有效场论简介

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2025年“微扰量子场论及其应用”前沿讲习班暨前沿研讨会

7.6 – 7.21，山东大学，济南

授课内容

1. EFT基本概念

2. Relevant, irrelevant, marginal interactions (Landau liquid and BCS)

3. 微扰匹配 (perturbative matching)

4. Heavy particle decoupling in $\overline{MS}$ scheme

5. 重夸克有效理论 (why sky is blue,重电子有效理论,卢瑟福散射之软极限)

6. 刻画非相对论性系统的non-relativistic EFT (NR scalar field theory, NRQCD/NRQED，Lamb shift)

7. 总结

参考文献

◊ J. Polchinski, Effective field theory and Fermi surface, hep-th/9210046

◊ H. Georgi, Effective field theory, Annu. Rev. Nucl. Part. Sci. 1993

◊ D. Kaplan, Effective field theories, nucl-th/9506035

◊ A. Manohar, Effective field theories, hep-ph/9606222

◊ P. Lepage, What is renormalization, hep-ph/0506330

◊ S. Davidson et al. (ed), Effective field theory in particle physics and cosmology, Les Houches 2017

Section 1. 有效场论(EFT)基本概念

朴素的经验事实

◊ **不同能标(距离）会呈现出新的物理现象;**

◊ **低能标（长程）物理对高能标（短程）物理的细节并不敏感**

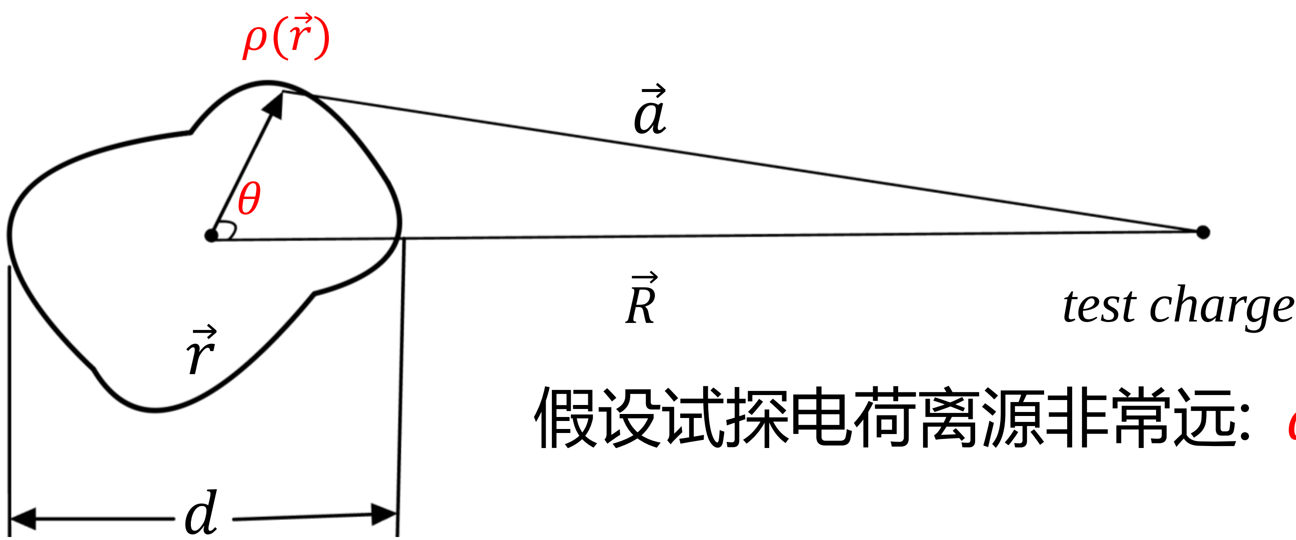
◊ **物理学的进展建立在做合理近似的基础上**

* 研究流体力学并不需要知道原子的微观结构;
* 研究原子，分子，凝聚态物理并不需要了解QCD及电弱统一理论;
* 研究原子核物理并不需要知道顶夸克及Higgs的精确知识;
* 研究TeV物理并不需要知道完整的量子引力理论(string theory?)

◊ **相比从最基本的理论框架出发做ab-initio计算，更加有效的研究方式是聚焦于最相关的低能自由度。高能标动力学的效应可以通过有限个参数来刻画**

静电势的多极矩展开 (multipole expansion)

定域的电荷分布源，电荷密度为$\rho(\vec{r})$



假设试探电荷离源非常远: $d \ll R$

源的尺寸为d

静电势的具体形式非常复杂！

$$V(R) = \int d^3\,\vec{r}\,\frac{\rho(\vec{r})}{|\vec{a}|} = \int d^3 r\,\frac{\rho(\vec{r})}{\sqrt{R^2 + 2Rrcos\theta + R^2}}$$

静电势的多极矩展开 (multipole expansion)

利用 $|\vec{r}| \sim d \ll R$，做Taylor expansion in $\frac{r}{R}$

$$V(R) \approx \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int d^3r\, r^n \rho(\vec{r}) P_n(cos\theta)$$

$$= \frac{q}{R} + \frac{p}{R^2} + \frac{Q}{R^3} + \cdots$$

monopole/charge　dipole　quadrupole

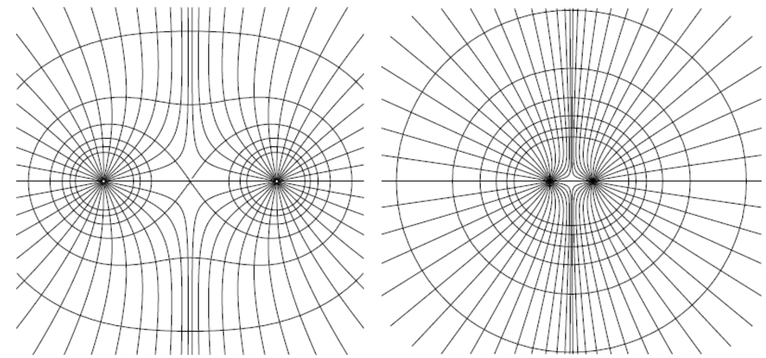


Fig. 2.1 The electric field and potential lines for two point charges of the same sign. The right figure is given by zooming out the left figure.

Wilson系数： q, p, Q, encode short-distance physics

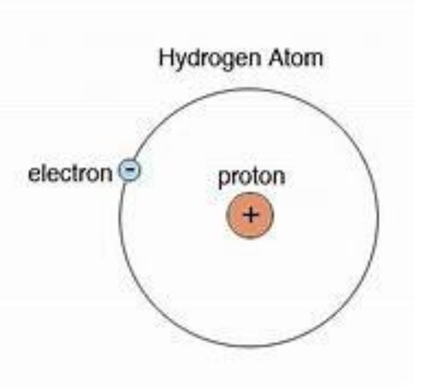
多级矩展开，级数收敛很快

长程物理对短程物理的细节（即$\rho(\vec{r})$的具体形式）并不敏感，
只敏感于电荷源的一些bulk properties:　q, p, Q, ...

氢原子

氢原子: 电子和质子通过电磁作用形成的非相对论性束缚态

基态束缚能 = **-13.6 eV**



【精细结构需要考虑电子自旋，旋轨耦合, Darwin项等相对论修正,
更高精度需要包括QED辐射修正(Lamb shift)】

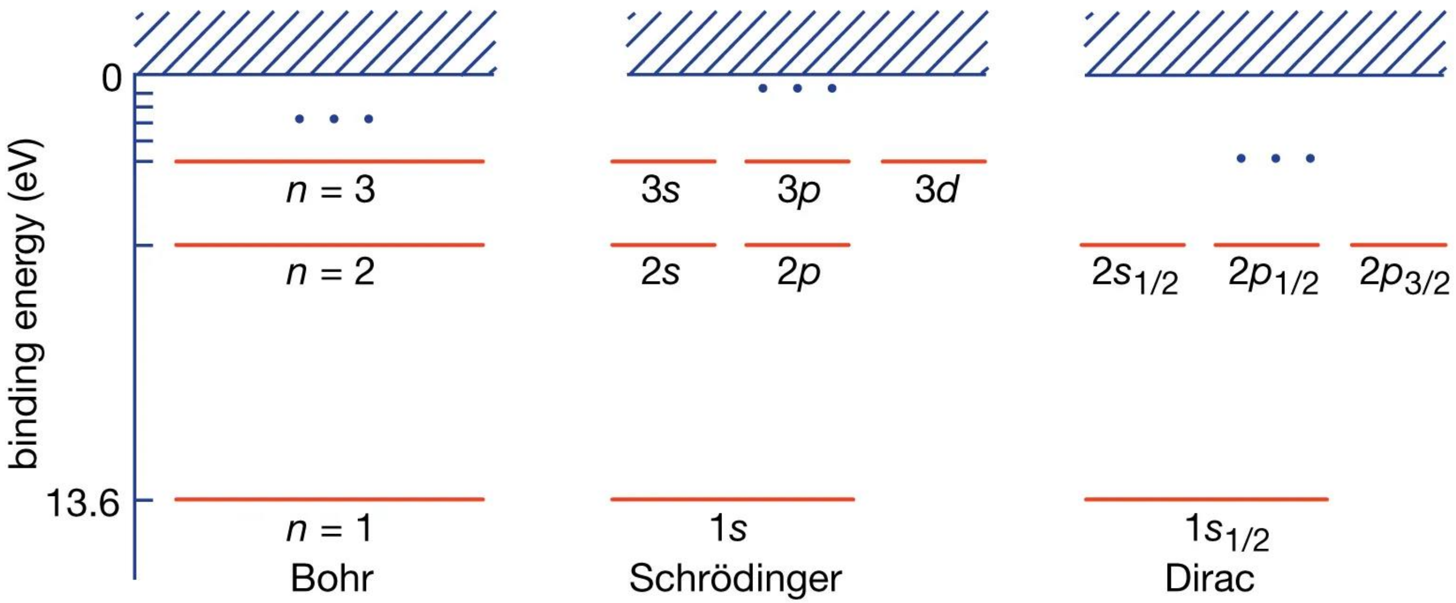
What about proton?

唯一需要知道质子的性质：

无穷重、静止的点粒子, 携带单位正电荷 (提供库伦势)

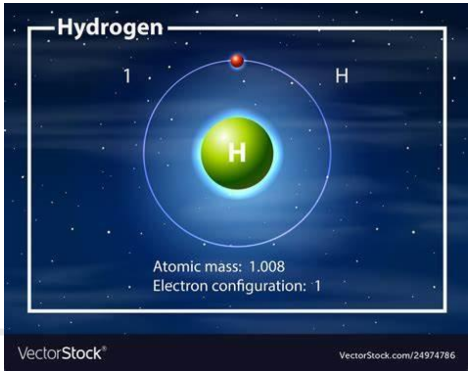
质子的质量、自旋及内部结构(夸克成分, 部分子分布)，charm/bottom夸克, 弱相互作用, 均为无关信息(irrelevant)

氢原子能级



binding energy (eV)
0
13.6
n = 3
n = 2
n = 1
Bohr
3s
3p
3d
2s
2p
1s
Schrödinger
2s1/2
2p1/2
2p3/2
1s1/2
Dirac

能级公式



薛定谔方程： $$\left[-\frac{\hbar\nabla^2}{2m_e}-\frac{Z\alpha}{r}\right]\Psi_{\mathrm{Sch}}(\mathrm{r}) = E\,\Psi_{Sch}(r)$$

$$E_n = -\frac{m\,\alpha^2}{2n^2} = \frac{13.6eV}{n^2}$$

狄拉克方程： $$\left(-i\hbar\boldsymbol{\alpha}\cdot\nabla+\beta m-\frac{Z\alpha}{r}\right)\Psi_{Dirac} = E\Psi_{Dirac}$$

$$E_{nj} = \frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n-j-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\alpha^2}\right)^2}}}$$

Quite accurate, but missing the O($m\,\alpha^5$ ln α) Lamb shift

氢原子是多标度的非相对论性束缚态

基态束缚能 $E_{binding} \sim m\,\alpha^2$

Virial theorem

$$< T > = -\frac{1}{2} < V > \;\sim\; mv^2$$

因此 $\frac{v}{c} \sim \alpha = \frac{1}{137} \ll 1$ 电子是相对论性的！

电子的特征动量(玻尔动量)： $p = \frac{\hbar}{a_0} \sim mv \sim m\,\alpha$

氢原子含有三个分得很开的特征能标：

$$m \gg mv \gg mv^2$$

MeV　　KeV　　eV

狄拉克方程作为基本理论

Underlying theory: Dirac equation in Coulomb potential

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\alpha}{r}\right]\Psi_{Dirac}=\left[-i\hbar c\boldsymbol{\alpha}\cdot\nabla+mc^2\beta\right]\Psi_{Dirac}$$

4分量

$$\Psi_{Dirac}=\begin{pmatrix}大\\小\end{pmatrix}$$

量子化的氢原子能级公式：Darwin, Gordon (1929)

$$E_{nj}=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n-j-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\alpha^2}\right)^2}}}$$

$$\approx mc^2\left[1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)+\cdots\right]$$

Bohr　　fine-structure

薛定谔-泡利方程作为低能有效理论

有效理论： Schrödinger-Pauli equation: **对Dirac方程做非相对论展开**

valid at $\mathbf{p \ll \Lambda < m_e}$

$$i\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{sch}} = Heff\,\Psi_{sch}$$

$$\Psi_{\mathrm{sch}} = \varphi_{space} \otimes \boldsymbol{\varphi_{spin}}$$

2分量

有效哈密顿量=Leading + O(v²)

$$H_{eff} = H_0 + \triangle H$$

$$\text{where } H_0 = \frac{\vec{P}^2}{2m_e} - \frac{\alpha}{r}$$

$$\triangle H = \frac{\vec{P}^4}{8{m_e}^3c^2} - \frac{e\hbar}{4{m_e}^2c^2}\vec{\sigma}\cdot\vec{E}\times\vec{P} - \frac{e\hbar^2}{8{m_e}^2c^2}\nabla\cdot\vec{E}$$

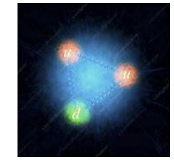
Rel. Corr. Spin-Orbital Darwin term

$$= -\frac{\vec{P}^4}{8{m_e}^3c^2} - \frac{\alpha}{2r^3{m_e}^2}\vec{L}\cdot\vec{S} + \frac{\pi\alpha}{2{m_e}^2c^2}\delta^{(3)}(\vec{r})$$

$\triangle H$作一阶微扰，可以重现Dirac方程预言的fine structure：

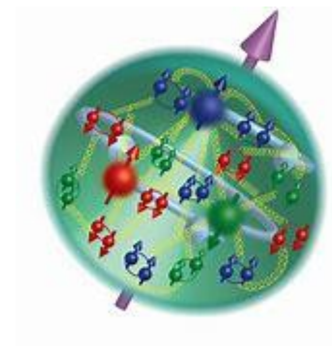
$$E_{nj} = m[1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \cdots]$$

更精确的预言：needs going beyond point-particle approximation for proton



更精确的预言需包括核子的反冲效应(折合质量)，需要输入质子质量

超精细（hyperfine）结构(21 cm谱线)需要考虑质子磁矩



更高精度甚至还需要考虑质子的**电荷半径**

proton charge radius puzzle:

alarming discrepancy between ep scattering measurement and muonic hydrogen atom spectroscopy

弱相互作用对能级贡献小到可以忽略不计：

$$\left(\frac{m_e\alpha}{M_W}\right)^2 \sim \left(5\times 10^{-8}\right)^2$$

但如过考虑atomic parity violation，弱作用给出领头阶贡献

什么是有效场论? Wilsonian perspective

Consider a QFT, with characteristic (fundamental) UV scale M

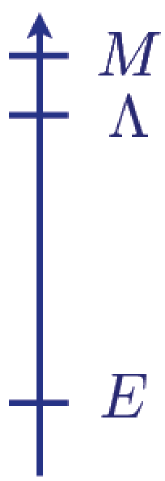
感兴趣的实验能标满足 E<<M

第一步: 选择cutoff **Λ** <M， 把所有量子场分解成高频(ω> **Λ**)和低频(ω<**Λ**)模式

$$\phi = \phi_L + \phi_H$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right)$$

$$\omega = E_k = \sqrt{\boldsymbol{k}^2 + m^2}$$



什么是有效场论? Wilsonian perspective

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低能物理完全由$\boldsymbol{\phi_L}$ 描述。低频场的Green函数可由如下生成泛函得到：

$$Z[J_L] = \int \mathcal{D}\phi_L \mathcal{D}\phi_H e^{iS(\phi_L,\phi_H)+i\int d^D x J_L(x)\phi_L(x)}$$

$$\langle 0|T\{\phi_L(x_1)\ldots\phi_L(x_n)\}|0\rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J_L(x_1)}\right)\ldots\left(-i\frac{\delta}{\delta J_L(x_n)}\right)\mathrm{Z}[\mathrm{J_L}]\Big|_{J_L=0}$$

什么是有效场论? Wilsonian perspective

第二步: 因为高频场ϕ_H并没有出现在生成泛函的最终表达式中，我们可以选择将其在路径积分中积掉(integrating out)

$$Z[J_L] \equiv \int \mathcal{D}\phi_L e^{iS_\Lambda(\phi_L)+i\int d^D x J_L(x)\phi_L(x)}$$

其中$S_\Lambda(\phi_L)$ 称作Wilsonian effective action

$$e^{iS_\Lambda(\phi_L)} = \int \mathcal{D}\,\phi_H\, e^{iS_\Lambda(\phi_L,\phi_H)}$$

注意：Wilson有效作用量显示依赖 Λ

什么是有效场论?
Wilsonian perspective

第三步：有效作用量是non-local on the scale $\Delta t \sim 1/\Lambda$, 对应于已经移除的高频模式的传播距离

显式保留低频场的能量$\omega<\Lambda$，由于分辨率有限，非定域的有效拉氏量可以展开成为无穷多个local operators的和:

$$S_\Lambda(\phi_L) = d^D x\, \mathcal{L}_\Lambda^{eff}(x)$$

Wilson系数

$$\mathcal{L}_\Lambda^{eff} = \sum c_i(\Lambda) O_i(\Lambda) = \mathcal{L}_{D\leq 4} + \frac{c_5(\Lambda) O_5(\Lambda)}{M} + \frac{c_6(\Lambda) O_6(\Lambda)}{M^2} + \cdots$$

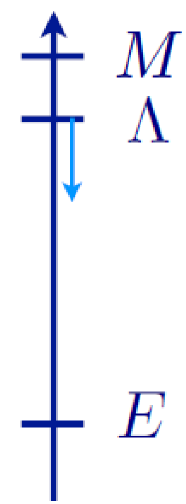
由于引入了高量纲算符，这种类型的QFT曾经被广泛认为认为是不可重整的，因此不是健康的(具有预言性)理论 .

