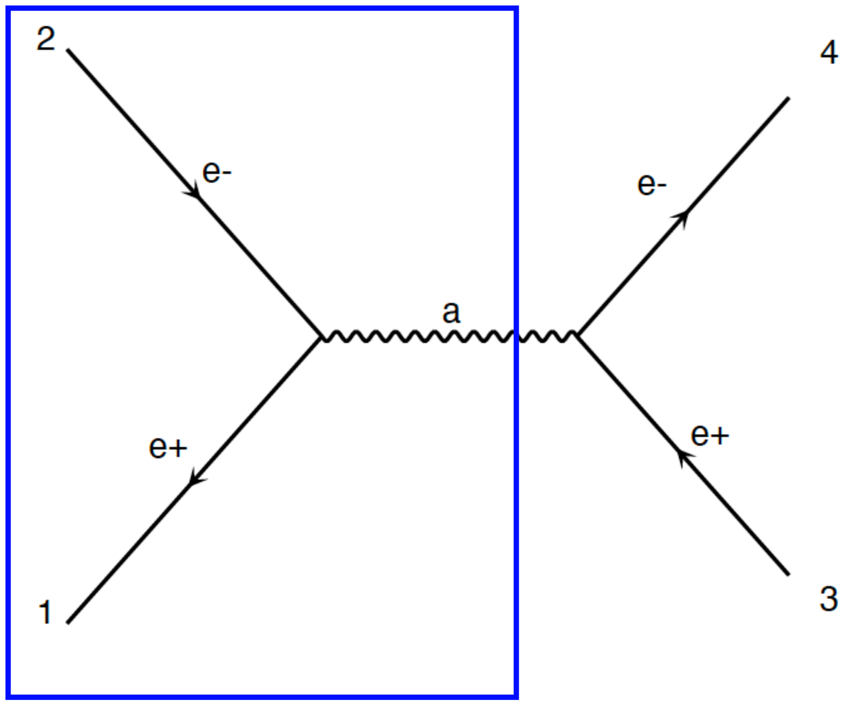

$$\mathcal{M}_n = \bar{u}_4 \, e\gamma^\mu \, v_3 \, \frac{g_{\mu\nu}}{(p_1+p_2)^2 - m_a^2 + i\Gamma_a m_a} \, \bar{v}_1 \qquad u_2$$

Matrices for given helicity and momenta of external legs

Calculations of propagators & vertices

Helicity Amplitudes

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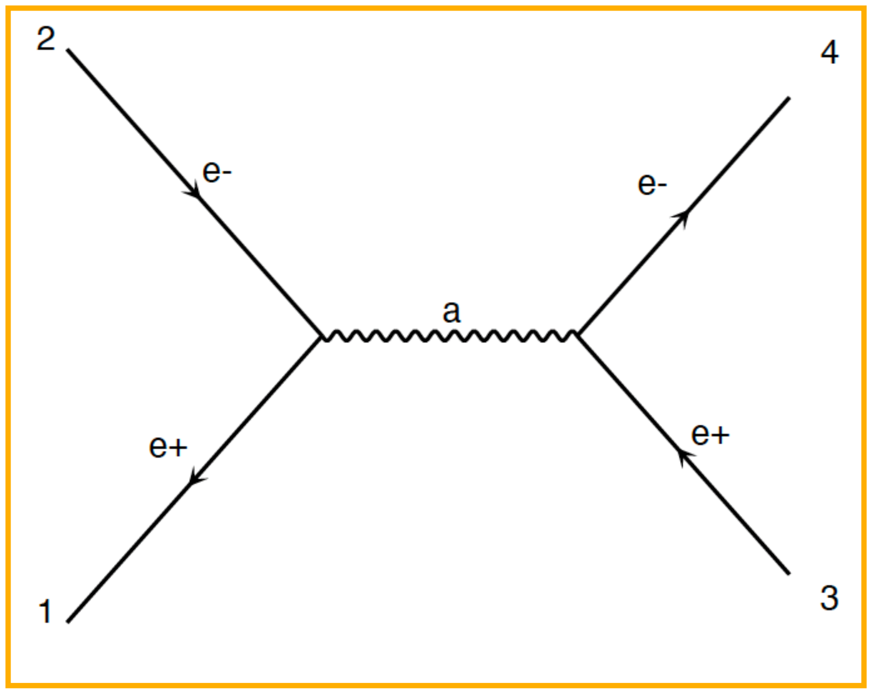

$$\mathcal{M}_n = \bar{u}_4 e\gamma^\mu v_3 \frac{g_{\mu\nu}}{(p_1+p_2)^2 - m_a^2 + i\Gamma_a m_a} \bar{v}_1 e\gamma^\nu u_2$$

Matrices for given helicity and momenta of external legs

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Evaluation of helicity amplitude (c-number)

Helicity Amplitudes

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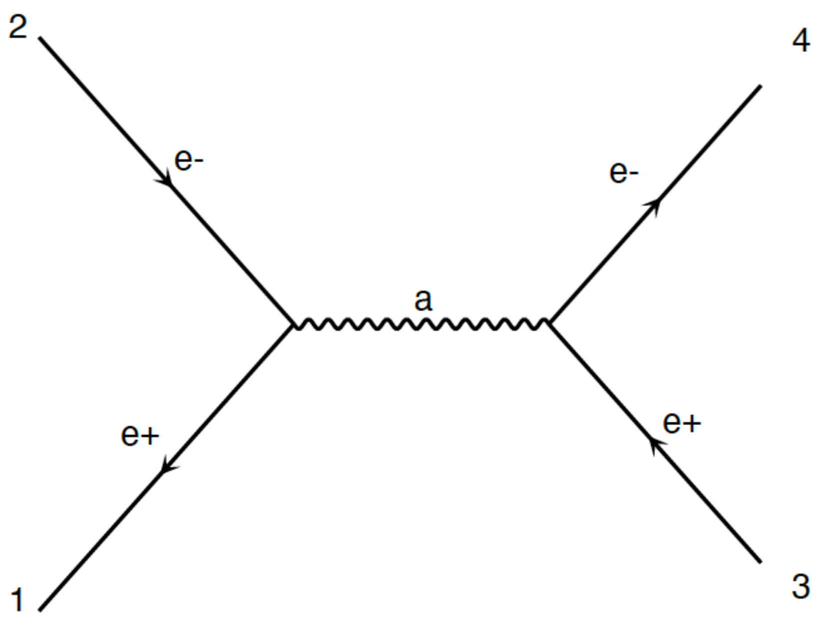

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Matrices for given helicity and momenta of external legs

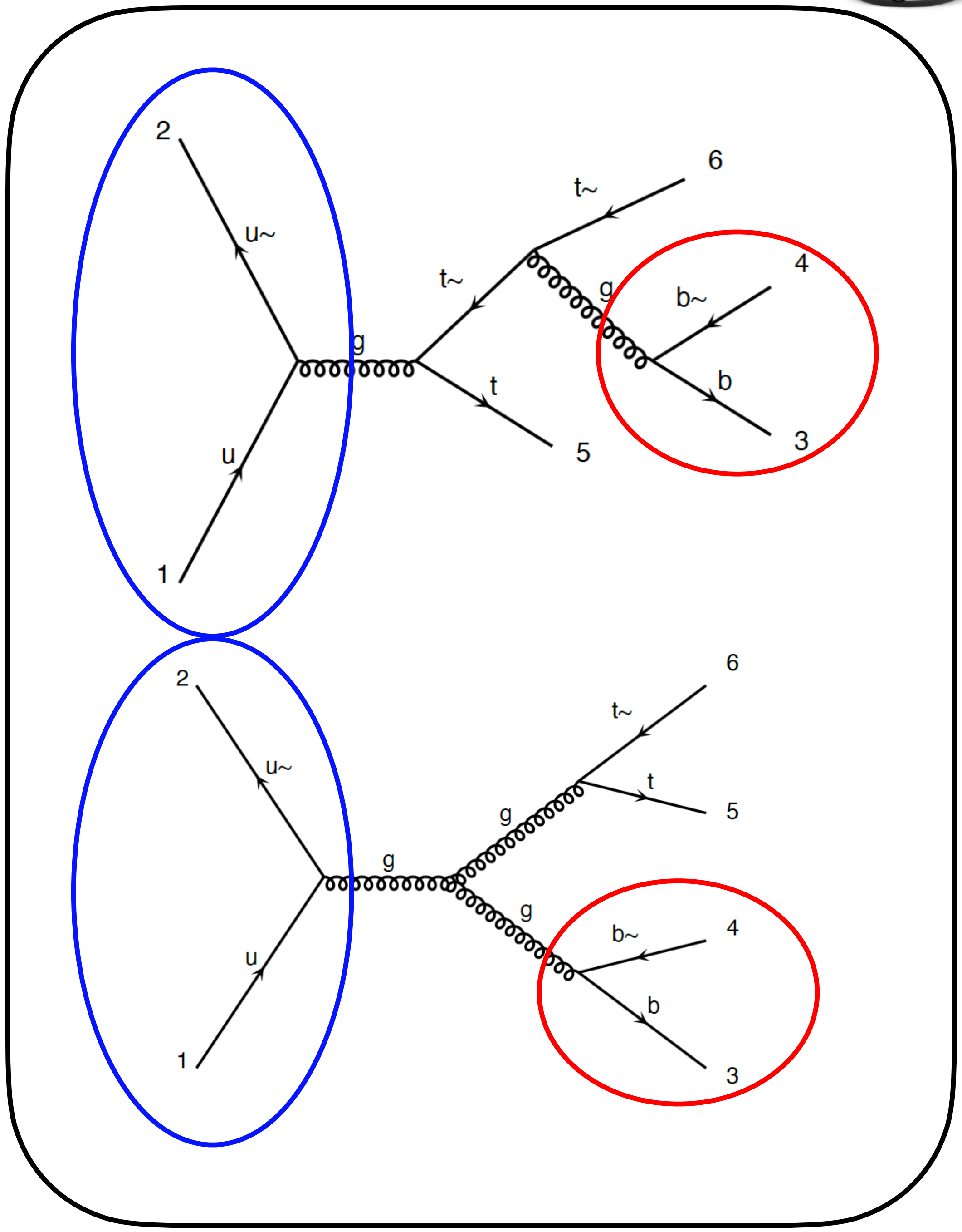
Calculations of propagators & vertices

Evaluation of helicity amplitude (c-number)

$$\boxed{\begin{aligned} \bar{v}_1 &= \mathrm{fct}(p_1, m_1) \\ u_2 &= \mathrm{fct}(p_2, m_2) \\ v_3 &= \mathrm{fct}(p_3, m_3) \\ \bar{u}_4 &= \mathrm{fct}(p_4, m_4) \\ W_a &= \mathrm{fct}(\bar{v}_1, u_2, m_a, \Gamma_a) \\ \mathcal{M}_n &= \mathrm{fct}(v_3, \bar{u}_4, W_a) \end{aligned}}$$

Helicity Amplitudes

	M diagrams	N particles	Ex: 2->6
Analytic	M^2	$(N!)^2$	$1.6 \cdot 10^9$
Helicity	M	$(N!)2^N$	$1.0 \cdot 10^7$
Recycling	M	$(N-1)!\ 2^{(N-1)}$	$6.5 \cdot 10^5$
Recursion	log(M)	$2^N 2^{(N-1)}$	$3.3 \cdot 10^4$



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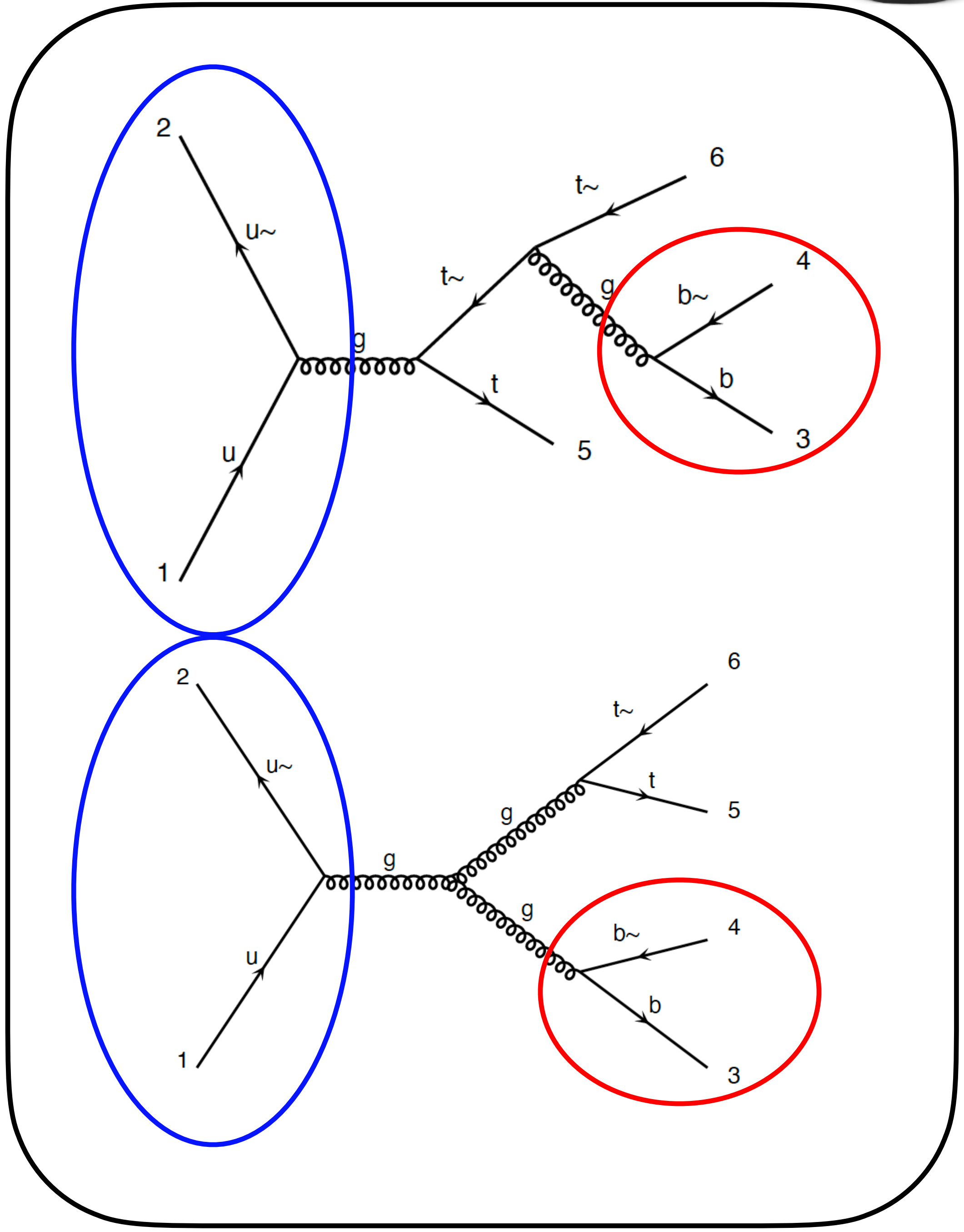


- We want to compute matrix element
  - for large number of final states
  - for any (B)SM theory
  - for loop(s)

Lecture 1

LO Event Simulations

Phase space: event generation

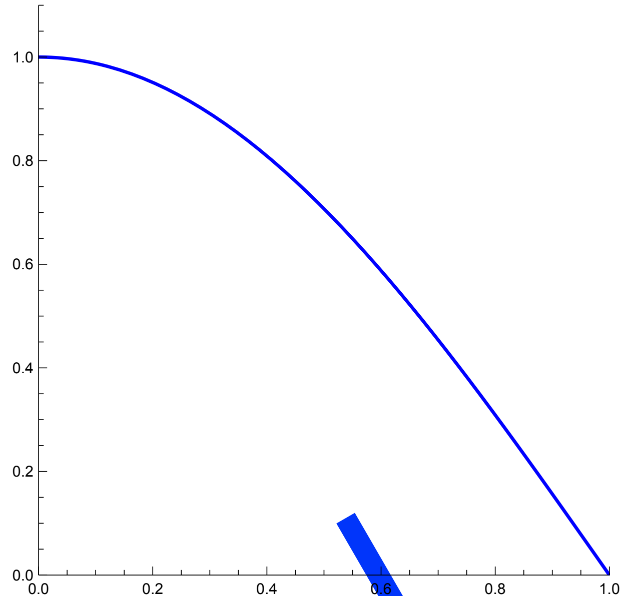
Phase Space Integration

- Challenges in numerical integration:
  - high-dimension phase space
  - Integrand is a very peaked function
  - General and flexible
  - Not only integration but also event generation

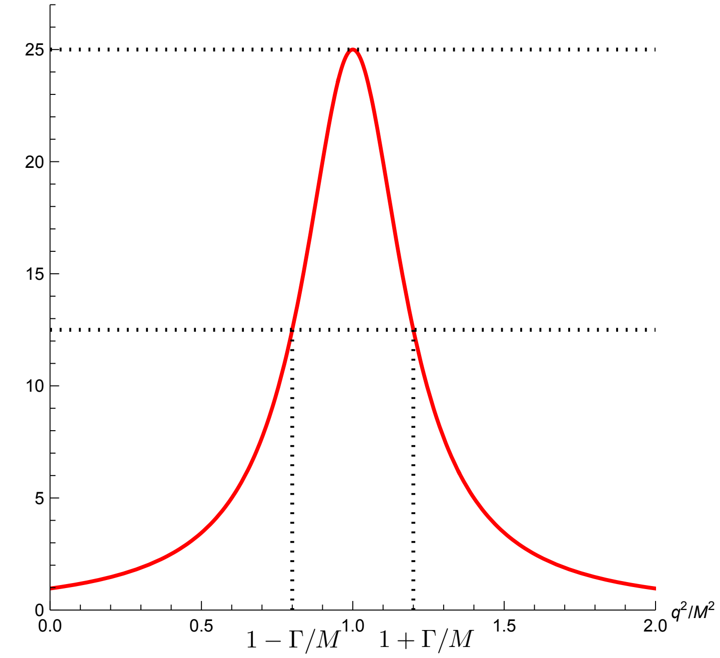
$$\boxed{\hat{\sigma} = \frac{1}{2s}\int d\Phi_n |\mathcal{M}_n|^2}$$

Numerical Integration

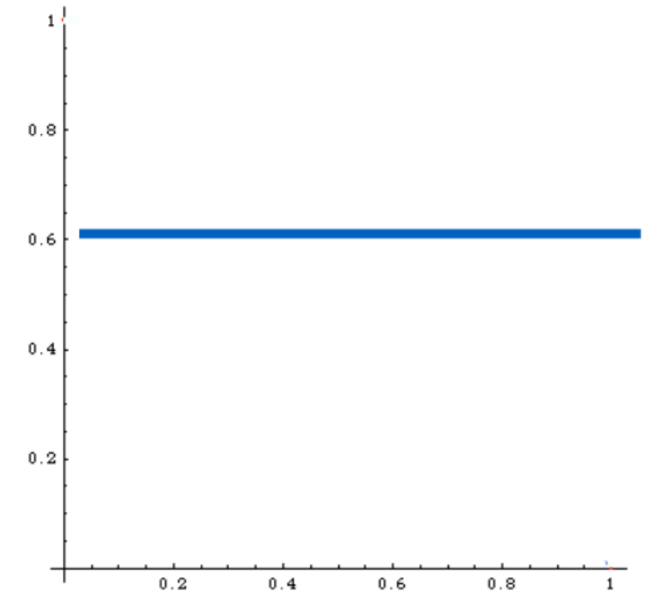
$$\int_0^1 dx \cos\left(\frac{\pi}{2}x\right)$$



$$\int \frac{dq^2}{(q^2-M^2)^2+M^2\Gamma^2}$$



$$\int dx C$$

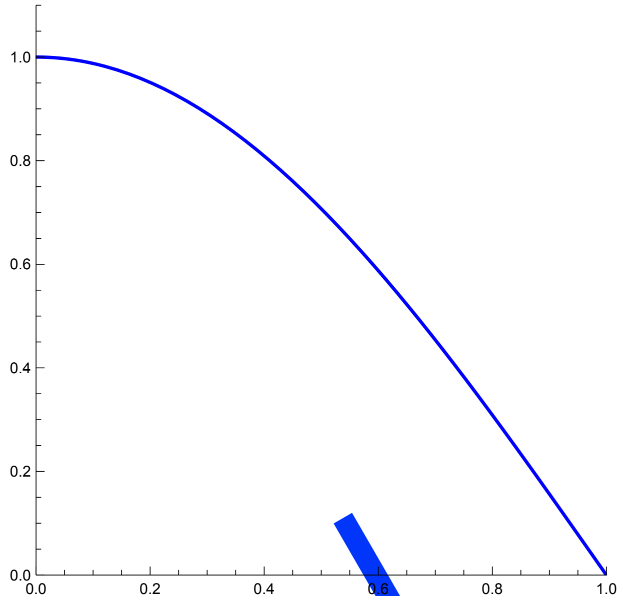


N	Simpson	MC
3	0.638	0.3
5	0.6367	0.8
20	0.63662	0.6
100	0.636619	0.65
1000	0.636619	0.636

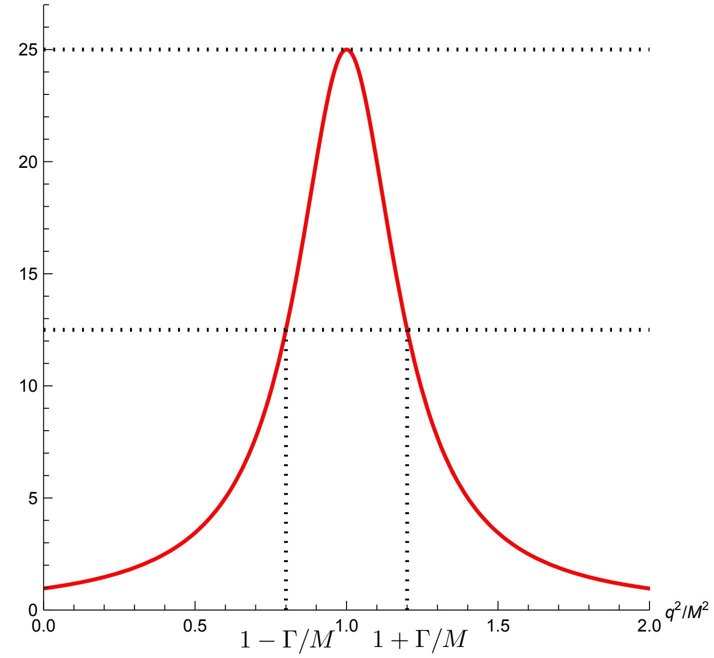
- Relative error:
  - MC $\frac{1}{\sqrt{N}}$ ✗
  - Simpson $\frac{1}{N^4}$ ✓

Numerical Integration

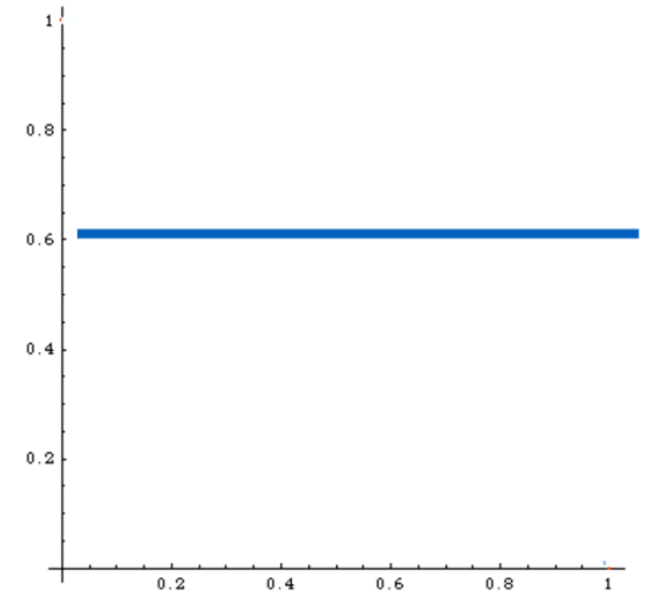
$$\int_0^1 dx \cos\left(\frac{\pi}{2}x\right)$$



$$\int \frac{dq^2}{(q^2 - M^2)^2 + M^2\Gamma^2}$$



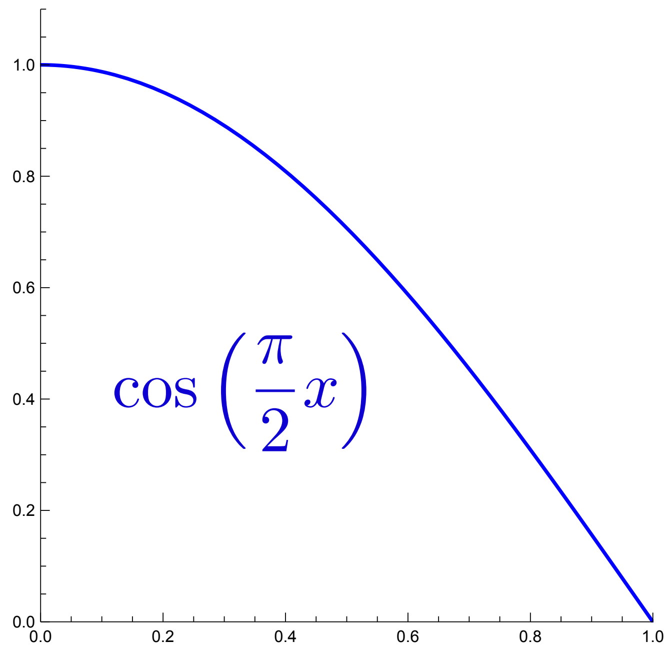
$$\int dx C$$



N	Simpson	MC
3	0.638	0.3
5	0.6367	0.8
20	0.63662	0.6
100	0.636619	0.65
1000	0.636619	0.636

- Relative error in d dim:
  - MC $\frac{1}{\sqrt{N}}$ ✔
  - Simpson $\frac{1}{N^{\frac{4}{d}}}$ ✘

Integration Variable Choices

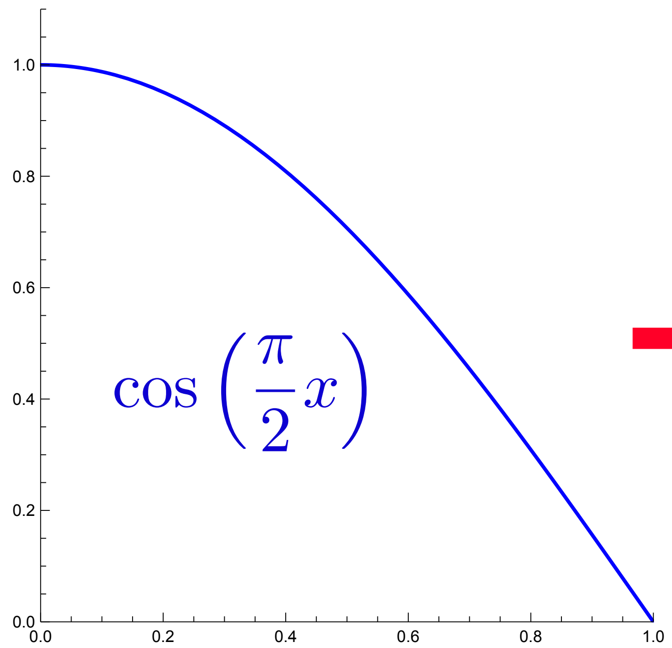


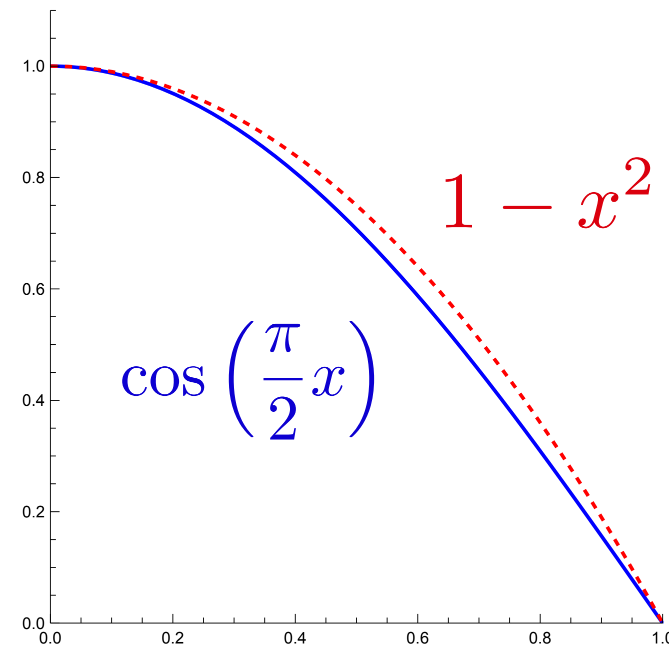
$$I = \int_0^1 dx \cos\left(\frac{\pi}{2}x\right)$$

$$I = I_N \pm \sqrt{\frac{V_N}{N}}$$

$$I = 0.637 \pm \frac{0.307}{\sqrt{N}}$$

Integration Variable Choices





$$I=\int_0^1 dx\cos\left(\frac{\pi}{2}x\right)$$

$$I=I_N\pm\sqrt{\frac{V_N}{N}}$$

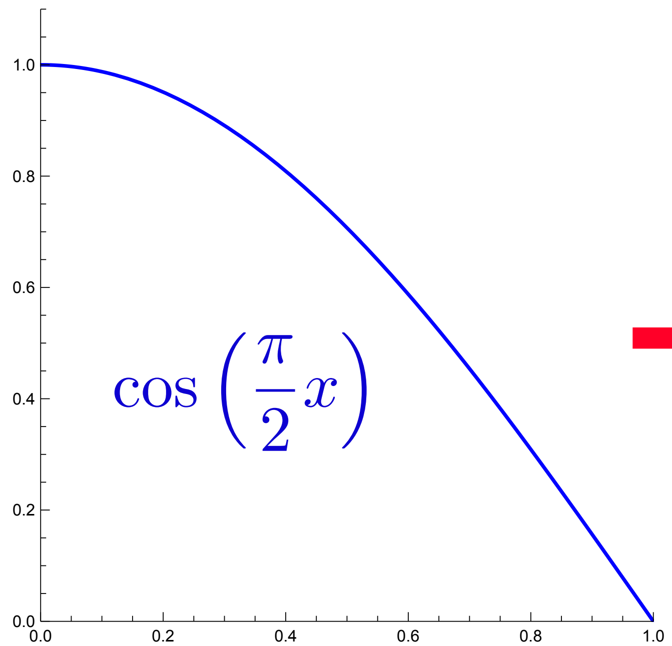
$$I=0.637\pm\frac{0.307}{\sqrt{N}}$$

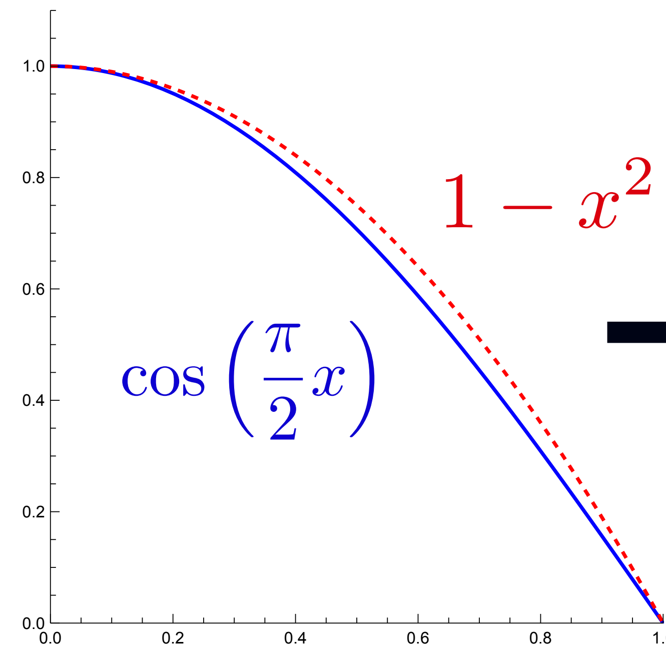
$$I=\int_0^1 dx(1-x^2)\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x^2}$$