Running couplings/Wilson coefficients

通常高频场ϕ_H代表着重粒子，其效应在低能退耦，可以安全被integrate out

ϕ_H也可以代表零质量粒子的高能激发， 因此也应该在低能EFT中被积掉

考虑降低截断Λ(不跨越重粒子阈值), 有效拉氏量中复合算符O_i的结构保持不变。由于Λ取不同值并不影响物理预言，因此降低截断Λ 的效应必须完全吸收进有效拉氏量中的耦合常数c_i 中



因此，Wilson系数$c_i = c_i(\Lambda)$是随Λ跑动的耦合常数！

量纲分析

考虑d维时空，有效拉氏量

$$\mathcal{L} = \sum c_i O_i = \sum \mathcal{L}_D$$

由洛伦兹不变、规范不变的定域算符组成

泛函积分中涉及 e^{iS}，因此作用量S量纲为0

d维时空中场和规范耦合的量纲

量纲背后对应的物理:　　标度变换

考虑d维时空的自由场论：

$$S = \int d^d x\, \bar{\psi}\, i\not{D}\, \psi, \qquad S = \int d^d \partial_\mu \phi \partial^\mu \phi$$

所以

$$0 = -d + 2[\psi] + 1, \qquad 0 = -d + 2[\phi] + 2$$

标量场，旋量场，协变导数，矢量场的量纲为:

$$[\phi] = (d-2)/2,\ [\psi] = (d-1)/2,\ [D] = 1,\ \left[gA_\mu\right] = 1$$

场强张量 $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \cdots$ 因此 A_μ 和标量场的量纲一样， 规范耦合常数的量纲为

$$[g] = 1 - (d-2)/2 = (4-d)/2$$

4维时空的可重整场论

当$d = 4$，

$$[\phi] = 1, \quad [\psi] = 3/2, \quad [A_\mu] = 1, \quad [D] = 1, \quad [g] = 0$$

可重整相互作用项要求拉氏量中算符系数的质量量纲≥ 0.
Only Lorentz invariant renormalizable interactions
(with $D \le 4$) are

$D = 0$: 1

$D = 1$: ϕ

$D = 2$: ϕ^2

$D = 3$: $\phi^3, \bar{\psi}\psi$

$D = 4$: $\phi^4, \ \phi\bar{\psi}\psi, \bar{\psi} \not{A} \psi, \ F_{\mu\nu}F^{\mu\nu}$

$D \le 3$ relevant operators
Super-renormalizable

$D = 4$ marginal: renormalizable

其它时空维数的可重整场论

可重整相互作用项要求拉氏量中算符系数的质量量纲$\geq 0.$

当$d = 2,$

$[\phi] = 0, \quad [\psi] = 1/2, \quad [A_\mu] = 0, \quad [D] = 1, \; [g] = 1$

任意的势函数$V(\phi)$ is可重整的。 4费米相互作用$(\bar{\psi}\psi)^2$ 也可重整

当$d = 6,$

$[\phi] = 2, \quad [\psi] = 5/2, \quad [A_\mu] = 2, \quad [D] = 1, \; [g] = -1$

只有 ϕ^3 理论是可重整的

4维时空的有效场论 (regarded as non-renormalizable in the old days)

Effective Lagrangian:

$$L_D = \frac{O_D}{M^{D-d}}$$

so in $d = 4$,

$$\mathcal{L}_{eff} = \mathcal{L}_{D\leq 4} + \frac{c_5\, O_5}{M} + \frac{c_6\, O_6}{M^2} + \cdots$$

An infinite number of terms (and parameters)

O_5 , O_6: Non-renormalizable interactions

$D > 4$ irelevant operators

Wilson coefficients have negative mass dimensions

Power Counting

考虑一个低能物理过程，特征动量为p, 如果忽略有效拉氏量中量纲为D及更高的算符，则EFT对于振幅预言的误差是

$$\left(\frac{p}{M}\right)^{D-4}$$

给定量纲D的算符数目是有限的, 抵消项的数目也有限

A non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy.

Usual renormalizable QFT given by taking $M \rightarrow \infty$.

Relevant, marginal and irrelevant

Relevant operators ($D < 4$): $E \to 0$ 时, 贡献越来越重要

irrelevant operators ($D > 4$): $E \to 0$ 时, 贡献越来越**不**重要

Marginal ($D = 4$): 介乎两者之间

算符伴随的Wilson系数也称为relevant, irrelevant, and marginal parameters/couplings

Examples for relevant parameters

QED中 $m_e\bar{\psi}\psi$ 是relevant算符($D=3$), 电子质量是relevant耦合

QED高能散射过程，电子质量可忽略，但低能时变得重要

类似，自旋为0玻色子质量也是 relevant参数

ϕ^3 interaction is also relevant coupling

单位算符也是relevant operators, 宇宙学常数是relevant parameters.

subject to fine-tuning problem

例子: 双标量场模型中relevant coupling

Two types of real scalar fields, one light, and one heavy

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\Phi^2 - \frac{1}{2}\kappa\phi^2\Phi \ .$$

Assuming $m \simeq \kappa \ll M$

$$[\phi] = [\Phi] = [\kappa] = 1$$

κ 是 relevant耦合

考虑 $\phi\phi \to \phi\phi$ 散射（简单起见，考虑高能情形，可忽略m)

$$\sigma_{2\phi\to 2\phi}\big|_{E_\phi \gg m, M, \kappa} \propto \left(\frac{\kappa}{E_\phi}\right)^4 \frac{1}{E_\phi^2}$$

E 增加, 截面迅速压低

$$\sigma_{2\phi\to 2\phi}\big|_{m \ll E_\phi \ll \kappa, M} \propto \left(\frac{\kappa}{M}\right)^4 \frac{1}{E_\phi^2}$$

Example for irrelevant interactions: 中微子散射截面

考虑弱相互作用的四费米子有效理论：

$$L = -\frac{4G_F}{\sqrt{2}}(\bar{e}\gamma^\mu P_L \nu_e)(\bar{\nu}_\mu \gamma_\mu P_L \mu) + \mathcal{O}(\frac{1}{M_W^4})$$

因为费米耦合常数量纲为负，$[G_F]$=-2 irrelevant interaction

中微子和带电轻子散射截面有如下的能量依赖

$$\sigma_\nu \simeq G^2 s$$

$E \to 0$ 时, 贡献越来越**小**: thus irrelevant

Characteristic of high dim. operators in any EFT

Myth of marginal coupling: $\lambda\phi^4$模型为例

λ量纲为0, 因此marginal coupling at classical level

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

考察截腿的1PI *n*点Green function (量纲=4-n):

$$\Gamma_n(p_1, ..., p_n; m, \lambda)$$

标度变换下： $\Gamma_n(sp; m, \lambda) = s^{4-n}\Gamma_n(p; m/s, \lambda)$

s<<1 case: m is relevant, λ is marginal

Renormalization of $\lambda\phi^4$模型

$$\mathcal{L}_{ren.} = \mathcal{L} + \mathcal{L}_{ct}$$

$$\mathcal{L}_{ren} = \tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \mu^{2\epsilon}\frac{\lambda}{4!}\phi^4 \ ,$$

$$\mathcal{L}_{ct} = \tfrac{1}{2}A(\partial\phi)^2 - \tfrac{1}{2}m^2 B\phi^2 - \mu^{2\epsilon}\frac{\lambda}{4!}C\phi^4 \ .$$

对场，质量和耦合常数做重整化

$$\phi_0 = \sqrt{1+A}\,\phi \equiv \sqrt{Z_\phi}\,\phi \ , \qquad m_0^2 = m^2(1+B)/Z_\phi \ , \qquad \lambda_0 = \lambda(1+C)/Z_\phi^2 \ .$$

$\lambda\phi^4$模型满足的重整化群方程

裸Green function and renormalized Green function满足

$$\Gamma_n^0(p_1,...,p_n;\lambda_0,m_0,\epsilon)=Z_\phi^{-n/2}\Gamma_n(p_1,...,p_n;\lambda,m,\mu,\epsilon)$$

利用 $\mathrm{d}\Gamma_n^0/\mathrm{d}\mu=0,$ 得到

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-n\gamma\right]\Gamma_n=0$$

$$\beta=\mu\partial\lambda/\partial\mu,\quad \gamma_m=\mu\partial m/\partial\mu,\quad \gamma=\frac{1}{2}\mu\partial\ln Z_\phi/\partial\mu.$$

单圈水平，得到

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2},\qquad \gamma_m=\frac{\lambda}{16\pi^2},\qquad \gamma=\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2$$

$\lambda\phi^4$模型, 重整化群方程

考虑量子修正，会增加一个新的含量纲参数：重整化能标μ

因此标度变换公式需改写为

$$\Gamma_n(sp; m, \lambda, \mu) = s^{4-n}\Gamma_n(p; m/s, \lambda, \mu/s)$$

此标度变换公式等价于以下微分方程

$$\left[s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} - (4-n)\right]\Gamma(sp; m, \lambda, \mu) = 0.$$

结合RGE方程, 可以得到对标度因子求导(消除对μ求导)的方程：

$$\left[-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial \lambda} + (\gamma_m - 1)m\frac{\partial}{\partial m} - n\gamma + 4 - n\right]\Gamma_n(sp; m, \lambda, \mu) = 0.$$

量子修正：标度破坏(反常量纲)

在mass-independent renormalization scheme (such as $\overline{\mathbf{MS}}$)

$\beta,\ \gamma_m$ and γ 只依赖于λ，并不依赖 m/ μ. 以上RGE的解为

$$\Gamma_n(sp; m, \lambda, \mu) = s^{4-n}\Gamma_n(p; \overline{m}(s), \overline{\lambda}(s), \mu)e^{-n\int_1^s \mathrm{d}s'\gamma(\overline{\lambda}(s'))/s'}$$

其中 $s\frac{\partial\overline{\lambda}(s)}{\partial s} = \beta(\overline{\lambda}(s))\ , \quad \overline{\lambda}(1) = \lambda \qquad s\frac{\partial\overline{m}(s)}{\partial s} = (\gamma_m - 1)\overline{m}(s)\ , \quad \overline{m}(1) = m$

At tree level, $\beta,\ \gamma_m$ and γ 为0，回到classical scaling rule

量子水平，如果只保留γ非0且假设是常数， e指数因子变为 $s^{-n\gamma}$

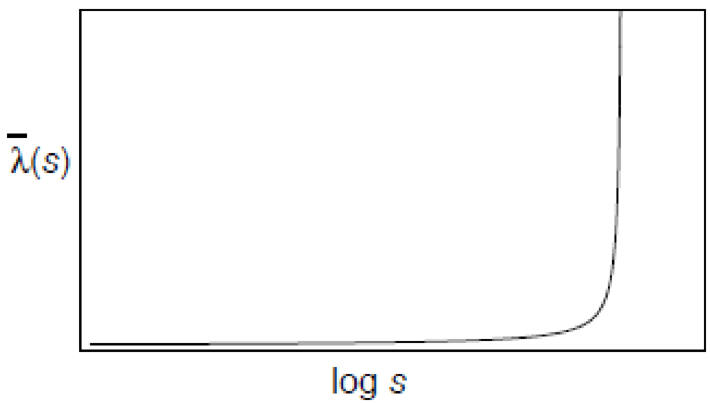
修改了经典的engineering dimension 4-n in s^{4-n}

所以称为 anomalous dimension

$\lambda\phi^4$模型中的跑动耦合常数

从β函数可以求解跑动耦合常数：

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \quad \longrightarrow \quad \bar{\lambda}(s) = \frac{\lambda}{1 - (3\lambda/16\pi^2)\ln s} \ .$$



Quantum correction turns marginal into irrelevant

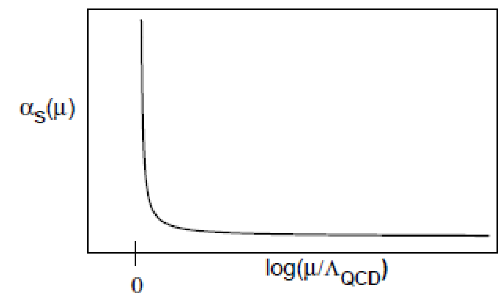
Similar pattern also occurs in QED

量子色动力学中的跑动强耦合常数

从单圈QCD β 函数（著名的负号）可以求解跑动强耦合常数：

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} = \frac{g^3}{16\pi^2}\left[-11 + \frac{2N_f}{3}\right] \equiv -\frac{b_0 g^3}{2}$$

$$\alpha_s(\mu) = \frac{1}{1/\alpha_s(\mu_0) + 4\pi b_0 \ln(\mu/\mu_0)} \equiv \frac{1}{4\pi b_0 \ln(\mu/\Lambda_{QCD})}$$



$$\Lambda_{QCD} = \mu_0 E^{1/4\pi b_0 \alpha(\mu_0)}$$

In this case, quantum correction changes marginal to relevant!

Similar pattern occur in 2D NR contact interaction and BCS

Lesson

以上例子说明, 不同于relevant及irrelevant耦合marginal interaction处于一种非常不稳定的情况。考虑量子修正后，marginal耦合要么变成irrelevant, 要么变为relevant.

除非有强大的对称性保证, β 函数恒为0 (N=4 Super Yang-Mills)

有效场论提供了关于重整化的现代理解

在 EFT框架下，**重整化**意味着即使不了解更基本的UV物理，我们依然可以做出模型无关的预言, 因为短程UV物理的效应是高度定域的, 且蕴藏在EFT中的**Wilson系数**中。

Nonrenormalizablity is no longer an issue.

Nonrenormalizable theories are renormalizable/predictive, provided one is content with a limited accuracy. (power counting is key)

Wilson系数可以

- ◊ **通过实验测量拟合: 手征微扰论 /SMEFT** **Bottom-up approach**
- ◊ **或从已知underlying理论推导(匹配)： HQET/NRQCD/SCET/LaMET** **top-down approach**

Wilsonian approach vs. continuum EFT: integrating out vs. matching

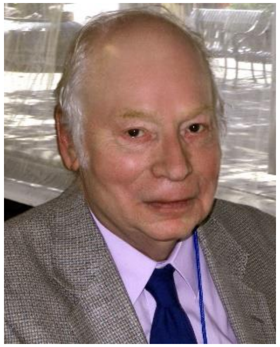
Wilsonian approach (using hard momentum cutoff or lattice spacing as UV regulator) is conceptually very clear, but practically difficult to do analytic calculation

In practice, it is much simpler to adopt the continuum EFT (using mass-independent subtraction scheme such as $\overline{\mathbf{MS}}$), with power counting much more transparent

In principle, these two approaches are equivalent

有效理论是物理学研究的现代方法论: 适合处理存在多个特征标度的物理系统

有效理论的五要素：

1. 能标分离: identify UV and IR scales of the system
 (确定展开参数)

2. 确定理论中需保留的最重要的active(有效)自由度

3. 施加紫外截断Λ（任何EFT仅在一定范围内适用）


4. 明确EFT中具备的对称性
 (作为指导原则写下对称性允许的所有可能的相互作用)

5. 数幂规则（power counting rule）
 （如何组织计算，使其满足小参数展开的指定精度）

有效理论必须完全重现underlying UV理论的（非解析）红外行为

使用有效理论的原因

1. Every theory is an effective theory

2.   Greatly simplifies the calculation by including only the most relevant interactions.

1

3. Deals with only one scale at a time

4. Makes symmetry manifest
   chiral symmetry, heavy quark spin/flavor symmetry

5. Sum large logs

6. Efficient and model-independent way to characterize new physics

7. Including nonperturbative effects

万物皆为有效理论

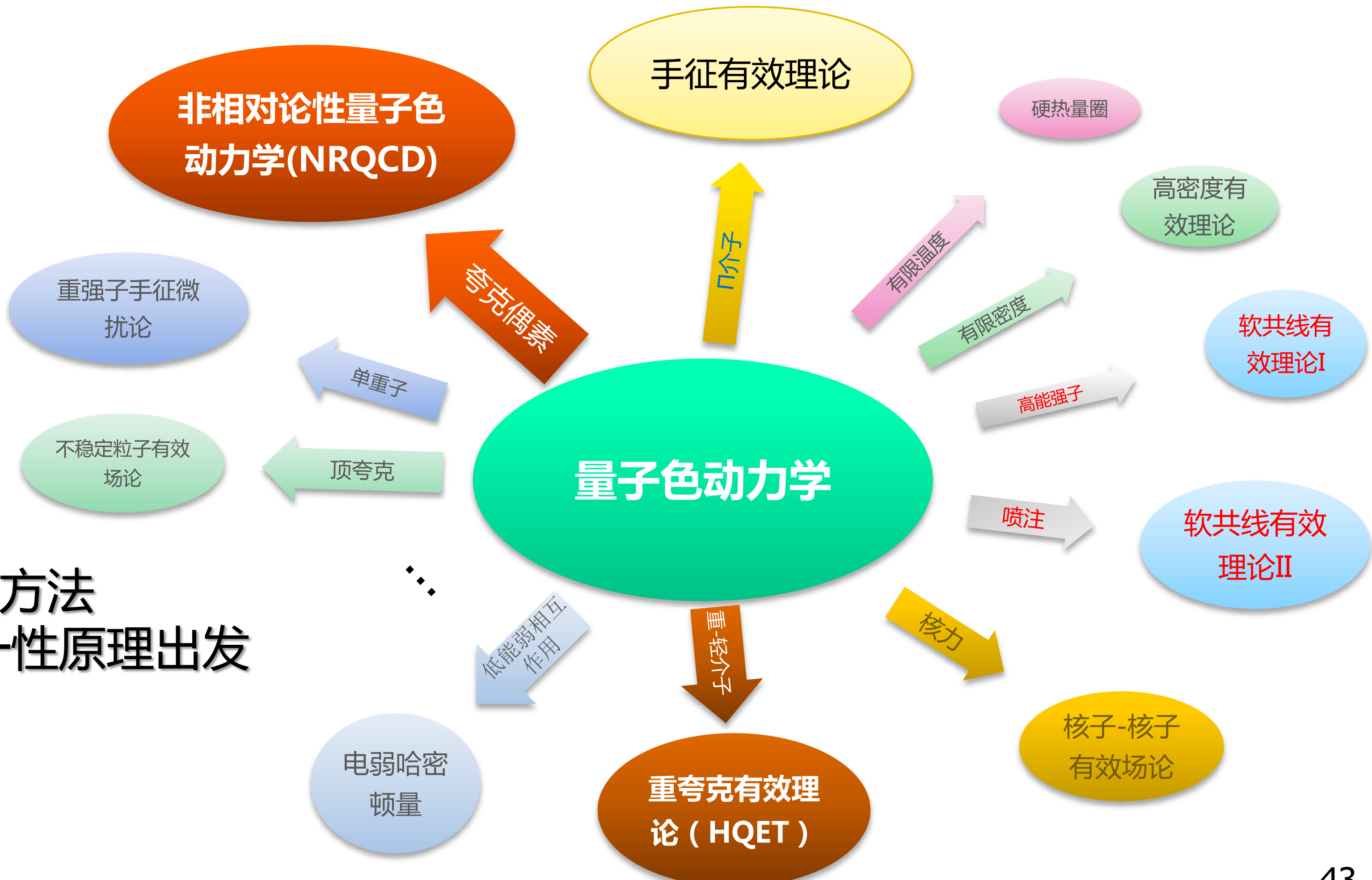
A tower of EFTs of fundamental physics theory