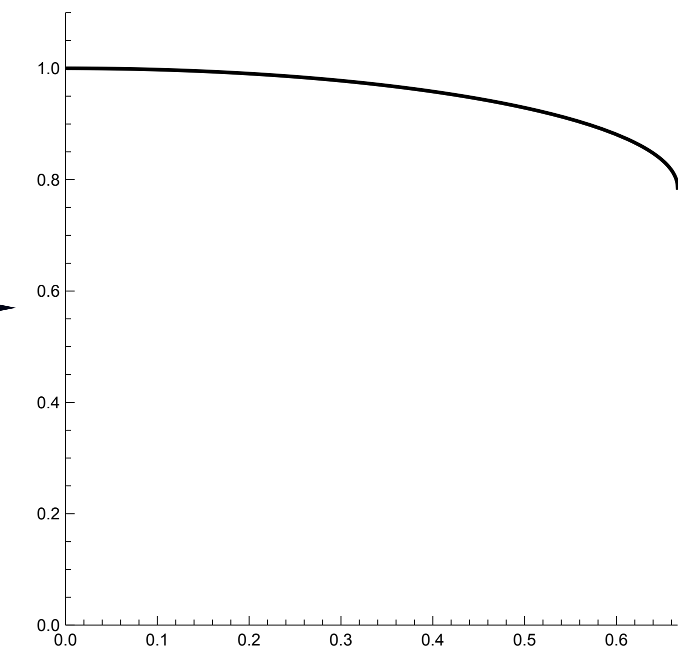
$$y=x-\frac{x^3}{3}$$

$$x\to x(y)$$

Integration Variable Choices







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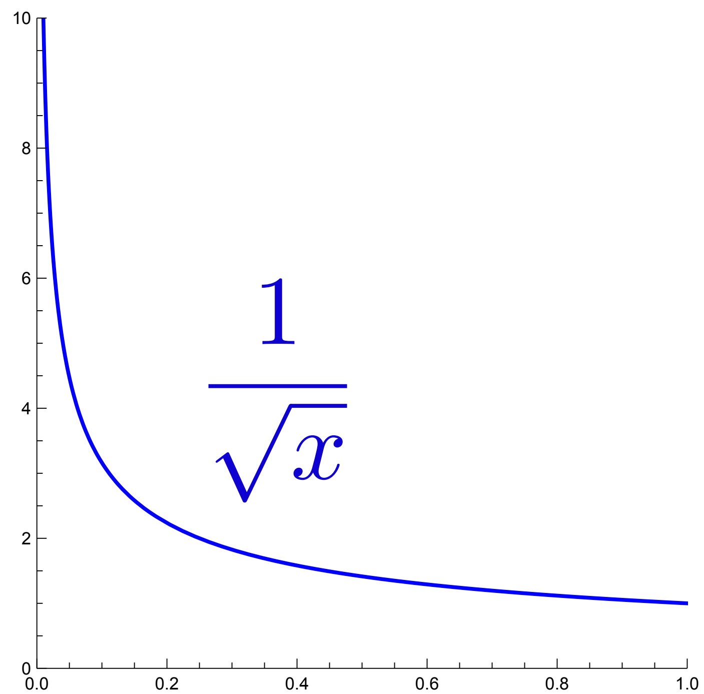
$$y = x - \frac{x^3}{3}$$

$$x \to x(y)$$

$$I = \int_0^{\frac{2}{3}} dy \frac{\cos\left(\frac{\pi}{2}x(y)\right)}{1-x^2(y)}$$

$$I = 0.637 \pm \frac{0.031}{\sqrt{N}}$$

Integration Variable Choices



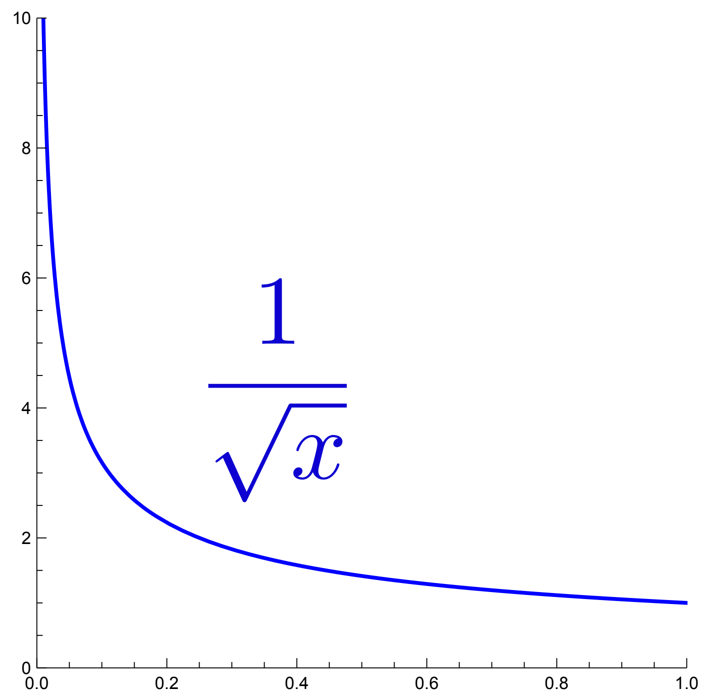
$$I = \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$$

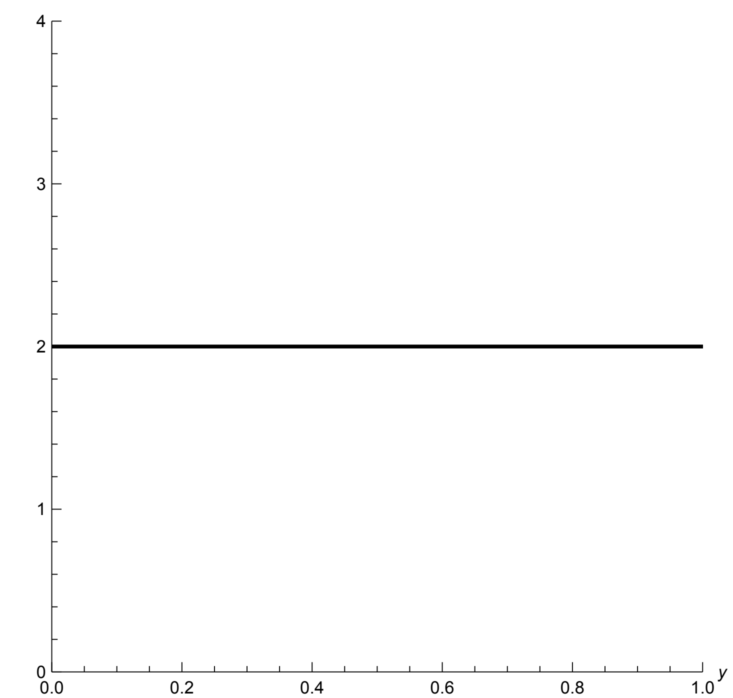
Integrand divergent at x=0

Variance divergent:

$$V = \int_0^1 \frac{dx}{x} - I^2 \propto \log(0)$$

Integration Variable Choices





$$I = \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$$

$$y = \sqrt{x}$$

$$I = 2\int_0^1 dy$$

Integrand divergent at x=0

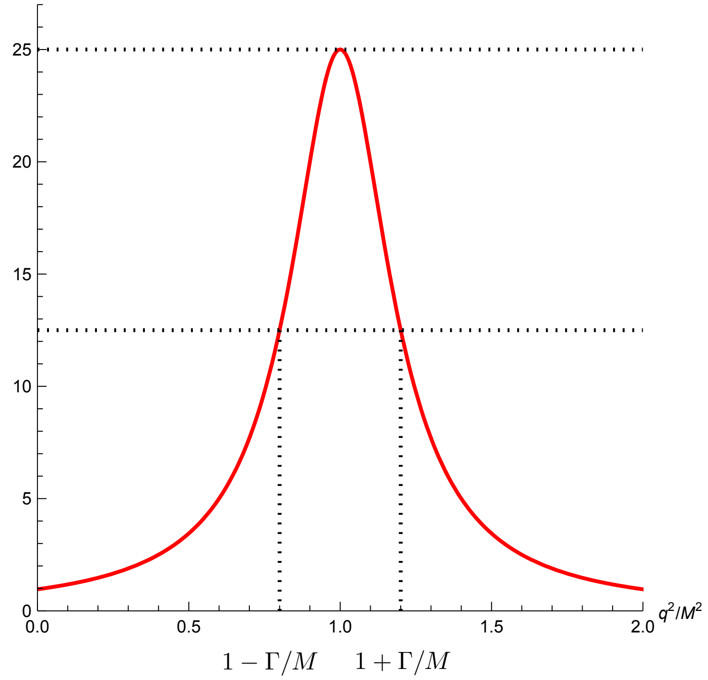
$$x = y^2$$

Variance divergent:

$$V = \int_0^1 \frac{dx}{x} - I^2 \propto \log(0)$$

$$I_N = 2 \pm \frac{0}{\sqrt{N}}$$

Importance Sampling

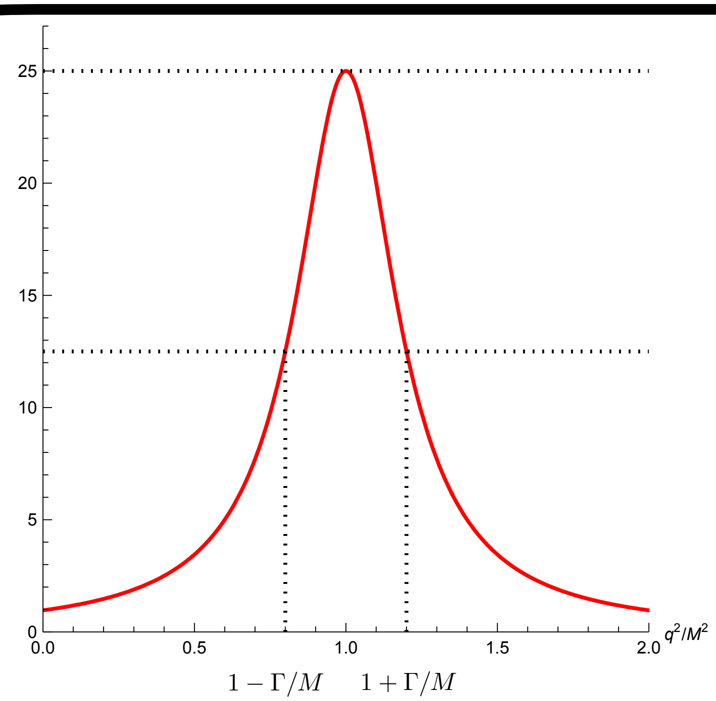


$$I = \int \frac{dq^2}{(q^2 - M^2) + M^2\Gamma^2}$$

Homework

Question: what is your good integration variable ?

Importance Sampling



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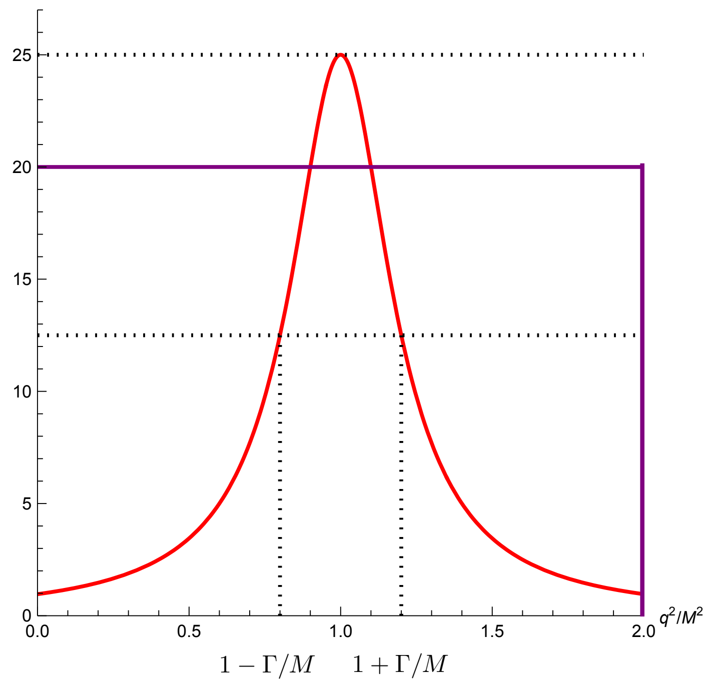
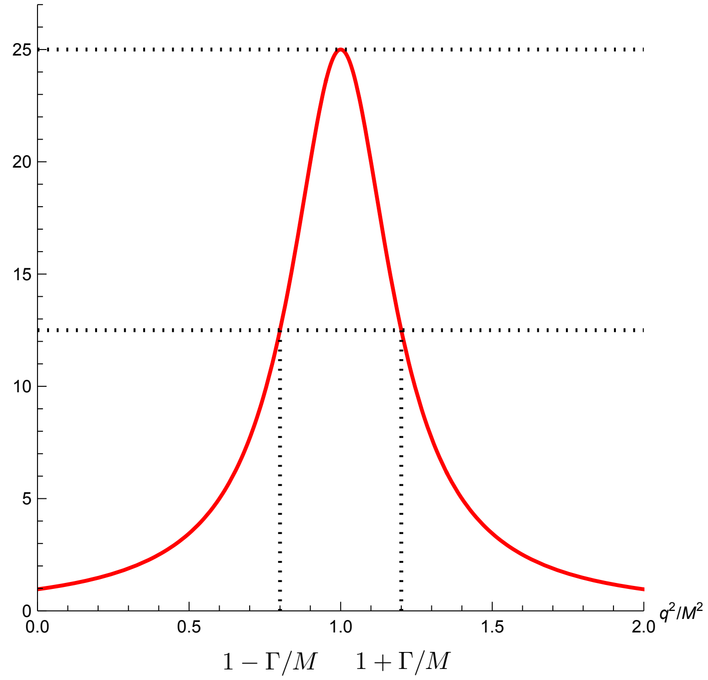


Importance Sampling



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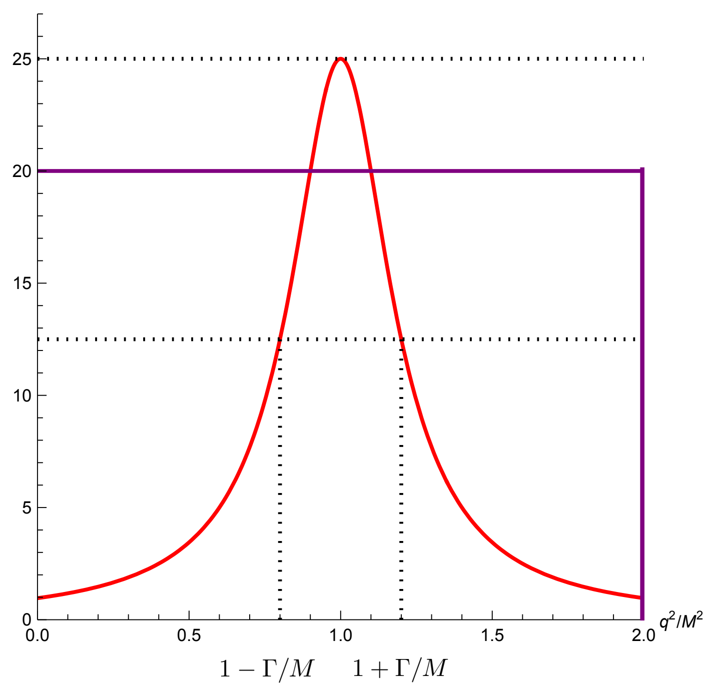
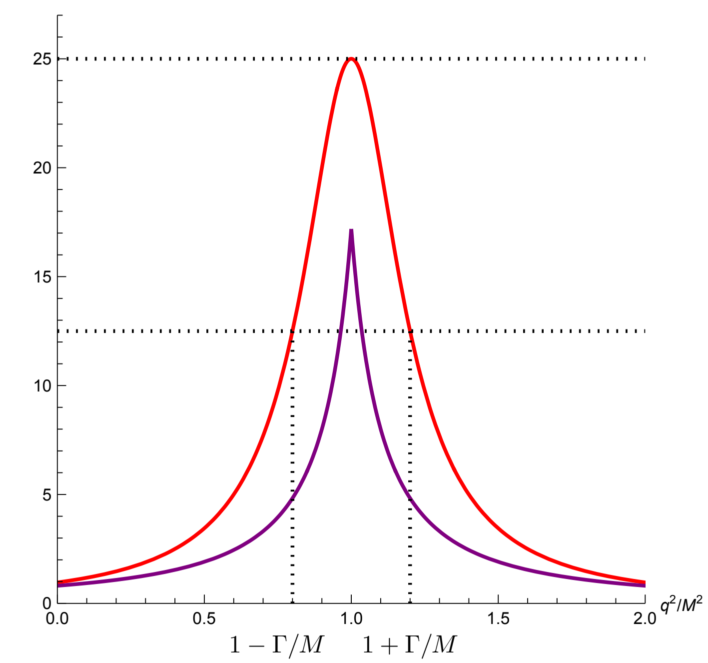


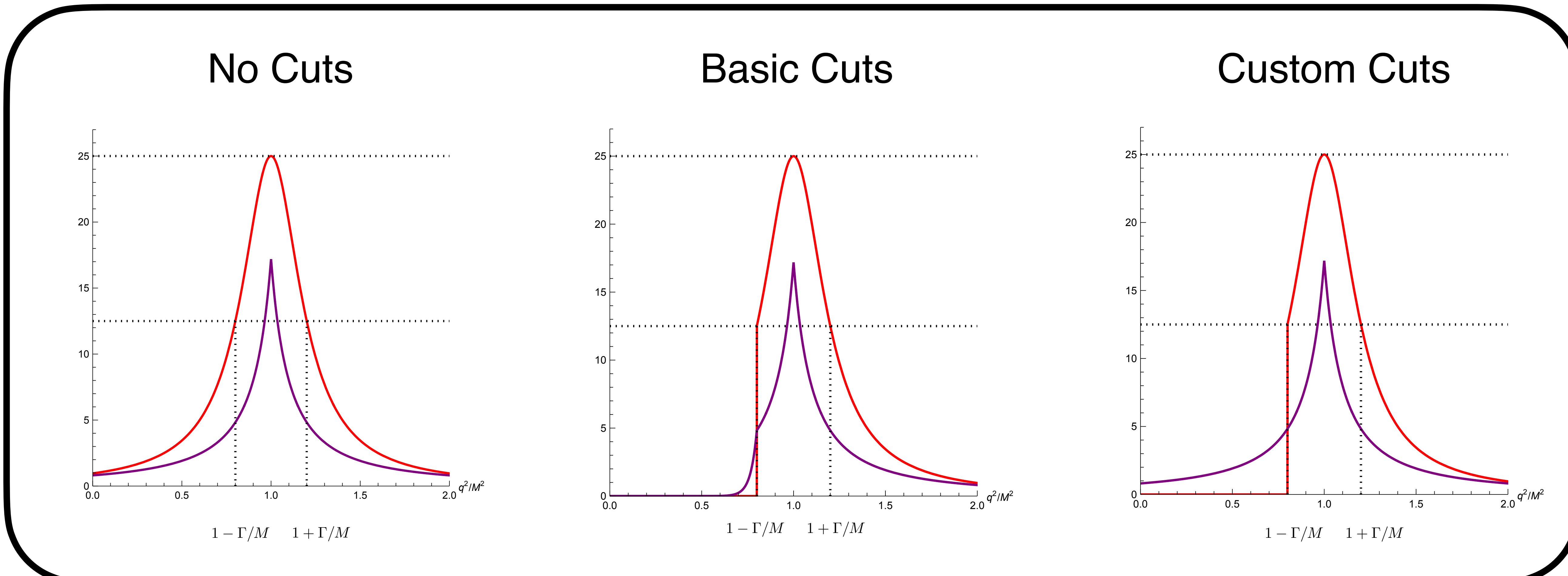
Probability of sampling random points over q^2

$p(q^2)$



Importance Sampling

- MC Integration is performed according to our best knowledge of the integrand
  - Basic cuts include in this knowledge
  - Custom cuts are ignored

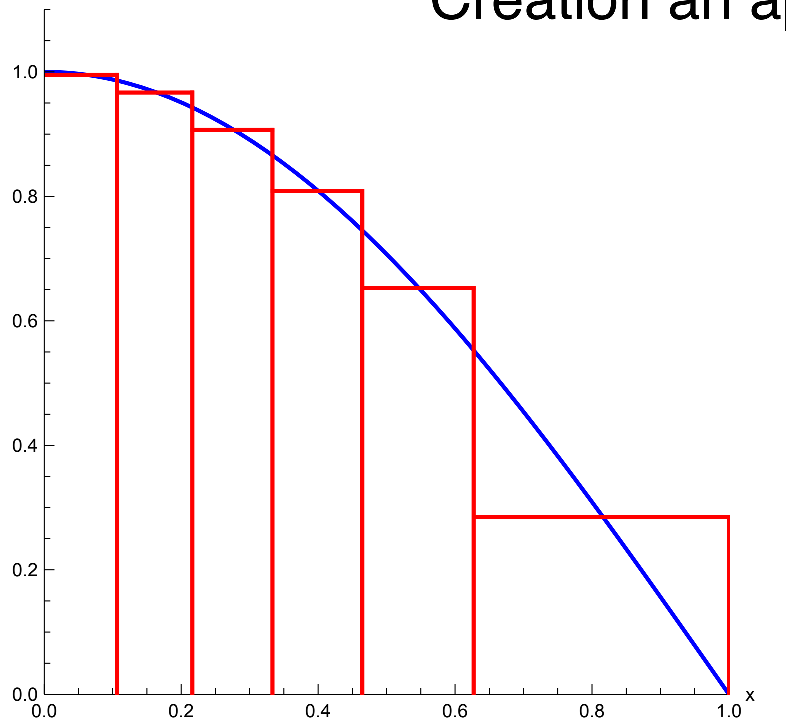


Importance Sampling

- Summary of key points
  - The phase-space parametrization is important for an efficient computation
  - The change of variable ensures that evaluation of integral is done where the integrand is largest !
  - Generate random points in a distribution which is close to the function to be integrated

Adaptative Monte Carlo integrator

Creation an approximation of the integrand on the fly



Algorithm:

- ★ Create bins such that each bin has the same contribution
- ★ Use the approximation for the importance sampling method

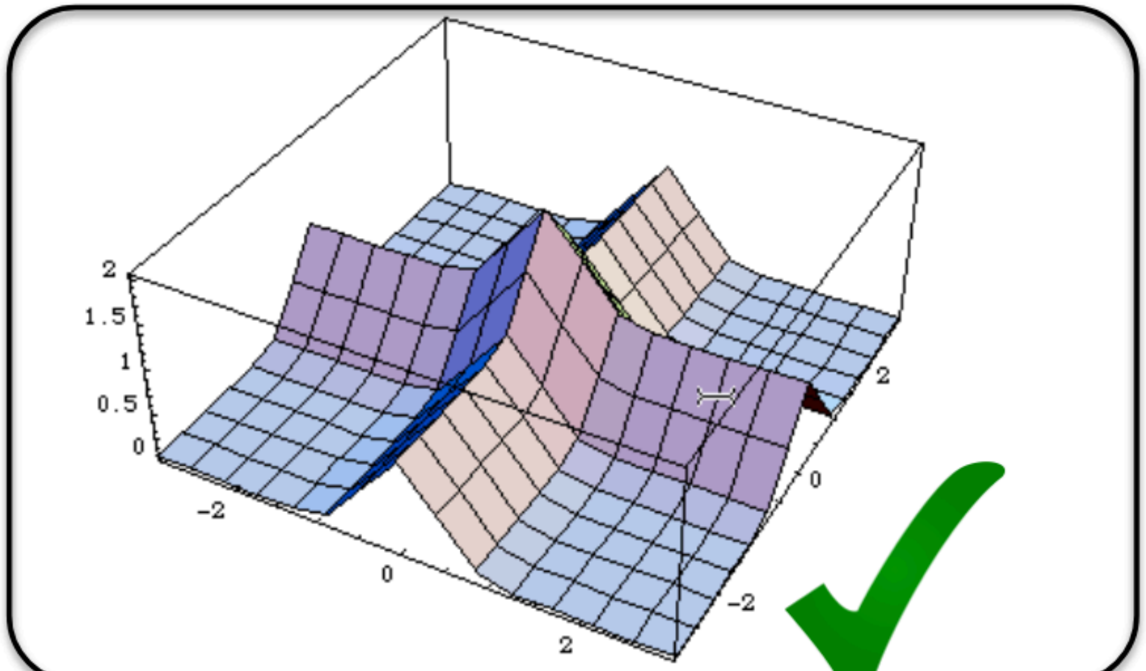
The VEGAS Algorithm

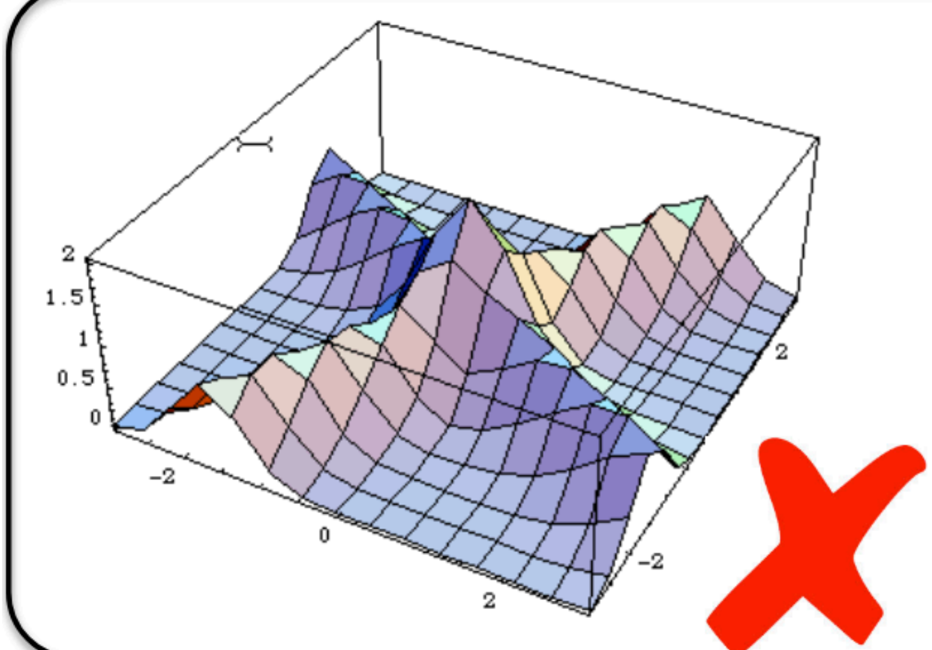
VEGAS for multi-dimensional integrations

Lepage (JCP'78)

- Use projection on the axis

$$p(x_1, x_2, \ldots, x_d) = p_1(x_1)p_2(x_2)\cdots p_d(x_d)$$





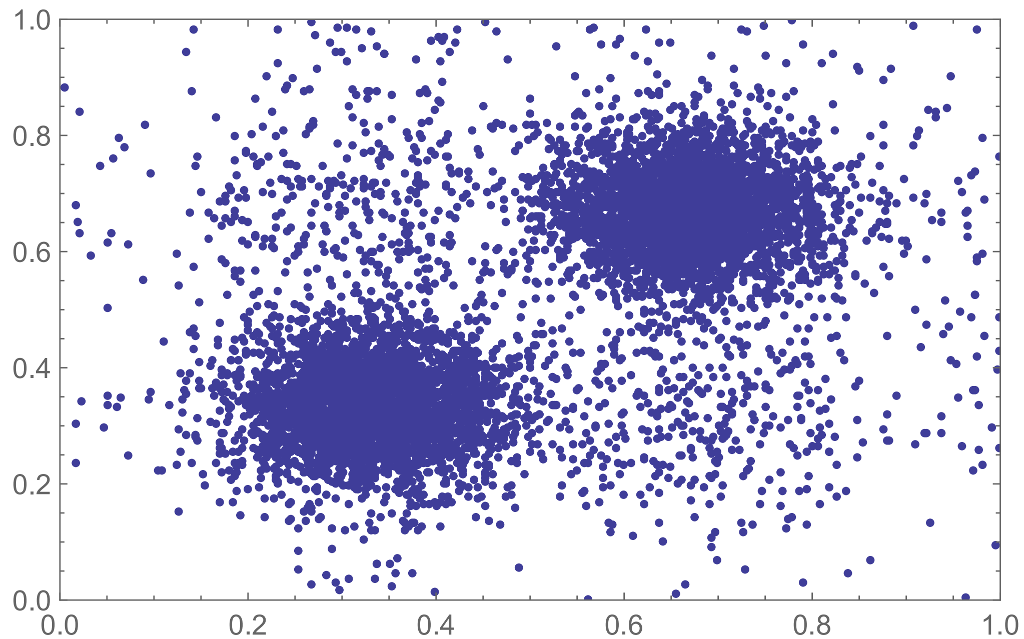
- We need to ensure the factorization !

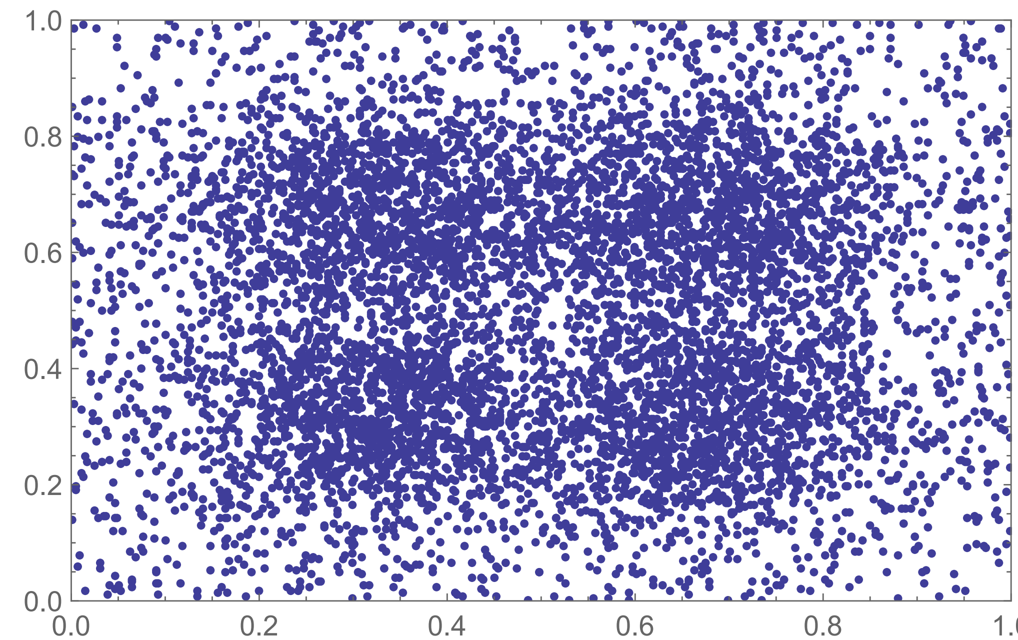
➡ Additional change of variable

Importance Sampling

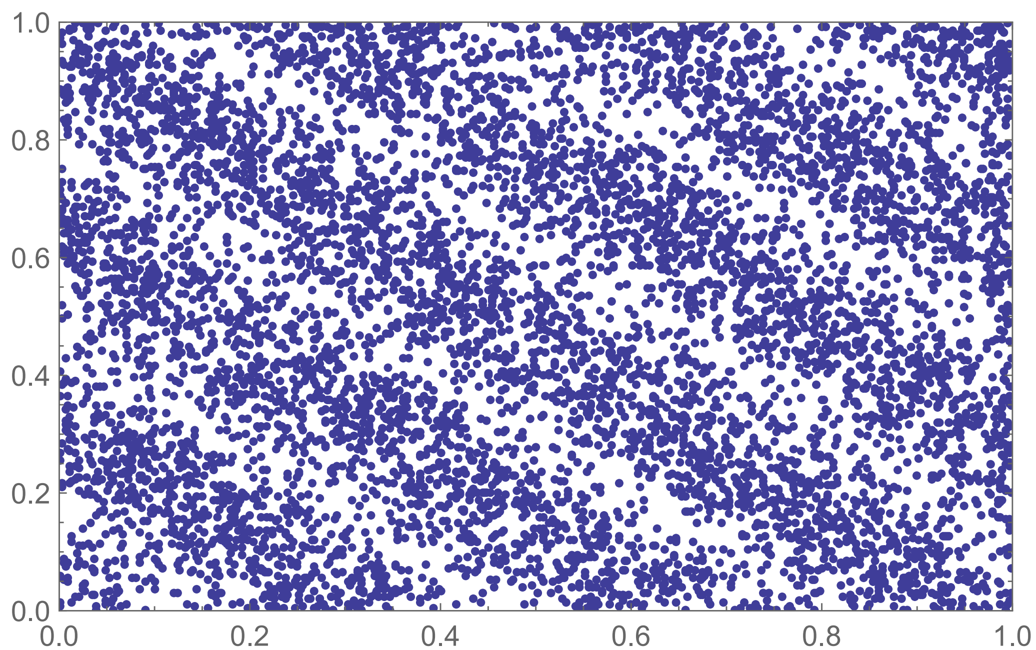
Truth | VEGAS

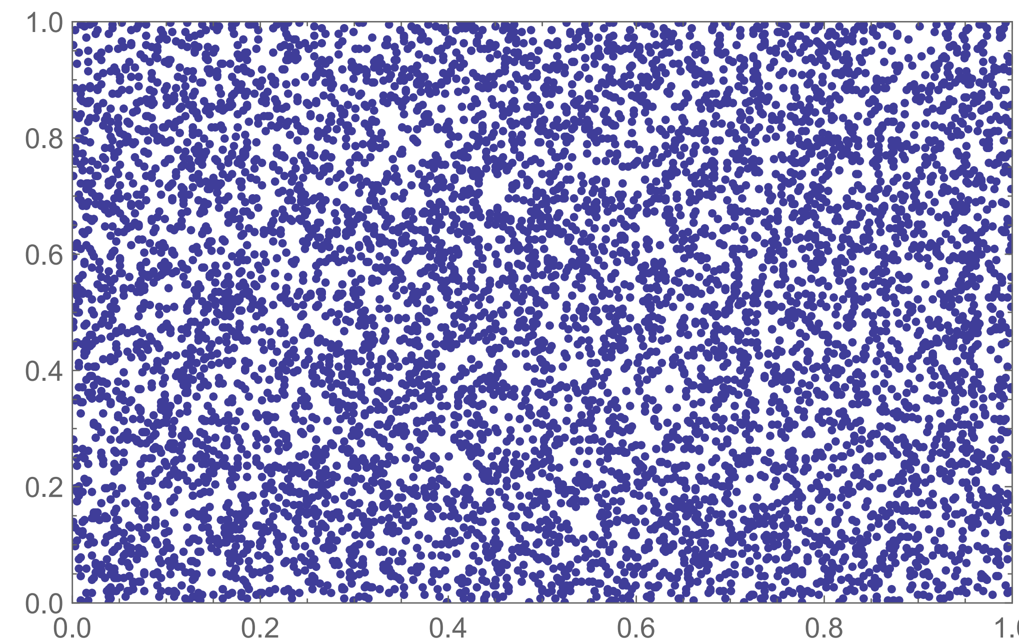
Double Gaussian Distribution





Oscillatory Distribution



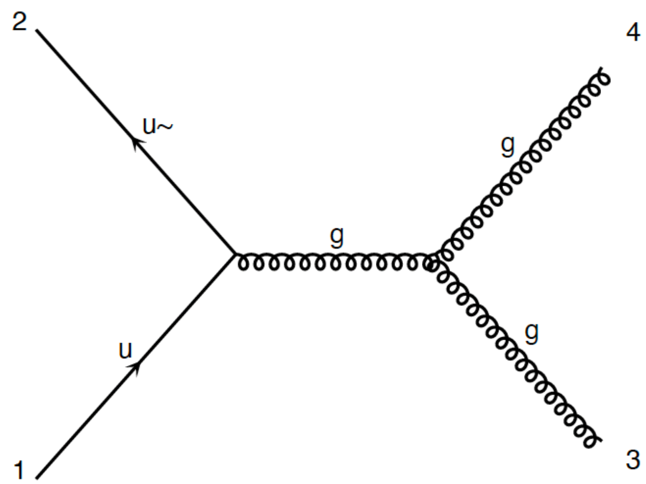


Phase Space Integrator

- An example: dijet production at the LHC

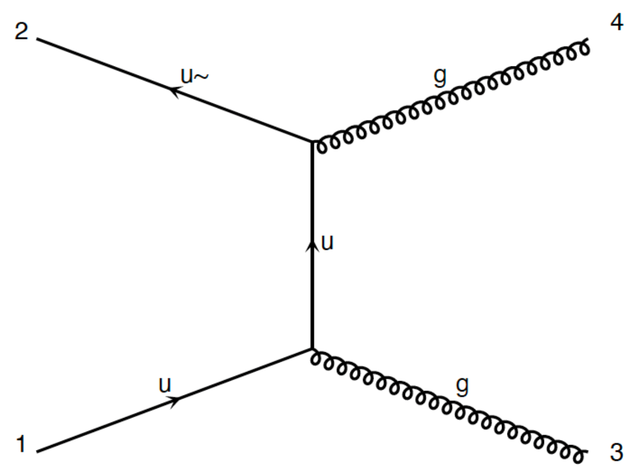
Main pole structure

s-channel



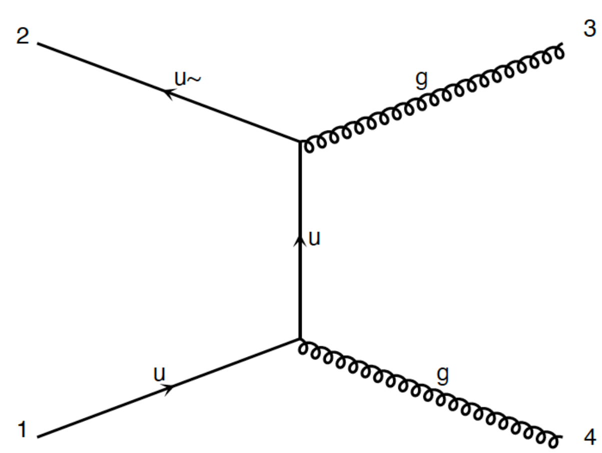
$$\mathcal{M}_s \propto \frac{1}{s} = \frac{1}{(p_1+p_2)^2}$$

t-channel



$$\mathcal{M}_t \propto \frac{1}{t} = \frac{1}{(p_1-p_3)^2}$$

u-channel



$$\mathcal{M}_u \propto \frac{1}{u} = \frac{1}{(p_1-p_4)^2}$$

Single-diagram enhanced multi-channelling

- Trick: split the complexity

Maltoni, Stelzer (JHEP'03)

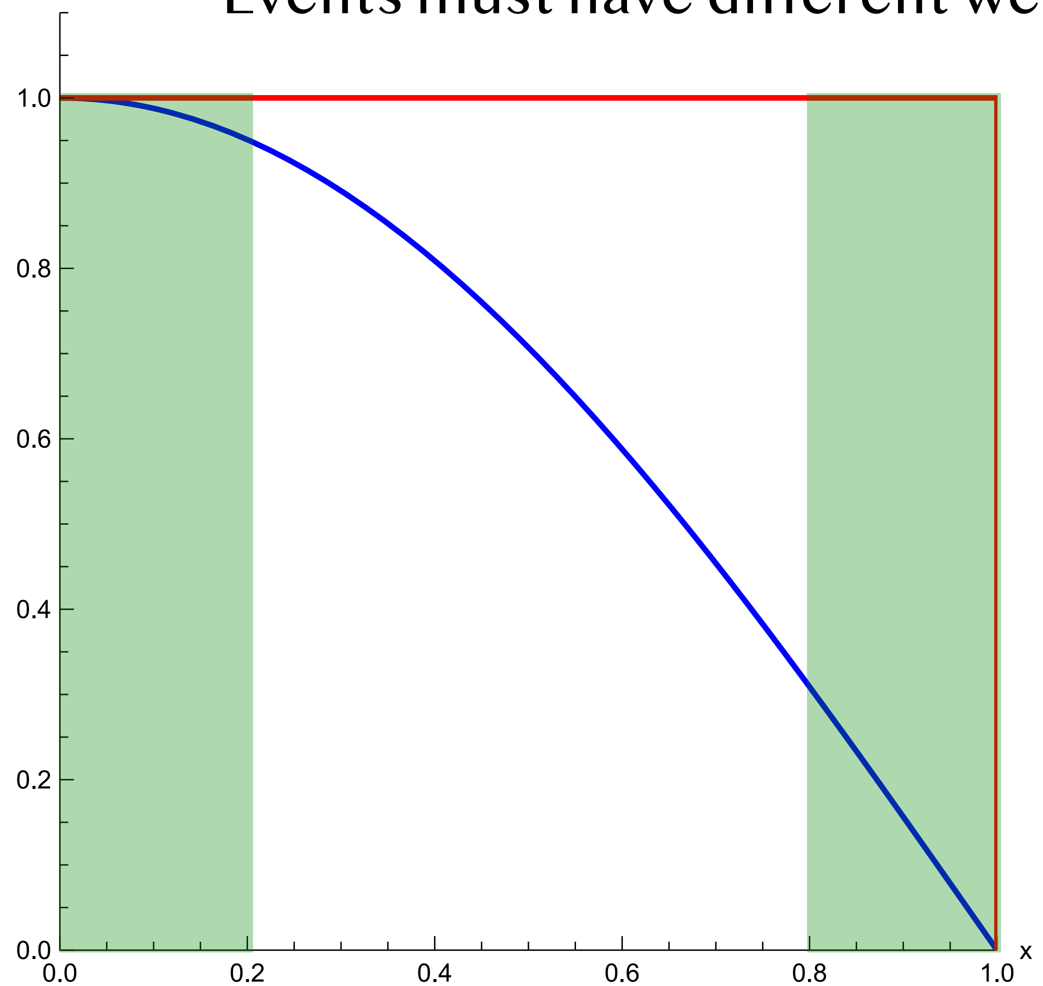
$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u$$

$$\hat{\sigma} = \frac{1}{2s}\int d\Phi_2 |\mathcal{M}|^2 = \frac{1}{2s}\sum_{i\in\{s,t,u\}}\int d\Phi_2 |\mathcal{M}_i|^2 \underbrace{\boxed{\frac{|\mathcal{M}|^2}{|\mathcal{M}_s|^2+|\mathcal{M}_t|^2+|\mathcal{M}_u|^2}}}_{\sim C}$$

- Any single diagram is "easy" to integrate (pole structures & good integration variables known)
- Divide the single integration into multi channels, based on diagrams/topologies
- All other peaks taken care of by denominator sum
- Errors in quadrature (no extra computational cost)
- Each diagram is calculated only once
- Paralle in nature

Event Generation

- Weighted events
  - Same number of events in areas of phase space with very different probabilities
    - ➡ Events must have different weights



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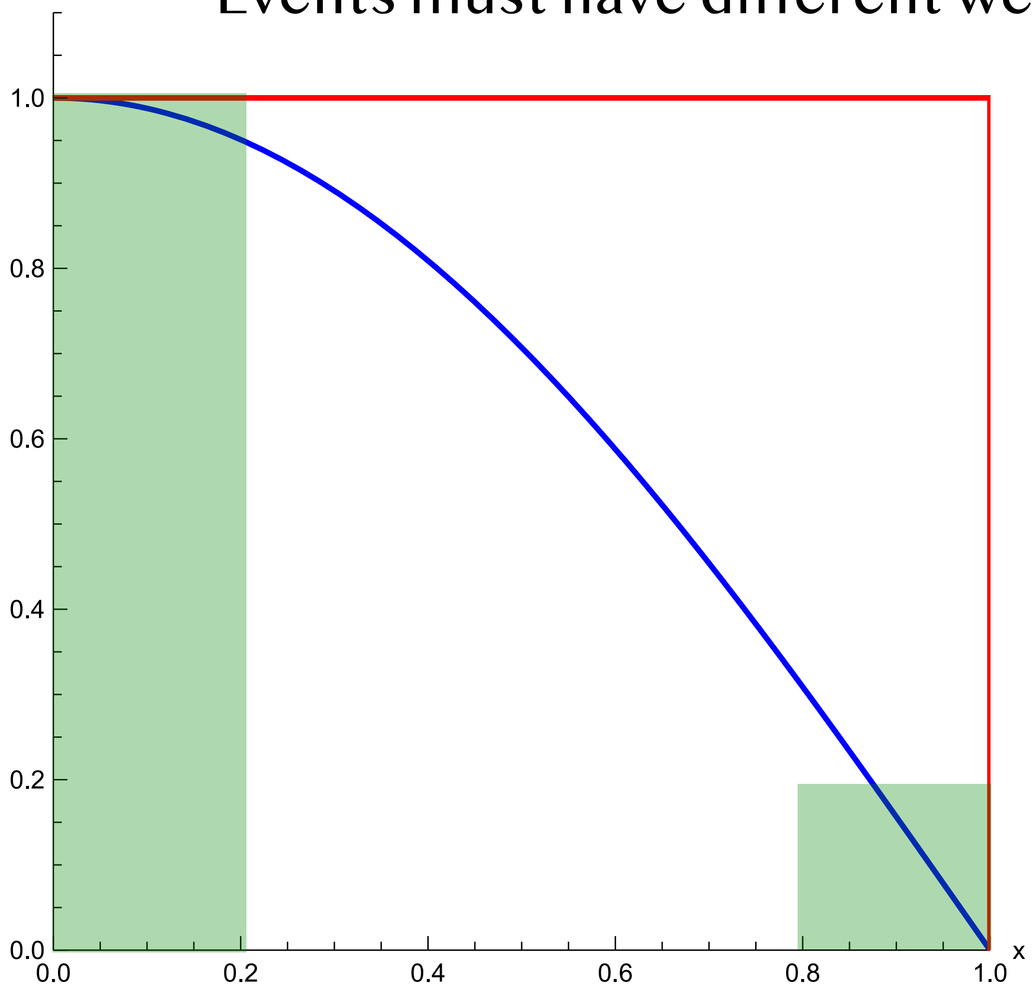

- Un-Weighted events
  - Number of events proportional to the probability of areas of phase space
    - ➡ Events have the same weight (“unweighted”)
    - ➡ Events distributed as in Nature

Event Generation

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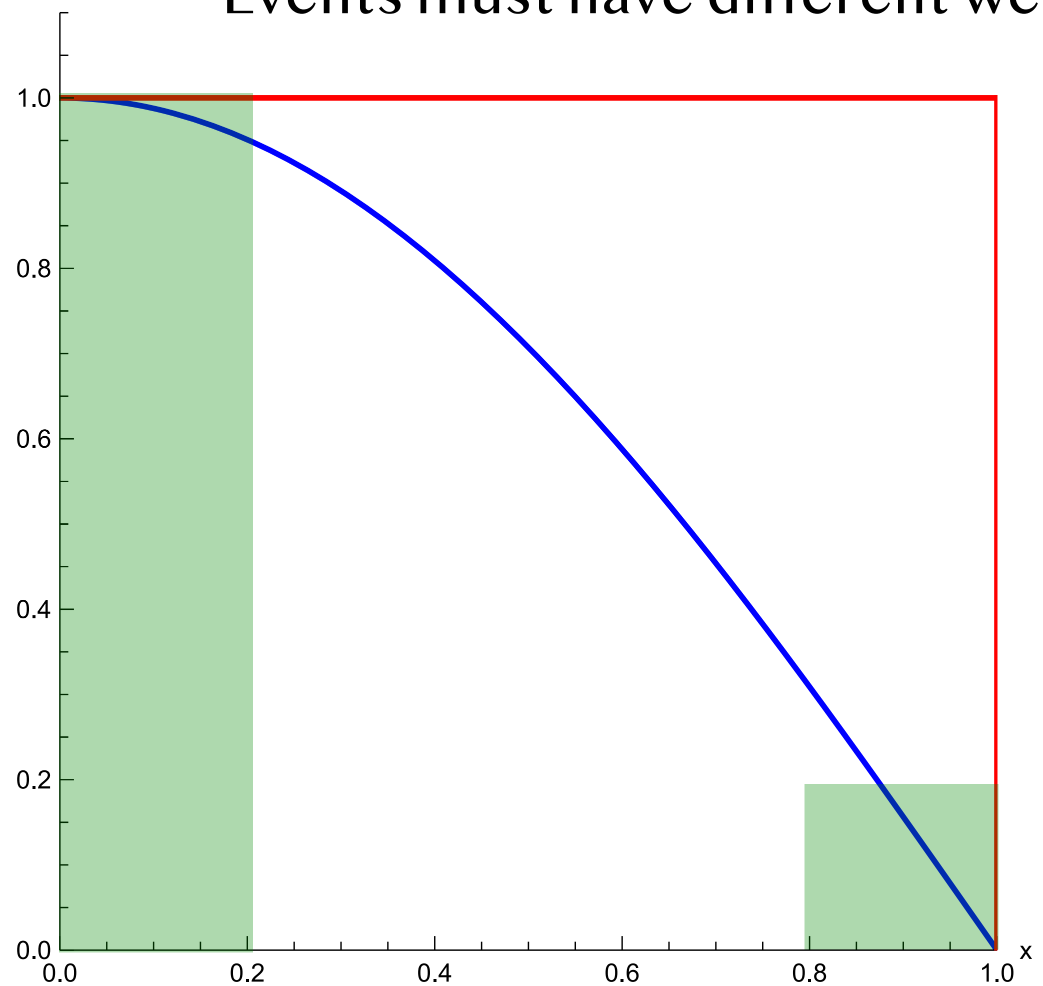


Unweight procedure
(acceptance-rejection method):

$$\int_0^1 dx f(x) = \frac{1}{N}\sum_{i=1}^{N} f(x_i) = \frac{1}{N}\sum_{i=1}^{N} P(x_i)\max(f)$$

$$P(x_i) = \frac{f(x_i)}{\max(f)}$$

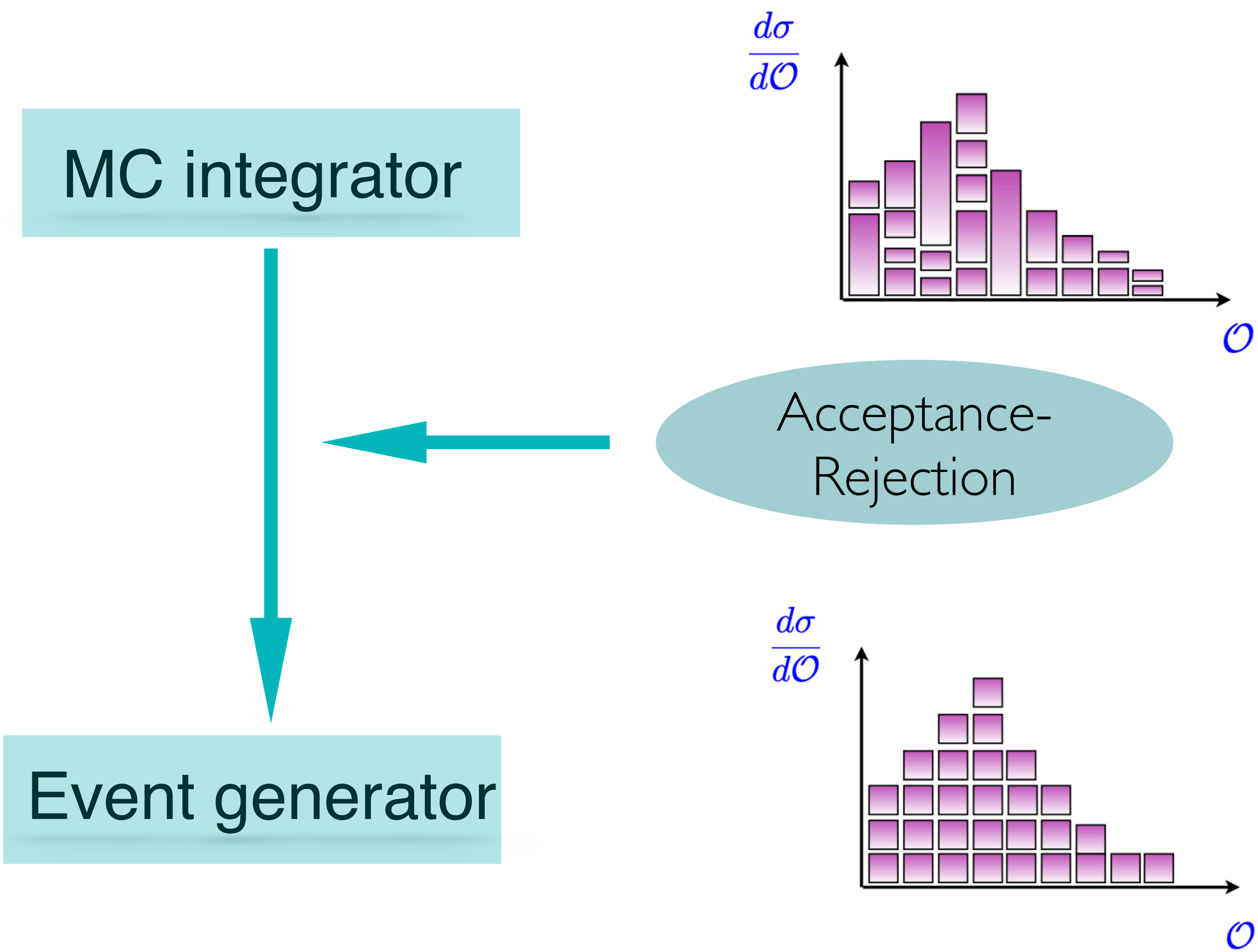
★ Generate a uniform random number x between [0,1]

★ Generate a second uniform random number y between [0,max(f)]

★ If y < f(x), accept x, otherwise reject x and repeat

- Un-Weighted events
  - Number of events proportional to the probability of areas of phase space
    - ➡ Events have the same weight (“unweighted”)
    - ➡ Events distributed as in Nature

Event Generation



☞ **This is possible only if f(x)<∞ AND has definite sign!**

Lecture 1

LO Event Simulations

Multi-jet merging techniques

How To Improve Theoretical Predictions ?

- **Common Principle:**
  - Adopt the advantages of different aspects

Hard interaction	Parton shower
Hard and Resolved	Soft and/or Collinear

  - Balance the accuracy over the steps in the simulation chain
  - Improve not only the single steps but also their merging

- **Two main directions:**
  - Matching
    - Avoid double counting between N^kLO ($k>0$) MEs and PSs
  - Merging
    - Include more real radiation MEs (formally higher order, improve kin.)
    - Subtract double counting of real radiation from MEs and PSs

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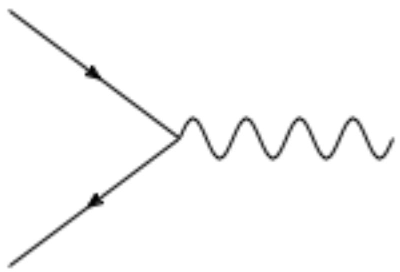
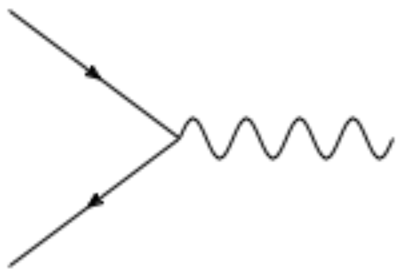
A Double-Counting Issue

$X + 0j$



A Double-Counting Issue

Parton shower



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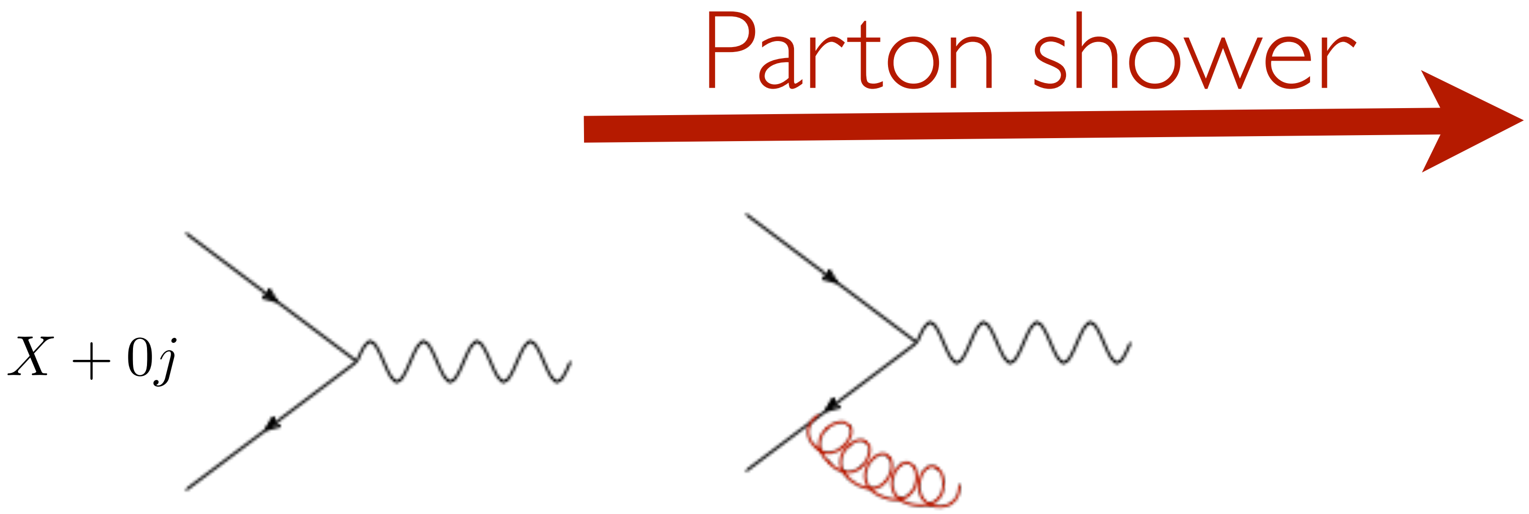

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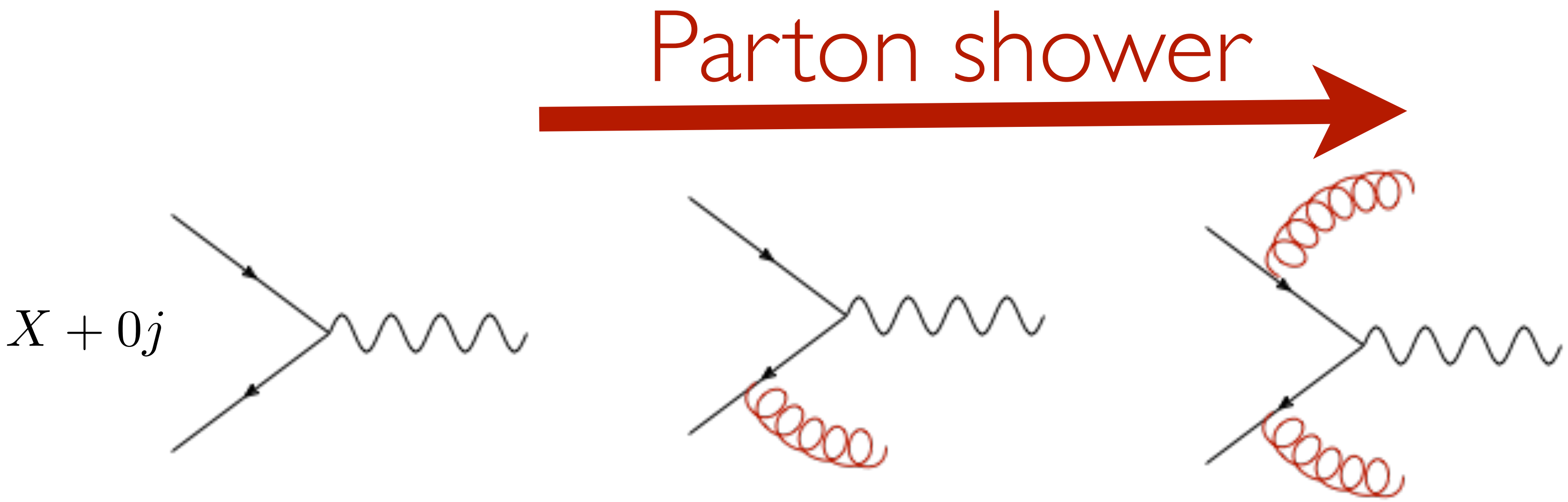

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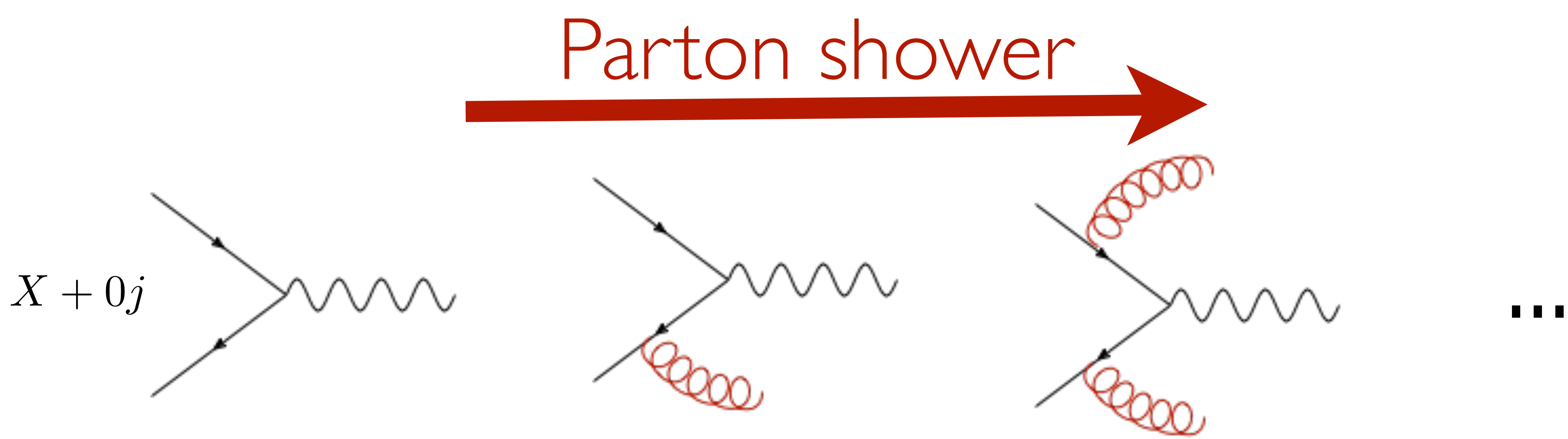

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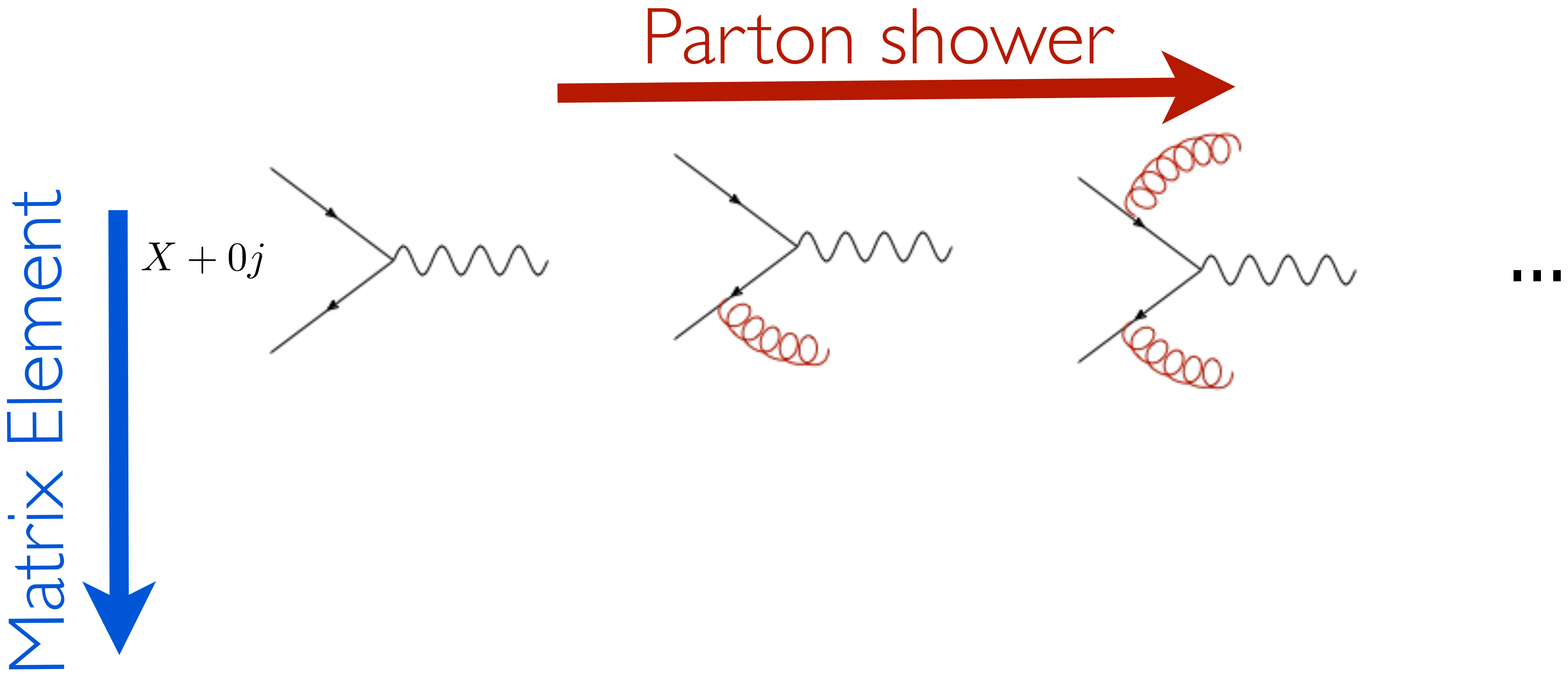