Examples of EFTs

High-energy theory	Fundamental scale	Low-energy theory
11-d M theory (?)	?	String theory (?)
String theory (?)	$M_S \sim 10^{18}$ GeV	QFT
GUT (?)	$M_{GUT} \sim 10^{16}$ GeV	SUSY (?)
SUSY (?)	$M_{SUSY} \sim 10^{4+x}$ GeV	SMEFT
SMEFT	$M_W \sim 10^2$ GeV	Fermi theory
QCD	$m_b \sim 5$ GeV	HQET, NRQCD
	$M_{\chi SB} \sim 1$ GeV	ChPT

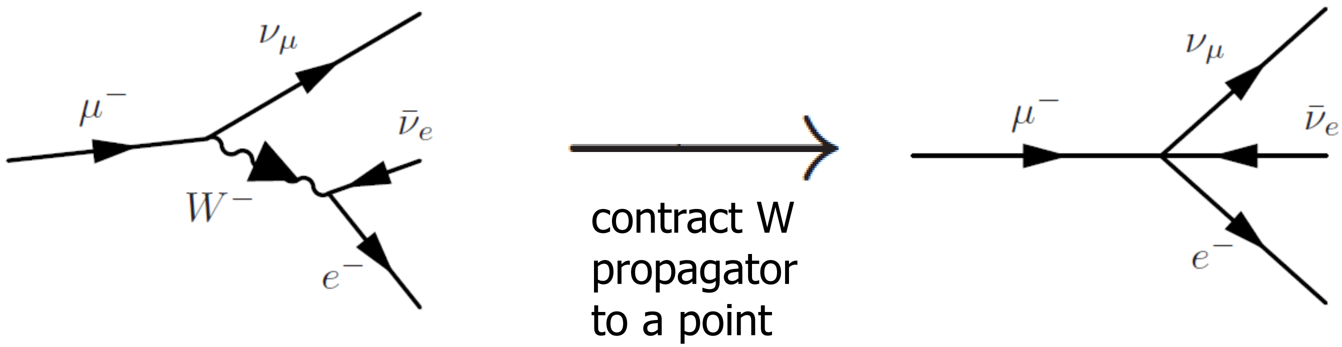
终极量子引力理论(string/M theory)不再是local QFT

量子色动力学的有效场论



模型无关的方法
从QCD第一性原理出发

弱作用中的四费米子有效场论



$$\mathcal{M}(\mu \to e\nu_\mu\bar{\nu}_e) = \left(\frac{g_2}{\sqrt{2}}\right)^2 [\bar{u}\left(p_{\nu_\mu}\right)\gamma^\mu P_L u(p_\mu)][\bar{u}(p_e)\gamma_\mu P_L v(p_{\bar{\nu}_e})]$$

$$\times \frac{1}{\left(p_\mu - p_{\nu_\mu}\right)^2 - M_W^2}$$

因为 $p_\mu^2 = m_\mu^2 \ll M_W^2$, W boson is highly virtual, can not propagate far,

$$\frac{1}{\left(p_\mu - p_{\nu_\mu}\right)^2 - M_W^2} \approx -\frac{1}{M_W^2}$$

弱作用中的四费米子有效场论

$$\mathcal{M}(\mu \to e\nu_\mu\bar{\nu}_e) = -\frac{4G_F}{\sqrt{2}}[\bar{u}(p_{\nu_\mu})\gamma^\mu P_L u(p_\mu)][\bar{u}(p_e)\gamma_\mu P_L v(p_{\bar{\nu}_e})]$$

费米耦合常数 $$\frac{G_F}{\sqrt{2}} \equiv \frac{g_2^2}{8M_W^2}$$

得到弱作用中著名的四费米子有效哈密顿量：

$$\mathcal{H}_{4-fermi} = -\mathcal{L}_W = \frac{4G_F}{\sqrt{2}}[\bar{\nu}_\mu\gamma^\mu P_L\mu][\bar{e}\gamma_\mu P_L\nu_e]$$

中微子质量 (dim-5 operator)

The lowest dimension operator in SMEFT which gives a neutrino mass is dimension 5 (Weinberg operator),

$$\mathcal{L} \sim \frac{(HL)^2}{M_S}$$

This gives a Majorana neutrino mass of

$$m_\nu \sim \frac{v^2}{M_S^2}$$

or a seesaw scale of 6×10^{15} GeV for $m_\nu \sim 10^{-2}$ eV.

Not far from GUT scale

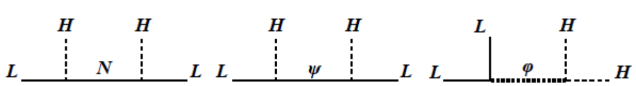


Figure 6: *Three ways the dimension 5 operator for neutrino masses in eq. (83) could arise from tree level exchange of a heavy particle: either from exchange of a heavy $SU(2) \times U(1)$ singlet fermion N, a heavy $SU(2) \times U(1)$ triplet fermion ψ, or else from exchange of a massive $SU(2)$ triplet scalar ϕ.*

质子衰变 (dim-6 operator)

The lowest dimension operator in the SMEFT which violates baryon number is dimension 6 (but conserves B-L),

$$L \sim \frac{qqql}{M_G^2}$$

This gives the proton decay rate $p \to e^+\pi^0$ as

$$\Gamma \sim \frac{m_p^5}{16\pi M_G^4}$$

Or $\tau \sim \left(\frac{M_G}{10^{15} GeV}\right)^4 \times 10^{30}$ years

中子-反中子振荡 (dim-9 operator)

In some BSM theories, B is violated but lepton number is not. Proton decay is forbidden. These theories do predict $n - \overline{n}$ oscillations, which only violates baryon number

The lowest dimension operator in the SMEFT which violates both B and B-L is dimension 9:

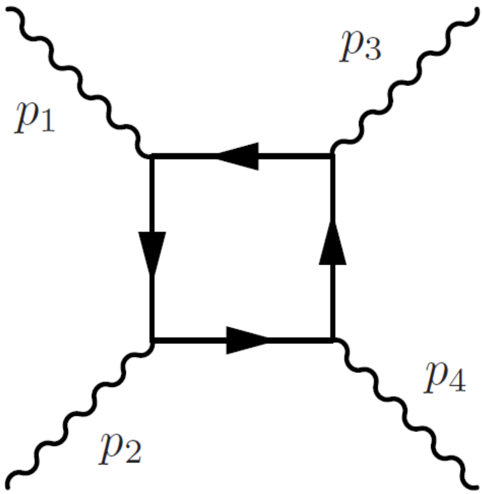
$$\mathscr{L} \sim \frac{q^6}{M_G^5},$$

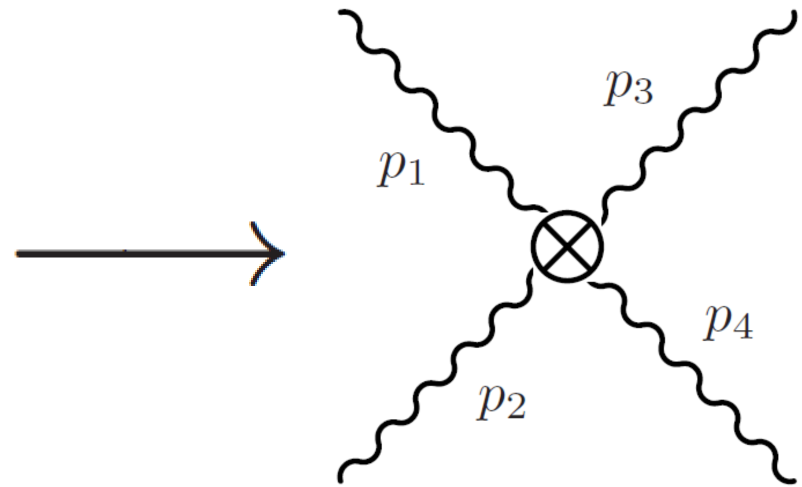
which gives oscillation amplitude

$$\mathscr{A} \sim \left(\frac{m_n}{M_G}\right)^5,$$

低能光子-光子散射

$\gamma\gamma \to \gamma\gamma$





光子能量为E_γ, 假设$E_\gamma \ll m_e$

我们可以从QED作用量中积掉电子场 [Euler, Heisenberg, Kockel, 1936]

$$\mathcal{L}_{QED}(\psi, \bar{\psi}, A_\mu) \to \mathcal{L}_{EH}(A_\mu)$$

低能光子-光子散射

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2}{m_e^4}\left[c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_1\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right]$$

$$F_{\mu\nu}F^{\mu\nu} \sim \vec{E}^2 - \vec{B}^2$$
$$F_{\mu\nu}\tilde{F}^{\mu\nu} \sim \vec{E}\cdot\vec{B}$$

展开参数为 $$\left(\frac{E_\gamma}{m_e}\right)^2$$

Terms with only three field strengths are forbidden by charge conjugation symmetry.

e^4 from vertices, and $1/16\pi^2$ from the loop.

低能光子-光子散射

An explicit computation gives

$$c_1 = \frac{1}{90}, \qquad c_2 = \frac{7}{90}.$$

Scattering amplitude

$$A \sim \frac{\alpha^2 \omega^4}{m_e^4}$$

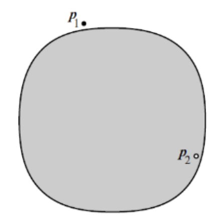
and

$$\sigma \sim \left(\frac{\alpha^2 \omega^4}{m_e^4}\right)^2 \frac{1}{\omega^2} \frac{1}{16\pi} \sim \frac{\alpha^2 \omega^6}{16\pi m_e^8} \times \frac{15568}{22275}$$

First observed in heavy-ion collision experiments, ATLAS, Nature Phys 13(2017)852

朗道液体和BCS

A large class of metals (conductors) can be described by Landau liquid model: ground state is the filled Fermi surface; low energy excitation are fermions (quasi-particle) with a complicated dispersion relation but no interaction!



Why this happens? EFT can give simple explanation

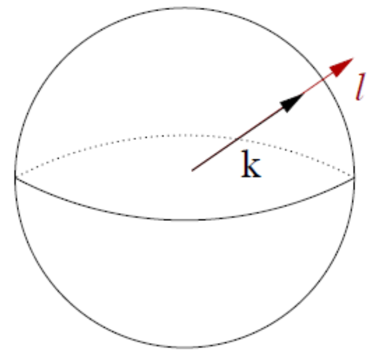
作用量(自由部分)　　　Polchinski, hep-th/9210046

$$S_{free} = \int dt \int d^3p \sum_{s=\pm\frac{1}{2}} \left[\psi_s(p)^\dagger i\partial_t \psi_s(p) - (\epsilon(p) - \epsilon_F)\psi_s^\dagger(p)\psi_s(p) \right]$$

朗道液体和BCS

如何判断场和各种变量的scaling behavior?
Different from familiar relativistic QFT

Like HQET, 分解动量 $\mathbf{p} = \mathbf{k} + \boldsymbol{\ell}$



Scaling transformation: $\ell \to r\ell.$

If an object scales as r^n , then say it has dimension n

$$[k] = 0,\ [\ell] = 1,\ \text{and}\ [\int d^3p = \int d^2k d\ell] = 1.$$

Fermi velocity $\mathbf{v}_F(\mathbf{k}) = \boldsymbol{\nabla}_k \epsilon(\mathbf{k})$

$$\epsilon(\mathbf{p}) - \epsilon_F = \ell \cdot \mathbf{v}_F(\mathbf{k}) + \mathcal{O}(\ell^2) \qquad [\epsilon - \epsilon_f] = 1 \qquad [\partial_t] = 1.$$

朗道液体和BCS

要求作用量的量纲为0, 则得到费米子场的量纲 $[\psi] = -\frac{1}{2}$.

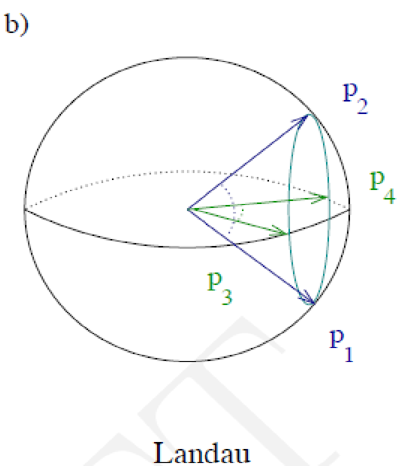
引入四费米子相互作用项：

$$S_{int} = \int dt \int \prod_{i=1}^{4} (d^2 \mathrm{k}_i d\ell_i) \delta^3(\mathrm{P}_{tot}) C(\mathrm{k}_1, \ldots, \mathrm{k}_4) \psi_s^\dagger(\mathrm{p}_1) \psi_s(\mathrm{p}_2) \psi_{s'}^\dagger(\mathrm{p}_3) \psi_{s'}(\mathrm{p}_4) \ .$$

容易得到 $[\delta^3(\mathrm{P}_{tot})C] = -1.$

For generic, not head-on collision

$$[\delta^3(\mathrm{P}_{tot})] = 0.$$



So [C]=-1, hence irrelevant, 低能极限下可忽略

System can be well described in terms of free fermions

朗道液体和BCS

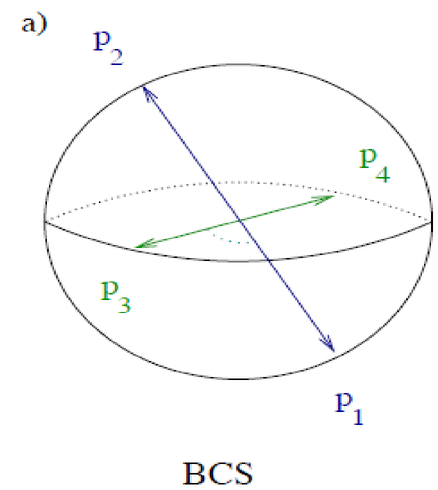
More interesting things happen for head-on collision

$$\mathbf{k}_1 + \mathbf{k}_2 = 0.$$

$$S_{int} = \int dt \int \prod_{i=1}^{4} (d^2\mathbf{k}_i d\ell_i) \delta^3(\mathbf{P}_{tot}) C(\mathbf{k}_1, \ldots, \mathbf{k}_4) \psi_s^\dagger(\mathbf{p}_1) \psi_s(\mathbf{p}_2) \psi_{s'}^\dagger(\mathbf{p}_3) \psi_{s'}(\mathbf{p}_4) \ .$$

$$[\delta^3(\mathbf{P}_{tot})C] = -1.$$

$$\delta^3(\mathbf{P}_{tot}) = \delta^2(\mathbf{k}_3 + \mathbf{k}_4)\, \delta(\ell_{tot}),$$



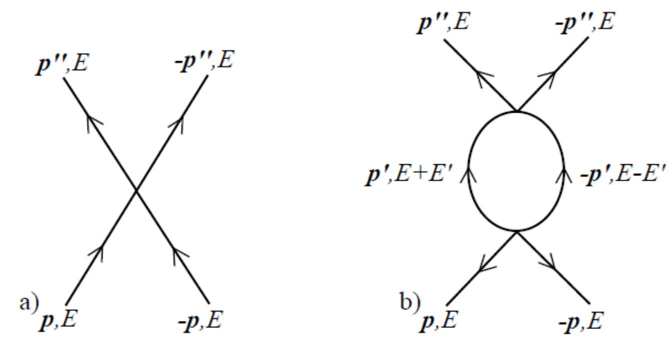
So [C]=0, hence marginal

Cooper pair/BCS superconductor

BCS: marginal turn to relevant

Marginal (BCS)：包括量子修正，求解重整化群方程

$$V(E)=\frac{V}{1+NV\ln(E_0/E)}.$$



A repulsive interaction (V>0) grows weaker as $E\to 0$

An attractive interaction (V<0) grows stronger as $E\to 0$

Cooper pair is indeed attractive-> BCS: marginal coupling becomes relevant

Similar to QCD

Section 2. hard momentum cutoff vs mass independent subtraction scheme ($\overline{\mathrm{MS}}$ scheme)

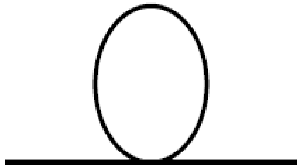
Hard momentum cutoff

考察仅含有费米子的EFT

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - m\bar{\psi}\psi - \frac{a}{\Lambda^2}(\bar{\psi}\psi)^2 - \frac{b}{\Lambda^4}(\bar{\psi}\psi)^3 - \ldots$$

自能修正

$$\frac{i}{\Lambda^2}\int \frac{d^4 q}{(2\pi)^4}\frac{m}{q^2 - m^2}\,.$$

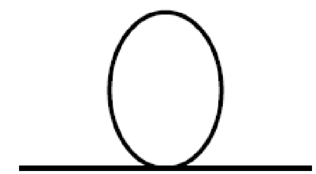


$$\delta m \sim (m/\Lambda^2) \times \Lambda^2 \sim m.$$

硬动量截断破坏 power counting: 4费米子算符是irrelevant算符！

质量无关的重整化方案($\overline{\text{MS}}$ scheme)

自能修正



$$\frac{\mu^{2\epsilon}}{\Lambda^2}\int \frac{d^4q}{(2\pi)^4}\frac{m}{q^2-m^2}\,.$$

维数正规化类似于*围道积分*

$$= \frac{m^3}{16\pi^2\Lambda^2}\left(-\frac{1}{\epsilon}+\gamma-1+\ln\left[\frac{m^2}{4\pi\mu^2}\right]\right)$$

积分依赖于μ
只能通过log μ

抵消项：
$$\frac{am^3}{16\pi^2\Lambda^2}\left(-\frac{1}{\epsilon}+\gamma-1-\ln 4\pi\right)$$

$$\delta m \sim \frac{am^3}{16\pi^2\Lambda^2}\ln\left[\frac{m^2}{\mu^2}\right]\,.$$

$$\frac{\delta m}{m} \propto \frac{m^2}{16\pi^2\Lambda^2}$$

Is small

满足power counting

Section 3. 微扰匹配 (perturbative matching)

什么是匹配？

如果UV理论已知，推导EFT的Wilson系数的手续称为匹配(matching)

匹配条件：

低能EFT给出和UV理论相同的物理预言, order by order in small EFT expansion parameter

微扰匹配和非微扰匹配

Underlying theory已知（比如QCD是强相互作用的基本理论)
通过 **top-down approach** 来定出低能有效场论中的Wilson系数

如果微扰论适用,

◊ **微扰匹配： HQET/NRQCD/NRQED/SCET/LaMET** √

如果微扰论失效(低能强作用),

◊ **非微扰匹配: chiral perturbation theory** **通过**lattice QCD数值模拟

A toy model with two scalar fields

Two types of real scalar fields, one light, and one heavy

$$\mathcal{L}_{UV} = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2 + (\partial\chi)^2 - M^2\chi^2 - \kappa\,\phi^2\chi\right)$$

其中ϕ是轻场， χ是重场

κ 是 relevant耦合

Assuming $m << M$, $\kappa \lesssim M$, $<\chi> = 0$

考虑 $2\phi \to 2\phi$ 散射， 质心能量E 远远低于M

构造只含ϕ场的低能EFT， integrate out heavy χ field:

$$e^{-i\frac{S_{EFT}(\phi)}{h}} = \int [d\chi]\, e^{-i\frac{S(\phi,\chi)}{h}}$$

The key terminology of EFT: **Matching and Running**

Large Scale
ϕ_j, χ

$\mathcal{L}_H(\chi, \phi) + \mathcal{L}(\phi)$ renormalization group

$\mu = M$
particle mass MATCHING

$\mathcal{L}(\phi) + \delta\mathcal{L}(\phi)$ renormalization group

Low Energy
ϕ_j

Matching **丽江摩梭族的走婚** **Running**

Matching is a loop expansion

Loop expansion is $\hbar$ expansion

可以通过欧拉公式

$$\underset{\text{loop}}{L} = \underset{\text{prop}}{\underset{\hbar^{+1}}{P}} - \underset{\text{vert}}{\underset{\hbar^{-1}}{V}} + 1$$

L-loop Feynman diagram contributes $\hbar^L$

The structure of EFT can be expresses as

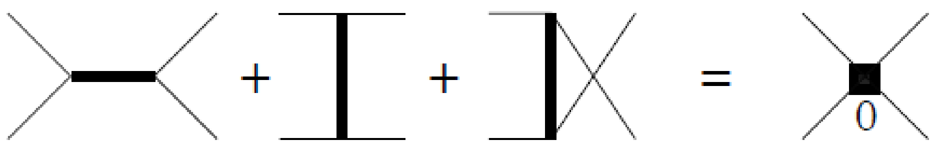
$$\mathcal{L}_{eft} = \mathcal{L}_{eft}^0 + \hbar \mathcal{L}_{eft}^1 + \hbar^2 \mathcal{L}_{eft}^2 + \cdots$$

树图匹配

Write down the most general operators in EFT

Symmetry: ϕ-> -ϕ

$$\mathcal{L}_{eft}^{0} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - c_0\frac{\kappa^2}{M^2}\frac{\phi^4}{4!} - d_0\frac{\kappa^2}{M^4}\frac{(\partial\phi)^2\phi^2}{4} + \cdots$$



匹配条件(强版本): equating 1LPI (one-light-particle irreducible) diagrams in full theory and EFT

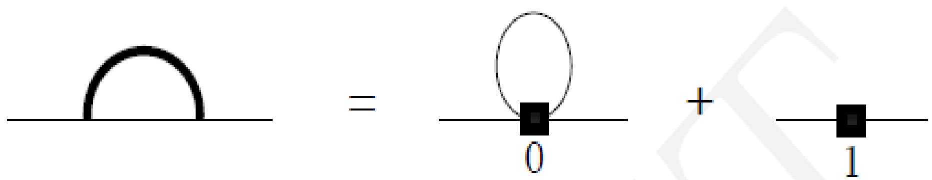
匹配条件(弱版本): equating on-shell amplitude in full theory and EFT

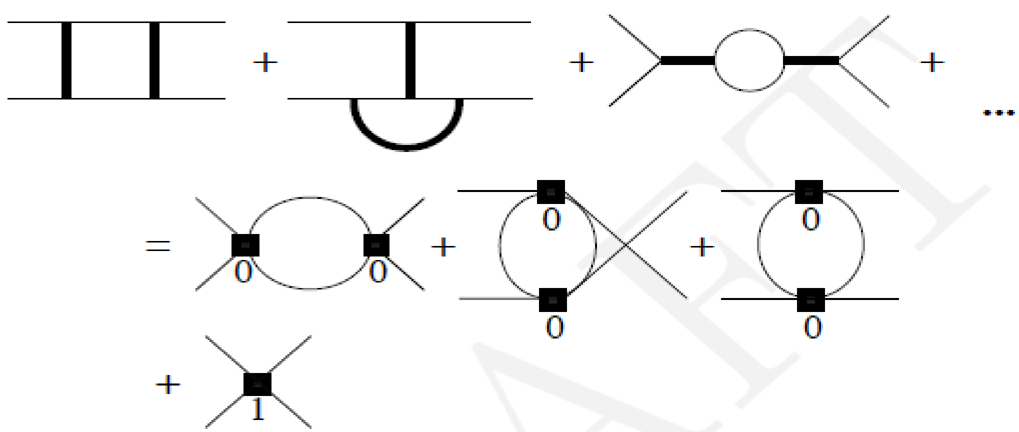
Equivalent，connected by field redefinition: lead to the same S matrix

单圈图匹配

$$\hbar\mathcal{L}^1_{eft} = \frac{1}{2}\left(a_1\frac{\kappa^2}{16\pi^2 M^2}\right)(\partial\phi)^2 - \frac{1}{2}\left(b_1\frac{\kappa^2}{16\pi^2} + b_1'\frac{m^2}{16\pi^2}\right)\phi^2$$

- $$-c_1\left(\frac{\kappa^4}{16\pi^2 M^4}\right)\frac{\phi^4}{4!} - d_1\left(\frac{\kappa^4}{16\pi^2 M^6}\right)\frac{(\partial\phi)^2\phi^2}{4} + \cdots$$





Some remarks

$$\mathcal{L}_{eff} = \frac{1}{2}\left(1 + a_1 \frac{\kappa^2}{16\pi^2 M^2}\right)(\partial\phi)^2 - \frac{1}{2}\left(m^2 + b_1 \frac{\kappa^2}{16\pi^2} + b_1' \frac{m^2}{16\pi^2}\right)\phi^2$$

$$-\left(c_0 \frac{\kappa^2}{M^2} + c_1 \frac{\kappa^4}{16\pi^2 M^4}\right)\frac{\phi^4}{4!} + \ldots$$

1. * Loop expansion is equivalent to $\boldsymbol{\kappa^2/16\pi^2 M^2 << 1}$, where perturbation theory works;

2. * Matching coeff. depends on regularization scheme. However physics does not depend

3. * In loop calculations in both full and EFT sides, the non-analytic terms depending on light particles such as $ln\, p^2$, $ln\, m^2$ must cancel, so the matching condition is a local expansion in 1/M

Some remarks

通常做场的重定义$\phi \to (1 - a_1 \frac{\kappa^2}{32\pi^2 M^2})\ \phi$把动能项变成正则形式，相互作用耦合也会有相应改变

Guaranteed by LSZ reduction formula, any field redefinition does change Green function, but never changes the physical S matrix!

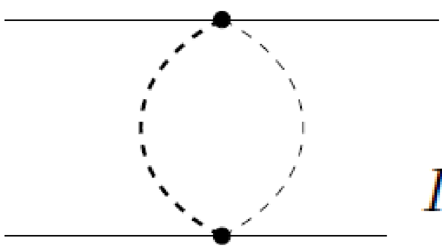
所有EFT都必须遵循的性质：

EFT重现UV理论的红外行为 (包括红外发散)

有效理论必须完全重现underlying UV理论的依赖于轻自由度的非解析项

单圈积分为例展示matching

细虚线：轻粒子 m; 重虚线：重粒子 M; $m << M$



$$I_F = g^2\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)(k^2-M^2)}$$
$$= \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \log\frac{\mu^2}{M^2} + \frac{m^2\log(m^2/M^2)}{M^2-m^2} + 1\right]$$

按1/M展开和圈积分次序并不对易

有效理论计算相当于假设 ***k ~ m << M***, 先对integrand做展开再积分

$$I_{\rm EFT} = g^2\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)}\left[-\frac{1}{M^2}-\frac{k^2}{M^4}-\dots\right]$$
$$= \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

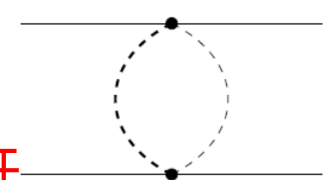
one-loop matching具体展示

细虚线：轻粒子 m; 重虚线：重粒子 M;　　　$m << M$

$$I_M = [I_F + I_{F,\mathrm{c.t.}}] - [I_{\mathrm{EFT}} + I_{\mathrm{EFT,c.t.}}]$$
$$= \frac{ig^2}{16\pi^2}\left[\left(\log\frac{\bar{\mu}^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\bar{\mu}^2}{M^2}+1\right)+\ldots\right].$$

Generic feature of EFTs:
完整理论和有效理论的UV发散不一样，
各自需要做重整化；

两个理论的红外非解析项***ln*** $\boldsymbol{m^2}$ 完全一样



Matching贡献是m^2 的解析函数，可以对其做local 1/M展开

Wilson系数中含有 log M/μ. Matching point μ around M to avoid大对数项

万有引力定律的量子修正

广义相对论场方程可以从Einstein-Hilbert作用量推出

$$S_{EH} = \int d^4x\sqrt{-g}(-\frac{2}{\kappa^2}R)$$

$$S = S_{EH} + S_{matter}$$

$$\delta S = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

R是曲率标量

$\kappa = \sqrt{32\pi G_N}$

G_N是牛顿万有引力常数

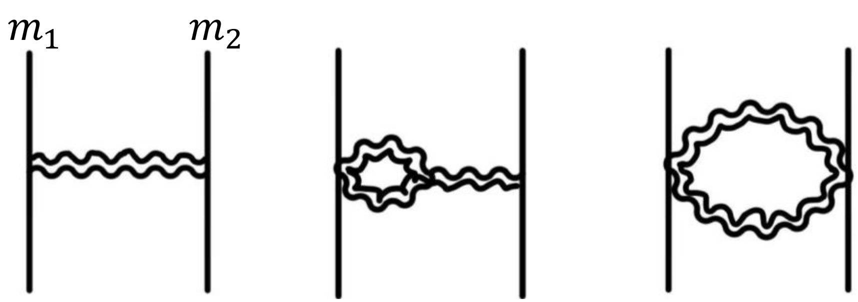
- 目前我们并不知道自洽的量子引力理论，但可以构造量子引力理论的低能有效理论 (symmetry: diffeomorphism = 广义坐标变换对称性)：

$$S_{eff} = \int d^4x\sqrt{-g}[-\Lambda - \frac{2}{\kappa^2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \cdots]$$

EFT 例5：万有引力定律的量子修正

广义相对论是量子引力的低能有效理论

- Symmetry: general coordinate transformation invariance
- In the spirit of EFT, write down the most general interaction terms allowed by symmetry also in matter sector.
- One-loop corrections to Newton's law



 代表 **引力子(graviton)**

Post Newtonian

Genuine quantum Correction

$$V(r) = -\frac{G_N m_1 m_2}{r}\left[1 + \frac{3G_N(m_1+m_2)}{rc^2} + \frac{41}{10\pi}\frac{\hbar G_N}{r^2c^3}\right] + c_1 G_N \delta^{(3)}(\vec{r})$$

non-analytic

[J. Donoghue,1995]

万有引力定律的量子修正

为什么虽然我们并不知道终极量子引力理论，但能确定万有引力定律的量子修正？

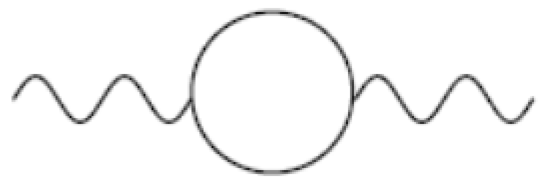
答案：**有效理论必须完全重现underlying UV理论的轻自由度的非解析行为**

$$
\begin{aligned}
&- \frac{\kappa^2}{4}\frac{1}{2m_1}V^{(1)}_{\mu\nu}(q)\left[iD^{\mu\nu,\alpha\beta}(q)+iD^{\mu\nu,\rho\sigma}i\Pi_{\rho\sigma,\eta\lambda}iD^{\eta\lambda,\alpha\beta}\right]V_{\alpha\beta}(q)\frac{1}{2m_2} \\
&\approx 4\pi G m_1 m_2\left[\frac{i}{\mathbf{q}^2}-\frac{i\kappa^2}{32\pi^2}\left[-\frac{127}{60}ln(\mathbf{q}^2)+\frac{\pi^2(m_1+m_2)}{2\sqrt{\mathbf{q}^2}}\right]+const\right]
\end{aligned}
$$

$$\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\frac{1}{q}=\frac{1}{2\pi^2r^2} \qquad \int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}ln\vec{q}^2=\frac{-1}{2\pi^2r^3}$$

Decoupling of heavy particle

- Heavy particles decouple from low energy physics.
- Obvious?
- Not explicit in a mass independent scheme such as $\overline{MS}$.



$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \int_0^1 dx\, x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right].$$

$$\equiv i\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\Pi(p^2)$$

and we want to look at $p^2 \ll m^2$.

- The graph is UV divergent.

Momentum Subtraction Scheme (MOM)

- Note that renormalization involves doing the integrals, and then performing a substraction using some scheme to render the amplitudes finite.
- Substract the value of the graph at the Euclidean momentum point $p^2 = -M^2$ (the $1/\epsilon$ drops out)

$$\Pi_{\text{mom}}(p^2, m^2, \mu_M^2) = \left[\int_0^1 dx\, x(1-x)\, \log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right].$$

$$\beta(e) = -\frac{e}{2} M \frac{d}{dM} \frac{e^2}{2\pi^2}\left[\int_0^1 dx\, x(1-x)\log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right]$$

$$= \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x)\frac{M^2 x(1-x)}{m^2 + M^2 x(1-x)}.$$

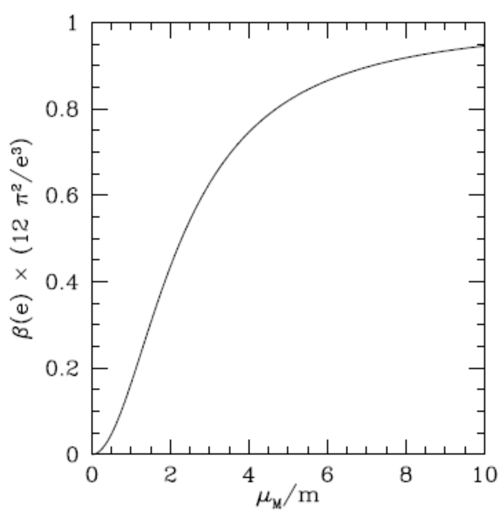
MOM重整化方案遵守退耦定理

- $m \ll M$ (light fermion)

$$\beta(e) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x) = \frac{e^3}{12\pi^2}.$$

$M \ll m$ (heavy fermion)

$$\beta(e) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x)\frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2}\frac{M^2}{m^2}.$$



质量无关的重整化方案($\overline{\mathrm{MS}}$ scheme) 重粒子并不退耦

- In the $\overline{MS}$ scheme

$$\Pi_{\overline{\mathrm{MS}}}(p^2, m^2, \overline{\mu}^2) = \left[\int_0^1 dx\, x(1-x)\ \log\frac{m^2-p^2x(1-x)}{\mu^2}\right].$$

$$\beta(e) = -\frac{e}{2}\mu\frac{d}{d\mu}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\, x(1-x)\ \log\frac{m^2-p^2x(1-x)}{\mu^2}\right]$$

$$= \frac{e^3}{2\pi^2}\int_0^1 dx\, x(1-x) = \frac{e^3}{12\pi^2},$$

Beta函数不依赖费米子质量； 所以top夸克和电子一样重要！

非物理，违背了退耦定理

解决方案：积掉重费米子

$$\Pi_{\overline{\text{MS}}}(0, m^2, \overline{\mu}^2) = \left[\int_0^1 dx\, x(1-x)\, \log\frac{m^2}{\mu^2}\right],$$

有限项含有大对数项

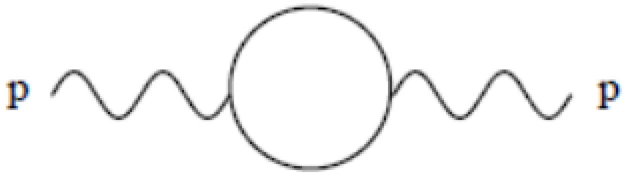
解决方案：integrate out heavy particles and go to an EFT.

$$\mathscr{L}^{(n_l+1)} \to \mathscr{L}^{(n_l)}$$

Full theory : Includes fermion with mass m.

EFT : drop the heavy fermion (it no longer contributes to β)

Full theory calculation:



Present in theory above m, but not in theory below m. Assume that $p \ll m$, so

$$\Pi_{\overline{\mathrm{MS}}}(0, m^2, \overline{\mu}^2) = \int_0^1 dx\, x(1-x)\, \log \frac{m^2 - p^2 x(1-x)}{\mu^2}$$

$$= \int_0^1 dx\, x(1-x) \left[\log \frac{m^2}{\mu^2} + \frac{p^2 x(1-x)}{m^2} + \cdots \right]$$

$$= \frac{1}{6} \log \frac{m^2}{\mu^2} + \frac{p^2}{30 m^2} + \cdots$$

Full theory calculation:

So in theory above m:

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \cdots\right] + c.t.$$

Counterterm cancels $1/\epsilon$ term (and also contributes to the β function).

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[-\frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \cdots\right]$$

Threshold correction

The log term gives

$$Z = 1 - \frac{e^2}{12\pi^2} \log \frac{m^2}{\mu^2}$$

So that in the effective theory,

$$\frac{1}{e_L^2(\mu)} = \frac{1}{e_H^2(\mu)} \left[1 - \frac{e_H^2(\mu)}{12\pi^2} \log \frac{m^2}{\mu^2} \right]$$

One usually integrates out heavy fermions at $\mu = m$, so that (at one loop), the coupling constant has no matching correction.

Threshold correction

The p^2 term gives the dimension six operator

$$-\frac{1}{4}\frac{e^2}{2\pi^2}\frac{1}{30m^2}F_{\mu\nu}\partial^2 F^{\mu\nu}$$

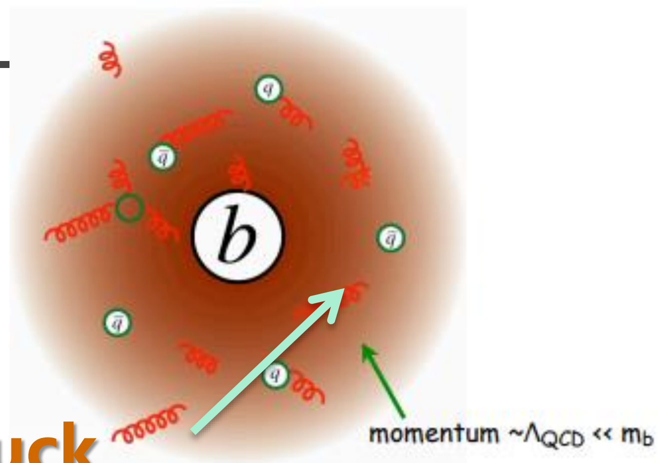
and so on.

仿照刚才提到的双标量场的例子，可以对电磁场做field redefinition, 回到正则的Maxwell动能项, 这样会把重粒子的效应吸收到EFT中的电磁耦合常数

Part 2. 重粒子有效理论(以HQET为例)

含单个重粒子束缚态的典型特征
(B介子，B重子，D介子，D介子，...)