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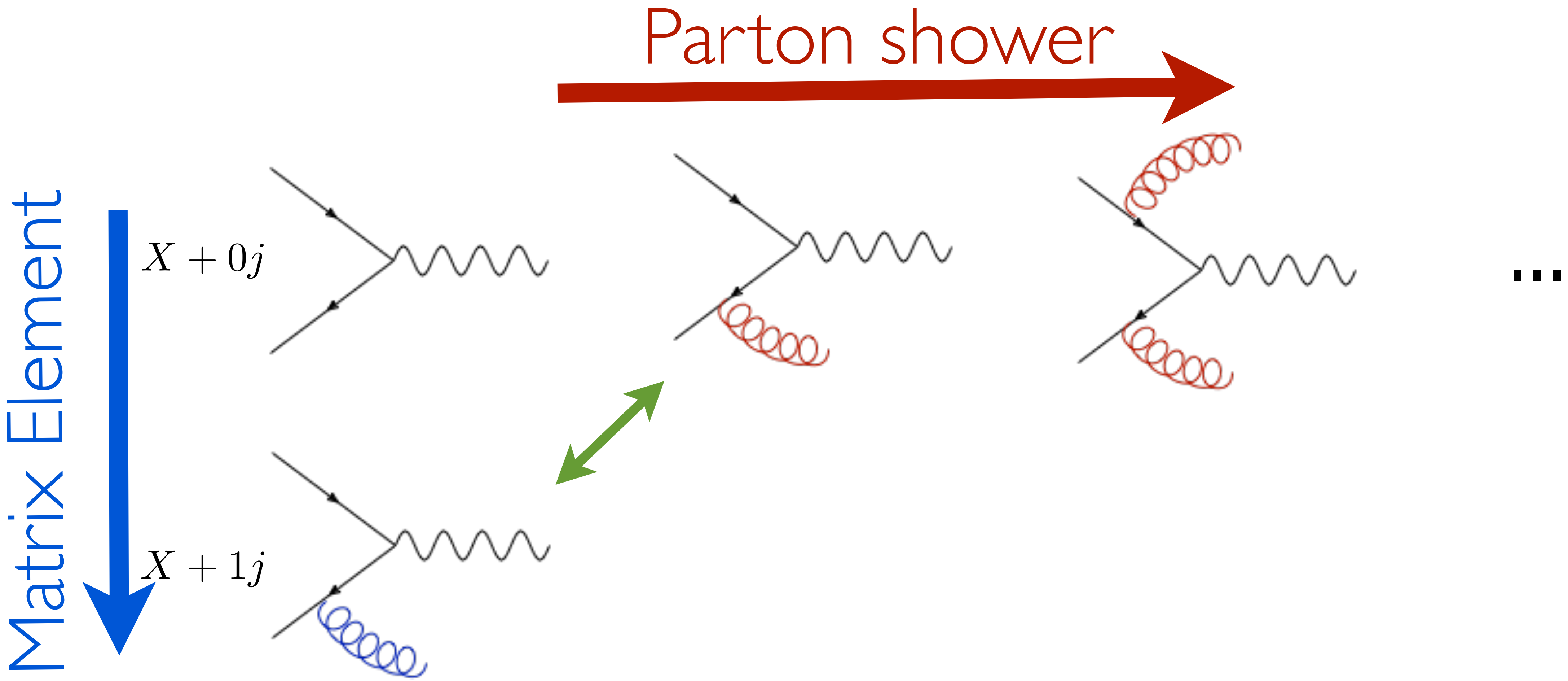

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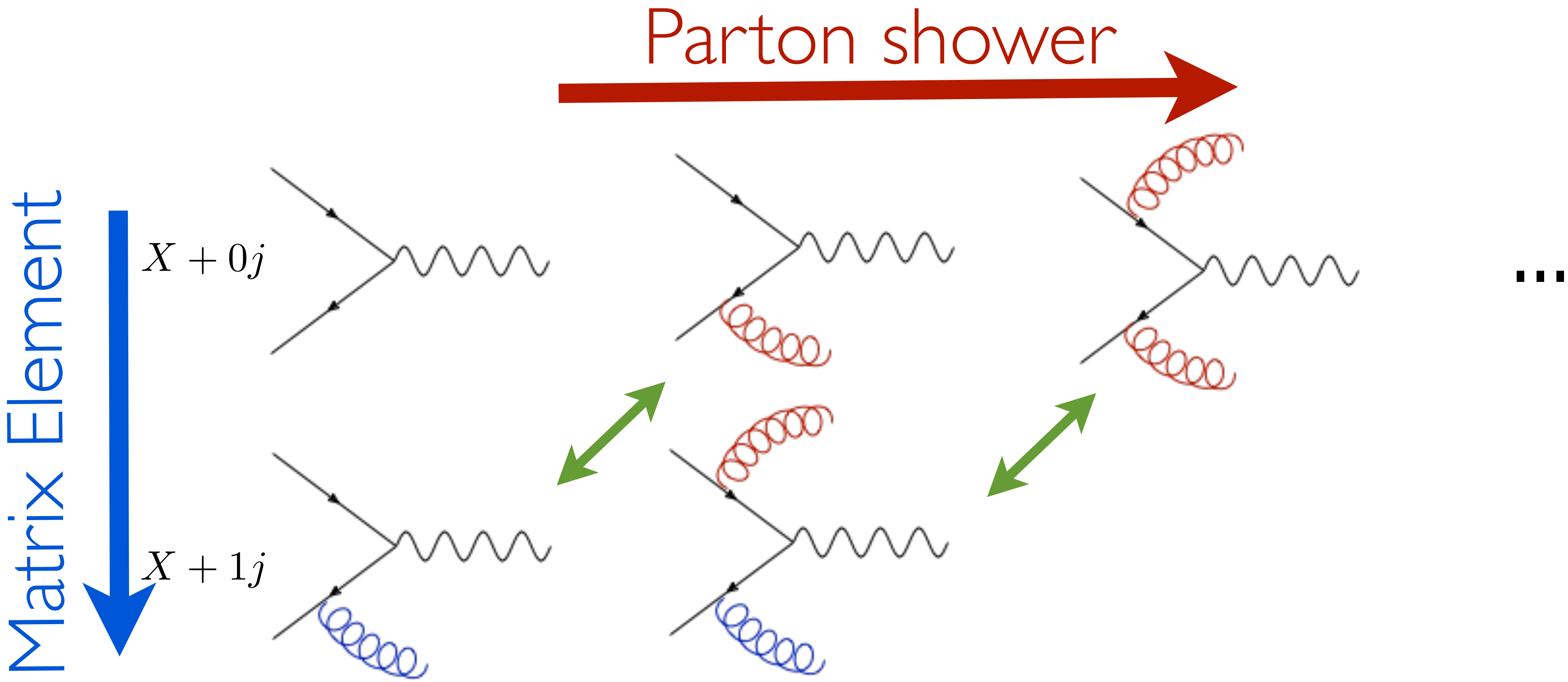

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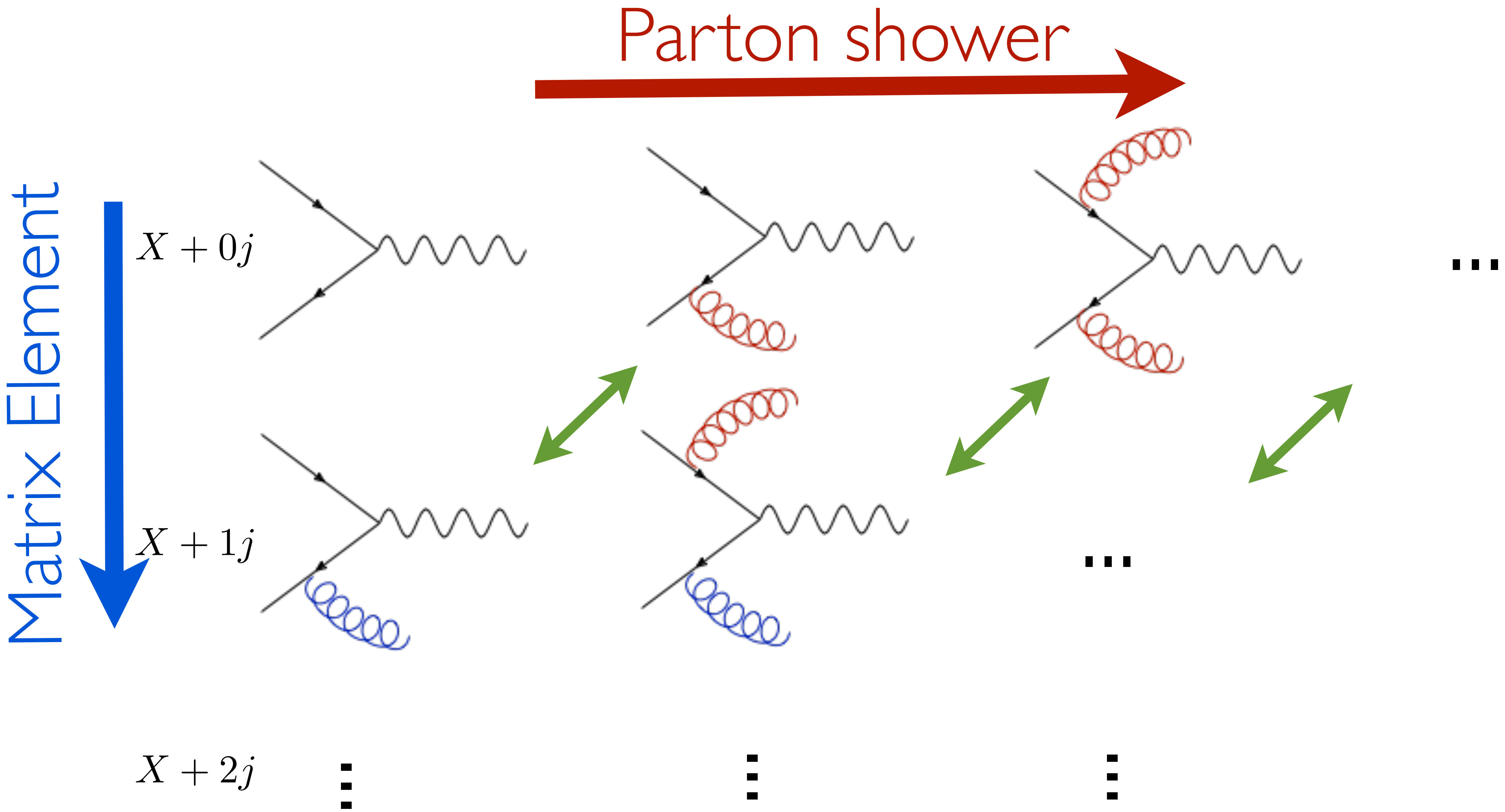

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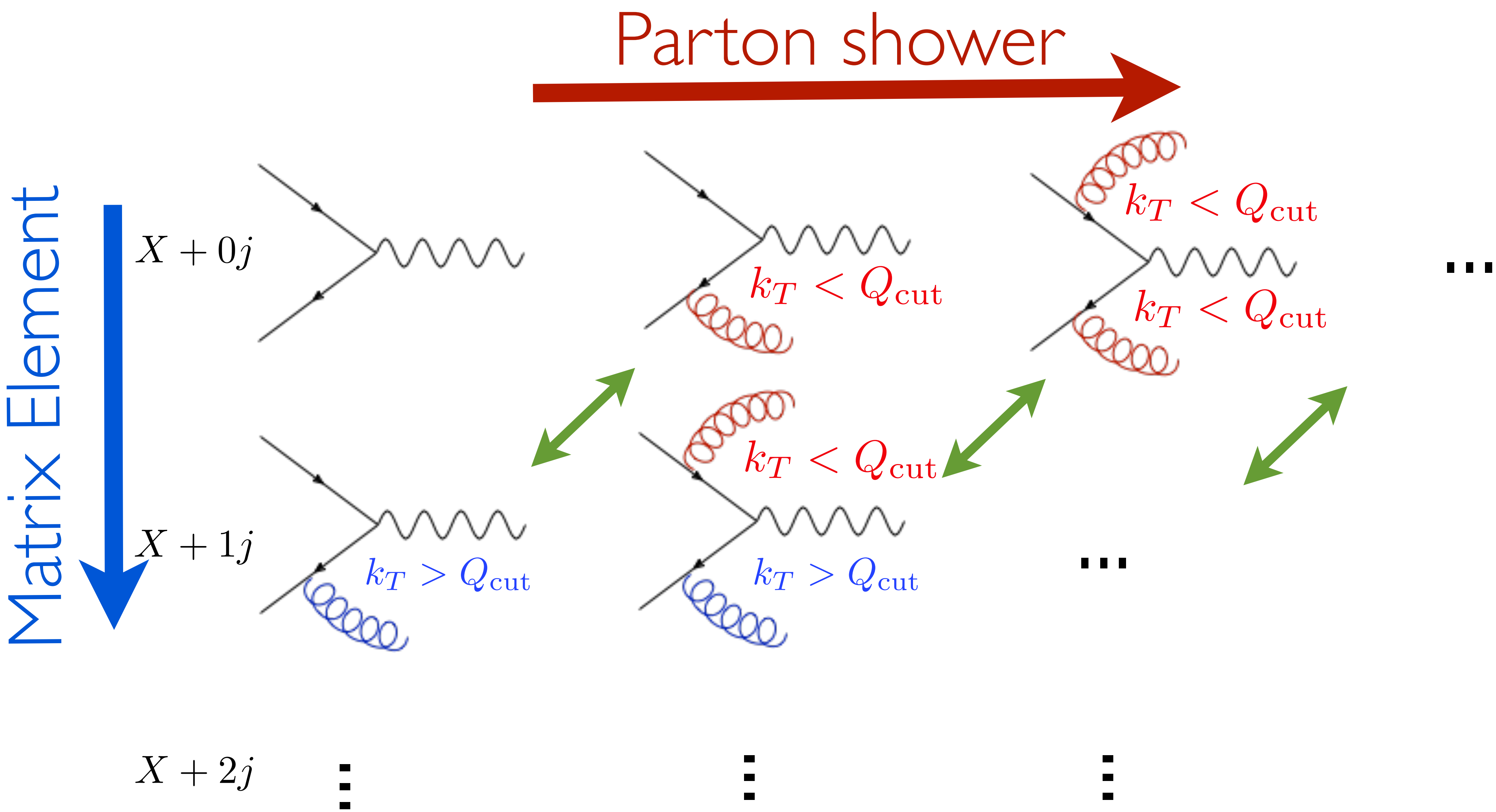

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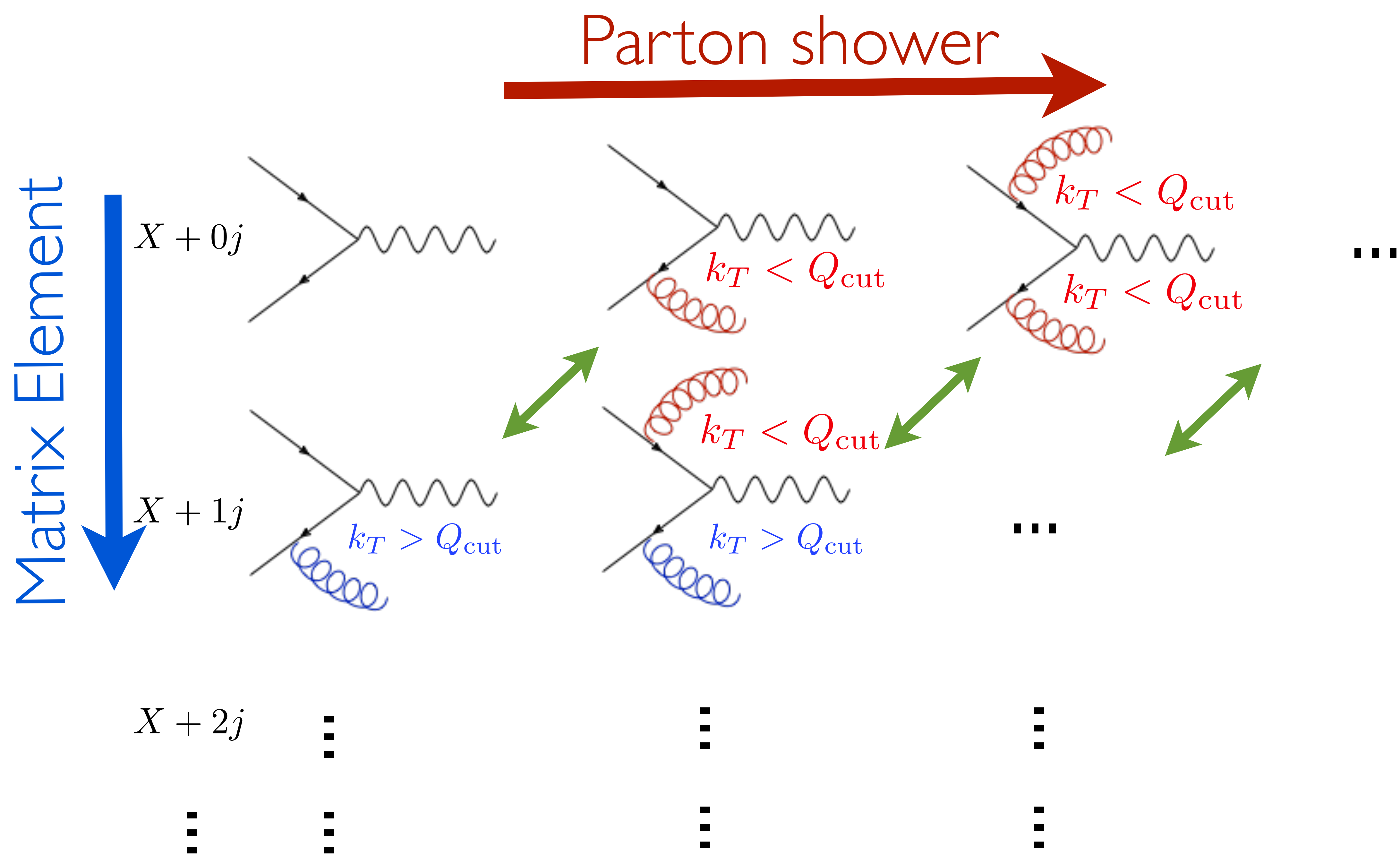
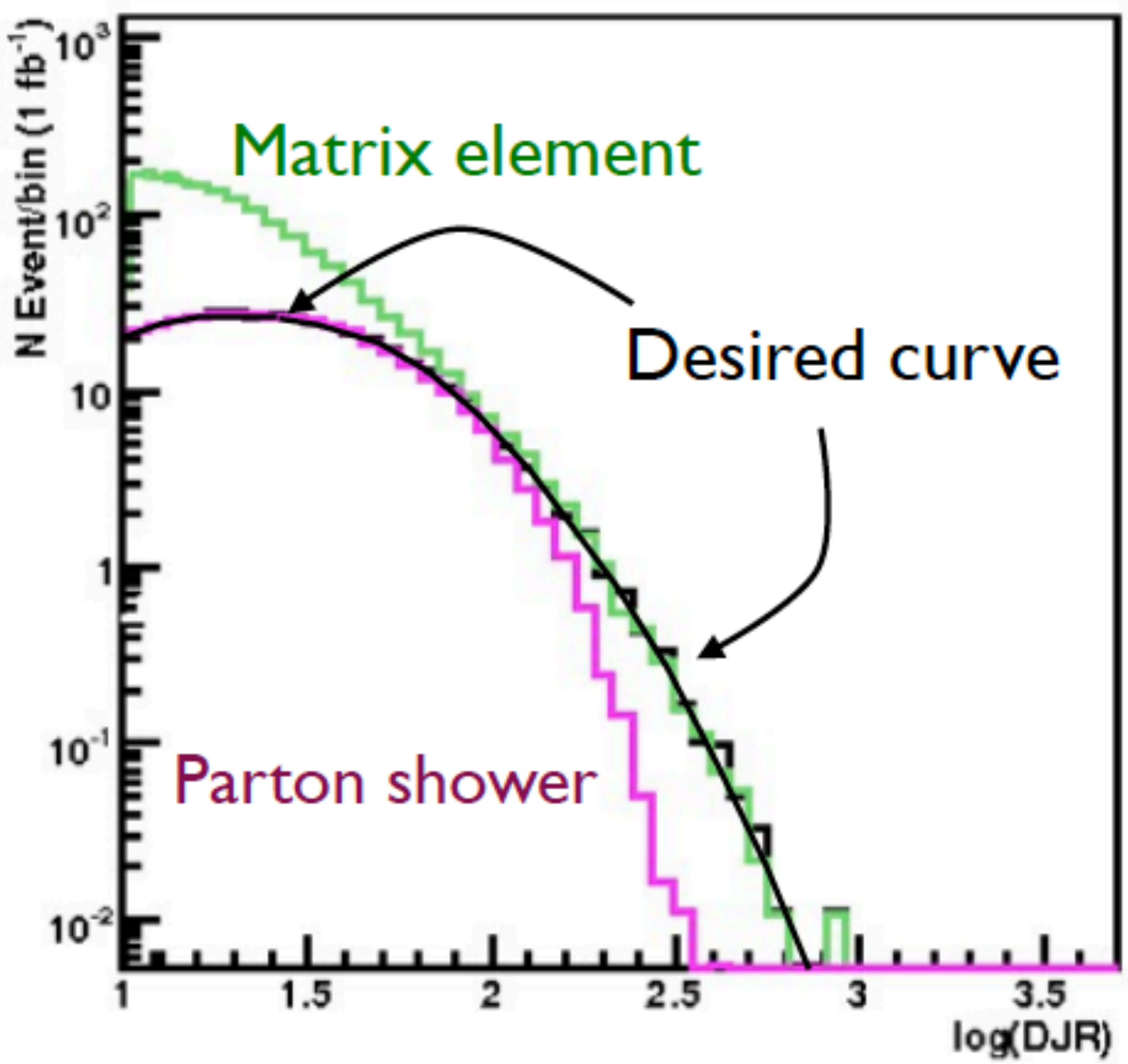


Multi-jet Merging

- Solve double-counting issue by dividing phase space in hard and soft regions
  - Generating ME samples with different jet multiplies with Q_{cut}
  - Q_{cut} is called merging scale (arbitrary but rule-of-thumb guidance)
  - Making exclusive by reweighting with no-emission probabilities
    - Also reweighing strong coupling and PDF ratios
  - Using normal parton shower in “soft region” below Q_{cut}
- Different multi-jet merging algorithms (will not go into details here)
  - CKKW(-L) merging scheme Catani, Krauss, Kuhn, Webber (JHEP'01); Lonnblad (JHEP'02)
  - MLM merging scheme Mangano (2002); Mangano, Moretti, Piccinini, Treccani (JHEP'07)
- Possible issues:
  - Merging scale dependence
  - Merging scale might not be defined in terms of shower evolution variable
  - Might break unitarity of shower

Multi-jet Merging

- Regularization of matrix element divergence
- Correction of the parton shower for large momenta
- Smooth jet distributions



2nd QCD radiation jet in top pair production at the LHC, using MadGraph + Pythia

Try to play/run MadGraph5_aMC@NLO according to 2 tutorials

1. Basic LO runs

https://indico.ihep.ac.cn/event/7822/contributions/98094/attachments/52130/60094/Tutorial_MG5aMC_Hefei2018_Basics.pdf

2. Multi-jet merging

https://indico.ihep.ac.cn/event/7822/contributions/98061/attachments/52120/60084/Tutorial_MG5aMC_Hefei2018_Merging.pdf

MG5aMC installation instruction

https://indico.ihep.ac.cn/event/7822/page/1116-tool-installation

Lecture 2

NLO Calculations &
Event Simulations at NLO QCD

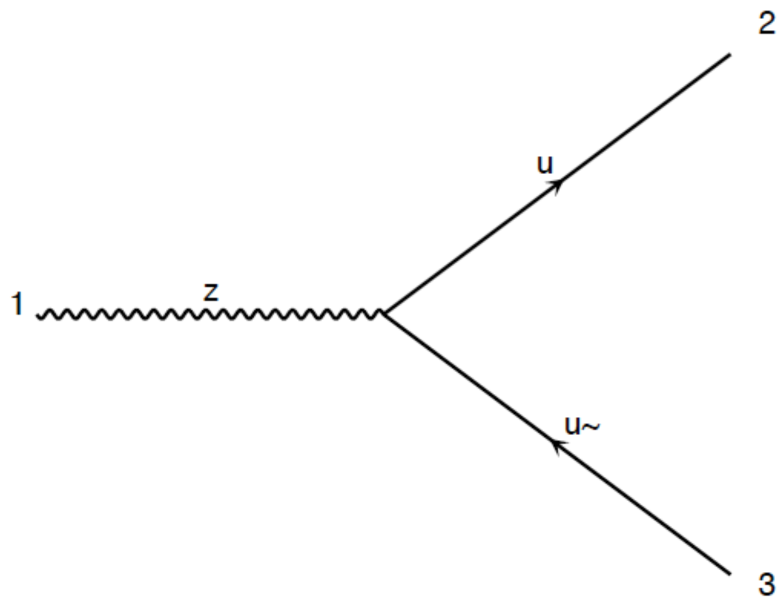
Lecture 2

NLO Calculations &
Event Simulations at NLO QCD

A NLO example

A NLO Example: Born

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Writing down Born amplitude according to Feynman rules**



For simplicity, we assume quarks are massless

$$\mathcal{A}_{\text{Born}} = -\delta_{c_q c_{\bar{q}}} \varepsilon_\mu(p_Z) \bar{u}(p_q).\Gamma^\mu_{Zq\bar{q}}.v(p_{\bar{q}})$$

$$\Gamma^\mu_{Zq\bar{q}} = ie\left(\frac{I_q}{\cos\theta_w \sin\theta_w} - Q_q \frac{\sin\theta_w}{\cos\theta_w}\right)\gamma^\mu P_L - ieQ_q\frac{\sin\theta_w}{\cos\theta_w}\gamma^\mu P_R$$

  - **Squaring amplitude, summing over colours and spins, and averaging the spin of the initial state**

$$\boxed{\alpha = \frac{e^2}{4\pi}}$$

$$\overline{\sum}|\mathcal{A}_{\text{Born}}|^2 = 8\pi\alpha m_Z^2\left(2Q_q^2\left(\frac{\sin\theta_w}{\cos\theta_w}\right)^2 - 2\frac{I_q Q_q}{\cos^2\theta_w} + \frac{I_q^2}{\cos^2\theta_w \sin^2\theta_w}\right)$$

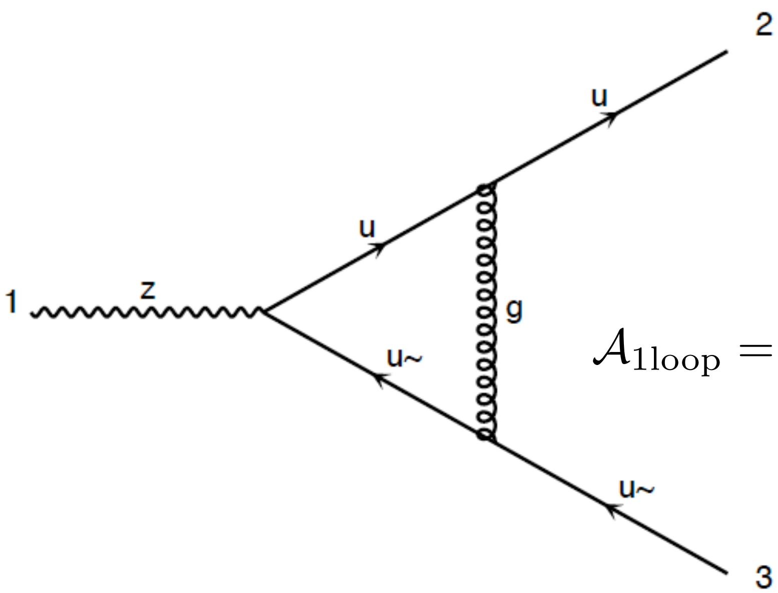
  - **Phase-space integration**

$$\Gamma_{\text{Born}}(Z\to q\bar{q}) = \frac{1}{2m_Z}\int (2\pi)^4\delta^4(p_Z - p_q - p_{\bar{q}})\frac{1}{(2\pi)^{3\times 2}}\frac{d^3p_q}{2E_q}\frac{d^3p_{\bar{q}}}{2E_{\bar{q}}}\overline{\sum}|\mathcal{A}_{\text{Born}}|^2$$
$$= \alpha m_Z\left(Q_q^2\frac{\sin^2\theta_w}{\cos^2\theta_w} - \frac{Q_q I_q}{\cos^2\theta_w} + \frac{I_q^2}{2\cos^2\theta_w\sin^2\theta_w}\right)$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Writing down one-loop amplitude according to Feynman rules**

For simplicity, we assume quarks are massless



$$\mathcal{A}_{1\text{loop}} = i g^{\nu\rho}\varepsilon_\mu(p_Z)\bar{u}(p_q).\left(-i g_s \gamma_\nu T^a_{c_q c}\right).\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{\bar{\not{l}}.\Gamma^\mu_{Zq\bar{q}}.\left(\bar{\not{l}}-\not{p}_Z\right)}{\bar{l}^2\left(\bar{l}-p_q\right)^2\left(\bar{l}-p_Z\right)^2}.\left(-i g_s \gamma_\rho T^a_{cc_{\bar{q}}}\right).v(p_{\bar{q}})$$

  - **Need to evaluate two tensor integrals**

$$I_1^\mu = \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{\bar{l}^\mu}{\bar{l}^2\left(\bar{l}-p_q\right)^2\left(\bar{l}-p_Z\right)^2} \qquad I_2^{\mu\nu} = \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{\bar{l}^\mu\bar{l}^\nu}{\bar{l}^2\left(\bar{l}-p_q\right)^2\left(\bar{l}-p_Z\right)^2}$$

according to Lorentz structures

$$I_1^\mu = p_q^\mu B_1 + p_Z^\mu B_2 \qquad I_2^{\mu\nu} = g^{\mu\nu}B_{00} + p_q^\mu p_q^\nu B_{11} + p_Z^\mu p_Z^\nu B_{22} + \left(p_q^\mu p_Z^\nu + p_Z^\mu p_q^\nu\right)B_{12}$$

Solving the coefficients B, e.g.

$$p_q\cdot I_1 = p_q^2 B_1 + p_q\cdot p_Z B_2 = p_q\cdot p_Z B_2 \qquad p_Z\cdot I_1 = p_q\cdot p_Z B_1 + p_Z^2 B_2 = p_q\cdot p_Z B_1 + m_Z^2 B_2$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Solving the coefficients B, e.g.

$$B_2 = \frac{p_q \cdot I_1}{p_q \cdot p_Z} \qquad B_1 = \frac{p_Z \cdot I_1 - m_Z^2 B_2}{p_q \cdot p_Z}$$

$$\begin{aligned}
p_q \cdot I_1 &= \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{p_q \cdot \bar{l}}{\bar{l}^2 \left(\bar{l}-p_q\right)^2 \left(\bar{l}-p_Z\right)^2} \\
&= \frac{1}{2} \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{\bar{l}^2 - \left(\bar{l}-p_q\right)^2}{\bar{l}^2 \left(\bar{l}-p_q\right)^2 \left(\bar{l}-p_Z\right)^2} \\
&= \frac{1}{2} \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\left(\bar{l}-p_q\right)^2 \left(\bar{l}-p_Z\right)^2} - \frac{1}{2} \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 \left(\bar{l}-p_Z\right)^2} \\
&= \frac{1}{2} \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 \left(\bar{l}-p_{\bar{q}}\right)^2} - \frac{1}{2} \int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 \left(\bar{l}-p_Z\right)^2}
\end{aligned}$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2\left(\bar{l}-p_{\bar{q}}\right)^2} = \int_0^1 dx \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\left[x\bar{l}^2+(1-x)\left(\bar{l}-p_{\bar{q}}\right)^2\right]^2}$$

Feynman parameterization !

$$= \int_0^1 dx \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\left(\bar{l}-(1-x)p_{\bar{q}}\right)^4}$$

Using on-shell condition !

$$= \int_0^1 dx \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\left(\bar{l}^2\right)^2}$$

Translational invariance !

$$= \int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\left(\bar{l}^2\right)^2}$$

Integration over x !