Voloshin, Shifman 87; Isgur, Wise 89



Brown muck

1. 束缚态中，重粒子only talks with 软胶子, 典型动量转移~$\Lambda_{QCD} \ll m_Q$

2. $m_Q \to \infty$ 极限下，重夸克速度很难被改变（硬胶子及电弱作用才能改变其速度）若仅考虑soft非微扰QCD效应(interacting with brown muck)，**velocity可以被认为是一个好量子数**。**(Warning: 对于正负电子偶素或重夸克偶素，这个图像并不成立)**

3. $m_Q \to \infty$极限下，重夸克自旋自由度退耦； $\frac{\sigma \cdot B}{m_Q}$

4. $m_Q \to \infty$极限下, 涌现出新的重夸克味道自旋对称性
   Emergent symmetry: combined spin x flavor **SU(2Nf)** symmetry

重夸克对称性的物理预言

重夸克自旋对称性 $m_{B^*}^2 - m_B^2 \approx m_{D^*}^2 - m_D^2 \approx \text{const.}$

实验值: $m_{B^*}^2 - m_B^2 \approx 0.49\ \text{GeV}^2, \qquad m_{D^*}^2 - m_D^2 \approx 0.55\ \text{GeV}^2.$

重夸克味道对称性:

$$m_{B_s} - m_B \approx m_{D_s} - m_D \approx 100\ \text{MeV}, \qquad m_{B_1} - m_B \approx m_{D_1} - m_D \approx 557\ \text{MeV},$$

$$m_{B_2^*} - m_B \approx m_{D_2^*} - m_D \approx 593\ \text{MeV},$$

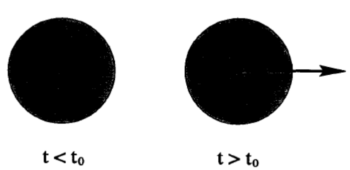
重味介子衰变常数: $\langle 0|\bar{q}\gamma^\mu\gamma_5 Q(0)\,|P(p)\rangle = -i f_P\, p^\mu.$ $\qquad \dfrac{f_B}{f_D} = \sqrt{\dfrac{m_D}{m_B}},$

B->D弱衰变形状因子

$$\frac{\langle D(p')|V^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_D}} = h_+(w)\,(v+v')^\mu + h_-(w)\,(v-v')^\mu,$$

$$\frac{\langle D^*(p',\epsilon)|V^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_{D^*}}} = h_V(w)\epsilon^{\mu\nu\alpha\beta}\epsilon_\nu^* v'_\alpha v_\beta,$$

$$\frac{\langle D^*(p',\epsilon)|A^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_{D^*}}} = -ih_{A_1}(w)(w+1)\epsilon^{*\mu} + ih_{A_2}(w)(\epsilon^*\cdot v)v^\mu + ih_{A_3}(w)(\epsilon^*\cdot v)v'^\mu.$$



$$h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w),$$
$$h_-(w) = h_{A_2}(w) = 0.$$

Isgur-wise函数 $\xi(1) = 1.$

重夸克有效理论(HQET)

E. Eichten, Hill, PLB 1990;   H. Georgi, PLB 1990

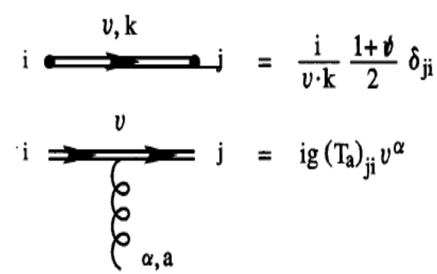
考察一个轻度离壳的重夸克 $p = m_Q v + k$, 剩余动量with $k \sim \Lambda_{QCD}$. 传播子可近似为

$$i\frac{\not{p} + m_Q}{p^2 - m_Q^2 + i\epsilon} = i\frac{m_Q \not{v} + m_Q + \not{k}}{2m_Q v\cdot k + k^2 + i\epsilon} = \frac{1+\not{v}}{2}\frac{i}{v\cdot k + i\epsilon}$$

把QCD中的重夸克场分解为大分量和小分量：

$$Q(x) = e^{im_Q v\cdot x}\,[Q_v(x) + \mathfrak{Q}_v(x)]$$

$$Q_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x) \qquad \mathfrak{Q}_v(x) = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x)$$



代回QCD拉氏量，在$1/m_Q$ 展开领头阶，HQET拉氏量只有一项:

$$\mathcal{L}_Q = \bar{Q}(i\not{D} - m_Q)Q = \boxed{\bar{Q}_v(iv\cdot D)Q_v + \mathcal{O}(m_Q^{-1})}$$

重夸克味道对称性和自旋对称性可以直接读出

HQET： 关于 $\frac{\Lambda_{QCD}}{m_Q}$ 的展开，是QCD中最简单的EFT

HQET

The full theory is QCD, the heavy quark part is

$$L = \bar{Q}\left(i\not{D} - m_Q\right)Q$$

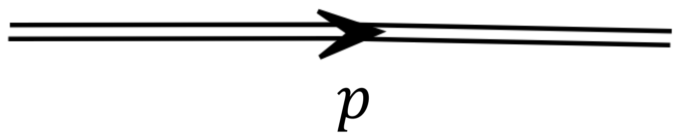
In the limit $m_Q \to \infty$, the heavy quark does not move when interacting with the light degrees of freedom.
Even though for finite m_Q, the quark does recoil, the EFT is constructed as a formal expansion in powers of $1/m_Q$, expanding about the $m_Q \to \infty$ limit. Recoil effects are taken care of by $1/m_Q$ corrections.

Quark moving with fixed four-velocity v_μ

$$p = m_Q v^\mu + k \qquad k \ll m_Q$$

Heavy quark propagator

- Look at the quark propagator:



$$i\,\frac{\not{p}+m_Q}{p^2-m_Q^2+i\epsilon}=i\,\frac{m_Q\not{v}+m_Q+\not{k}}{2m_Q v\cdot k+k^2+i\epsilon}$$

- Expanding this in the limit $k \ll m_Q$ gives

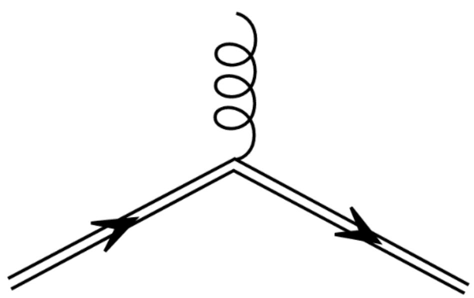
$$\rightarrow i\,\frac{1+\not{v}}{2v\cdot k+i\epsilon}+\mathcal{O}\left(\frac{k}{m_Q}\right)=i\,\frac{P_+}{v\cdot k+i\epsilon}+\mathcal{O}\left(\frac{k}{m_Q}\right)$$

- With a well defined limit

$$P_+\equiv\frac{1+\not{v}}{2}$$

Gluon Vertex

- The quark-gluon vertex



$$-igT^a\gamma^\mu \rightarrow -igT^a v^\mu$$

- using the spinors and keeping the leading terms in $1/m_Q$.
- In the rest frame: the coupling is purely that of an electric charge.

HQET $\mathcal{L}_0$

- HQET Lagrangian:

$$\mathcal{L} = \bar{h}_v(x)(iv \cdot D)h_v$$

- $h_v(x)$ is the quark field in the effective theory and satisfies

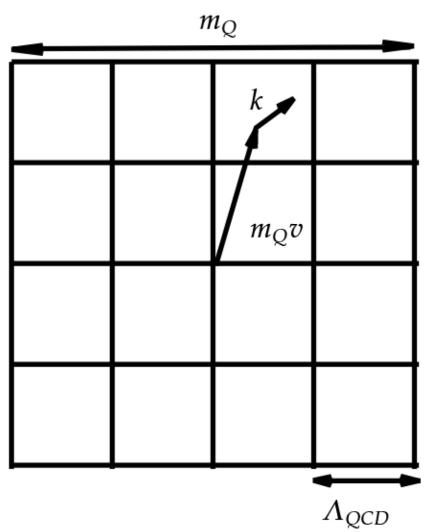
$$P_+ h_v(x) = h_v(x)$$

- h_v annihilates quarks with velocity v, but does not create antiquarks

- Mainfest spin-flavor symmetry of $\mathcal{L}$

Dividing up momentum space

- v appears explicitly in the HQET Lagrangian.
- h_v describes quarks with velocity v, and momenta within Λ_{QCD} of $m_Q v$



- quarks with velocity $v' \neq v$ are far away in the EFT.
- EFT: look at only one box. Full: All of momentum space.

重夸克有效理论(HQET)

E. Eichten, Hill, PLB 1990;　　H. Georgi, PLB 1990

To 1/**m** order,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots, \qquad \mathcal{L}_1 = -\bar{Q}_v \frac{D_\perp^2}{2m_Q} Q_v - g\,\bar{Q}_v \frac{\sigma_{\mu\nu}G^{\mu\nu}}{4m_Q} Q_v.$$

破坏了flavor and spin symmetry

To 1/**m³** order, see Manohar, PRD 1997

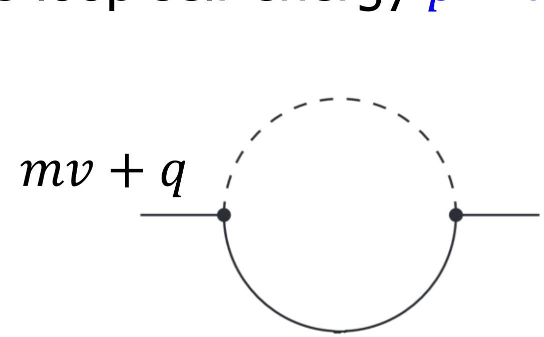
$$\begin{aligned}\mathcal{L}_v = \bar{Q}_v \Big\{ & iD\cdot v - c_2\frac{D_\perp^2}{2m} + c_4\frac{D_\perp^4}{8m^3} - c_F g\frac{\sigma_{\alpha\beta}G^{\alpha\beta}}{4m} - c_D g\frac{v^\alpha[D_\perp^\beta G_{\alpha\beta}]}{8m^2} + ic_S g\frac{v_\lambda\sigma_{\alpha\beta}\{D_\perp^\alpha, G^{\lambda\beta}\}}{8m^2} + c_{W1}g\frac{\{D_\perp^2, \sigma_{\alpha\beta}G^{\alpha\beta}\}}{16m^3} \\ & - c_{W2}g\frac{D_\perp^\lambda\sigma_{\alpha\beta}G^{\alpha\beta}D_{\perp\lambda}}{8m^3} + c_{p'p}g\frac{\sigma^{\alpha\beta}(D_\perp^\lambda G_{\lambda\alpha}D_{\perp\beta} + D_{\perp\beta}G_{\lambda\alpha}D_\perp^\lambda - D_\perp^\lambda G_{\alpha\beta}D_{\perp\lambda})}{8m^3} - ic_M g\frac{D_{\perp\alpha}[D_{\perp\beta}G^{\alpha\beta}] + [D_{\perp\beta}G^{\alpha\beta}]D_{\perp\alpha}}{8m^3} \\ & + c_{A1}g^2\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3} + c_{A2}g^2\frac{G_{\mu\alpha}G^{\mu\beta}v^\alpha v_\beta}{16m^3} + c_{A3}g^2\mathrm{Tr}\left(\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3}\right) + c_{A4}g^2\mathrm{Tr}\left(\frac{G_{\mu\alpha}G^{\mu\beta}v_\alpha v_\beta}{16m^3}\right) - ic_{B1}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha}, G_\mu{}^\beta]}{16m^3} \\ & - ic_{B2}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha}, G^{\nu\beta}]v_\mu v_\nu}{16m^3} \Big\} Q_v, \end{aligned} \tag{7}$$

v取静止系，HQET形式上和NRQCD形式上一样。但是这两个EFT的power counting非常不同。HQET 是关于 $\frac{\Lambda_{\mathrm{QCD}}}{m_Q}$ 的展开；NRQCD是关于v/c的展开。NRQCD远比HQET复杂

单圈费曼图角度理解 HQET = soft interaction

考虑一个slightly off-shell heavy quark one-loop self-energy $p = m_Q v + q$

Obeying $q <<< m_Q$



$$F_\Gamma(m\,, q; d) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{((mv + q - k)^2 - m^2)\, k^2}$$

$$= \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{(-2mv \cdot q - q^2 - i\epsilon)}{1-\epsilon}\, {}_2F_1[-\epsilon, \epsilon; 2-\epsilon; \frac{-1+\lambda}{\lambda}]$$

$$= \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} [\left(\frac{1}{1-2\epsilon} - \frac{v \cdot q}{m} \frac{1}{1-2\epsilon}\right) \frac{1}{m^{2\epsilon}} + \frac{\Gamma(1-\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(\epsilon)} \frac{(-2v \cdot q)^{1-2\epsilon}}{m}]$$

① ② ③

$1/m_Q$ Lagrangian

$$\mathcal{L} = \bar{h}_v(iv \cdot D)h_v + c_k \frac{1}{2m_Q} \bar{h}_v(iD_\perp)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta}\, h_v$$

- The $(iD_\perp)^2$ term violates flavor symmetry at order $1/m_Q$
- $g\sigma_{\alpha\beta}G^{\alpha\beta}$ term violates spin and flavor symmetry at order $1/m_Q$
- The coefficients c_k, c_F are fixed by matching, and are one at tree-level.
- One can carry out the expansion to higher order in $1/m_Q$.

利用区域展开方法重现重夸克单圈自能图

将圈动量k分为hard ($k \sim m$)及soft ($k \sim q$) 区域

Beneke and Smirnov NPB 1998

硬区域贡献

$$= \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{1}{m^{2\epsilon}} \frac{1}{1-2\epsilon} - \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{v \cdot q}{m^{1+2\epsilon}} \frac{1}{1-2\epsilon}$$

① ②

Wilson coeff.
=hard mode

软区域贡献

$$F_{soft} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 2mv \cdot (q-k)} + \cdots$$

$$= \frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{(-2v \cdot q - \mathrm{i}\epsilon)^{1-2\epsilon}}{m} \frac{\Gamma(1-\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(\epsilon)}$$

③

HQET contrit.
= soft mode

两项相加，重现上一页完整圈积分的渐近展开结果

HQET拉氏量中单圈水平Wilson系数

$$c_2 = 1$$

动能项

$$c_D = 1 + \frac{\alpha_s}{\pi}\left[\left(\frac{8}{3}\ln\frac{m}{\mu}\right)C_F + \left(\frac{1}{2} + \frac{2}{3}\ln\frac{m}{\mu}\right)C_A\right]$$

达尔文项

$$c_F = 1 + \frac{\alpha_s}{\pi}\left[\frac{1}{2}C_F + \left(\frac{1}{2} - \frac{1}{2}\ln\frac{m}{\mu}\right)C_A\right]$$

费米项

$$c_S = 1 + \frac{\alpha_s}{\pi}\left[C_F + \left(1 - \ln\frac{m}{\mu}\right)C_A\right]$$

自旋轨道耦合项

c_2=1 to all orders, protected by SR dispersion relation

$c_S = 2c_F - 1$， *exact relation*

重参数化不变性 (Reparametrization invariance) Manohar and Luke 1995

$$p_Q = m_Q v + k$$

$$v \to v + \varepsilon / m_Q, \quad k \to k - \varepsilon.$$

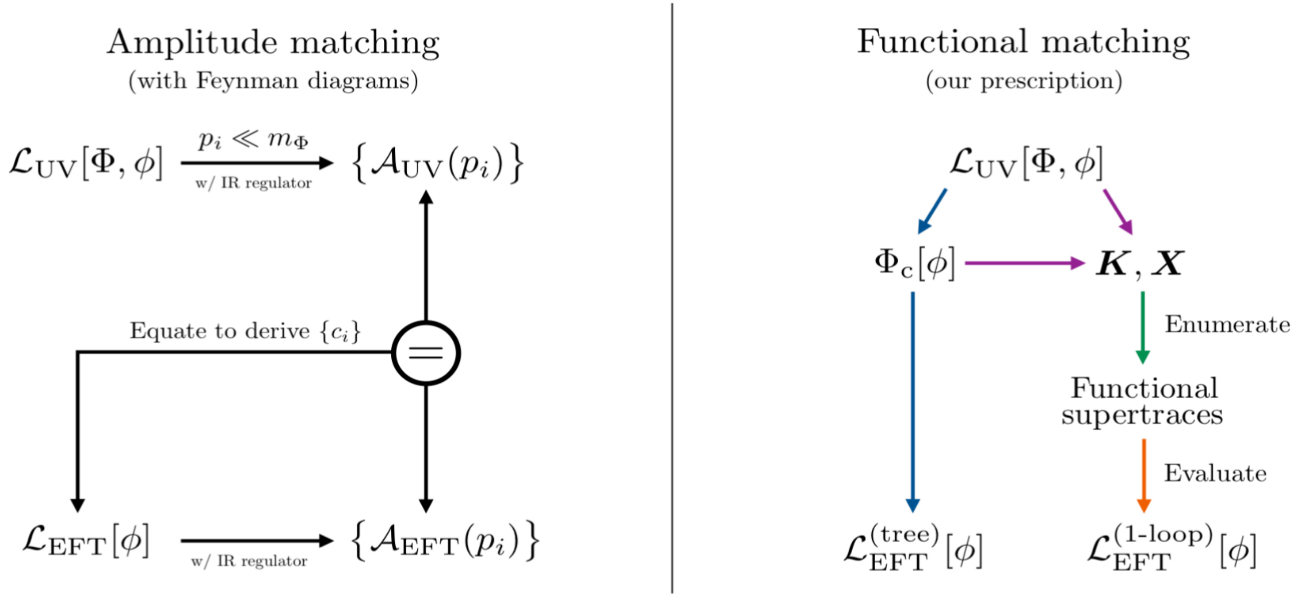
$$Q_v \to e^{i\varepsilon\cdot x}\left(1 + \frac{\not\varepsilon}{2m_Q}\right)Q_v$$

$\mathcal{L}_0 + \mathcal{L}_1$, is reparameterization invariant

$$E = \sqrt{m^2 + \vec{p}^2} = m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}$$

单圈HQET匹配的新方法：泛函积分

Cohen, Freytsis and Lu, JHEP 2020



- Requires knowledge of a basis of EFT operators
- Amplitudes needs to be computed twice from the UV theory and the EFT
- Breaks symmetries in the intermediate steps

- No need to know the EFT operators in advance
- Direct derivation of the Wilson coefficients
- Manifests the symmetries in a transparent way

单圈HQET匹配的新方法：泛函积分

Cohen, Freytsis and Lu, JHEP 2020

Matching condition:

$$S_{\rm EFT}^{(0)} = \int \mathrm{d}^4x \sum_i C_i^{(0)}\, \mathcal{O}_i(\phi) = S_{\rm UV}[\phi,\Phi]\big|_{\Phi=\Phi_c[\phi]}\,,$$

$$S_{\rm EFT}^{(1)} = \int \mathrm{d}^4x \sum_i \left(C_{i,\rm heavy}^{(1)} + C_{i,\rm mixed}^{(1)}\right) \mathcal{O}_i(\phi)\,,$$

$$S_{\rm EFT}^{\text{non-local}}[\phi] = S_{\rm UV}[\phi,\Phi]\big|_{\Phi=\Phi_c[\phi]}\,,$$

$$\int \mathrm{d}^4x \sum_i C_{i,\rm heavy}^{(1)}\, \mathcal{O}_i(\phi) = \frac{i}{2}\ln \mathrm{Sdet}\left(-\frac{\delta^2 S_{\rm UV}[\phi,\Phi]}{\delta\Phi^2}\bigg|_{\Phi=\Phi_c[\phi]}\right),$$

$$\int \mathrm{d}^4x \sum_i C_{i,\rm mixed}^{(1)}\, \mathcal{O}_i(\phi) = \frac{i}{2}\ln \mathrm{Sdet}\left(-\frac{\delta^2 S_{\rm EFT}^{\text{non-local}}[\phi]}{\delta\phi^2}\right) - \frac{i}{2}\ln \mathrm{Sdet}\left(-\frac{\delta^2 S_{\rm EFT}^{(0)}[\phi]}{\delta\phi^2}\right)$$

$$= \frac{i}{2}\ln \mathrm{Sdet}\left(-\frac{\delta^2 S_{\rm EFT}^{\text{non-local}}[\phi]}{\delta\phi^2}\right)\bigg|_{\rm hard}.$$

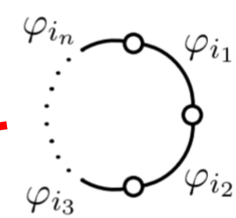
$$\mathcal{L}_{\rm QCD} \supset \bar{Q}\left(i\not{D} - m_Q\right) Q\,,$$

$$\mathcal{L}_{\rm HQET}^{\text{non-local}} \supset \bar{h}_v \left(iv\cdot D + i\not{D}_\perp \frac{1}{iv\cdot D + 2m_Q} i\not{D}_\perp\right) h_v\,.$$

Calculation of supertraces:

$$\int \mathrm{d}^dx\, \mathcal{L}_{\rm EFT}^{(1\text{-loop})}[\phi] = \frac{i}{2}\log \mathrm{Sdet}(\boldsymbol{K}-\boldsymbol{X})\Big|_{\rm hard} = \frac{i}{2}\mathrm{STr}\log(\boldsymbol{K}-\boldsymbol{X})\Big|_{\rm hard}$$

$$= \frac{i}{2}\mathrm{STr}\log \boldsymbol{K}\Big|_{\rm hard} - \frac{i}{2}\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{STr}\left[\left(\boldsymbol{K}^{-1}\boldsymbol{X}\right)^n\right]\Big|_{\rm hard}$$



Covariant Derivative Expansion (CDE)

$$\mathrm{STr}\, Q(P_\mu, U_k) = \pm \int \mathrm{d}^dx \int \frac{\mathrm{d}^dp}{(2\pi)^d}\, \mathrm{tr}\, Q\big(p_\mu + i\tilde{G}_{\mu\nu}\,\partial_p^\nu, \tilde{U}_k(x)\big)\,,$$

$$\tilde{G}_{\mu\nu} \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{(n+2)n!}\left(D_{\{\alpha_1,\ldots\alpha_n\}} G_{\mu\nu}\right) \partial_p^{\alpha_1}\cdots\partial_p^{\alpha_n}\,, \qquad D_{\{\mu_1,\cdots\mu_n\}} \equiv \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} D_{\mu_{\sigma(1)}}\cdots D_{\mu_{\sigma(n)}}\,.$$

HQET之应用

Why sky is blue

天空为什么是蓝的-Rayleigh散射

可见光和大气中的原子散射，涉及多个长度尺度.

- 可见光波长$\lambda \sim 5000$ Å $\gg$ 原子的尺寸~几个Å, 因此原子可以作为点粒子处理;
- 可见光子能量 $\ll$ 原子的激发能, 因此只需考虑光子和中性原子的弹性散射;
- $E_\gamma \ll \Delta E \ll a_0^{-1} \ll M_{atom}$

- 原子几乎是静止的, 不用考虑反冲，可以用HQET描述

天空为什么是蓝的-Rayleigh散射

有效理论描述无穷重的原子和软光子相互作用.

原子电中性，忽略其反冲

引入速度标签： $v^{\mu}=(1,0,0,0)$

$$\mathcal{L}_{eff}=\phi_v^{\dagger}(iv\cdot\partial)\phi_v-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{int}$$
$$=\phi_v^{\dagger}(i\partial_t)\phi_v-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{int}$$

Symmetry:

原子场的U(1)相位不变性，电磁规范对称性及Parity

天空为什么是蓝的-Rayleigh散射

$$\mathcal{L}_{\text{int}} = C_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} + C_2 \phi_v^\dagger \phi_v v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu}$$

$$\mathcal{L}_{\text{int}} = a_0^3 \phi_v^\dagger \phi_v \left(c_1 \vec{E}^2 + c_2 \vec{B}^2 \right) + \cdots$$

a_0 是原子的尺寸

dimensionless Wilson coeffs.

$$F_{\mu\nu} = (\vec{E}, \vec{B})$$

$$\vec{E} \sim -\dot{\vec{A}}$$
$$\vec{B} \sim \nabla \times \vec{A}$$

$$[\mathcal{L}] = 4\,, \qquad [\partial_\mu] = 1, \qquad [F_{\mu\nu}] = 2, \qquad [\phi_v] = \frac{3}{2}$$

天空为什么是蓝的-Rayleigh散射

$$\sigma_{Rayleigh} \propto a_0^6 \, E_\gamma^4$$

通过HQET, symmetry, 及简单的量纲分析Rayleigh散射的散射截面依赖于E_γ^4. 因此解释了天空为什么是蓝的！

Part 3B) 重粒子有效理论之selected topics

重电子有效理论: 软光子辐射

重电子有效理论(HEET vs. HQET)

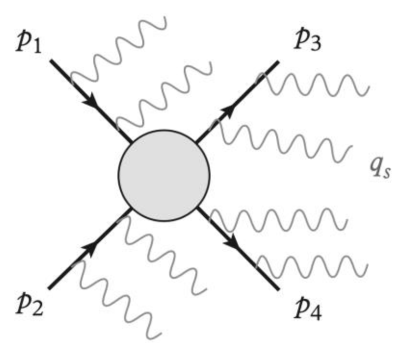
考虑电子-电子散射过程，可以伴随任意多unresolved软光子：

$$e^-(p_1) + e^-(p_2) \to e^-(p_3) + e^-(p_4) + X_s(q_s)$$

假设软光子的总能量满足限制： $E_s \ll m_e$

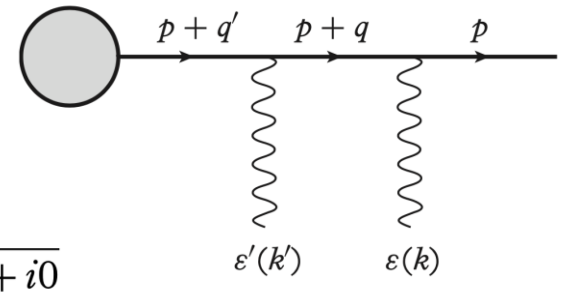
Goal： 构造恰当的EFT来有效描述该过程。要求展开参数是 $\lambda = E_\gamma / m_e$

显然，将电子场完全积掉的$\mathcal{L}_{EH}$不足以完成此任务，因为入射和出射电子作为有效自由度必须显式包含在该EFT中



软光子辐射 & eikonal近似

令出射电子4动量为 $p^\mu = m_e v^\mu$



软光子极限下，毗邻外线的电子传播子可以近似为

$$\Delta_F(p+q) = i\frac{\not{p}+\not{q}+m_e}{(p+q)^2-m_e^2+i0} = i\frac{\not{p}+m_e}{2p\cdot q+i0} = \frac{\not{v}+1}{2}\frac{i}{v\cdot q+i0}$$
$$\equiv P_v\frac{i}{v\cdot q+i0},$$

其中投影算符为 $P_v = \frac{1+\not{v}}{2}$

辐射多个软光子的振幅可以近似为 $\bar{u}(p)P_v\frac{i}{v\cdot q}(-ie\varepsilon\cdot v)P_v\frac{i}{v\cdot q'}(-ie\varepsilon'\cdot v)\ldots.$

其中电子和软光子的耦合顶角可近似为 $-iev^\mu$

问题： 上述传播子和顶角的费曼规则能否直接从某个EFT中直接推导出来？

答案： Yes, 重电子有效理论(HEET) exactly fulfills this goal!

HEET：描述电子电子散射过程的软光子辐射

Heavy Electron Effective Theory (HEET):

$$\mathcal{L}_{\rm eff} = \sum_{i=1}^{4} \bar{h}_{v_i}(x)\, i v_i \cdot D\, h_{v_i}(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Delta\mathcal{L}_{\rm int}\,, \qquad D_\mu = \partial_\mu + ieA_\mu$$

h_v : HEET中的电子场算符, 类似于HQET中的b夸克场

必须引入4个不同速度标签对应的电子场， 对应4个入射出射的电子 $v_i^\mu = p_i^\mu / m_e$

Super-selection rule: 场的velocity标签被认为是一个好量子数

Expansion parameter: $\lambda = E_\gamma / m_e$

Underlying symmetry: U(1) gauge invariance + heavy electron symmetry …

相互作用项：4费米子contact interaction

领头阶的相互作用拉氏密度可参数化为量纲为6的4电子contact interaction:

$$\Delta\mathcal{L}_{\rm int} = \sum_i C_i(v_1,v_2,v_3,v_4,m_e)\,\bar{h}_{v_3}(x)\Gamma_i h_{v_1}(x)\,\bar{h}_{v_4}(x)\Gamma_i h_{v_2}(x)\,,$$

Wilson系数C包含高能标物理(>m_e)，而此EFT刻画低能标物理 $E_s \ll m_e$

只包含领头阶算符。原则上可以写下任意规范不变的高量纲算符 (with covariant derivatives or more fields),但其效应是lambda幂次压低的。

思考：为什么不包括如下量纲为3的规范不变的算符(仅包含两个电子场)?

$$\Delta\mathcal{L}'_{\rm int} = C_{\alpha\beta}(v_1,v_3)\,h^{\alpha}_{v_1}(x)\bar{h}^{\beta}_{v_3}(x)\,,$$

原因：v1 not equal to v3，被电子速度标签的super-selection rule禁戒！

软规范链（soft Wilson line)

在领头幂次，HQET刻画自旋无关的重粒子和规范玻色子相互作用

semi-classical picture: 等价于soft Wilson line

$$S_i(x) = \exp\left[-ie\int_{-\infty}^{0} ds\, v_i \cdot A(x + s v_i)\right].$$

无穷重粒子以 $v_i^\mu = p_i^\mu / m_e$ 做匀速直线运动，轨迹由 $y^\mu(s) = x^\mu + s v_i^\mu$ 刻画

(由于电子能量无穷大，不需考虑辐射软光子引起的反冲效应)

考虑如下的真空-单光子矩阵元，并将Wilson线展开到e的第一阶，可以重复单光子辐射的eikonal结构：

$$\begin{aligned}\langle \gamma(k)|S_i(0)|0\rangle &= -ie\int_{-\infty}^{0} ds\, v_i^\mu \langle \gamma(k)|A_\mu(s v_i)|0\rangle \\ &= -ie\int_{-\infty}^{0} ds\, v_i \cdot \varepsilon(k) e^{isv_i\cdot k} = e\frac{v_i\cdot\varepsilon(k)}{-v_i\cdot k + i0}.\end{aligned}$$

可以推广到任意多个软光子辐射

场的重新定义/退耦变换
(Field redefinition/decoupling transformation)

对重电子场做field redefinition: $h_{v_i}(x) = S_i(x)\, h_{v_i}^{(0)}(x)\,,$ 引入sterile HEET场h_v^0

HEET拉氏量中的二次型变成自由场论(电子和光子退耦)

$$\begin{aligned}\bar{h}_{v_i}(x)\, iv_i \cdot D\, h_{v_i}(x) &= \bar{h}_{v_i}^{(0)}(x)\, S_i^\dagger(x)\, iv_i \cdot D\, S_i(x)\, h_{v_i}^{(0)}(x) \\ &= \bar{h}_{v_i}^{(0)}(x)\, S_i^\dagger(x)\, S_i(x)\, iv_i \cdot \partial\, h_{v_i}^{(0)}(x) \\ &= \bar{h}_{v_i}^{(0)}(x)\, iv_i \cdot \partial\, h_{v_i}^{(0)}(x)\,.\end{aligned}$$

$$v_i \cdot D\, S_i(x) = 0$$

注意两个电子场的速度标签一样，所以Wilson线才能被抵消

电子-电子散射过程中的软光子效应完全由相互作用4电子相互作用项刻画：
(体现在沿着4个不同速度的软Wilson线中)

$$\Delta\mathcal{L}_{\rm int} = \sum_i C_i(v_1, v_2, v_3, v_4)\, \bar{h}_{v_3}^{(0)}\, \bar{S}_3^\dagger\, \Gamma_i\, S_1\, h_{v_1}^{(0)}\;\; \bar{h}_{v_4}^{(0)}\, \bar{S}_4^\dagger\, \Gamma_i\, S_2\, h_{v_2}^{(0)}$$

电子-电子散射过程的因子化公式

经过退耦变换, 电子-电子散射(伴随着任意数目软光子辐射)对应的振幅可以因子化为如下形式：

$$\mathcal{M} = \sum_i C_i \, \bar{u}(v_3) \, \Gamma_i \, u(v_1) \, \bar{u}(v_4) \, \Gamma_i \, u(v_2) \, \langle X_s(k) | \bar{S}_3^\dagger \, S_1 \, \bar{S}_4^\dagger \, S_2 | 0 \rangle$$

$$= \mathcal{M}_{ee} \langle X_s(k) | \bar{S}_3^\dagger \, S_1 \, \bar{S}_4^\dagger \, S_2 | 0 \rangle ,$$

振幅可以因子化为Wilson系数 x 自由费米子矩阵元 x 软Wilson线的矩阵元

类似的推导可以推广到QCD中软胶子辐射， 额外复杂性源于Wilson线是定义在色空间的矩阵

截面的因子化公式及演化方程

相应的截面可以因子化为

$$\sigma = \mathcal{H}(m_e, \{\underline{v}\})\, \mathcal{S}(E_s, \{\underline{v}\}),$$

其中硬函数

$$\mathcal{H}(m_e, \{\underline{v}\}) = \frac{1}{2E_1 2E_2 |\vec{v}_1 - \vec{v}_2|} \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} |\mathcal{M}_{ee}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4),$$

软函数

$$\mathcal{S}(E_s, \{\underline{v}\}) = \sum_{X_s}\!\!\!\!\!\!\!\int \left| \langle X_s | \bar{S}_3^\dagger S_1 \bar{S}_4^\dagger S_2 | 0 \rangle \right|^2 \theta(E_s - E_{X_s}).$$

硬函数和软函数都依赖于速度标签：$\{\underline{v}\} = \{v_1, \ldots, v_4\}$

高阶QED修正展示，硬函数包含红外发散，软函数包含紫外发散，但inclusive截面一定保证有限。

通过硬函数中的红外发散(或软函数中的紫外发散)，可以抽取出HEET的Wilson系数对应的紫外发散(出现在硬函数中)。通过其对重整化能标依赖得到演化方程，进而可以重求和大对数项

Part 3C) 重粒子有效理论之selected topics

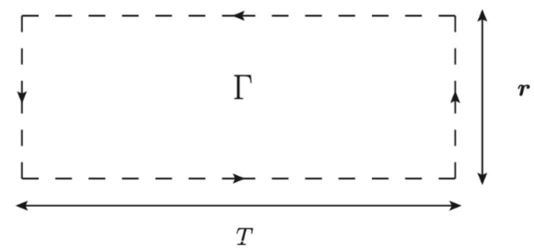
QCD静态势 (QCD static potential)

静止的无穷重夸克和无穷重反夸克之间的色单态势能

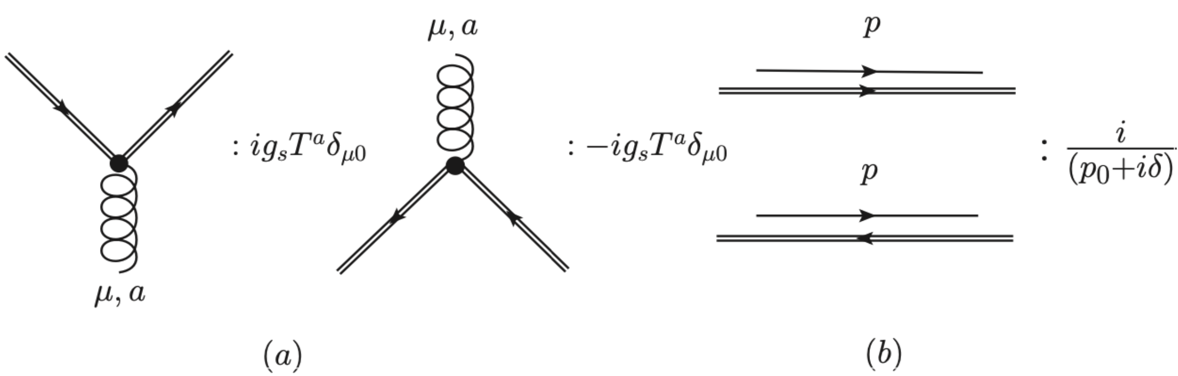
(color-singlet static potential)

规范不变的非微扰定义-- rectangular Wilson loop的真空期望值:

$$V(r) = -\lim_{T\to\infty}\frac{1}{T}\ln\left\langle \mathrm{tr}\mathscr{P}\exp\left(ig\oint_{\Gamma}dx_{\mu}A_{\mu}\right)\right\rangle$$



费曼规则具有eikonal结构 ~ identical to HQET with v^mu=(1,0)

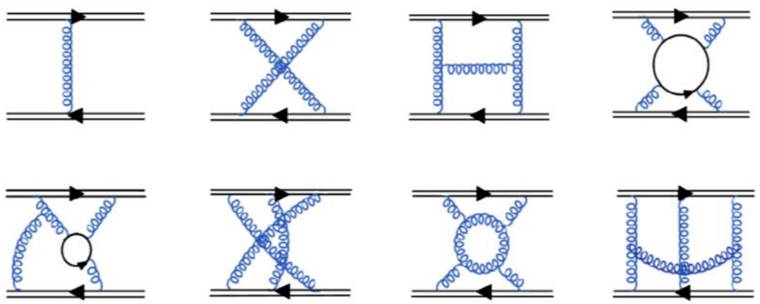


Feynman rules for (anti-)source propagator and (anti-)source-gluon vertices.