$$= \int \frac{d\bar{l}_0 d^{d-1}\vec{\bar{l}}}{(2\pi)^d}\frac{1}{\left(\bar{l}_0^2-|\vec{\bar{l}}|^2\right)^2}$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 \left(\bar{l}-p_{\bar{q}}\right)^2} \overset{\bar{l}_0 \to i\bar{l}_0}{=} \frac{i}{(2\pi)^d} \int d\Omega_d \int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Wick rotation & spherical coordinate !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Integration over solid angle !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \left(\int_0^1 d|\bar{l}||\bar{l}|^{d-5} + \int_1^{+\infty} d|\bar{l}||\bar{l}|^{d-5} \right)$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2\left(\bar{l}-p_{\bar{q}}\right)^2} \stackrel{\bar{l}_0 \to i\bar{l}_0}{=} \frac{i}{(2\pi)^d} \int d\Omega_d \int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Wick rotation & spherical coordinate !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Integration over solid angle !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \left(\underline{\int_0^1 d|\bar{l}||\bar{l}|^{d-5}} + \underline{\int_1^{+\infty} d|\bar{l}||\bar{l}|^{d-5}}\right)$$

$|\bar{l}| \to 0$ (IR): the integral is divergent when $d \leq 4$

$|\bar{l}| \to +\infty$(UV): the integral is divergent when $d \geq 4$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2\left(\bar{l}-p_{\bar{q}}\right)^2} \overset{\bar{l}_0\to i\bar{l}_0}{=} \frac{i}{(2\pi)^d}\int d\Omega_d \int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Wick rotation & spherical coordinate !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\int_0^{+\infty} d|\bar{l}||\bar{l}|^{d-5}$$

Integration over solid angle !

$$= \frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\left(\underline{\int_0^{1} d|\bar{l}||\bar{l}|^{d-5}} + \underline{\int_1^{+\infty} d|\bar{l}||\bar{l}|^{d-5}}\right)$$

$|\bar{l}|\to 0$ (IR): the integral is divergent when $d\leq 4$

$|\bar{l}|\to +\infty$(UV): the integral is divergent when $d\geq 4$

Regularisations:

$$d=4-2\epsilon_{\rm IR},\epsilon_{\rm IR}\to 0-$$
$$d=4-2\epsilon_{\rm UV},\epsilon_{\rm UV}\to 0+$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2\left(\bar{l}-p_{\bar{q}}\right)^2}=\frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\left(-\frac{1}{2\epsilon_{\mathrm{IR}}}+\frac{1}{2\epsilon_{\mathrm{UV}}}\right)$$

  - **Squaring with Born amplitude, summing over colours and spins, and averaging the spin of the initial state**

$$\overline{\sum}2\Re\{\mathcal{A}_{1\mathrm{loop}}\mathcal{A}^*_{\mathrm{Born}}\}=\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\overline{\sum}|\mathcal{A}_{\mathrm{Born}}|^2\right)\frac{\alpha_s}{\pi}\left[\frac{2}{3\epsilon_{\mathrm{UV}}}-\frac{4}{3\epsilon^2_{\mathrm{IR}}}-\frac{4}{3\epsilon_{\mathrm{IR}}}\left(1-\log\frac{m_Z^2}{4\pi^2\mu_R^2}\right)-\frac{2}{3}\left(5-\pi^2-\log\frac{m_Z^2}{4\pi^2\mu_R^2}+\log^2\frac{m_Z^2}{4\pi^2\mu_R^2}\right)\right]$$

$$\boxed{\alpha_s=\frac{g_s^2}{4\pi}}$$

  - **The UV divergence needs renormalisation**

$$\overline{\sum}2\Re\{\mathcal{A}_{\mathrm{UV}}\mathcal{A}^*_{\mathrm{Born}}\}=\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\overline{\sum}|\mathcal{A}_{\mathrm{Born}}|^2\right)\frac{\alpha_s}{\pi}\left[-\frac{2}{3\epsilon_{\mathrm{UV}}}+\frac{2}{3\epsilon_{\mathrm{IR}}}\right]$$

A NLO Example: Virtual Correction

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Need to evaluate two tensor integrals**

Evaluating the scalar integrals, e.g.

$$\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2\left(\bar{l}-p_{\bar{q}}\right)^2}=\frac{i2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\left(-\frac{1}{2\epsilon_{\rm IR}}+\frac{1}{2\epsilon_{\rm UV}}\right)$$

- **Squaring with Born amplitude, summing over colours and spins, and averaging the spin of the initial state**

$$\overline{\sum}2\Re\{\mathcal{A}_{\rm 1loop}\mathcal{A}^*_{\rm Born}\}=\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\overline{\sum}|\mathcal{A}_{\rm Born}|^2\right)\frac{\alpha_s}{\pi}\left[\cancel{\frac{2}{3\epsilon_{\rm UV}}}-\frac{4}{3\epsilon^2_{\rm IR}}-\frac{4}{3\epsilon_{\rm IR}}\left(1-\log\frac{m_Z^2}{4\pi^2\mu_R^2}\right)\right.$$
$$\left.-\frac{2}{3}\left(5-\pi^2-\log\frac{m_Z^2}{4\pi^2\mu_R^2}+\log^2\frac{m_Z^2}{4\pi^2\mu_R^2}\right)\right]$$

$$\boxed{\alpha_s=\frac{g_s^2}{4\pi}}$$

- **The UV divergence needs renormalisation**

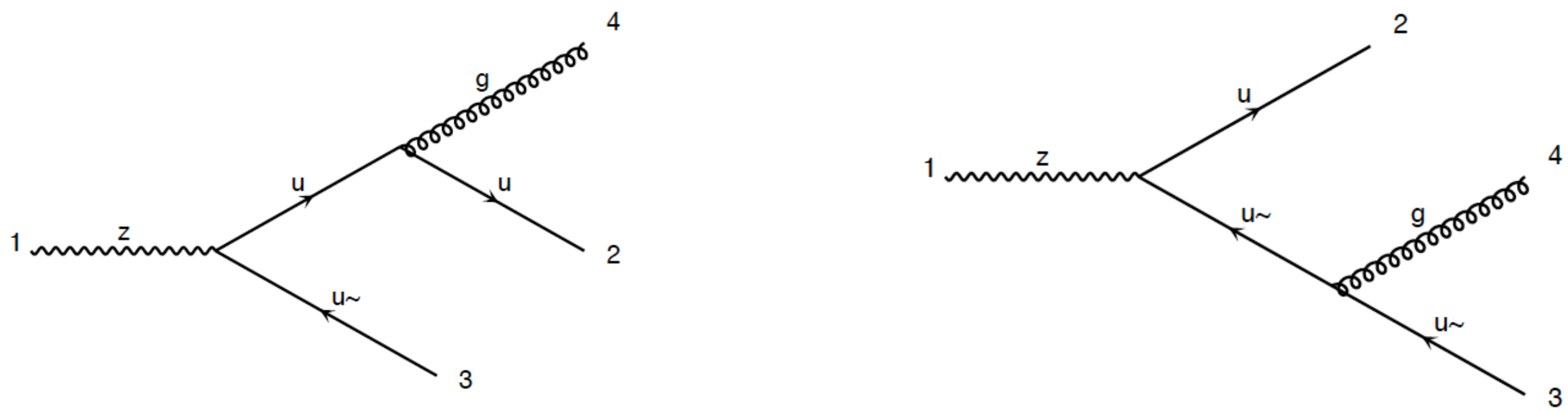
$$\overline{\sum}2\Re\{\mathcal{A}_{\rm UV}\mathcal{A}^*_{\rm Born}\}=\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\overline{\sum}|\mathcal{A}_{\rm Born}|^2\right)\frac{\alpha_s}{\pi}\left[\cancel{-\frac{2}{3\epsilon_{\rm UV}}}+\frac{2}{3\epsilon_{\rm IR}}\right]$$

- **The virtual matrix element is:**

$$\mathcal{V}=\overline{\sum}2\Re\{\mathcal{A}_{\rm 1loop}\mathcal{A}^*_{\rm Born}\}+\overline{\sum}2\Re\{\mathcal{A}_{\rm UV}\mathcal{A}^*_{\rm Born}\}$$

A NLO Example: Real Emission

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Writing down real amplitude according to Feynman rules**



  - **Squaring amplitude, summing over colours and spins, and averaging the spin of the initial state**

$$\overline{\sum}|\mathcal{A}_{\mathrm{real}}|^2=\left(\overline{\sum}|\mathcal{A}_{\mathrm{Born}}|^2\right)\alpha_s\frac{8\pi(d-2)}{3m_Z^2 s_{24}s_{34}}$$

$$\times\left[(d-2)s_{24}^2+2(d-4)s_{24}s_{34}+(d-2)s_{34}^2-4m_Z^2(s_{24}+s_{34})+4m_Z^4\right]$$

$$s_{24}=(p_q+p_g)^2,\, s_{34}=(p_{\bar{q}}+p_g)^2$$

A NLO Example: Real Emission

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **3-body phase-space integration**

$$\Gamma_{\rm real}=\frac{1}{2m_Z}\int(2\pi)^d\,\delta^d\left(p_Z-p_q-p_{\bar{q}}-p_g\right)\frac{1}{(2\pi)^{3(d-1)}}\frac{d^{d-1}\vec{p}_q}{2E_q}\frac{d^{d-1}\vec{p}_{\bar{q}}}{2E_{\bar{q}}}\frac{d^{d-1}\vec{p}_g}{2E_g}\overline{\sum}|\mathcal{A}_{\rm real}|^2$$

$$y=\frac{s_{34}}{m_Z^2},1-y-z=\frac{s_{24}}{m_Z^2}$$

$$\begin{aligned}d\Phi^{(2)}(p_Z\to p_q,p_{\bar{q}})&=(2\pi)^d\delta^d(p_Z-p_q-p_{\bar{q}})\frac{1}{(2\pi)^{2(d-1)}}\frac{d^{d-1}\vec{p}_q}{2E_q}\frac{d^{d-1}\vec{p}_{\bar{q}}}{2E_{\bar{q}}}\\&=\frac{(4\pi)^{2\epsilon}}{8(2\pi)^2}\frac{1}{m_Z^{2\epsilon}}d\Omega_d\end{aligned}$$

$$\begin{aligned}d\Phi^{(3)}(p_Z\to p_q,p_{\bar{q}},p_g)&=(2\pi)^d\delta^d(p_Z-p_q-p_{\bar{q}}-p_g)\frac{1}{(2\pi)^{3(d-1)}}\frac{d^{d-1}\vec{p}_q}{2E_q}\frac{d^{d-1}\vec{p}_{\bar{q}}}{2E_{\bar{q}}}\frac{d^{d-1}\vec{p}_g}{2E_g}\\&=\frac{(4\pi)^{3\epsilon}}{32(2\pi)^4\Gamma(1-\epsilon)}(m_Z^2)^{1-2\epsilon}d\Omega_d\\&\quad\times\int_0^1dzz^{-\epsilon}\int_0^{1-z}dyy^{-\epsilon}(1-z-y)^{-\epsilon}\\&=d\Phi^{(2)}(p_Z\to p_q,p_{\bar{q}})\times\frac{(4\pi)^{\epsilon}}{16\pi^2\Gamma(1-\epsilon)}(m_Z^2)^{1-\epsilon}\\&\quad\times\int_0^1dzz^{-\epsilon}\int_0^{1-z}dyy^{-\epsilon}(1-z-y)^{-\epsilon}\end{aligned}$$

A NLO Example: Real Emission

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **3-body phase-space integration**

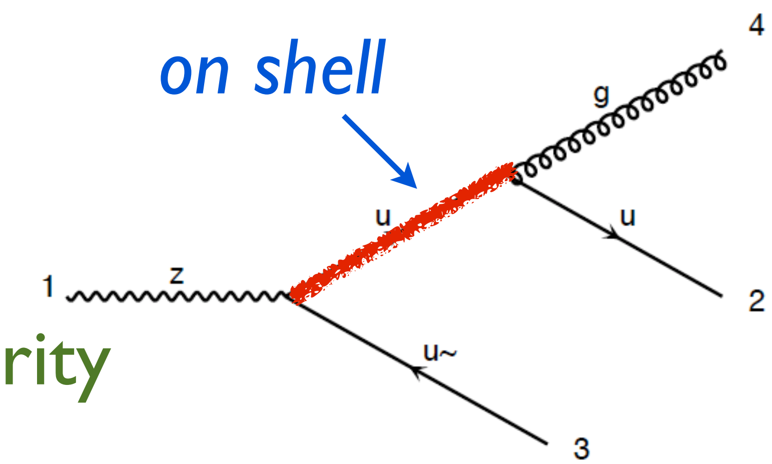
$$\overline{\sum}|\mathcal{A}_{\rm real}|^2 = \left(\overline{\sum}|\mathcal{A}_{\rm Born}|^2\right)\alpha_s \frac{8\pi(d-2)}{3m_Z^2 y(1-z-y)}\left[(d-2)(1-z)^2+4y^2-4y(1-z)+4z\right]$$

The integration over y is divergent when $d \leq 4\ (\epsilon \geq 0)$

$y \to 1-z\ (s_{24} \to 0)$

(1) $p_g \to 0\ (z \to 0)$ soft singularity

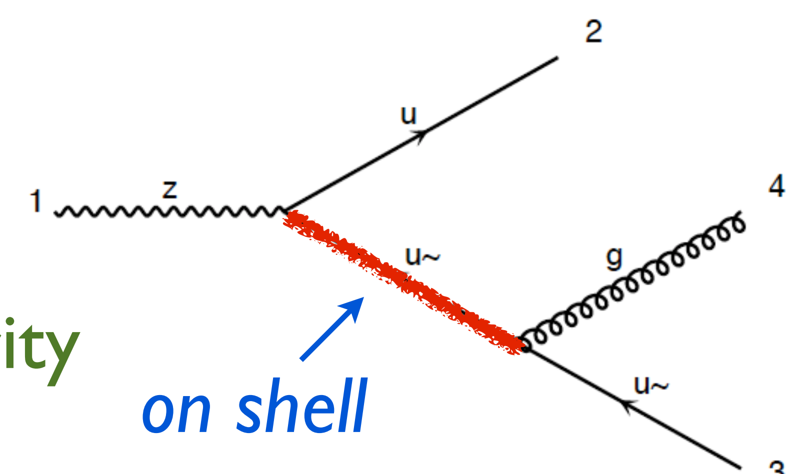
(2) $p_g || p_q$ collinear singularity



$y \to 0\ (s_{34} \to 0)$

(1) $p_g \to 0\ (z \to 0)$ soft singularity

(2) $p_g || p_{\bar{q}}$ collinear singularity



A NLO Example: Real Emission

- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **3-body phase-space integration**

$$\begin{aligned}\Gamma_{\mathrm{real}} &= \frac{1}{2m_Z}\int d\Phi^{(3)}(p_Z \to p_q, p_{\bar{q}}, p_g)\overline{\sum}|\mathcal{A}_{\mathrm{real}}|^2 \\ &= \frac{1}{2m_Z}\int d\Phi^{(2)}(p_Z \to p_q, p_{\bar{q}})\left(\overline{\sum}|\mathcal{A}_{\mathrm{Born}}|^2\right) \\ &\quad \times\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{\pi}\left[\frac{4}{3\epsilon_{\mathrm{IR}}^2}+\frac{2}{3\epsilon_{\mathrm{IR}}}\left(1-2\log\frac{m_Z^2}{4\pi^2\mu_R^2}\right)\right. \\ &\quad \left.+\frac{1}{3}\left(2\log^2\frac{m_Z^2}{4\pi^2\mu_R^2}-2\log\frac{m_Z^2}{4\pi^2\mu_R^2}-2\pi^2+13\right)\right]\end{aligned}$$

  - **Sum real and virtual**

$$\Gamma_{\mathrm{virtual}} = \frac{1}{2m_Z}\int d\Phi^{(2)}(p_Z \to p_q, p_{\bar{q}})\mathcal{V}$$

$$\Gamma_{\mathrm{virtual}}+\Gamma_{\mathrm{real}} = \frac{1}{2m_Z}\int d\Phi^{(2)}(p_Z \to p_q, p_{\bar{q}})\left(\overline{\sum}|\mathcal{A}_{\mathrm{Born}}|^2\right)\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\alpha_s}{\pi}$$

A NLO Example: NLO

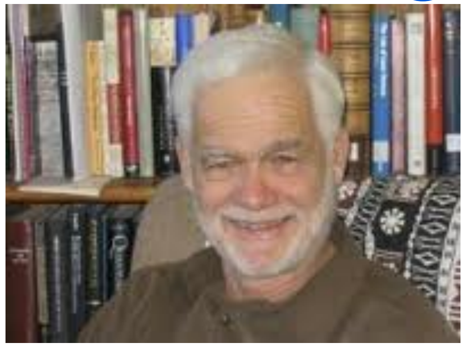
- **Let us calculate NLO QCD of Z -> q qbar decay**
  - **Sum real and virtual**

All remaining IR poles cancel (in general KLN theorem)

Kinoshita

Lee

Nauenberg



$$\Gamma_{\text{virtual}} + \Gamma_{\text{real}} = \frac{1}{2m_Z}\int d\Phi^{(2)}(p_Z \to p_q, p_{\bar{q}})\left(\overline{\sum}|\mathcal{A}_{\text{Born}}|^2\right)\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}\frac{\alpha_s}{\pi}$$

$$\stackrel{\epsilon\to 0}{=} \Gamma_{\text{Born}}(Z \to q\bar{q})\frac{\alpha_s}{\pi}$$

$$\boxed{\Gamma_{\text{NLO}}(Z \to q\bar{q} + X) = \Gamma_{\text{Born}}(Z \to q\bar{q})\left(1 + \frac{\alpha_s}{\pi}\right)}$$

We finally get a well-known result !

Homework

Filling all the gaps that I did not show !

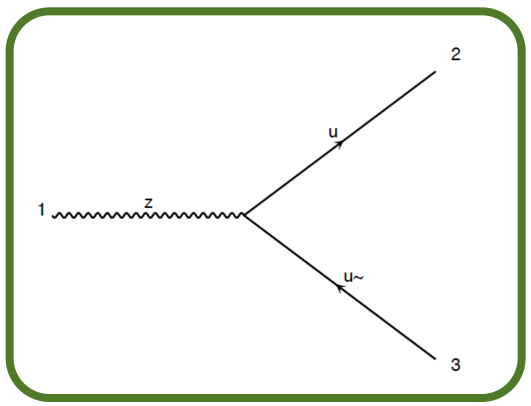
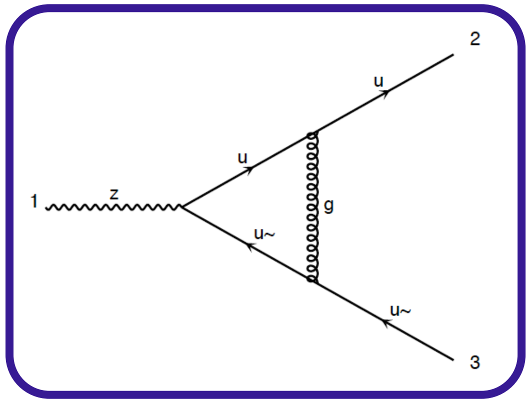
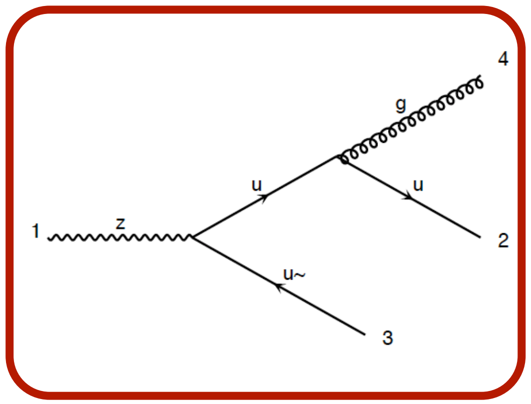
In general, NLO calculations are complex (and tedious, error-prone). Let us work with the aid of a computer and MadGraph5_aMC@NLO.

NLO Anatomy

- **Three parts need to be computed in a NLO calculation**

$$\sigma_{\text{NLO}} = \int d\Phi^{(n)}\mathcal{B} + \int d\Phi^{(n)}\mathcal{V} + \int d\Phi^{(n+1)}\mathcal{R}$$

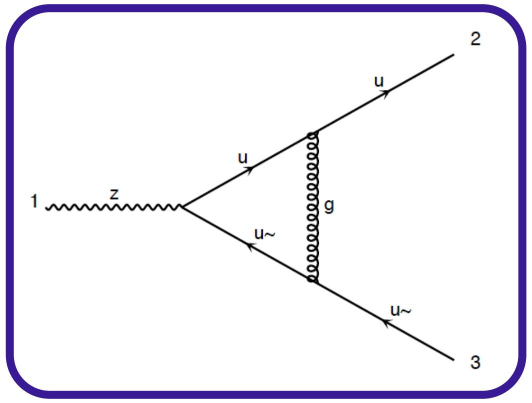
$$\mathcal{O}(\alpha_s^b) \qquad \mathcal{O}(\alpha_s^{b+1}) \qquad \mathcal{O}(\alpha_s^{b+1})$$

Born cross section	Virtual correction	Real correction
		
Finite	Divergent	Divergent
MadEvent	**MadLoop**	**MadFKS**

Lecture 2

NLO Calculations & Event Simulations at NLO QCD

Virtual=Loop+UV

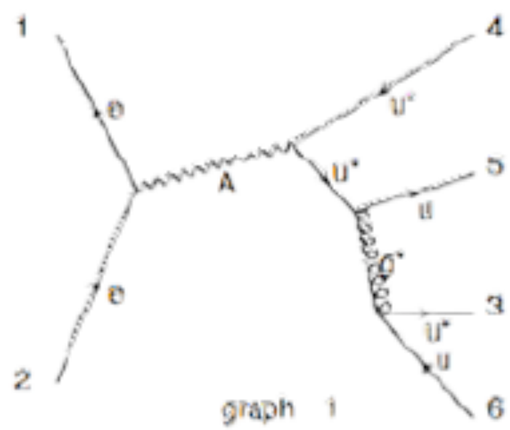


One-Loop Diagram Generation

- No external tool for loop diagram generation:
  Reuse MG5_aMC efficient tree level diagram generation!

- Cut loops have two extra external particles

Trees ($e^+e^- \rightarrow$ u u~ u u~) ≡ Loops ($e^+e^- \rightarrow$ u u~)



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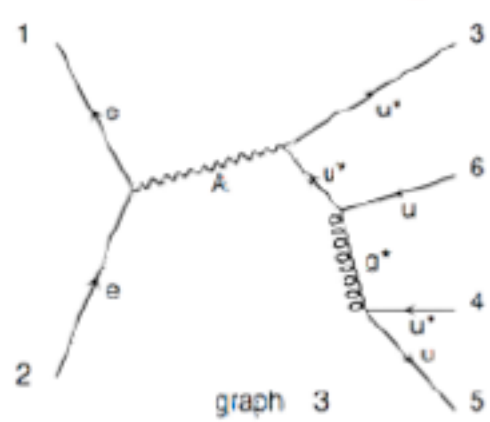

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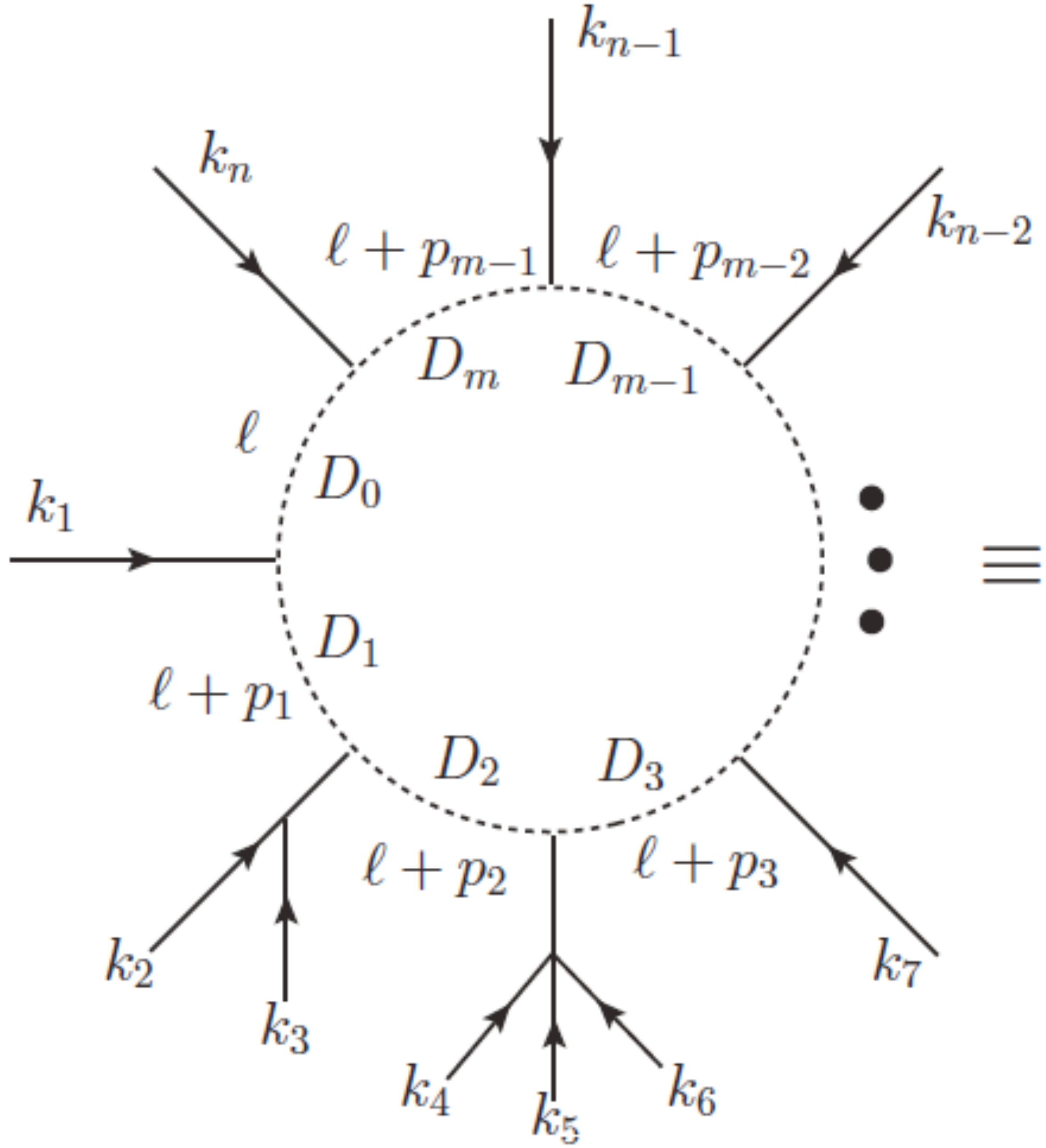
One-Loop Integral Evaluation



- Consider this m-point loop diagram with n external momenta

$$\equiv \int \frac{d^d \ell}{(2\pi)^d} \frac{\mathcal{N}(\ell)}{D_0 D_1 D_2 D_3 \cdots D_{m-2} D_{m-1}}$$

with $D_i = (\ell + p_i)^2 - m_i^2$

We will denote by $\mathcal{C}$ this integral.

One-Loop Integral Evaluation

$$\mathcal{C}^{\text{1-loop}} = \sum_{i_0<i_1<i_2<i_3} d_{i_0i_1i_2i_3}\text{Box}_{i_0i_1i_2i_3} + \sum_{i_0<i_1<i_2} c_{i_0i_1i_2}\text{Triangle}_{i_0i_1i_2} + \sum_{i_0<i_1} b_{i_0i_1}\text{Bubble}_{i_0i_1} + \sum_{i_0} a_{i_0}\text{Tadpole}_{i_0} + R + \mathcal{O}(\epsilon)$$

$$\text{Box}_{i_0i_1i_2i_3} = \int d^dl \frac{1}{D_{i_0}D_{i_1}D_{i_2}D_{i_3}}$$

$$\text{Triangle}_{i_0i_1i_2} = \int d^dl \frac{1}{D_{i_0}D_{i_1}D_{i_2}}$$

$$\text{Bubble}_{i_0i_1} = \int d^dl \frac{1}{D_{i_0}D_{i_1}}$$

$$\text{Tadpole}_{i_0} = \int d^dl \frac{1}{D_{i_0}}$$

The a, b, c, d and R coefficients depend only on external parameters and momenta.