静态势在短程（大动量转移）可通过微扰论计算
当前世界纪录: 三圈解析

Anzai, Kiyo, Sumino, PRL 2010; Smirnov, Smirnov, Steinhauser, PRL 2010
Lee,Smirnov,Smirnov,Steinhauser, PRD 2016

典型的费曼图(up to three loop)



静态势的微扰展开(动量空间)

https://mediatum.ub.tum.de/doc/1121280/1121280.pdf

$$\tilde{V}^{(0)}(|\vec{q}\,|) = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}} \left\{ 1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu_{\rm IR}^2}{\vec{q}^{\,2}}\right) + \mathcal{O}\left(\alpha_s^4\right) \right\}$$

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_f, \tag{3.3}$$

$$a_2 = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right) C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3)\right) C_A T_F n_f - \left(\frac{55}{3} - 16\zeta(3)\right) C_F T_F n_f + \left(\frac{20}{9} T_F n_f\right)^2, \tag{3.4}$$

where $C_F = 4/3$, $C_A = 3$, $T_F = 1/2$ for SU(3) and n_f is the number of light quark flavors. At three-loop order, infrared-singular contributions proportional to $\ln(\mu_{\rm IR}^2/\vec{q}^{\,2})$ start to play a role (see, e.g., [22]). The accompanying constant

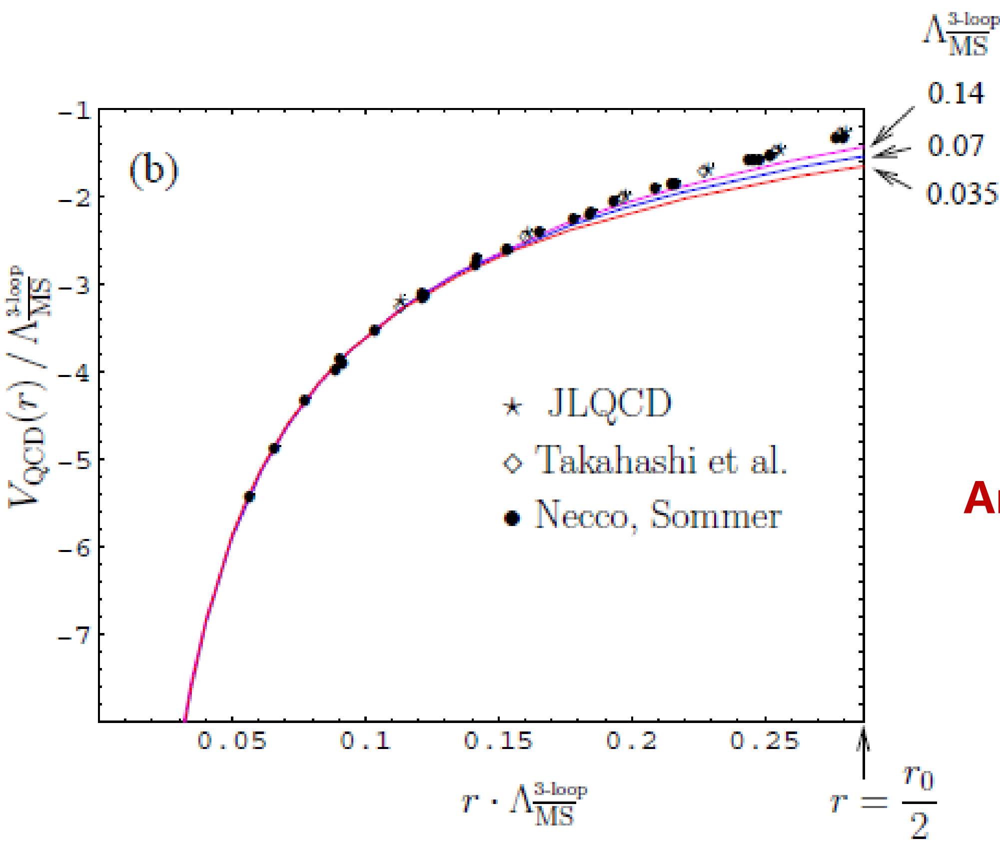
$$a_3 = 64\left(209.884(1) - 51.4048 n_f + 2.9061 n_f^2 - 0.0214 n_f^3\right) \tag{3.5}$$

has been calculated independently in [23] and [24].

色库伦势及其量子修正

$$V(|\vec{q}\,|) = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}} \Bigg[1 + \frac{\alpha_s}{\pi}(2.5833 - 0.2778 n_l) + \left(\frac{\alpha_s}{\pi}\right)^2 (28.5468 - 4.1471 n_l + 0.0772 n_l^2) + \left(\frac{\alpha_s}{\pi}\right)^3 (209.884(1) - 51.4048 n_l + 2.9061 n_l^2 - 0.0214 n_l^3) + \cdots \Bigg], \tag{4}$$

色单态静态势: 比较微扰论预言和格点QCD测量提供了一种确定alpha_s的重要手段



Anzai, Kiyo, Sumino, PRL 2010

r的取值范围上限大致对于Upsilon(1S)的半径

Warning: r很大时，格点模拟给出线性势, 意味着色禁闭, 微扰论完全失效

Part 3D) 重粒子有效理论之selected topics

卢瑟福散射之软极限

卢瑟福散射的软极限—重靶粒子质量展开

1909: Geiger & Marsden : alpha粒子轰击金箔实验

1911: 卢瑟福从经典力学出发提出卢瑟福公式，并引入原子核的概念

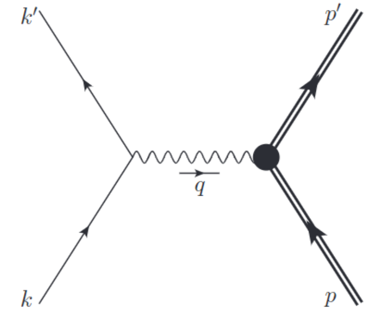
考虑入射电子能量远小于原子核质量，交换的虚光子携带很长的波长，探针相当粗糙，无法辨别复合靶粒子的精细内部结构

软极限 = 多级展开

Gell-Mann和Low在50年代研究了康普顿散射的软极限。靶粒子质量展开领头阶, 微分截面和靶粒子自旋无关;展开下一阶, 依赖于靶粒子磁矩

让我们考察卢瑟福散射的低能极限

Jia and Zhang, 2303.18243



复合靶粒子的电磁形状因子(自旋从0到2)

独立的形状因子个数=2s+1

$$\langle N(p',\lambda')|J^\mu|N(p,\lambda)\rangle_{s=0} = 2P^\mu F_{1,0}\left(\frac{q^2}{M^2}\right),$$

$$\langle N(p',\lambda')|J^\mu|N(p,\lambda)\rangle_{s=\frac{1}{2}} = \bar{u}(p',\lambda')\left[2P^\mu F_{1,0}\left(\frac{q^2}{M^2}\right) + i\sigma^{\mu\nu}q_\nu F_{2,0}\left(\frac{q^2}{M^2}\right)\right]u(p,\lambda),$$

$$\langle N(p',\lambda')|J^\mu|N(p,\lambda)\rangle_{s=1} = -\varepsilon^*_{\alpha'}(p',\lambda')\Big\{2P^\mu\left[g^{\alpha'\alpha}F_{1,0}\left(\frac{q^2}{M^2}\right) - \frac{q^{\alpha'}q^\alpha}{2M^2}F_{1,1}\left(\frac{q^2}{M^2}\right)\right] - \left(g^{\mu\alpha'}q^\alpha - g^{\mu\alpha}q^{\alpha'}\right)F_{2,0}\left(\frac{q^2}{M^2}\right)\Big\}\varepsilon_\alpha(p,\lambda),$$

$$\langle N(p',\lambda')|J^\mu|N(p,\lambda)\rangle_{s=\frac{3}{2}} = -\bar{u}_{\alpha'}(p',\lambda')\Big\{2P^\mu\left[g^{\alpha'\alpha}F_{1,0}\left(\frac{q^2}{M^2}\right) - \frac{q^{\alpha'}q^\alpha}{2M^2}F_{1,1}\left(\frac{q^2}{M^2}\right)\right] + i\sigma^{\mu\nu}q_\nu\left[g^{\alpha'\alpha}F_{2,0}\left(\frac{q^2}{M^2}\right) - \frac{q^{\alpha'}q^\alpha}{2M^2}F_{2,1}\left(\frac{q^2}{M^2}\right)\right]\Big\}u_\alpha(p,\lambda),$$

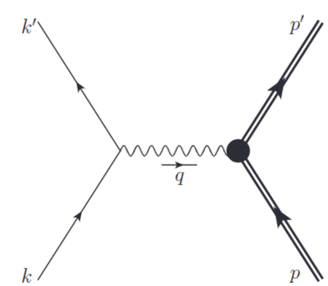
$$\langle N(p',\lambda')|J^\mu|N(p,\lambda)\rangle_{s=2} = \varepsilon^*_{\alpha'_1\alpha'_2}(p',\lambda')\Big\{2P^\mu\Big[g^{\alpha'_1\alpha_1}g^{\alpha'_2\alpha_2}F_{1,0}\left(\frac{q^2}{M^2}\right) - \frac{q^{\alpha'_1}q^{\alpha_1}}{2M^2}g^{\alpha'_2\alpha_2}F_{1,1}\left(\frac{q^2}{M^2}\right) + \frac{q^{\alpha'_1}q^{\alpha_1}}{2M^2}\frac{q^{\alpha'_2}q^{\alpha_2}}{2M^2}F_{1,2}\left(\frac{q^2}{M^2}\right)\Big] - \left(g^{\mu\alpha'_2}q^{\alpha_2} - g^{\mu\alpha_2}q^{\alpha'_2}\right) \times \left[g^{\alpha'_1\alpha_1}F_{2,0}\left(\frac{q^2}{M^2}\right) - \frac{q^{\alpha'_1}q^{\alpha_1}}{2M^2}F_{2,1}\left(\frac{q^2}{M^2}\right)\right]\Big\}\varepsilon_{\alpha_1\alpha_2}(p,\lambda).$$

$F_{1,0}(0)=Z$: 总电荷(以e为单位)

$F_{1,0}(0)+F_{1,1}(0)$: 电四极矩(以e/(2M)^2为单位)

$F_{2,0}(0)$: 磁矩(以e/(2M)为单位)

$F_{2,0}(0)+F_{2,1}(0)$: 磁八极矩(以e/(2M)^3为单位)



Cotogno, Lorce et al., PRD 2020

低能自旋1/2零质量入射粒子

对微分截面做v/c及1/M的双重展开

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}=\frac{\pi\alpha^2 Z^2\cos^2\frac{\theta}{2}}{2\mathbf{k}^2\sin^4\left(\frac{\theta}{2}\right)}-\frac{\pi\alpha^2 Z^2\cos^2\frac{\theta}{2}}{M|\mathbf{k}|\sin^2\left(\frac{\theta}{2}\right)}+\mathcal{O}\left(\frac{1}{M^2}\right)$$

1/M展开的头两阶universal

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)^{s=0}_{\mathrm{NNLO}}=-\frac{4\pi\alpha^2}{M^2\sin^2\frac{\theta}{2}}\left(F'_{1,0}Z\cos^2\frac{\theta}{2}+\frac{1}{8}Z^2\cos^2\theta-\frac{1}{8}Z^2\right)$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)^{s=\frac{1}{2}}_{\mathrm{NNLO}}=-\frac{4\pi\alpha^2}{M^2\sin^2\frac{\theta}{2}}\left[\frac{1}{16}F^2_{2,0}\left(\cos\theta-3\right)+\frac{1}{4}\cos^2\frac{\theta}{2}\left(4F'_{1,0}Z+F_{2,0}Z+Z^2\cos\theta-\frac{3}{2}Z^2\right)\right]$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)^{s=1}_{\mathrm{NNLO}}=-\frac{4\pi\alpha^2}{M^2\sin^2\frac{\theta}{2}}\left[\frac{1}{24}F^2_{2,0}\left(\cos\theta-3\right)+\frac{1}{4}\cos^2\frac{\theta}{2}\left(4F'_{1,0}Z-\frac{2}{3}F_{1,1}Z+\frac{2}{3}F_{2,0}Z+Z^2\cos\theta-\frac{5}{3}Z^2\right)\right]$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)^{s=\frac{3}{2}}_{\mathrm{NNLO}}=-\frac{4\pi\alpha^2}{M^2\sin^2\frac{\theta}{2}}\left[\frac{5}{144}F^2_{2,0}\left(\cos\theta-3\right)+\frac{1}{4}\cos^2\frac{\theta}{2}\left(4F'_{1,0}Z-\frac{2}{3}F_{1,1}Z+F_{2,0}Z+Z^2\cos\theta-\frac{13}{6}Z^2\right)\right]$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)^{s=2}_{\mathrm{NNLO}}=-\frac{4\pi\alpha^2}{M^2\sin^2\frac{\theta}{2}}\left[\frac{1}{32}F^2_{2,0}\left(\cos\theta-3\right)+\frac{1}{4}\cos^2\frac{\theta}{2}\left(4F'_{1,0}Z-\frac{2}{3}F_{1,1}Z+\frac{2}{3}F_{2,0}Z+Z^2\cos\theta-\frac{7}{3}Z^2\right)\right]$$

次次领头阶universality被破坏(靶粒子自旋依赖)，但依然存在一些明显可以辨识的pattern(why?)

低能非相对论性的自旋1/2入射粒子

对微分截面做v/c及1/M的双重展开

$$\left(\frac{d\sigma}{d\cos\theta}\right)^s_{(v^0)} = \frac{2\pi Z^2\alpha^2}{\mathbf{k}^4}\frac{m^2(M+m)^2\left(\sqrt{M^2-m^2\sin^2\theta}+m\cos\theta\right)^2}{M\sqrt{M^2-m^2\sin^2\theta}\left(M-\cos\theta\sqrt{M^2-m^2\sin^2\theta}+m\sin^2\theta\right)^2}$$
$$= \frac{8\pi Z^2\alpha^2 m^2}{\mathbf{k}^4\sin^4\frac{\theta}{2}} - \frac{\pi Z^2\alpha^2 m^4}{M^2\mathbf{k}^4} + \mathcal{O}\left(\frac{m^6}{M^4\mathbf{k}^4}\right).$$

$$\left(\frac{d\sigma}{d\cos\theta}\right)^s_{(v^2)} = \frac{\pi\alpha^2}{\mathbf{k}^2\sin^2\frac{\theta}{2}}\left[\frac{Z^2\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} - \frac{Z^2 m\cos^2\frac{\theta}{2}}{M} + \frac{Zm^2}{4M^2}f^s_{\rm NNLO} + \mathcal{O}\left(\frac{1}{M^3}\right)\right]$$

$$f^{s=0}_{\rm NNLO} = 16F'_{1,0} + Z\cos\theta - Z,$$
$$f^{s=\frac{1}{2}}_{\rm NNLO} = 16F'_{1,0} + Z\cos\theta + 4F_{2,0} - 3Z,$$
$$f^{s=1}_{\rm NNLO} = 16F'_{1,0} + Z\cos\theta - \frac{8}{3}F_{1,1} + \frac{8}{3}F_{2,0} - \frac{11}{3}Z,$$
$$f^{s=\frac{3}{2}}_{\rm NNLO} = 16F'_{1,0} + Z\cos\theta - \frac{8}{3}F_{1,1} + 4F_{2,0} - \frac{17}{3}Z,$$
$$f^{s=2}_{\rm NNLO} = 16F'_{1,0} + Z\cos\theta - \frac{8}{3}F_{1,1} + \frac{8}{3}F_{2,0} - \frac{19}{3}Z.$$

Universality首次在O(v^2/M^2)阶被破坏，但依然存在一些可以明确被辨识的pattern(why?)

重粒子有效理论(HPET)可以重现soft limit

HPET描述重的自旋1/2复合粒子和软光子的相互作用。Up to 1/M^2阶，拉氏量为(形式上类似HQET, 但这里物质场不再是点粒子)

$$\mathcal{L}_{\rm HPET} = \bar{h}_v \left(iD_0 + c_2 \frac{\mathbf{D}^2}{2M} + c_F e \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2M} + c_D e \frac{[\boldsymbol{\nabla}\cdot\mathbf{E}]}{8M^2} + i c_S e \frac{\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})}{8M^2} \right) h_v + \mathcal{O}(1/M^3)$$

Wilson系数和电磁形状因子的关系

$$c_F = F_{2,0}, \quad c_D = 2F_{2,0} + 8F'_{1,0} - F_{1,0},$$

从HPET出发，可以严格重复前面用QED暴力计算展开后得到的结果,并理解为什么靶粒子自旋依赖首次在NNLO阶出现

$$\begin{aligned}\mathcal{M}_{\rm HPET} &= -\sqrt{1+c_2\frac{\mathbf{p}'^2}{2M^2}}\frac{e^2}{q^2}\Big\{ -Z\bar{u}_{\rm NR}u_{\rm NR}\bar{u}(k')\gamma^0 u(k) + \frac{c_2 Z}{2M}\bar{u}_{\rm NR}u_{\rm NR}\bar{u}(k')\mathbf{p}'\cdot\boldsymbol{\gamma}u(k) \\ &\quad -\frac{c_F}{4M}\bar{u}_{\rm NR}\left[\not{q},\gamma^\mu\right]u_{\rm NR}\bar{u}(k')\gamma_\mu u(k) - \frac{c_D q^2}{8M^2}\bar{u}_{\rm NR}u_{\rm NR}\bar{u}(k')\gamma^0 u(k)\Big\} \\ &= \frac{e^2}{q^2}\Big\{ Z\bar{u}_{\rm NR}u_{\rm NR}\bar{u}(k')\gamma^0 u(k) + \frac{c_F}{4M}\bar{u}_{\rm NR}\left[\not{q},\gamma^\mu\right]u_{\rm NR}\bar{u}(k')\gamma_\mu u(k) + \frac{c_D q^2}{8M^2}\bar{u}_{\rm NR}u_{\rm NR}\bar{u}(k')\gamma^0 u(k)\Big\}\end{aligned}$$

$$\begin{aligned}\left.\frac{d\sigma}{d\cos\theta}\right|_{\rm EFT} &= \frac{\pi\alpha^2 Z^2\cos^2\frac{\theta}{2}}{2\mathbf{k}^2\sin^4\frac{\theta}{2}} - \frac{\pi\alpha^2 Z^2\cos^2\frac{\theta}{2}}{M|\mathbf{k}|\sin^2\frac{\theta}{2}} \\ &\quad - \frac{\pi\alpha^2}{8M^2\sin^2\frac{\theta}{2}}\left[Z^2(\cos2\theta-1) + c_D Z(\cos\theta+1) + c_F^2(\cos\theta-3)\right]\end{aligned}$$

Part 4. 刻画非相对论性系统的NREFT (以NR scalar field theory, NRQCD/NRQED为范例)

Why NREFT ?