Reduction of the loop to these scalar coefficients can be achieved using either Tensor Integral Reduction or Reduction at the integrand level

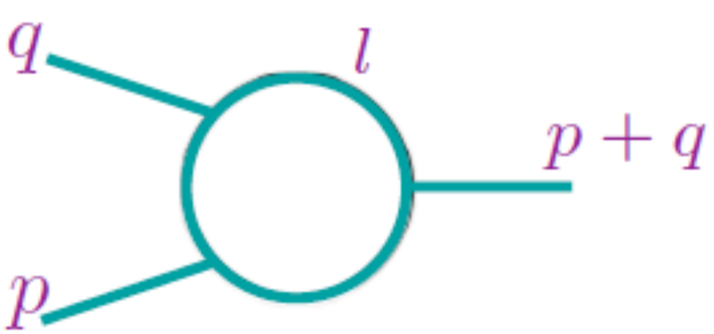
Tensor Integral Reduction

- Passarino-Veltman reduction:

$$\int d^d l \, \frac{N(l)}{D_0 D_1 D_2 \cdots D_{m-1}} \rightarrow \sum_i \mathrm{coeff}_i \int d^d l \, \frac{1}{D_0 D_1 \cdots}$$

  - Reduce a general integral to "scalar integrals" by "completing the square"

- Example:
  Application of PV to this triangle rank-1 integral



$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)}$$

- Implemented in codes such as:

COLLIER [A. Denner, S .Dittmaier, L. Hofer, 1604.06792]
GOLEM95 [T. Binoth, J.Guillet, G. Heinrich, E.Pilon, T.Reither, 0810.0992]

Tensor Integral Reduction

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2-m_1^2)((l+p)^2-m_2^2)((l+q)^2-m_3^2)}$$

- The only independent four vectors are p^μ and q^μ . Therefore, the integral must be proportional to those. We can set-up a system of linear equations and try to solve for C_1 and C_2

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2-m_1^2)((l+p)^2-m_2^2)((l+q)^2-m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

We can solve for C_1 and C_2 by contracting with p and q

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l\cdot p] \\ [2l\cdot q] \end{pmatrix} = G \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p\cdot p & 2p\cdot q \\ 2p\cdot q & 2q\cdot q \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $[2l\cdot p] = \int \frac{d^n l}{(2\pi)^n} \frac{2l\cdot p}{l^2(l+p)^2(l+q)^2}$ (For simplicity, the masses are neglected here)

- By expressing $2l.p$ and $2l.q$ as a sum of denominators we can express R_1 and R_2 as a sum of simpler integrals, *e.g.*

$$R_1 = \int \frac{d^n l}{(2\pi)^n} \frac{2l\cdot p}{l^2(l+p)^2(l+q)^2} = \int \frac{d^n l}{(2\pi)^n} \frac{(l+p)^2-l^2-p^2}{l^2(l+p)^2(l+q)^2}$$

$$= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2(l+q)^2} - \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p)^2(l+q)^2} - p^2 \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2(l+p)^2(l+q)^2}$$

Tensor Integral Reduction

- And similarly for R_2

$$R_2 = \int \frac{d^nl}{(2\pi)^n} \frac{2l\cdot q}{l^2(l+p)^2(l+q)^2} = \int \frac{d^nl}{(2\pi)^n} \frac{(l+q)^2 - l^2 - q^2}{l^2(l+p)^2(l+q)^2}$$
$$= \int \frac{d^nl}{(2\pi)^n} \frac{1}{l^2(l+p)^2} - \int \frac{d^nl}{(2\pi)^n} \frac{1}{(l+p)^2(l+q)^2} - q^2 \int \frac{d^nl}{(2\pi)^n} \frac{1}{l^2(l+p)^2(l+q)^2}$$

- Now we can solve the equation

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l\cdot p] \\ [2l\cdot q] \end{pmatrix} = G \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p\cdot p & 2p\cdot q \\ 2p\cdot q & 2q\cdot q \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

by inverting the "Gram" matrix G

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = G^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

- We have re-expressed, reduced, our original integral

$$\int \frac{d^nl}{(2\pi)^n} \frac{l^\mu}{(l^2-m_1^2)((l+p)^2-m_2^2)((l+q)^2-m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

in terms of known, simpler *scalar* integrals

Integrand Reduction

Ossola, Papadopulos, Pittau (NPB'06)

TIR

- The decomposition to the basis scalar integrals works at the level of the integrals

$$
\begin{aligned}
\mathcal{C}^{\text{1-loop}} = & \sum_{i_0<i_1<i_2<i_3} d_{i_0 i_1 i_2 i_3} \mathrm{Box}_{i_0 i_1 i_2 i_3} \\
& + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \mathrm{Triangle}_{i_0 i_1 i_2} \\
& + \sum_{i_0<i_1} b_{i_0 i_1} \mathrm{Bubble}_{i_0 i_1} \\
& + \sum_{i_0} a_{i_0} \mathrm{Tadpole}_{i_0} \\
& + R + \mathcal{O}(\epsilon)
\end{aligned}
$$

OPP

- Knowing a relation directly at the integrand level, we would be able to manipulate the reduction without doing the the integrals

$$
\begin{aligned}
N(l) = & \sum_{i_0,i_1,i_2,i_3} \left(d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}\right) \prod_{i \neq i_0,i_1,i_2,i_3} D_i \\
& + \sum_{i_0,i_1,i_2} \left(c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}\right) \prod_{i \neq i_0,i_1,i_2} D_i \\
& + \sum_{i_0,i_1} \left(b_{i_0 i_1} + \tilde{b}_{i_0 i_1}\right) \prod_{i \neq i_0,i_1} D_i \\
& + \sum_{i_0} \left(a_{i_0} + \tilde{a}_{i_0}\right) \prod_{i \neq i_0} D_i \\
& + \tilde{P}(l) \prod_i D_i + \mathcal{O}(\varepsilon)
\end{aligned}
$$

Integrand Reduction

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& + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \mathrm{Triangle}_{i_0 i_1 i_2} \\
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$$
\begin{aligned}
N(l) = & \sum_{i_0,i_1,i_2,i_3} (d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}) \prod_{i \neq i_0,i_1,i_2,i_3} D_i \\
& + \sum_{i_0,i_1,i_2} (c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}) \prod_{i \neq i_0,i_1,i_2} D_i \\
& + \sum_{i_0,i_1} (b_{i_0 i_1} + \tilde{b}_{i_0 i_1}) \prod_{i \neq i_0,i_1} D_i \\
& + \sum_{i_0} (a_{i_0} + \tilde{a}_{i_0}) \prod_{i \neq i_0} D_i \\
& + \tilde{P}(l) \prod_i D_i + \mathcal{O}(\varepsilon)
\end{aligned}
$$

Spurious term

Integrand Reduction

- The functional form of the spurious terms is known (it depends on the rank of the integral and the number of propagators in the loop) [del Aguila, Pittau 2004]
  - for example, a box coefficient from a rank 1 numerator is

$$\tilde{d}_{i_0 i_1 i_2 i_3}(l) = \tilde{d}_{i_0 i_1 i_2 i_3}\, \epsilon^{\mu\nu\rho\sigma}\, l^{\mu} p_1^{\nu} p_2^{\rho} p_3^{\sigma}$$

  (remember that p_i is the sum of the momentum that has entered the loop so far, so we always have $p_0 = 0$)

  - The integral is zero

$$\int d^d l \frac{\tilde{d}_{i_0 i_1 i_2 i_3}(l)}{D_0 D_1 D_2 D_3} = \tilde{d}_{i_0 i_1 i_2 i_3} \int d^d l \frac{\epsilon^{\mu\nu\rho\sigma}\, l^{\mu} p_1^{\nu} p_2^{\rho} p_3^{\sigma}}{D_0 D_1 D_2 D_3} = 0$$

Integrand Reduction

- **Take Box (4-point) coefficients as an example**

$$N(l^{\pm}) = d_{0123} + \tilde{d}_{0123}(l^{\pm}) \prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{\pm})$$

- Two values are enough given the functional form for the spurious term. We can immediately determine the Box coefficient

$$d_{0123} = \frac{1}{2}\left[\frac{N(l^{+})}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{+})} + \frac{N(l^{-})}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{-})}\right]$$

- By choosing other values for l, that set other combinations of 4 "denominators" to zero, we can get all the Box coefficients

- **In general:**

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$

$$+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i$$

$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$

$$+\tilde{P}(l) \prod_{i}^{m-1} D_i$$

$$= 0$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

For example, choosing *l* such that

$$D_0(l^\pm) = D_1(l^\pm) = \\ = D_2(l^\pm) = D_3(l^\pm) = 0$$

sets all the terms in this equation to zero except the first line

There are two (complex) solutions to this equation due to the quadratic nature of the propagators

Integrand Reduction

- **In general:**

$$
\begin{aligned}
N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i} \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

$$= 0$$

Now we choose l such that

$$D_0(l^i) = D_1(l^i) = D_2(l^i) = 0$$

sets all the terms in this equation to zero except the first and second line

Coefficient computed in a previous step

Integrand Reduction

- **In general:**

$$
\begin{aligned}
N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i} \\
& + \boxed{\sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i} \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

$$= 0$$

Now, choosing l such that

$$D_0(l^i) = D_1(l^i) = 0$$

sets all the terms in this equation to zero except the first, second and third line

Coefficient computed in a previous step

Integrand Reduction

- **In general:**

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \boxed{\tilde{P}(l) \prod_{i}^{m-1} D_i} \\
& = 0
\end{aligned}
$$

Now, choosing l such that

$$D_1(l^i) = 0$$

sets the last line to zero

Coefficient computed in a previous step

Integrand Reduction

- **In general:**

$$N(l) = \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i}$$
$$+ \boxed{\sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i}$$
$$+ \boxed{\sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i}$$
$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$
$$\boxed{+\tilde{P}(l) \prod_{i}^{m-1} D_i}$$
$$= 0$$

Now, choosing *l* such that

$$D_1(l^i) = 0$$

sets the last line to zero

Coefficient computed in a previous step

D-dimensional Complex

- **The previous expression should in fact be written in d dimensions**

$$\int \frac{d^d\bar{l}}{(2\pi)^d} \frac{N(\bar{l},\epsilon)}{\bar{D}_0\bar{D}_1\bar{D}_2\cdots\bar{D}_{m-1}}$$

$$\bar{D}_i = \left(\bar{l}+p_i\right)^2 - m_i^2, \quad p_0 = 0$$

D-dimensional Complex

- **The previous expression should in fact be written in d dimensions**

$$\int \frac{d^d\bar{l}}{(2\pi)^d} \frac{N(\bar{l},\epsilon)}{\bar{D}_0\bar{D}_1\bar{D}_2\cdots\bar{D}_{m-1}}$$

$$\bar{D}_i = \left(\bar{l}+p_i\right)^2 - m_i^2, \quad p_0 = 0$$

- **In numerical calculations, it is very convenient to perform the following decomposition**

$$\underset{d-\mathrm{dim}}{\bar{l}^\mu} = \underset{4-\mathrm{dim}}{l^\mu} + \underset{(-2\epsilon)-\mathrm{dim}}{\tilde{l}^\mu} \qquad \mu = \underbrace{0,1,2,3}_{\substack{4d \text{ spacetime} \\ \text{physical}}},\underbrace{\cdots,3-2\epsilon}_{\substack{(-2\epsilon)d \text{ space} \\ \text{abstract}}}$$

$$l^\mu = 0, \mu \in (-2\epsilon)d \text{ space} \qquad \tilde{l}^\mu = 0, \mu \in 4d \text{ spacetime}$$

D-dimensional Complex

- **The previous expression should in fact be written in d dimensions**

$$\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{N(\bar{l},\epsilon)}{\bar{D}_0\bar{D}_1\bar{D}_2\cdots\bar{D}_{m-1}}$$

$$\bar{D}_i=\left(\bar{l}+p_i\right)^2-m_i^2,\quad p_0=0$$

- **In numerical calculations, it is very convenient to perform the following decomposition**

$$\bar{l}^\mu=l^\mu+\tilde{l}^\mu \qquad \mu=\underbrace{0,1,2,3,}_{\substack{4d\text{ spacetime}\\ \text{physical}}}\underbrace{\cdots,3-2\epsilon}_{\substack{(-2\epsilon)d\text{ space}\\ \text{abstract}}}$$

$d-\text{dim}$ $\quad$ $4-\text{dim}$ $\quad$ $(-2\epsilon)-\text{dim}$

$$l^\mu=0,\mu\in(-2\epsilon)d\text{ space}\qquad \tilde{l}^\mu=0,\mu\in 4d\text{ spacetime}$$

$$N(\bar{l},\epsilon)=N(l)+\tilde{N}(l,\tilde{l},\epsilon)$$

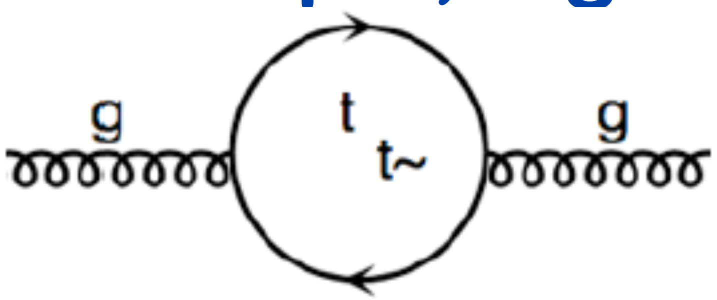
Suitable for numerical calc. $\quad$ Complement with special CT R_2

D-dimensional Complex

- **Compute the remaining loop part in terms of rational functions of external momentum invariants and masses**

$$\boxed{R_2 = \lim_{\epsilon \to 0} \int \frac{d^d \bar{l}}{(2\pi)^d} \frac{\tilde{N}(l, \tilde{l}, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}}$$

- **For example, a gluon self-energy diagram:**



$$N(\bar{l}, \epsilon) = -2\pi\alpha_s \delta_{ab} \mathrm{Tr}\left[\gamma^\mu \left(\bar{\not{l}} + m_t\right) \gamma^\nu \left(\bar{\not{l}} + \not{p}_g + m_t\right)\right] \varepsilon_\mu \varepsilon_\nu$$

- **After performing some Dirac algebra, we have**

$$\tilde{N}(l, \tilde{l}, \epsilon) = 8\pi\alpha_s \delta_{ab} g^{\mu\nu} \tilde{l}^2 \varepsilon_\mu \varepsilon_\nu$$

- **Using the integration**

$$\int \frac{d^d \bar{l}}{(2\pi)^d} \frac{\tilde{l}^2}{(\bar{l}^2 - m_t^2)\left((\bar{l} + p_g)^2 - m_t^2\right)} = -\frac{i}{32\pi^2}\left(2m_t^2 - \frac{p_g^2}{3}\right) + \mathcal{O}(\epsilon)$$

- **We have R_2 term**

$$R_2 = -\frac{i\alpha_s}{4\pi} \delta_{ab} \left(2m_t^2 - \frac{p_g^2}{3}\right) g^{\mu\nu} \varepsilon_\mu \varepsilon_\nu$$

D-dimensional Complex

- **It has been proven that R_2 is only UV related. Therefore, like renormalisation counterterms, they can be reexpressed into R_2 Feynman rules**

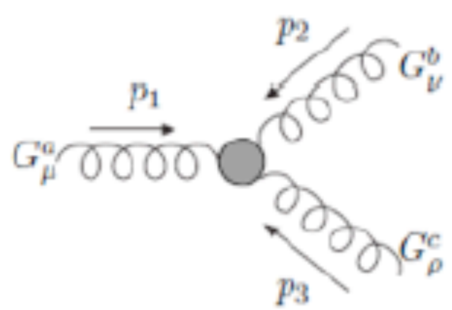
QCD R_2 Feynman Rules

$= \mathrm{Vert}(G^a_\mu, G^b_\nu)$

$$\mathrm{Vert}(G^a_\mu, G^b_\nu) = \frac{i g_s^2 N_c}{48\pi^2}\,\delta^{ab}\left[\frac{p^2}{2} g_{\mu\nu} + \lambda_{HV}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right) + \sum_Q \frac{p^2 - 6m_Q^2}{N_c} g_{\mu\nu}\right]$$

$= \mathrm{Vert}(Q^i_l, \bar{Q}^j_m)$

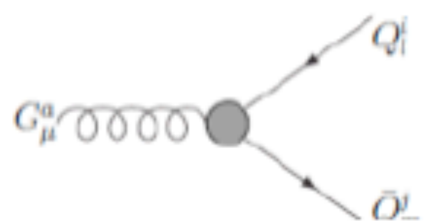
$$\mathrm{Vert}(Q^i_l, \bar{Q}^j_m) = \frac{i g_s^2}{16\pi^2}\,\frac{N_c^2-1}{2N_c}\,\delta^{ij}\,\delta_{lm}\,\left(-\not{p} + 2m_{Q_l}\right)\lambda_{HV}\,.$$



$= \mathrm{Vert}(G^a_\mu, G^b_\nu, G^c_\rho)$

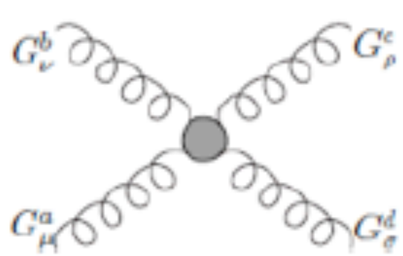
$$\mathrm{Vert}(G^a_\mu, G^b_\nu, G^c_\rho) = -\frac{g_s^3 N_c}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + \frac{2N_f}{N_c}\right) f^{abc}\, V_{\mu\nu\rho}(p_1, p_2, p_3)$$

$$V_{\mu\nu\rho}(p_1, p_2, p_3) = g_{\mu\nu}(p_2 - p_1)_\rho + g_{\nu\rho}(p_3 - p_2)_\mu + g_{\rho\mu}(p_1 - p_3)_\nu\,.$$



$= \mathrm{Vert}(G^a_\mu, Q^i_l, \bar{Q}^j_m)$

$$\mathrm{Vert}(G^a_\mu, Q^i_l, \bar{Q}^j_m) = \delta_{lm}\,\frac{i g_s^3}{16\pi^2}\,T^a_{ji}\,\frac{N_c^2-1}{2N_c}\,\gamma_\mu\left(1 + \lambda_{HV}\right)$$



$= \mathrm{Vert}\left(G^a_\mu, G^b_\nu, G^c_\rho, G^d_\sigma\right)$

$$\mathrm{Vert}\left(G^a_\mu, G^b_\nu, G^c_\rho, G^d_\sigma\right) = \frac{i g_s^4}{48\pi^2}\left(C_1\, g_{\mu\nu}g_{\rho\sigma} + C_2\, g_{\mu\rho}g_{\nu\sigma} + C_3\, g_{\mu\sigma}g_{\nu\rho}\right),$$

$$\begin{aligned} C_1 = {} & Tr(\{T^a, T^b\}\{T^c, T^d\})\,\left(5N_c + 2\lambda_{HV}N_c + 6N_f\right) \\ & - \left(Tr(T^aT^cT^bT^d) + Tr(T^aT^dT^bT^c)\right)\left(12N_c + 4\lambda_{HV}N_c + 10N_f\right) \\ & - \left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right),\quad C_2 = C_1(b \leftrightarrow c) \quad C_3 = C_1(b \leftrightarrow d) \end{aligned}$$

Draggiotis, Garzelli, Papadopoulos, Pittau (JHEP'09); HSS, Zhang, Chao (JHEP'11)

D-dimensional Complex

- **In integrand reduction, additional rational terms R_1 are needed !**

$$
\begin{aligned}
N(l) = & \sum_{i_0,i_1,i_2,i_3} (d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}) \prod_{i \neq i_0,i_1,i_2,i_3} D_i \\
& + \sum_{i_0,i_1,i_2} (c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}) \prod_{i \neq i_0,i_1,i_2} D_i \\
& + \sum_{i_0,i_1} (b_{i_0 i_1} + \tilde{b}_{i_0 i_1}) \prod_{i \neq i_0,i_1} D_i \\
& + \sum_{i_0} (a_{i_0} + \tilde{a}_{i_0}) \prod_{i \neq i_0} D_i \\
& + \tilde{P}(l) \prod_i D_i + \mathcal{O}(\varepsilon)
\end{aligned}
$$

$$\frac{1}{D_i} \to \frac{1}{\bar{D}_i} = \frac{1}{D}\left(1 - \frac{\tilde{l}^2}{D_i}\right)$$

integration of this piece gives rise R_1

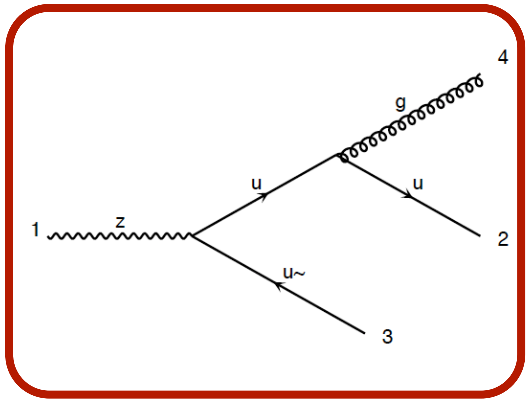
$\boxed{4d \text{ couterparts}}$

- **Can be included in OPP reduction**
- **Not needed in TIR reduction**

Lecture 2

NLO Calculations &
Event Simulations at NLO QCD

Real



NLO Anatomy

- **Three parts need to be computed in a NLO calculation**

$$\sigma_{\mathrm{NLO}} = \int d\Phi^{(n)}\mathcal{B} + \int d\Phi^{(n)}\mathcal{V} + \int d\Phi^{(n+1)}\mathcal{R}$$

Born cross section | Virtual correction | Real correction

$$\mathrm{Virtual} = -\frac{A}{\epsilon^2} - \frac{B}{\epsilon} + V \quad \mathrm{Real} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + R$$

NLO Anatomy

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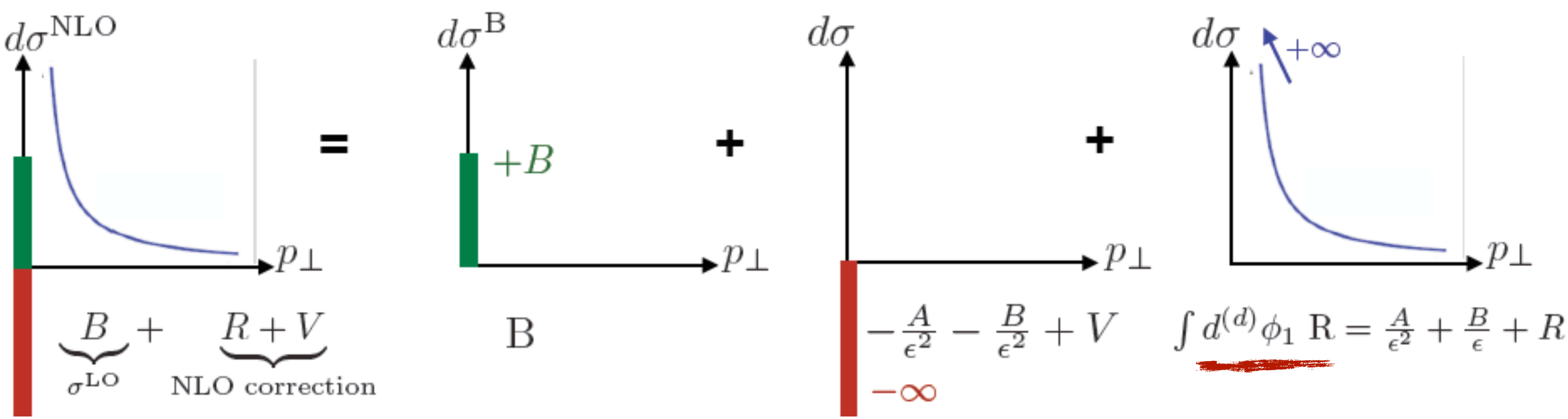
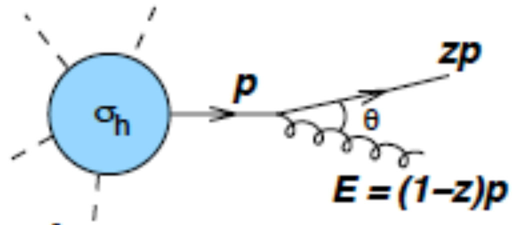


Branching: To Be or Not To Be

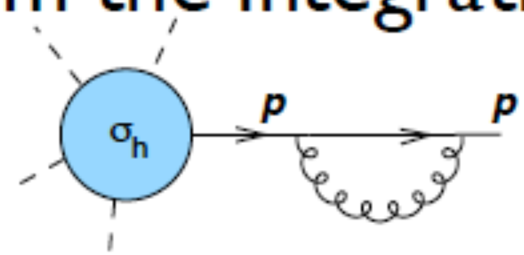
- Let us consider the branching of a gluon from a quark



$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

  Where k_t is the transverse momentum of the gluon $k_t = E\sin\theta$. It diverges in the soft ($z \to 1$) and collinear ($k_t \to 0$) region

- These singularities cancel with the virtual contribution, which comes from the integration of the loop momentum



$$\sigma_{h+V} \simeq -\sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

- The cancelation happens if we cannot distinguish between the case of no branching, and of a soft or collinear branching

Infrared Safety

- **In order to have meaningful fixed-order predictions in perturbation theory, observables must be IR-safe, i.e. not sensitive to the emission of soft/collinear partons**

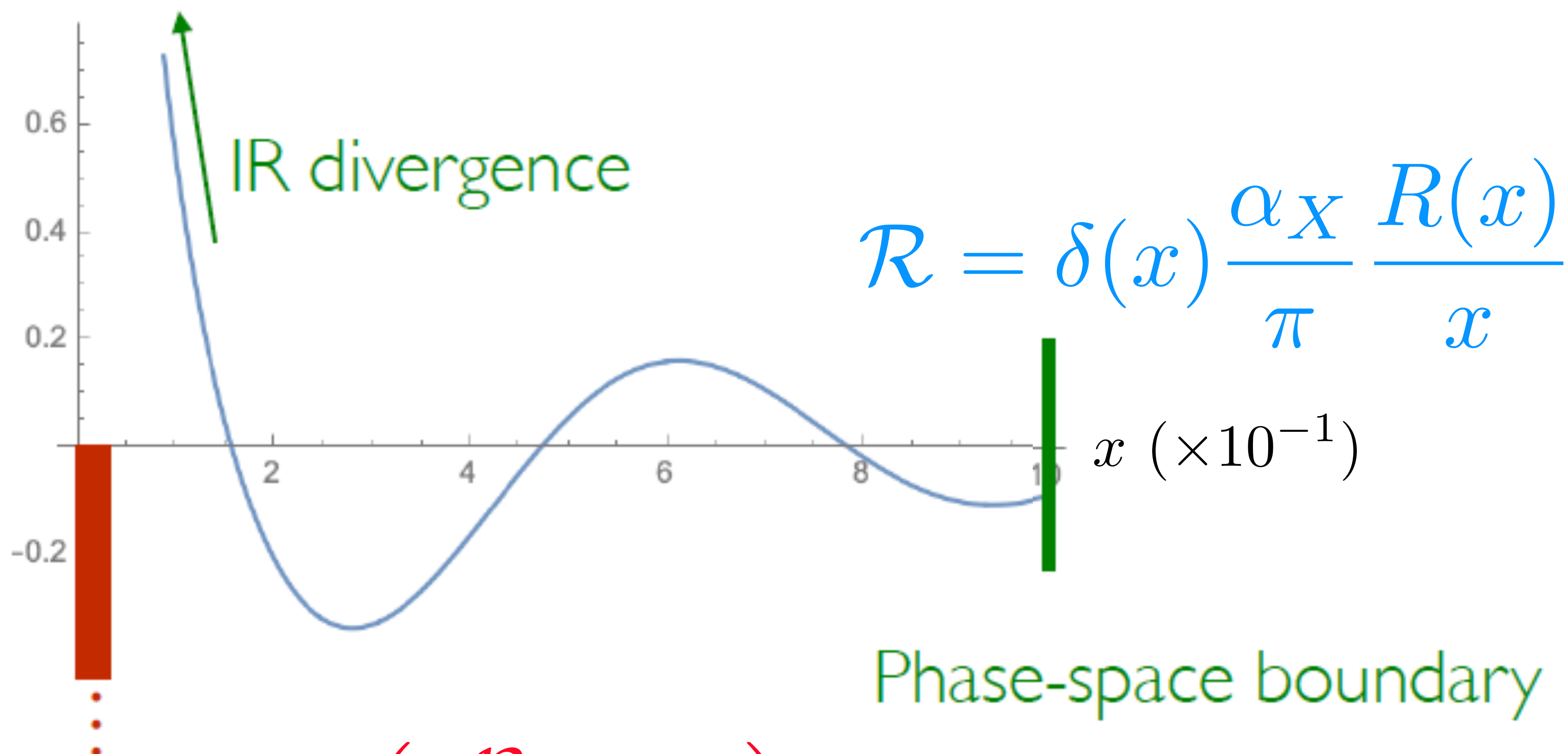
$$\lim_{p_i || p_j} \mathcal{O}\left(1,\cdots,i,\cdots,j-1,j,j+1,\cdots,n\right) = \mathcal{O}\left(1,\cdots,ij,\cdots,j-1,j+1,\cdots,n\right)$$

$$\lim_{p_i \to 0} \mathcal{O}\left(1,\cdots,i-1,i,i+1,\cdots,n\right) = \mathcal{O}\left(1,\cdots,i-1,i+1,\cdots,n\right)$$

- **For example,**
  - **The number of gluons is NOT IR safe.**
  - **The leading p_T/energy particle is NOT IR safe (soft or collinear unsafe ?).**
  - **The colour in a given cone is NOT IR safe (soft or collinear unsafe ?).**
  - **The transverse energy sum is IR safe.**

A Toy Example

- **Assuming the phase space integration can be casted into a one-dimensional case $x \in [0,1]$:**



$$\mathcal{R} = \delta(x)\frac{\alpha_X}{\pi}\frac{R(x)}{x}$$

$$\delta(x)\mathcal{V} = \delta(x)\frac{\alpha_X}{\pi}\left(\frac{\mathcal{B}}{2\epsilon_{\mathrm{IR}}} + V\right)$$

$\forall\ \mathcal{O}(x)$ IR safety $\lim_{x\to 0}\mathcal{O}(x)R(x) = \mathcal{O}(0)\mathcal{B}$

A Toy Example

$$\mathcal{O}(0)\mathcal{V} + \int_0^1 dx x^{-2\epsilon_{\rm IR}} \mathcal{O}(x)\mathcal{R}$$

Dimensionally regularise in x !

$$= \frac{\alpha_X}{\pi}\left[\mathcal{O}(0)\left(\frac{\mathcal{B}}{2\epsilon_{\rm IR}} + V\right) + \int_0^1 dx x^{-1-2\epsilon_{\rm IR}} \mathcal{O}(x) R(x)\right]$$

$$= \frac{\alpha_X}{\pi}\left[\mathcal{O}(0)\left(\frac{\mathcal{B}}{2\epsilon_{\rm IR}} + V\right) + \left(-\mathcal{O}(0)\frac{\mathcal{B}}{2\epsilon_{\rm IR}} + \int_0^1 dx \left(\frac{1}{x}\right)_+ \mathcal{O}(x) R(x)\right)\right]$$

$$= \frac{\alpha_X}{\pi}\left[\mathcal{O}(0) V + \int_0^1 dx \left(\frac{1}{x}\right)_+ \mathcal{O}(x) R(x)\right]$$

- **We have used:**

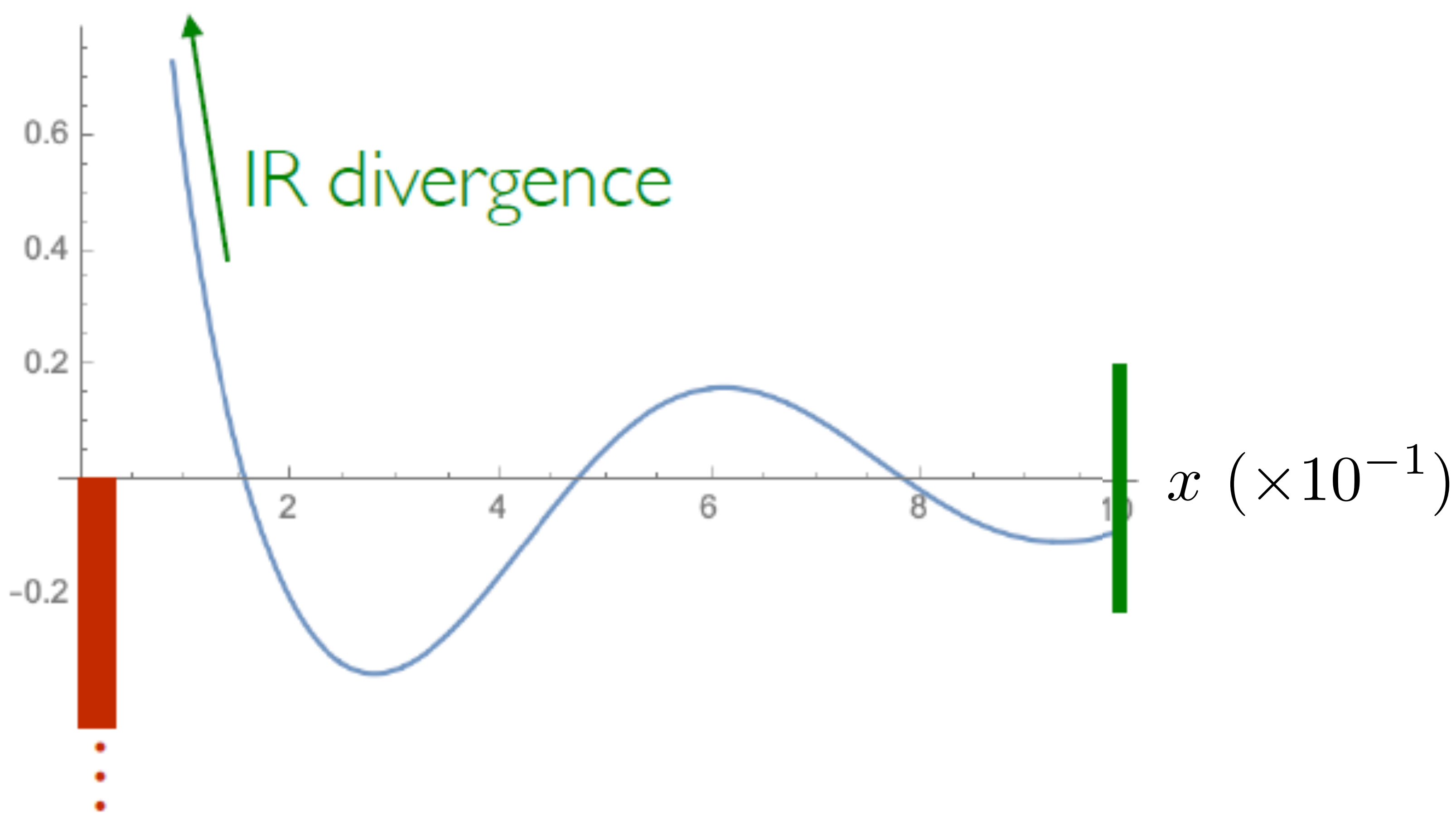
$$x^{-1-2\epsilon_{\rm IR}} = -\frac{1}{2\epsilon_{\rm IR}}\delta(x) + \left(\frac{1}{x}\right)_+ + \epsilon_{\rm IR} \text{ term}$$

$$\left(\frac{1}{x}\right)_+ f(x) \equiv \frac{f(x) - f(0)}{x} \qquad\qquad \forall\ f(x)$$

Phase Space Slicing

- **In general, the phase-space integration over real matrix element is very hard. Dedicated general approaches are developed !**
  - **Phase-space slicing**

$$\int_0^1 dx x^{-1-2\epsilon_{\rm IR}} \mathcal{O}(x) R(x)$$



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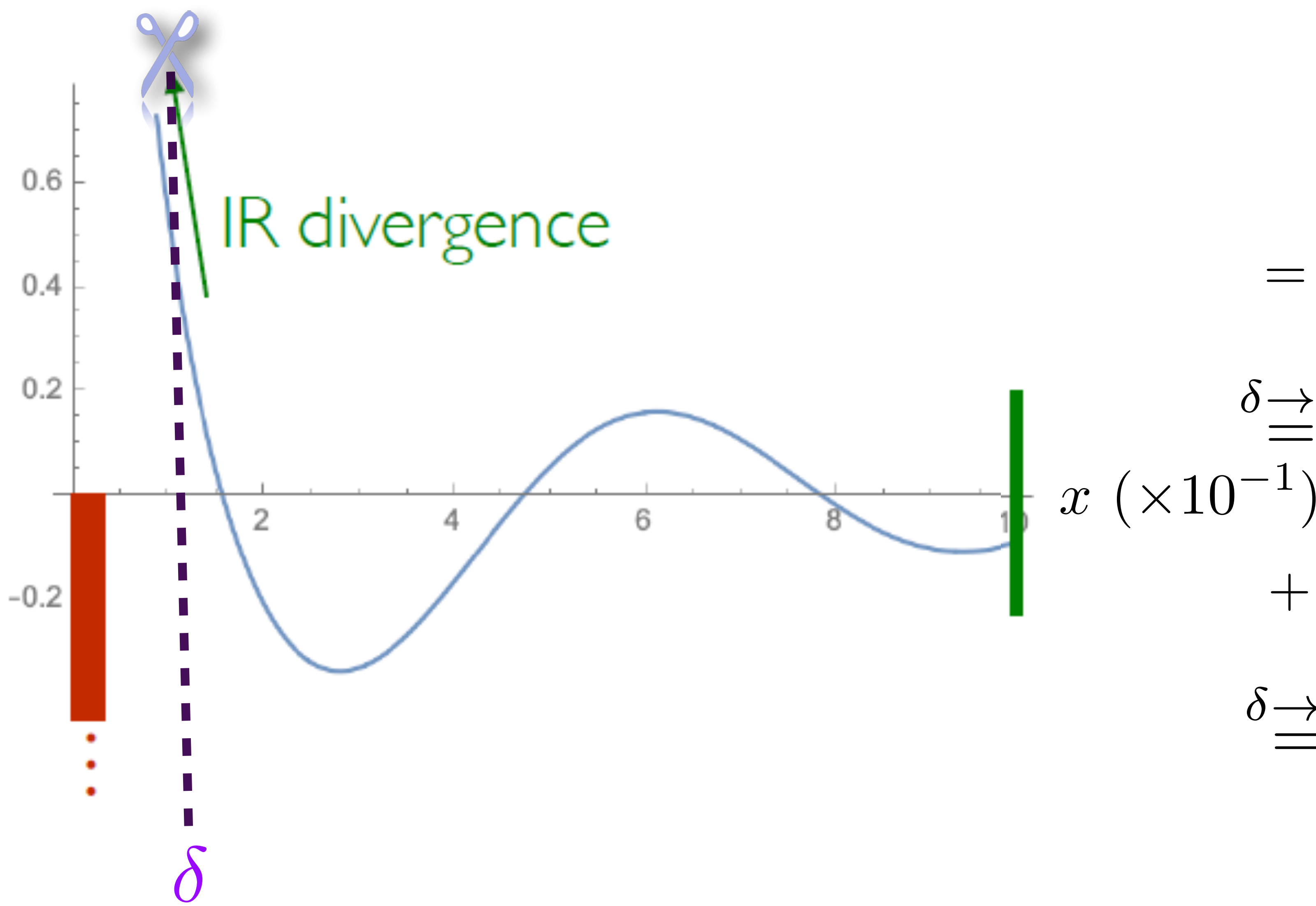

$$
\begin{aligned}
&\int_0^1 dx x^{-1-2\epsilon_{\rm IR}} \mathcal{O}(x) R(x) \\
&= \left(\int_0^\delta + \int_\delta^1\right) dx x^{-1-2\epsilon_{\rm IR}} \mathcal{O}(x) R(x) \\
&\overset{\delta\to 0}{=} \left(-\mathcal{O}(0)\frac{\mathcal{B}}{2\epsilon_{\rm IR}}\delta^{-2\epsilon_{\rm IR}}\right) \\
&\quad + \left(\int_\delta^1 dx x^{-1}\mathcal{O}(x)R(x) + \epsilon_{\rm IR}\ \text{term}\right) \\
&\overset{\delta\to 0}{=} -\mathcal{O}(0)\mathcal{B}\left(\frac{1}{2\epsilon_{\rm IR}} - \log\delta\right) \\
&\quad + \int_\delta^1 dx x^{-1}\mathcal{O}(x)R(x)
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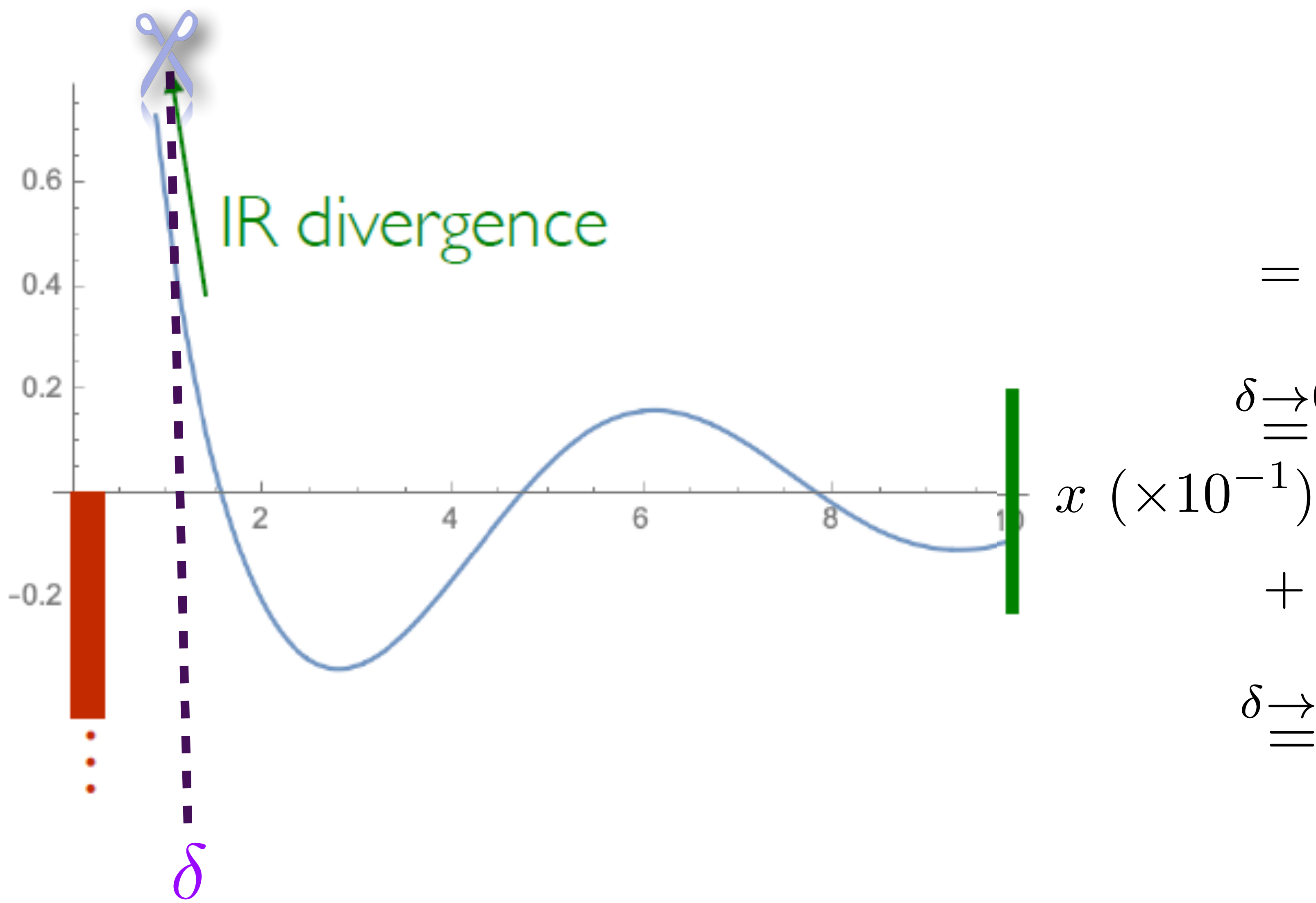

finite integral
(can be computed numerically)

$$\int_0^1 dx x^{-1-2\epsilon_{\mathrm{IR}}} \mathcal{O}(x) R(x)$$

$$= \left(\int_0^\delta + \int_\delta^1\right) dx x^{-1-2\epsilon_{\mathrm{IR}}} \mathcal{O}(x) R(x)$$

$$\overset{\delta\to 0}{=} \left(-\mathcal{O}(0)\frac{\mathcal{B}}{2\epsilon_{\mathrm{IR}}}\delta^{-2\epsilon_{\mathrm{IR}}}\right)$$

$$+ \left(\int_\delta^1 dx x^{-1} \mathcal{O}(x) R(x) + \epsilon_{\mathrm{IR}}\ \mathrm{term}\right)$$

$$\overset{\delta\to 0}{=} -\mathcal{O}(0)\mathcal{B}\left(\frac{1}{2\epsilon_{\mathrm{IR}}} - \log\delta\right)$$

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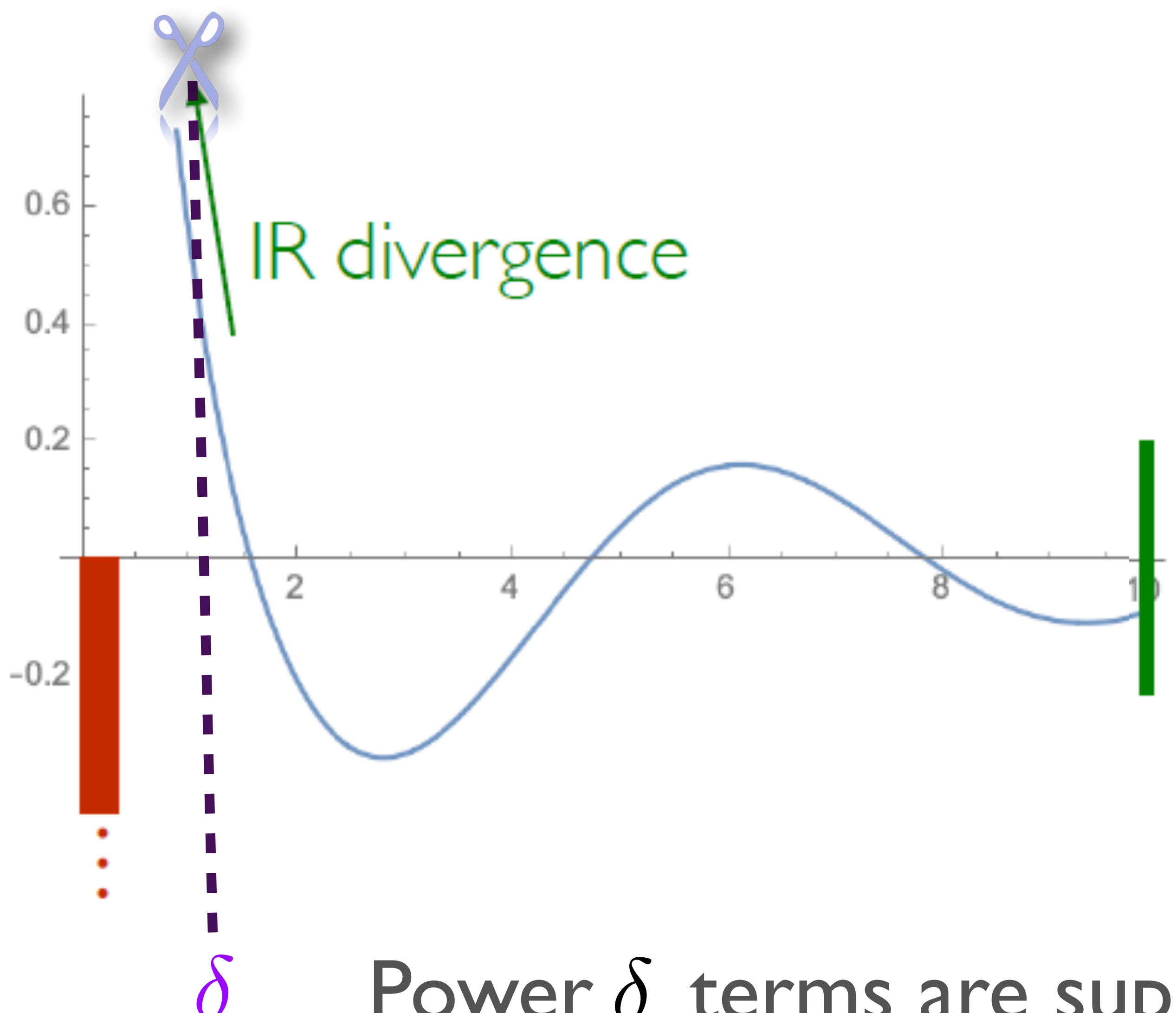

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Power δ terms are suppressed !
Large numerical cancellations !

Local IR Subtraction

- **In general, the phase-space integration over real matrix element is very hard. Dedicated general approaches are developed !**
  - **Subtraction method**
    - Find a generic simple function S has exactly same IR singularity as real matrix element

$$\lim_{p_i||p_j} \mathcal{O}(x)S = \lim_{p_i||p_j} \mathcal{O}(x)\mathcal{R} \qquad \lim_{p_i\to 0} \mathcal{O}(x)S = \lim_{p_i\to 0} \mathcal{O}(x)\mathcal{R}$$

    - ... but much easier to integrate analytically.

$$\mathcal{O}(0)\mathcal{V} + \int_0^1 dx x^{-2\epsilon_{\mathrm{IR}}} \mathcal{O}(x)\mathcal{R}$$
$$= \left(\mathcal{O}(0)\mathcal{V} + \int_0^1 dx x^{-2\epsilon_{\mathrm{IR}}} \mathcal{O}(x)S\right) + \int_0^1 dx x^{-2\epsilon_{\mathrm{IR}}} \mathcal{O}(x)\left(\mathcal{R} - S\right)$$

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Analytically known

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Analytically known

Integrating numerically in 4d

Local IR Subtraction

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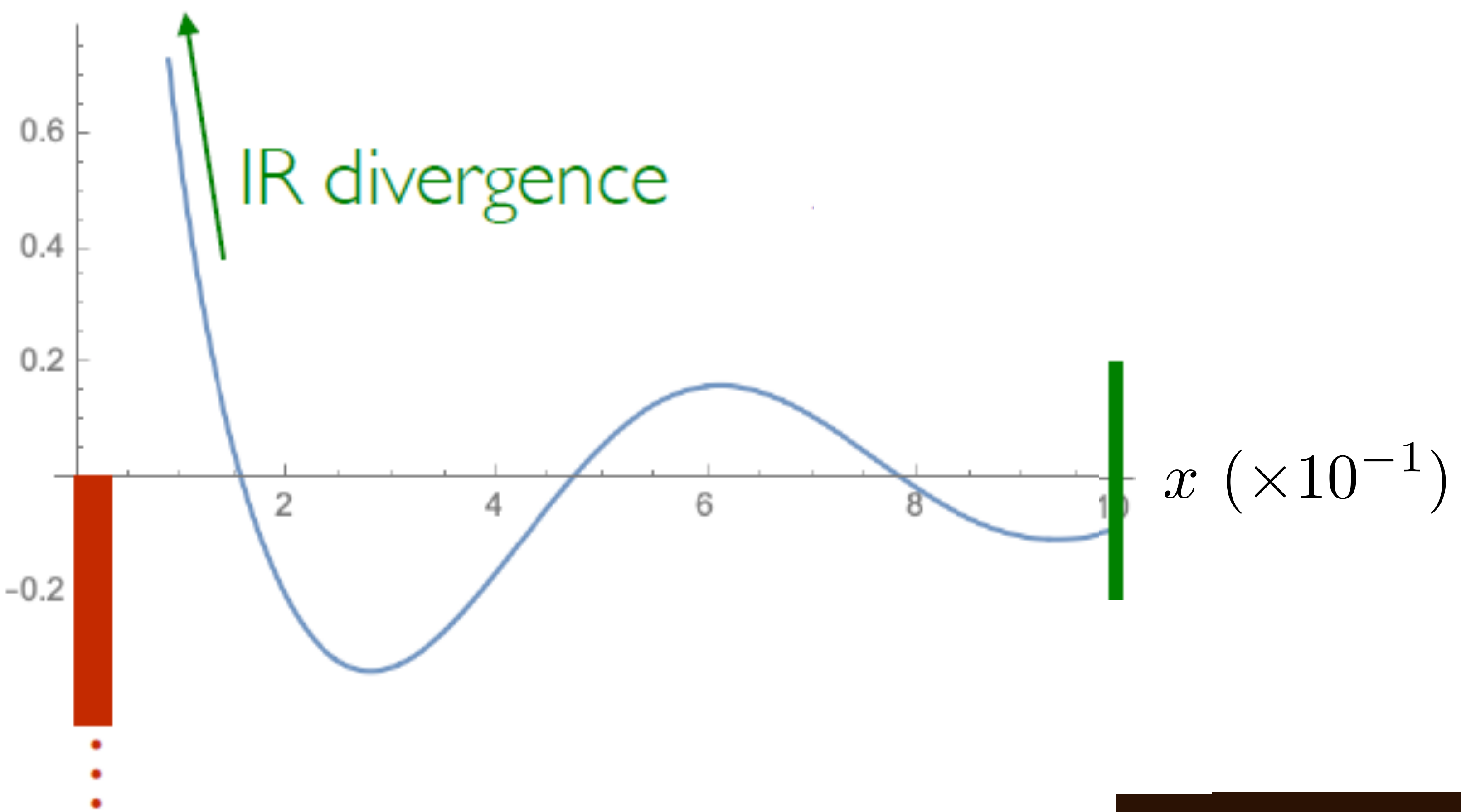

Local IR Subtraction

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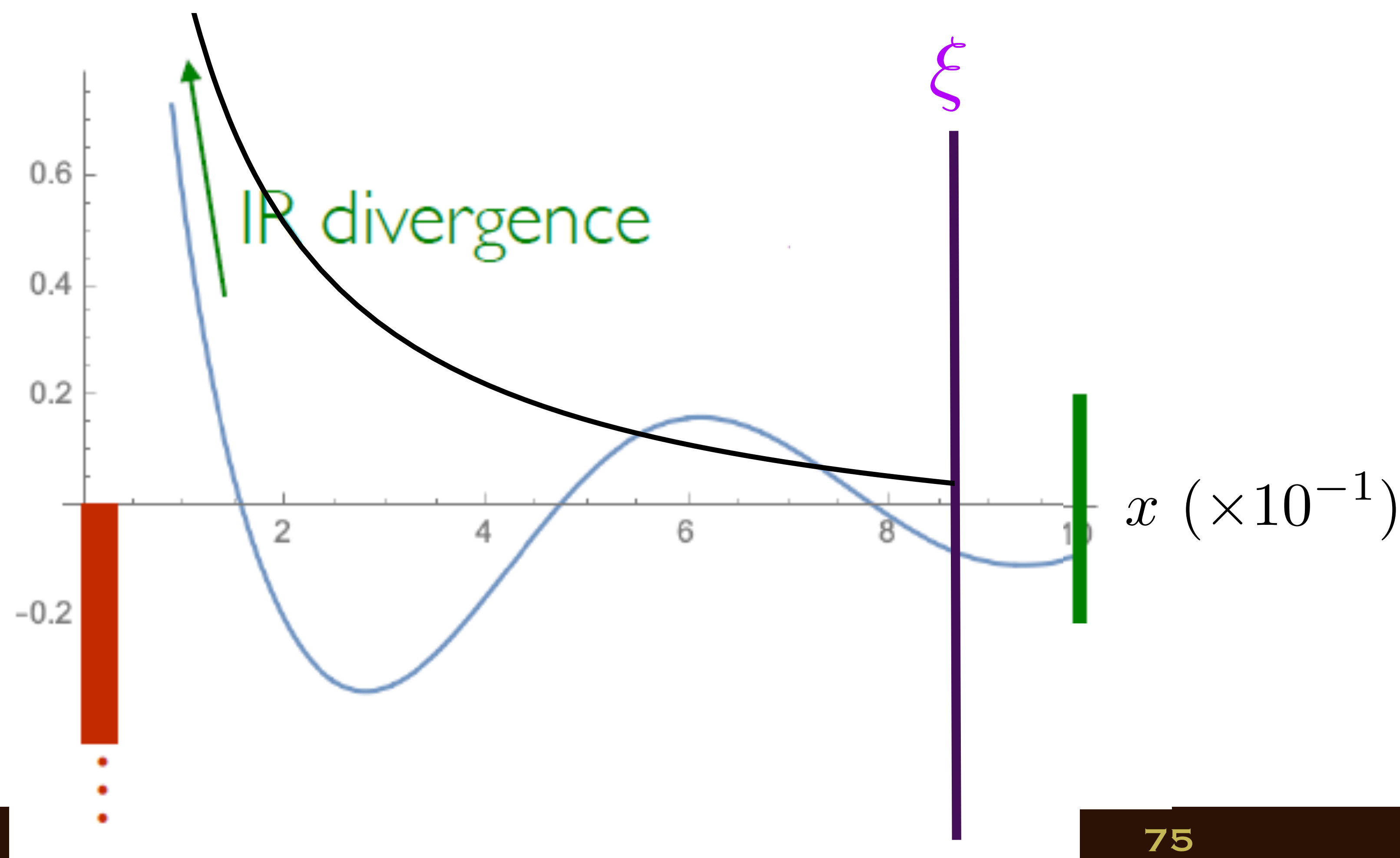
  - **Subtraction method**
    - In above toy example

$$S = \frac{\alpha_X}{\pi}\frac{\mathcal{B}\Theta(\xi - x)}{x} \qquad \forall\ \xi \in\ (0, 1]$$



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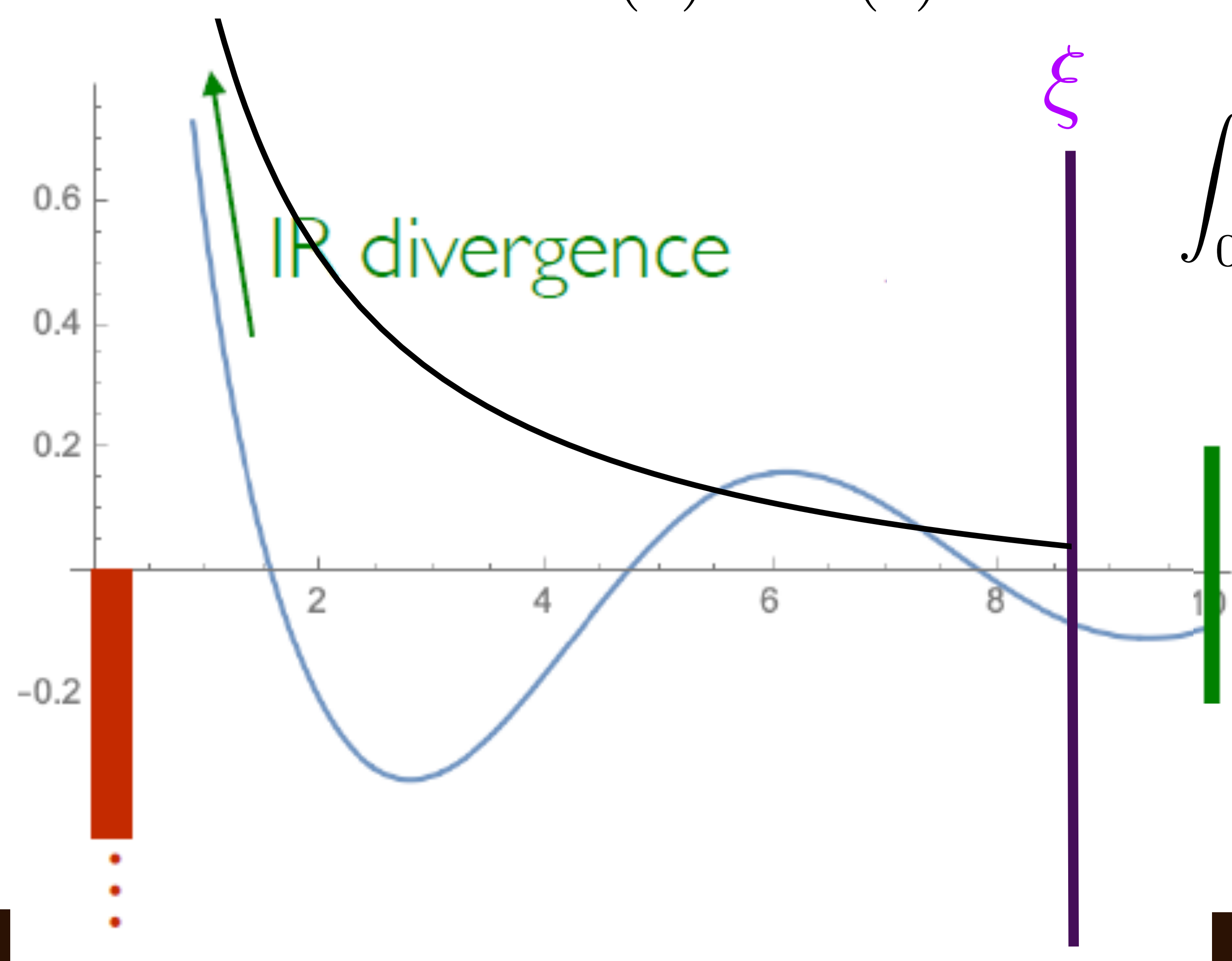

$$\int_0^1 dx x^{-2\epsilon_{\rm IR}}\mathcal{O}(x)S = -\frac{\alpha_X}{\pi}\mathcal{O}(0)\mathcal{B}\left(\frac{1}{2\epsilon_{\rm IR}} - \log\xi\right)$$

$x\ (\times 10^{-1})$

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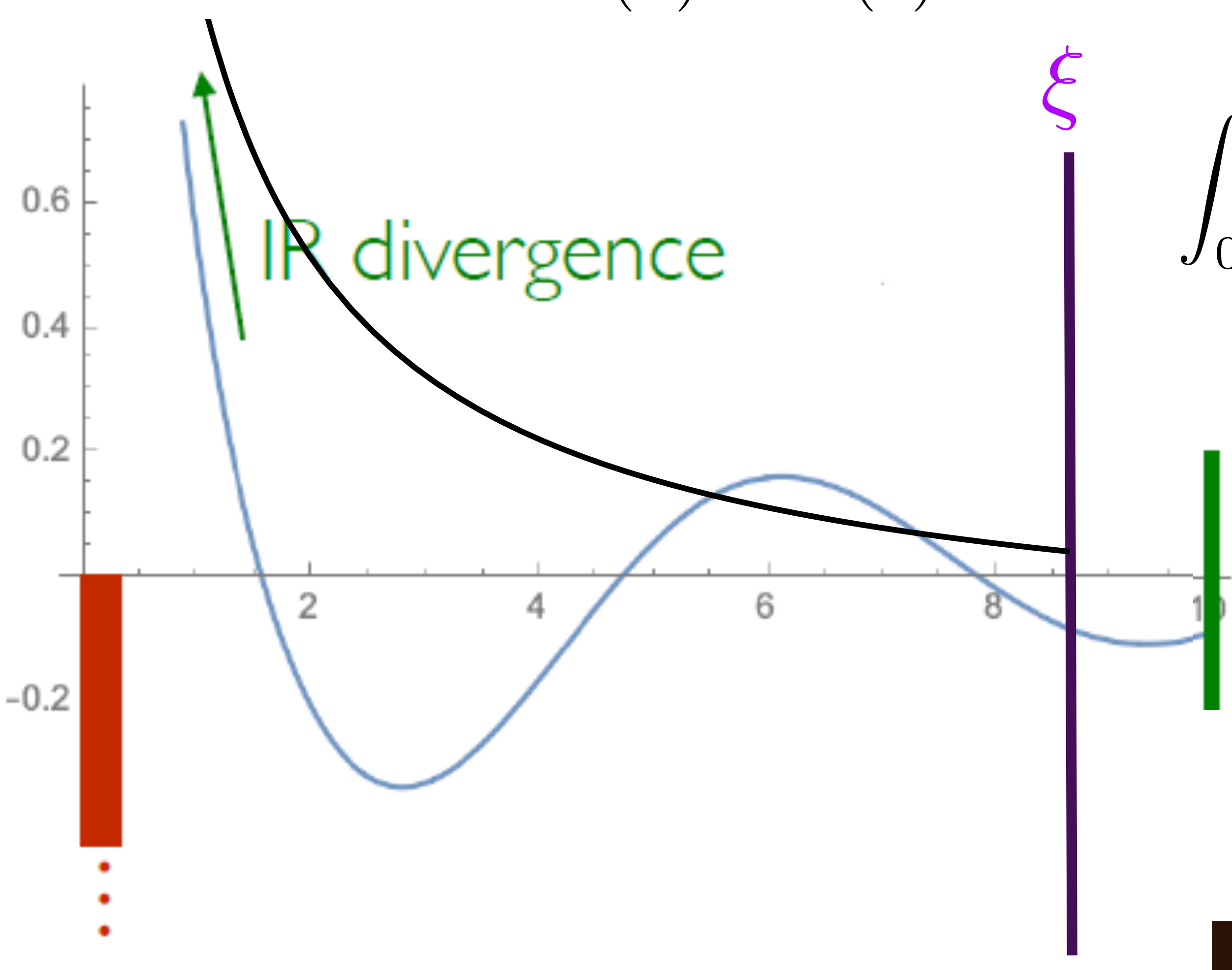


$$\int_0^1 dx x^{-2\epsilon_{\rm IR}}\mathcal{O}(x)S = -\frac{\alpha_X}{\pi}\mathcal{O}(0)\mathcal{B}\left(\frac{1}{2\epsilon_{\rm IR}} - \log\xi\right)$$

No approximation !

Numerical cancellations mitigated !

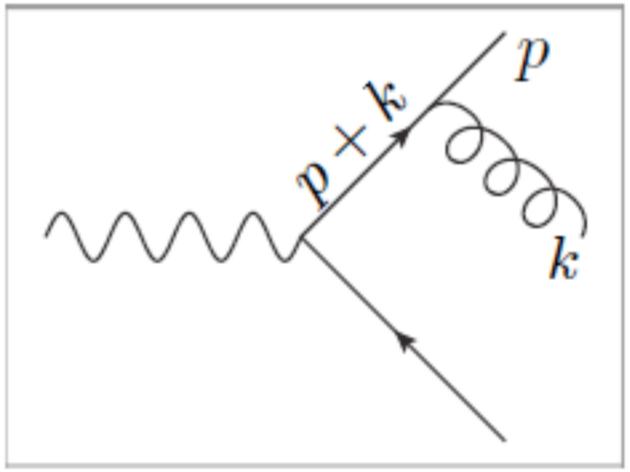
Local IR Subtraction

- **Master formula:**

$$\sigma_{\rm NLO} = \int d\Phi^{(n)}\mathcal{B} + \int d\Phi^{(n)}\mathcal{V} + \int d\Phi^{(n+1)}\mathcal{R}$$

$$= \int d\Phi^{(n)}\mathcal{B} + \int d\Phi^{(n)}\left[\mathcal{V} + \int d\Phi^{(1)} S\right] + \int d\Phi^{(n+1)}\left[\mathcal{R} - S\right]$$

- **The subtraction counterterm S should be chosen:**
  - **It exactly matches the singular behaviour of real ME**
  - **It can be integrated numerically in a convenient way**
  - **It can be integrated exactly in d dimension**
  - **It is process independent (overall factor times Born ME)**
- **In gauge theory, the singular structure is universal**



$$(p+k)^2 = 2E_pE_k(1-\cos\theta_{pk})$$

- Collinear singularity:

$$\lim_{p//k} |M_{n+1}|^2 \simeq |M_n|^2 \, P^{AP}(z)$$

- Soft singularity:

$$\lim_{k\to 0} |M_{n+1}|^2 \simeq \sum_{ij} |M_n^{ij}|^2 \frac{p_i p_j}{p_i k \; p_j k}$$

Two Widely-Used Subtraction Schemes

Dipole subtraction

Catani, Seymour, hep-ph/9602277 & hep-ph/9605323

- Most used method
- Recoil taken by one parton $\rightarrow N^3$ scaling
- Method evolves from cancelation of soft divergences
- Proven to work for simple and complicated processes
- Automated in MadDipole, AutoDipole, Sherpa, Helac-NLO, …

FKS subtraction

Frixione, Kunszt, Signer, hep-ph/9512328

- Less known method
- Recoil distributed among all particles $\rightarrow N^2$ scaling
- Probably (?) more efficient because less subtraction terms are needed
- Method evolves from cancelation of collinear divergences
- Proven to work for simple and complicated processes
- Automated in MadGraph5_aMC@NLO and in the Powheg box/Powhel

FKS Subtraction Scheme

- **The real ME singular as**

$$\mathcal{R} \xrightarrow{\text{IR limit}} \frac{1}{\xi_i}\frac{1}{1-y_{ij}} \qquad \xi_i = \frac{E_i}{\sqrt{\hat{s}}} \qquad y_{ij} = \cos\theta_{ij}$$

- **Partition the phase space in order to have at most one soft and/or one collinear singularity**

$$\mathcal{R}d\Phi^{(n+1)} = \sum_{ij} S_{ij}\mathcal{R}d\Phi^{(n+1)} \qquad \sum_{ij} S_{ij} = 1$$

$$S_{ij} \to 1 \text{ if } p_i \cdot p_j \to 0$$

$$S_{ij} \to 0 \text{ if } p_m \cdot p_n \to 0,\ \{m,n\} \neq \{i,j\}$$

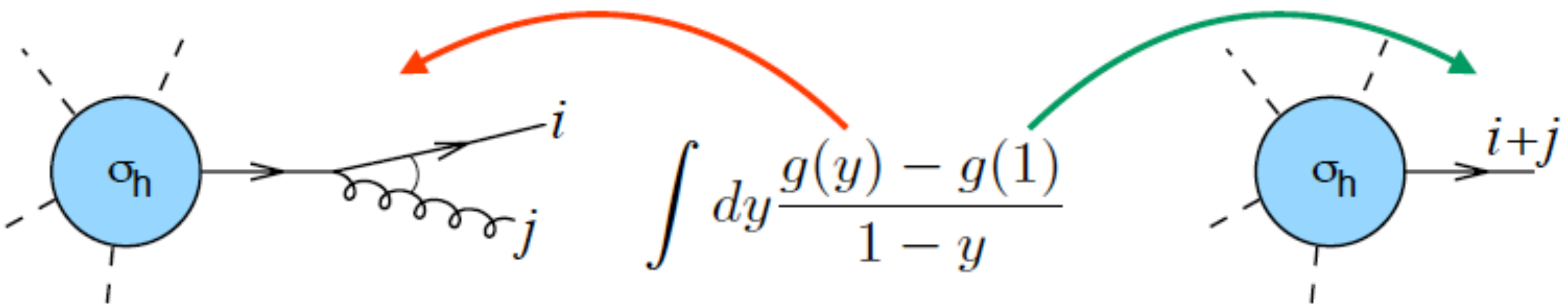
- **Use plus prescriptions to subtract the divergences**

$$d\sigma_{\tilde{R}} = \sum_{ij} \left(\frac{1}{\xi_i}\right)_+ \left(\frac{1}{1-y_{ij}}\right)_+ \xi_i\,(1-y_{ij})\,S_{ij}\mathcal{R}d\Phi^{(n+1)}$$

$$\int d\xi \left(\frac{1}{\xi}\right)_+ f(\xi) = \int d\xi \frac{f(\xi)-f(0)}{\xi} \qquad \int dy \left(\frac{1}{1-y}\right)_+ g(y) = \int dy \frac{g(y)-g(1)}{1-y}$$

FKS Subtraction Scheme

- **Counterevents:**
  - Soft counterevent ($\xi_i \to 0$)
  - Collinear counterevents ($y_{ij} \to 1$)
  - Soft-collinear counterevents ($\xi_i \to 0$ and $y_{ij} \to 1$)



Real emission

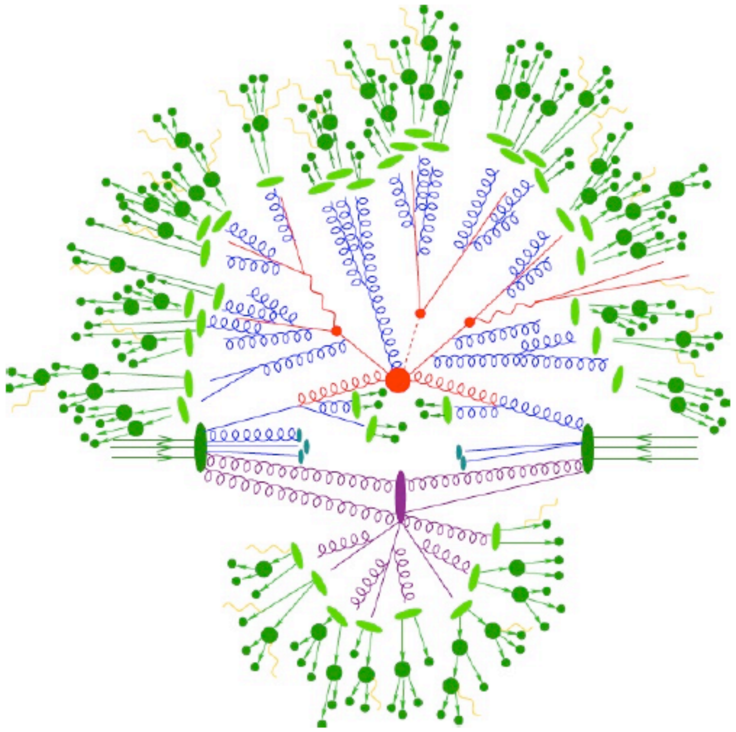
Subtraction term

- If i and j are on-shell in the event, for the counterevent the combined particle $i+j$ must be on shell
- $i+j$ can be put on shell only be reshuffling the momenta of the other particles
- It can happen that event and counterevent end up in different histogram bins
  - Use IR-safe observables and don't ask for infinite resolution!

Lecture 2

NLO Calculations & Event Simulations at NLO QCD

NLO QCD + PS



How To Improve Theoretical Predictions ?

- **Common Principle:**
  - Adopt the advantages of different aspects

Hard interaction	Parton shower
Hard and Resolved	Soft and/or Collinear

  - Balance the accuracy over the steps in the simulation chain
  - Improve not only the single steps but also their merging

- **Two main directions:**
  - Matching
    - Avoid double counting between N^kLO ($k>0$) MEs and PSs
  - Merging
    - Include more real radiation MEs (formally higher order, improve kin.)
    - Subtract double counting of real radiation from MEs and PSs

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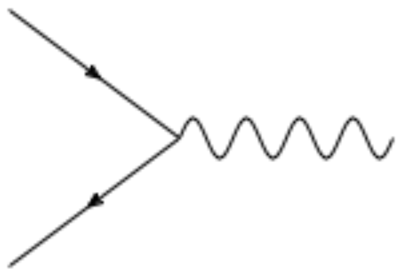
Another Double-Counting Issue

- **Matching to parton showers:** avoid double counting



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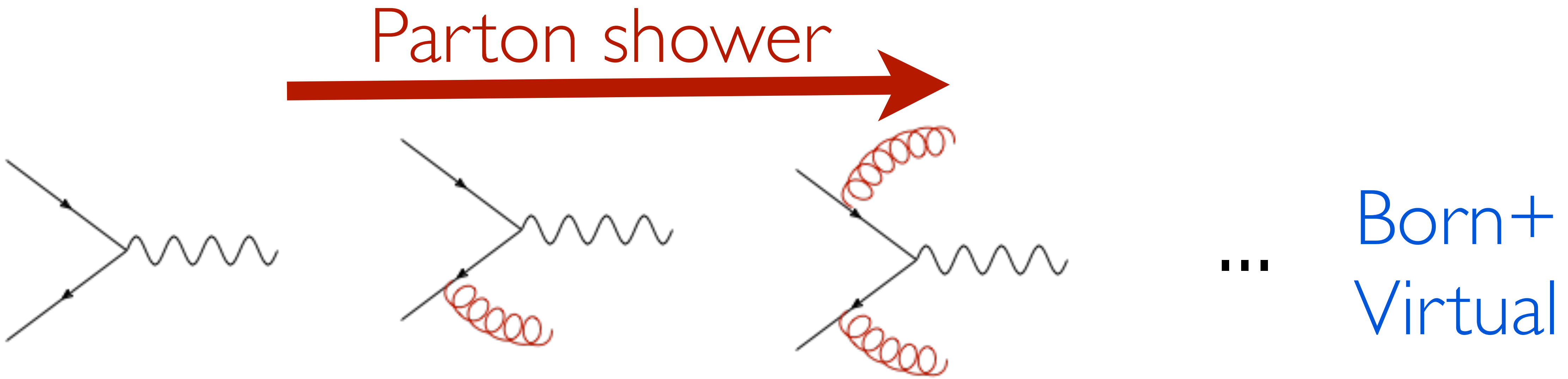

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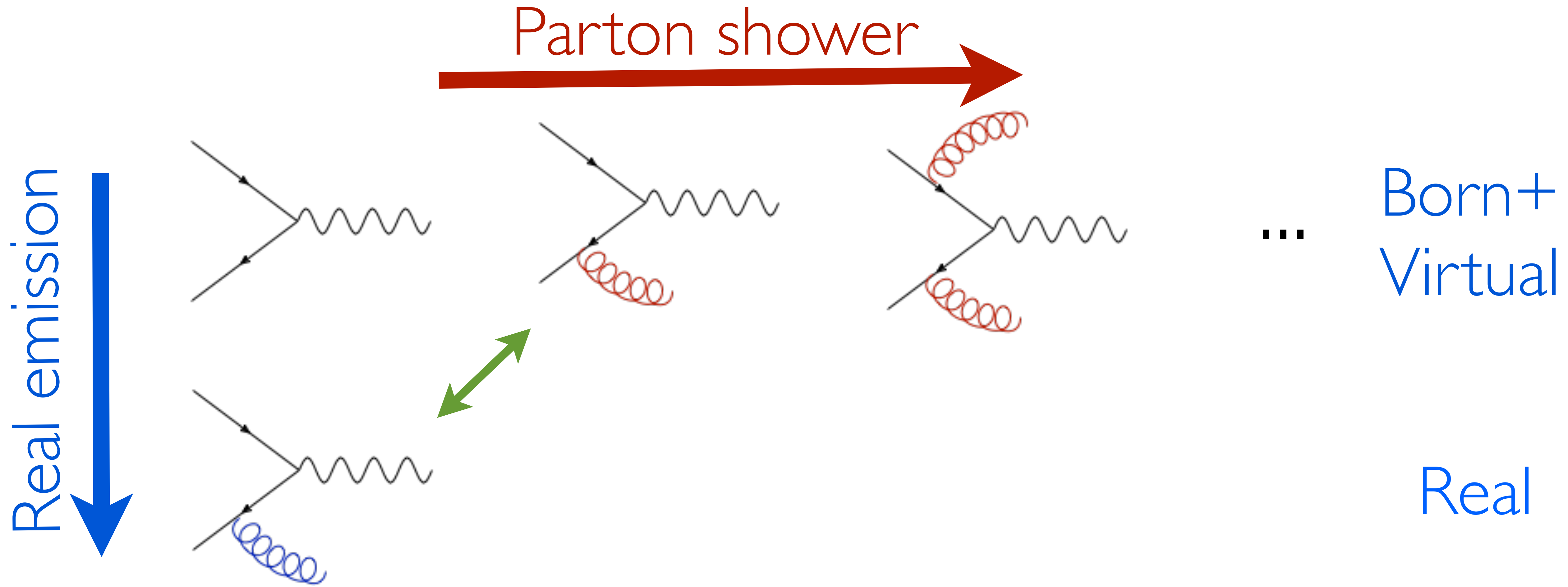

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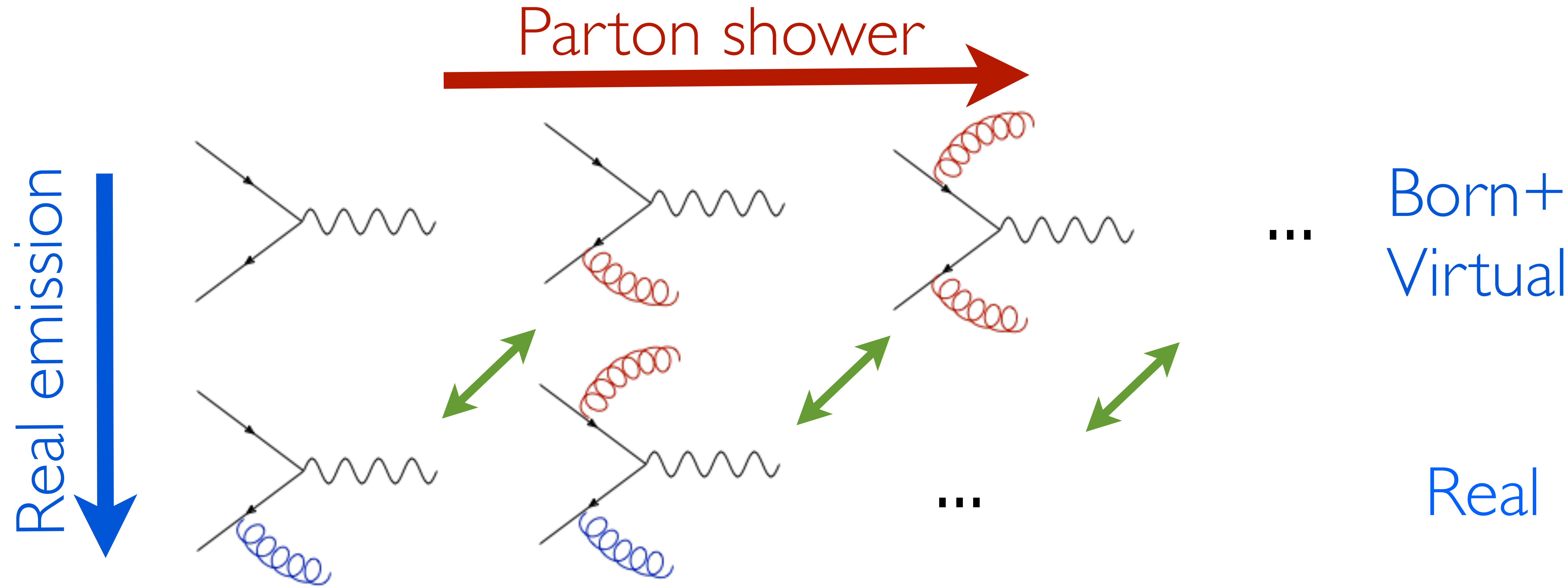
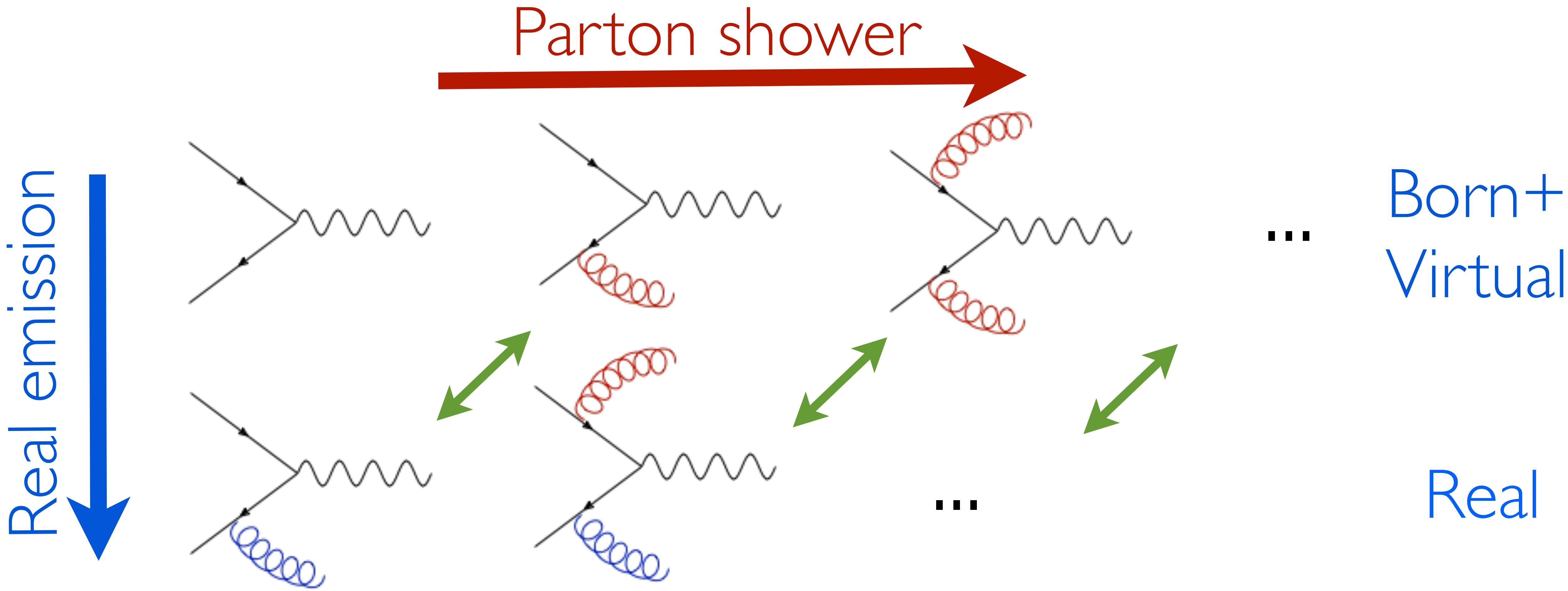


Another Double-Counting Issue

- **Matching to parton showers:** avoid double counting



- **Double counting between real emission and parton shower**
- **Double counting between virtual corrections and the non-emission probability via the Sudakov factor in parton shower**

Another Double-Counting Issue

- **Like LO, let us wrongly generate events separately from Born, virtual and real parts, and then pass these events to a parton shower:**

$$d\sigma_{\rm NLO+PS}^{\rm naive} = \left[\mathcal{B}+\mathcal{V}\right] d\Phi^{(n)} I_{\rm MC}^{(n)} + \mathcal{R} d\Phi^{(n+1)} I_{\rm MC}^{(n+1)}$$

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Parton shower operators

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  - Let us check ...

$$I_{\text{MC}} = \Delta_a + \Delta_a d\Phi^{(1)} \frac{\alpha_s}{2\pi} P_{a\to bc}$$

$$\Delta_a = \exp\left(-\int d\Phi^{(1)} \frac{\alpha_s}{2\pi} P_{a\to bc}\right) = 1 - \int d\Phi^{(1)} \frac{\alpha_s}{2\pi} P_{a\to bc} + \mathcal{O}(\alpha_s^2)$$

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$$d\sigma^{\text{naive}}_{\text{NLO+PS}} = (\mathcal{B}+\mathcal{V})\, d\Phi^{(n)} + \mathcal{R} d\Phi^{(n+1)}$$

$$+\mathcal{B} d\Phi^{(n)} \left(d\Phi^{(1)} \frac{\alpha_s}{2\pi} P_{a\to bc} - \int d\Phi^{(1)} \frac{\alpha_s}{2\pi} P_{a\to bc}\right) + \mathcal{O}(\alpha_s^{b+2})$$

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The MC@NLO Method

Frixione, Webber (JHEP'02)

- **In the MC@NLO formalism, double counting can be cured by the so-called Monte Carlo counterterms**

$$\Delta = \exp\left(-\int d\Phi^{(1)} MC\right)$$
$$I_{\rm MC} = \Delta + \Delta d\Phi^{(1)} MC = 1 - \int d\Phi^{(1)} MC + d\Phi^{(1)} MC + \mathcal{O}(\alpha_s^2)$$

- **The MC@NLO cross section is:**

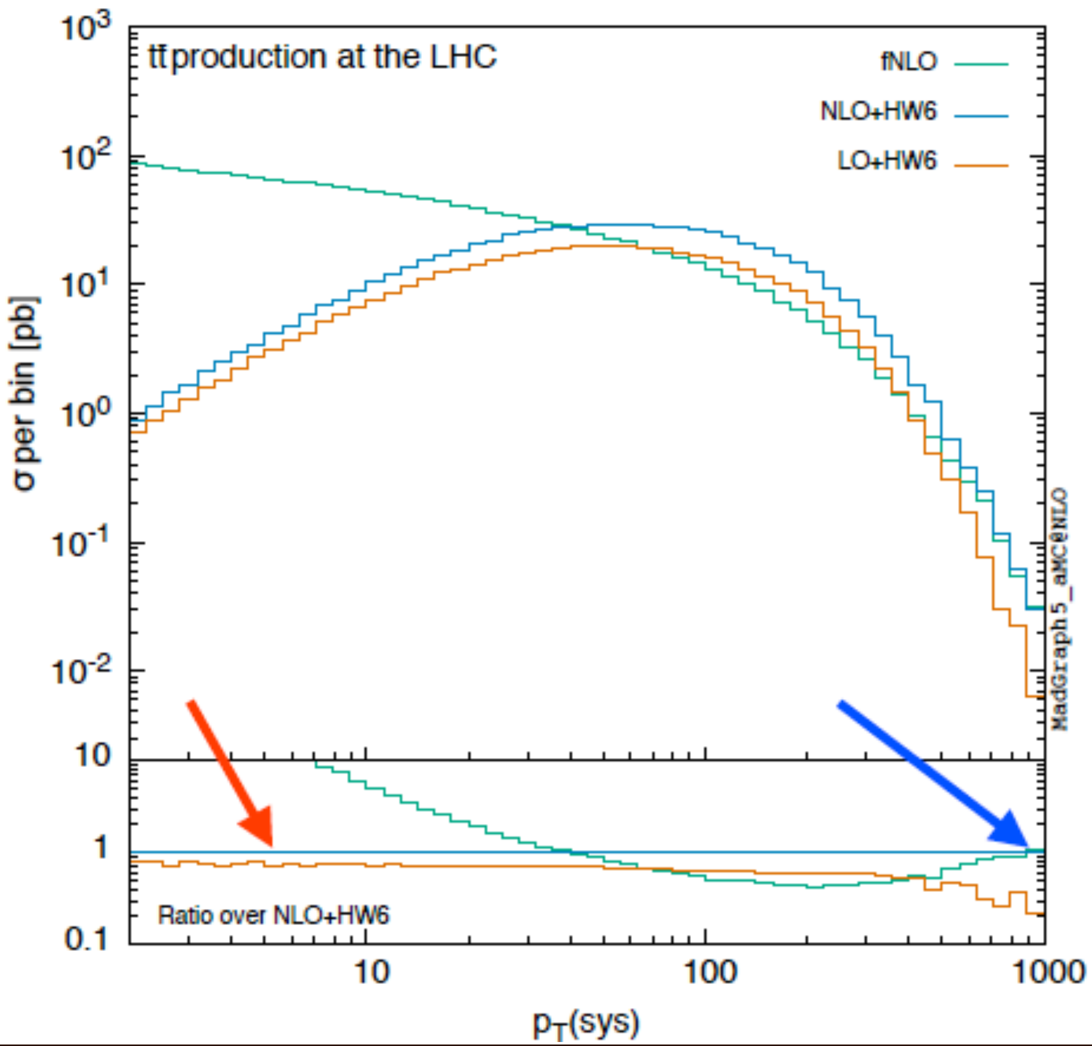
$$d\sigma^{\rm MC@NLO}_{\rm NLO+PS} = \left(\mathcal{B} + \mathcal{V} + \mathcal{B}\int d\Phi^{(1)} MC\right) d\Phi^{(n)} I^{(n)}_{\rm MC} + \left(\mathcal{R} - \mathcal{B}MC\right) d\Phi^{(n+1)} I^{(n+1)}_{\rm MC}$$

- **Expanding the Sudakov up to NLO:**

$$\begin{aligned} d\sigma^{\rm MC@NLO}_{\rm NLO+PS} = & \left(\mathcal{B} + \mathcal{V} + \mathcal{B}\int d\Phi^{(1)} MC\right) d\Phi^{(n)} + \left(\mathcal{R} - \mathcal{B}MC\right) d\Phi^{(n+1)} \\ & + \mathcal{B}\left(d\Phi^{(1)} MC - \int d\Phi^{(1)} MC\right) d\Phi^{(n)} + \mathcal{O}(\alpha_s^{b+2}) \\ = & \; d\sigma_{\rm NLO} + \mathcal{O}(\alpha_s^{b+2}) \end{aligned}$$

The MC@NLO Method

- **The MC counterterm has remarkable properties:**
  - **Avoiding double counting**
  - **Matching the IR singular behaviour of the real ME, making it possible to generate unweighted events (up to a sign though)**
  - **A smooth matching between PS and ME: in the IR (hard) region, same shape as PS (ME)**



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S-event H-event

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S-event H-event

Without showering, NLO events from LHE file is NOT physical.

The POWHEG Method

Nason (JHEP'04)

- **In the POWHEG formalism, it modifies the Sudakov for the first emission.**

$$\tilde{\Delta}(Q,Q_0) = \exp\left(-\int_{Q_0}^{Q} d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}\right)$$

$$\tilde{I}_{\rm MC} = \tilde{\Delta}(Q,Q_0) + \tilde{\Delta}(Q,t)d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}$$

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Where t is the scale at which R/B is evaluated

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- **The POWHEG cross section is:**

$$d\sigma^{\rm POWHEG}_{\rm NLO+PS}=\left(\mathcal{B}+\mathcal{V}+\int d\Phi^{(1)}\mathcal{R}\right)d\Phi^{(n)}\tilde{I}_{\rm MC}$$

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$$d\sigma^{\mathrm{POWHEG}}_{\mathrm{NLO+PS}}=\left(\mathcal{B}+\mathcal{V}+\int d\Phi^{(1)}\mathcal{R}\right)d\Phi^{(n)}\tilde{I}_{\mathrm{MC}}$$

- **Verifying there is no double counting.**

$$\tilde{\Delta}(Q,t)d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}=\frac{d\tilde{\Delta}(Q,t)}{dt}\quad\Longrightarrow\quad\int_{Q_0}^{Q}dt\tilde{\Delta}(Q,t)d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}=\tilde{\Delta}(Q,Q)-\tilde{\Delta}(Q,Q_0)=1-\tilde{\Delta}(Q,Q_0)$$

t integration goes to 1

$$d\sigma^{\mathrm{POWHEG}}_{\mathrm{NLO+PS}}=\left(\mathcal{B}+\mathcal{V}+\int d\Phi^{(1)}\mathcal{R}\right)d\Phi^{(n)}\left(1-\int d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}+d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}+\mathcal{O}(\alpha_s^2)\right)$$

$$=d\sigma_{\mathrm{NLO}}+\mathcal{O}(\alpha_s^{b+2})$$

The POWHEG Method

$$d\sigma^{\rm POWHEG}_{\rm NLO+PS} = \boxed{\left(\mathcal{B}+\mathcal{V}+\int d\Phi^{(1)}\mathcal{R}\right)} d\Phi^{(n)}\left(\boxed{\tilde{\Delta}(Q,Q_0)} + \tilde{\Delta}(Q,t)d\Phi^{(1)}\frac{\mathcal{R}}{\mathcal{B}}\right)$$

global K factor

modified Sudakov for 1st emission

- Note that when matching to PS one has to veto emissions harder than t (in the Powheg formalism, is has to be interpreted as transverse momentum), even for showers with a different ordering variable
  - Formula to be modified for angular-ordered PS in order to keep color coherence
- MC@NLO and Powheg are formally equivalent at NLO level. In practice, there are many differences between the two

MC@NLO vs POWHEG

- **The two methods can be cast into a single formula**

$$d\sigma_{\mathrm{NLO+PS}} = \overline{\mathcal{B}}^s \left(\Delta^s(Q,Q_0) + \Delta^s(Q,t) d\Phi^{(1)} \frac{\mathcal{R}^s}{\mathcal{B}} \right) d\Phi^{(n)} + \mathcal{R}^f d\Phi^{(n+1)}$$

$$\overline{\mathcal{B}}^s = \mathcal{B} + \mathcal{V} + \int d\Phi^{(1)} \mathcal{R}^s$$

$$\mathcal{R} = \boxed{\mathcal{R}^s} + \boxed{\mathcal{R}^f}$$

singular finite

MC@NLO $\mathcal{R}^s = \mathcal{B}MC$

Powheg $\mathcal{R}^s = F\mathcal{R}, \mathcal{R}^f = (1-F)\mathcal{R}$

default F=1,
but can be tuned in order to
suppress non-singular part of R

MC@NLO vs POWHEG

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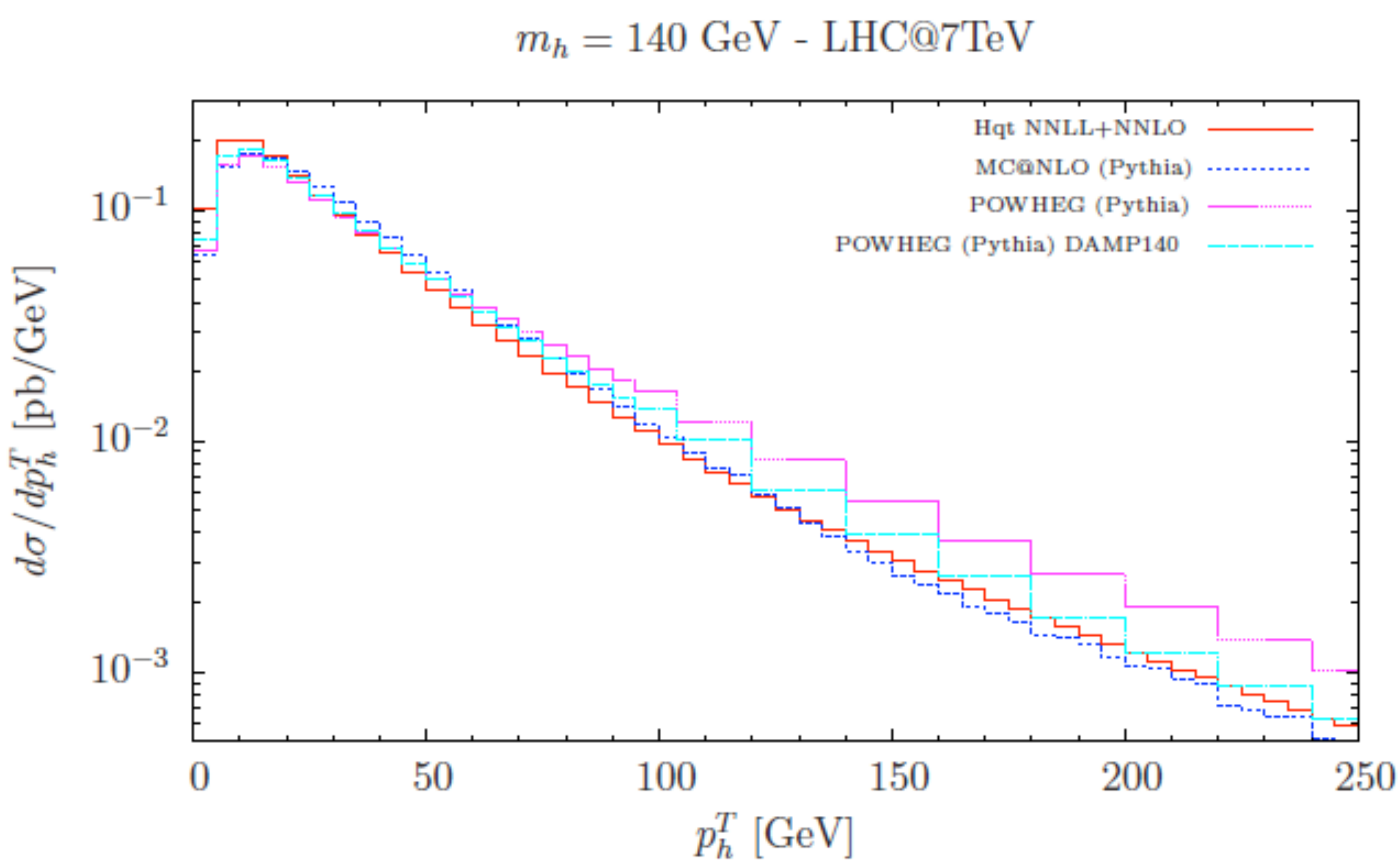
$$d\sigma_{\mathrm{NLO+PS}} = \overline{\mathcal{B}}^s \left(\Delta^s(Q, Q_0) + \Delta^s(Q, t) d\Phi^{(1)} \frac{\mathcal{R}^s}{\mathcal{B}^s} \right) d\Phi^{(n)} + \mathcal{R}^f d\Phi^{(n+1)}$$

$$F = \frac{h^2}{h^2 + p_T^2}$$ $p_T \gg h$ are suppressed



MC@NLO naturally matches analytic resummation+FO curve at large p_T
Powheg (without damping) overshoots the FO
Damping recovers matching at large p_T

to
of R

MC@NLO vs POWHEG

	MC@NLO	POWHEG
Parton showers are (usually) not exact in the soft limit: MC@NLO needs an artificial smoothing	☹	☺
MC@NLO does not exponentiate the non-singular part of the real emission amplitudes	☺	☹
MC@NLO does not require any tricks for treating Born zeros	☺	☹
POWHEG is independent from the parton shower (although, in general the shower should be a truncated vetoed)	☹	☺
POWHEG has (almost) no negatively weighted events	☹	☺
Automation of the methods: http://amcatnlo.cern.ch, http://powhegbox.mib.infn.it, http://www.sherpa-mc.de	☺	☺

Try to play/run MadGraph5_aMC@NLO according to the tutorial

3. NLO runs

https://indico.ihep.ac.cn/event/7822/contributions/98095/attachments/52139/60103/Tutorial_MG5aMC_Hefei2018_NLO.pdf

MG5aMC installation instruction

https://indico.ihep.ac.cn/event/7822/page/1116-tool-installation

The code becomes too large that no one can understand everything.
If you have any questions, please post your questions on the launchpad !

https://launchpad.net/mg5amcnlo