◊ **深入人心的概念：QFT = Special Relativity + Quantum Mechanics**
相对论性量子场论有不少far-reaching consequences:
SR+QM → 反粒子, 自旋-统计定理, CPT定理，...
粒子物理(描述高能、高速、粒子产生湮灭现象)的基本工作语言

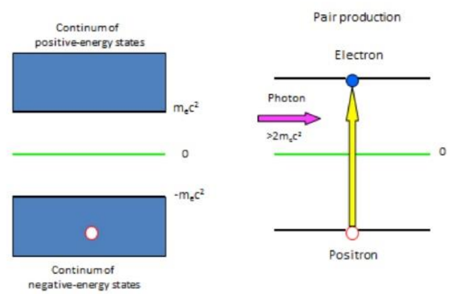
◊ **然而,大多数物理领域研究对象均为非相对论性系统。比如核物理, 原子分子物理，量子化学，凝聚态物理……甚至暗物质，双黑洞并合导致引力波辐射, GPS校准(通过NRGR做后牛顿近似), ...**

* 非相对论极限意味着taking **c** → **∞** limit
* NREFT可认为是Relativistic QFT的低能(速)有效场论
* NREFT的展开参数是v/c，or simply 1/c

构建NREFT背后的物理考量

相对论性QFT作为underlying UV theory：因果性要求粒子和反粒子必须以平权的方式包括在理论中:

根据狄拉克海的图像，粒子和反粒子（空穴）存在能隙>2 mc^2; 在非相对论(低能)极限下，粒子和反粒子阴阳相隔，彼此退耦



非相对论极限下粒子数和反粒子(空穴)数目必须分别守恒

Dirac场的平面波分解

$$\Psi(x) = \int_{|\vec{p}|<\Lambda\sim m_c} \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sum_s (a^s_{\vec{p}}\, u^s_p\, e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}}\, v^s_p\, e^{ip\cdot x})$$

正频部分　　负频部分

非相对论极限下, 最自然的做法是退耦场的正频和负频部分，即退耦粒子和反粒子(空穴)

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \chi(x) \end{pmatrix} \quad \psi(x) = \int_{|\vec{p}|<\Lambda} \frac{d^3\vec{p}}{(2\pi)^3} \sum_{s=1}^{2} a^s_{\vec{p}} \xi^s\, e^{-ip\cdot x} \quad \chi(x) = \int_{|\vec{p}|<\Lambda} \frac{d^3\vec{p}}{(2\pi)^3} \sum_{s=1}^{2} b^{s\dagger}_{\vec{p}} \eta^s\, e^{ip\cdot x})$$

d维时空薛定谔场论的标度变换

$$SSchr = \int dt \int d^{d-1}x \, \mathcal{L}_{\text{Schr}}$$

$$\mathcal{L}_{\text{Schr}} = \phi^* (i\partial_t \phi + \frac{\nabla^2}{2m})\phi$$

Scaling transformation: 时空不对称

$$t \to e^{2\lambda} t, \qquad x \to e^{\lambda} x, \qquad \phi \to e^{-\frac{d-1}{2}\lambda} \phi$$

Recall scaling transformation in relativistic QFT

Klein-Gordon: $x \to e^{\lambda} x,\ \phi \to e^{-\frac{d-2}{2}\lambda} \phi$

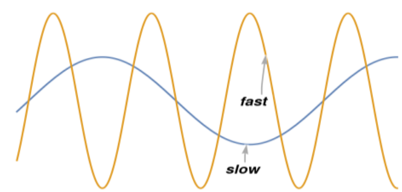
Dirac: $x \to e^{\lambda} x,\ \psi \to e^{-\frac{d-1}{2}} \psi$

Warm-up: 实标量场的NREFT

Jia, hep-th/0401171

假设如下标量场toy model是underlying theory:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,m^2\,\phi^2 - \frac{\lambda}{4!}\,\phi^4\,,$$



将实标量场分解为 $\phi = \frac{1}{\sqrt{2m}}\left(e^{-imt}\Psi + e^{imt}\Psi^*\right)\,,$ 带回原始拉氏量,
扔掉快速随时间振荡的项($\exp(\pm 2imt)$), 即得到如下NREFT:

$$\mathcal{L}_{\rm NREFT} = \Psi^*\left(i\partial_t + \frac{\nabla^2}{2M_H} - \frac{\partial_t^2}{2M_H}\right)\Psi - \frac{C_0}{4}(\Psi^*\Psi)^2 - \frac{C_2}{8}\nabla(\Psi^*\Psi)\cdot\nabla(\Psi^*\Psi) + \cdots\,,$$

非相对论极限下粒子数守恒 -> U(1)相位不变性

动能项含二阶时间导数。若做场的重新定义 $\Psi = \left(1 + \frac{\nabla^2}{4m^2} + \frac{5\nabla^4}{32m^4} + \cdots\right)\Psi' = \left(\frac{m}{\sqrt{m^2+\mathbf{k}^2}}\right)^{1/2}\Psi'$

并利用运动方程，可得到如下正则形式的NREFT:

$$\mathcal{L}'' = \Psi'^*\left(i\partial_t + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3} + \frac{\nabla^6}{16m^5} + \cdots\right)\Psi' + \cdots$$

Field redefinition and NREFT lagrangian

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{1}{2m} \frac{i}{E - \frac{\mathbf{k}^2}{2m} + \frac{E^2}{2m} + i\epsilon},$$

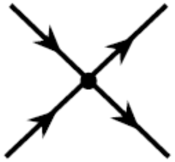
$$\mathcal{L}' = \Psi^* \left(i\partial_t + \frac{\nabla^2}{2m} - \frac{\partial_t^2}{2m} \right) \Psi - \frac{\lambda}{16m^2} (\Psi^* \Psi)^2 .$$

$$\Psi = \left(1 + \frac{\nabla^2}{4m^2} + \frac{5\nabla^4}{32m^4} + \cdots \right) \Psi' = \left(\frac{m}{\sqrt{m^2 + \mathbf{k}^2}} \right)^{1/2} \Psi',$$

$$\mathcal{L}_{\mathrm{NREFT}} = \Psi^* \left(i\partial_t + \frac{\nabla^2}{2M_H} - \frac{\partial_t^2}{2M_H} \right) \Psi - \frac{C_0}{4} (\Psi^* \Psi)^2 - \frac{C_2}{8} \nabla(\Psi^* \Psi) \cdot \nabla(\Psi^* \Psi) + \cdots ,$$

Tree-level matching

$$T_0 = 4m^2\,\mathcal{A}_0\,,$$



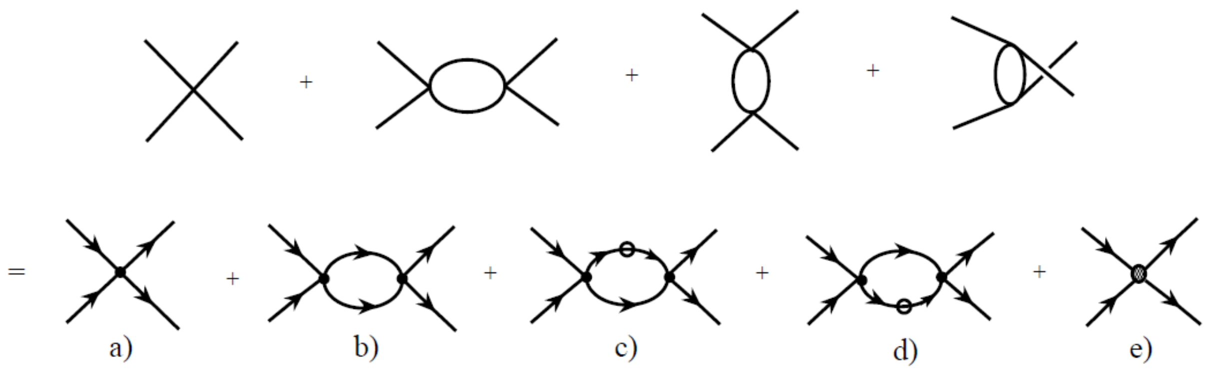
$$\mathcal{A}_0^{\text{tree}} = -C_0 - \frac{C_2}{4}[(\mathbf{k}_1 - \mathbf{k}_1')^2 + (\mathbf{k}_1 - \mathbf{k}_2')^2] - \cdots\,,$$

$$\mathcal{A}_0'^{\,\text{tree}} = -C_0\left(1 - \frac{k^2}{m^2} + \cdots\right) - C_2 k^2 - \cdots\,.$$

$$C_0 = \frac{\lambda}{4m^2} + O(\lambda^2)\,,$$

$$C_2 = 0 + O(\lambda^2)\,.$$

One-loop matching



Full theory:

$$T^{1-\text{loop}} = -\lambda + \frac{\lambda^2}{2\,m^{4-D}}\frac{\Gamma[2-D/2]}{(4\pi)^{D/2}}\int_0^1 dx\,\Big\{\Big[1-x(1-x)s/m^2 - i\epsilon\Big]^{D/2-2} + (s\to t) + (s \to u)\Big\} - \delta\lambda\,,$$

NREFT (S-wave amplitude):

$$\mathcal{A}_0^{1-\text{loop}} = -C_0 - C_0(I_0 + \tilde{I}_0)\,C_0 - C_2\,k^2 - \delta C_0 - \delta C_2\,k^2\,.$$

2+1维时空的zero-range interaction (平面上的非相对论性短程相互作用)

D=3 rich physics occurs

Point particle coupled with Chern-Simons gauge field:

Aharonov-Bohm effect, Fractional Quantum Hall effect

即使不施加外场，也有有趣的物理涌现：

Quantum scaling violation,

dimensional transmutation (维度蜕变)

Just like QCD

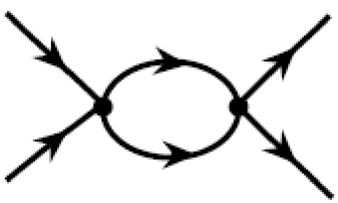
3D one-loop amplitude in Full theory and EFT

Jia, hep-th/0401171

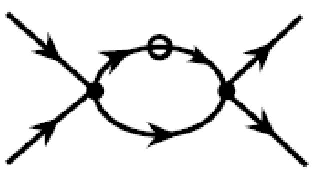
Full theory one-loop: UV finite (super-renormalizable)

$$T_0^{1-\mathrm{loop}} = -\lambda + \frac{\lambda^2}{16\pi m}\left[2 + \left(1 - \frac{k^2}{2m^2}\right)\left[\ln\left(\frac{2m}{k}\right) + \frac{i\pi}{2}\right] - \frac{k^2}{12m^2}\right].$$

NREFT one-loop: UV logarithmically divergent



$$I_0 = -\frac{m}{2}\left(\frac{e^{\gamma}\mu^2}{4\pi}\right)^{\epsilon}\int \frac{d^{2-2\epsilon}\mathbf{q}}{(2\pi)^{2-2\epsilon}}\frac{1}{\mathbf{q}^2 - 2mE - i\epsilon}$$
$$= -\frac{m}{8\pi}\left[\frac{1}{\epsilon} + \ln\mu^2 - \ln(-2mE - i\epsilon)\right],$$



c)

$$\tilde{I}_0 = -\frac{1}{8m}\left(\frac{e^{\gamma}\mu^2}{4\pi}\right)^{\epsilon}\int \frac{d^{2-2\epsilon}\mathbf{q}}{(2\pi)^{2-2\epsilon}}\frac{3k^4 - 2k^2\mathbf{q}^2}{(\mathbf{q}^2 - k^2 - i\epsilon)^2}$$
$$= \frac{m}{8\pi}\left(\frac{k^2}{2m^2}\right)\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\mu}{k}\right) + i\pi + \frac{1}{2}\right].$$

One-loop matching in 3D:

One-loop amplitude in NREFT

$$\mathcal{A}_0^{1-\mathrm{loop}} = -C_0 + C_0^2\left(\frac{m}{4\pi}\right)\left(1-\frac{k^2}{2m^2}\right)\left[\ln\left(\frac{\mu}{k}\right)+\frac{i\pi}{2}\right] - C_2 k^2\,,$$

Counter-terms in MS scheme:

$$\delta C_0 = \frac{m}{8\pi}\,\frac{C_0^2}{\epsilon}\,,$$

$$\delta C_2 = -\frac{1}{2(8\pi)m}\,\frac{C_0^2}{\epsilon}\,.$$

Wilson coefficients at one-loop order

$$C_0(\mu) = \frac{\lambda}{4m^2} - \left(\frac{\lambda}{4m^2}\right)^2\left(\frac{m}{4\pi}\right)\left[2+\ln\left(\frac{2m}{\mu}\right)\right] + O(\lambda^3)\,,$$

$$C_2(\mu) = \left(\frac{\lambda}{4m^2}\right)^2\left(\frac{1}{8\pi m}\right)\left[\frac{1}{6}+\ln\left(\frac{2m}{\mu}\right)\right] + O(\lambda^3)\,.$$

Renormalization group in 3D NREFT

$$C_0^B = \mu^{2\epsilon}\left[C_0 + \frac{m}{8\pi}\frac{C_0^2}{\epsilon} + \left(\frac{m}{8\pi}\right)^2 \frac{C_0^3}{\epsilon^2} + \cdots\right]$$
$$= \frac{\mu^{2\epsilon} C_0}{1 - \frac{m}{8\pi}\frac{C_0}{\epsilon}} .$$

Using the chain rule, we find the following beta function:

$$\beta(C_0, \epsilon) = -\frac{2\epsilon\, C_0^B(C_0, \epsilon)}{\partial C_0^B / \partial C_0|_\epsilon}$$
$$= -2\epsilon\, C_0 + \frac{m}{4\pi} C_0^2 .$$

Dimensional transmutation

$$C_0(\mu) = \left[\frac{1}{C_0(\Lambda)} + \frac{m}{4\pi}\ln\left(\frac{\Lambda}{\mu}\right)\right]^{-1}$$

Similar to the Lambda_QCD in QCD, we introduce a new characterisic scale

$$\rho = \Lambda \exp\left[\frac{4\pi}{m\, C_0(\Lambda)}\right]$$

$$C_0(\mu) = \frac{4\pi}{m}\ln^{-1}\left(\frac{\rho}{\mu}\right)$$

RGE for higher-dimensional operator

$$C_2^B = \mu^{2\epsilon}\left[\frac{C_2}{\left(1-\frac{m}{8\pi}\frac{C_0}{\epsilon}\right)^2} - \frac{1}{2(8\pi)m}\frac{C_0^2}{\epsilon} - \frac{1}{(8\pi)^2}\frac{C_0^3}{\epsilon^2} - \frac{3m}{2(8\pi)^3}\frac{C_0^4}{\epsilon^3} - \cdots\right]$$

$$= \mu^{2\epsilon}\frac{C_2 - \frac{1}{2(8\pi)m}\frac{C_0^2}{\epsilon}}{\left(1-\frac{m}{8\pi}\frac{C_0}{\epsilon}\right)^2}\,.$$

$$\beta(C_2,\epsilon) = -2\epsilon\, C_2 + \frac{m}{2\pi}C_0C_2 - \frac{C_0^2}{8\pi m}\,.$$

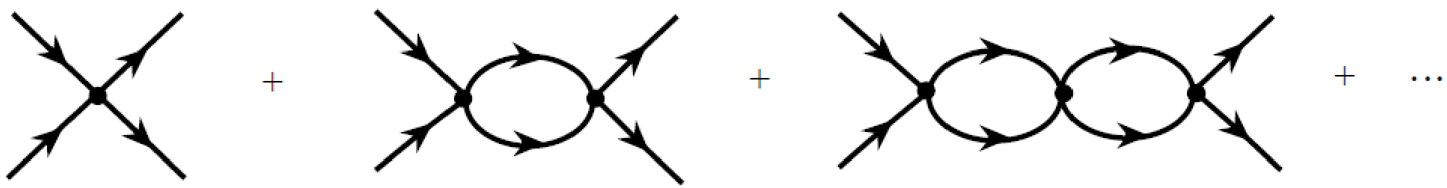
$$\mu\frac{d}{d\mu}\left(\frac{C_2}{C_0^2}\right) = -\frac{1}{8\pi m}$$

$$\frac{C_2(\mu)}{C_0^2(\mu)} = \frac{C_2(\rho)}{C_0^2(\rho)} + \frac{1}{8\pi m}\ln\left(\frac{\rho}{\mu}\right)$$

In the $\mu \to 0$ limit,

$$C_2(\mu) \approx \frac{2\pi}{m^3}\ln^{-1}\left(\frac{\rho}{\mu}\right)$$

Binding energy (attractive)



$$\mathcal{A}_0^{\rm sum} = -\left[\frac{1}{C_0 + C_2\, k^2 + \cdots} + \frac{m}{4\pi}\left(1 - \frac{k^2}{2m^2} + \cdots\right)\left[\ln\left(\frac{\mu}{k}\right) + \frac{i\pi}{2}\right]\right]^{-1} .$$

$$E_B \approx -\frac{\Lambda^2}{m}\, \exp\left[-\frac{8\pi}{m|C_0(\Lambda)|}\right] .$$

More precisely
$$E_B = -\frac{\rho^2}{m}\left[1 + \frac{8\pi}{m}\,\frac{C_2(\rho)}{C_0^2(\rho)}\,\rho^2 + \frac{\rho^2}{4m^2} + O(\rho^4)\right] .$$

标量场NREFT的一个应用 (just for fun)：两个上帝粒子之间的短程力

Feng, Jia, Sang, 1312.1944

SM Higgs sector作为underlying theory:

$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}M_H^2 H^2 - \frac{\lambda v}{4}H^3 - \frac{\lambda}{16}H^4 + \frac{2M_W^2}{v}W^{+\mu}W_\mu^- H + \frac{M_W^2}{v^2}W^{+\mu}W_\mu^- H^2 + \frac{M_Z^2}{v}Z^\mu Z_\mu H + \frac{M_Z^2}{2v^2}Z^\mu Z_\mu H^2 - \frac{m_t}{v}\bar{t}tH + \cdots,$$

可将此UV理论微扰匹配到如下的NREFT, 并定出Wilson系数**C_0** and **C_2**

$$\mathcal{L}_{\rm NREFT} = \Psi^*\left(i\partial_t + \frac{\nabla^2}{2M_H} - \frac{\partial_t^2}{2M_H}\right)\Psi - \frac{C_0}{4}(\Psi^*\Psi)^2 - \frac{C_2}{8}\nabla(\Psi^*\Psi)\cdot\nabla(\Psi^*\Psi) + \cdots,$$

Tree-level matching:

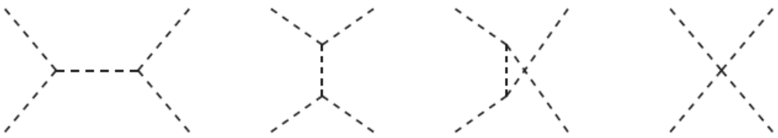


FIG. 1: The tree-level diagrams for $HH \to HH$ in SM.

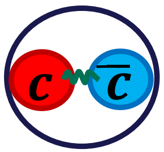
$$\mathcal{A}^{(0)}_{\rm S-wave,\,NREFT} = -C_0 - C_2 k^2 + \cdots.$$

$$C_0^{(0)} = -\frac{3}{v^2},\; C_2^{(0)} = \frac{8}{M_H^2 v^2}.$$

两个希格斯玻色子之间存在**短程弱吸引力**

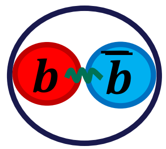
强相互作用中的“正负电子偶素”：重夸克偶素 (heavy quarkonium)

重夸克偶素是由重夸克和反重夸克形成的非相对论性束缚态



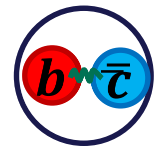
粲偶素

J/ψ, η_c



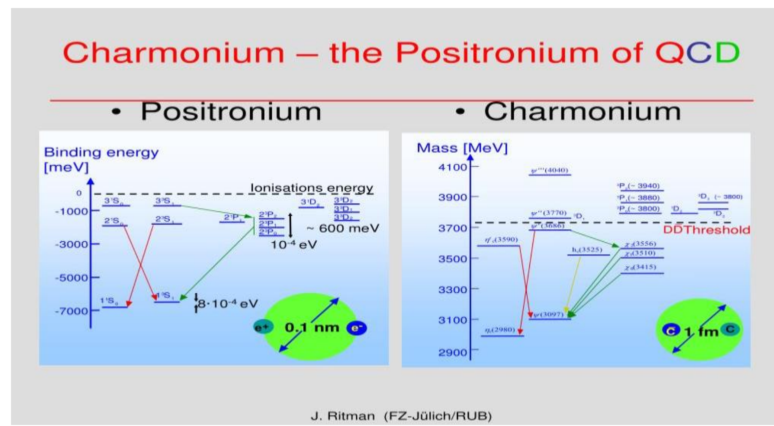
底偶素

ϒ, η_b



B_c , B^*_c

重夸克偶素是QCD中最简单的强子, 类比于QED中的正负电子偶素, 远比pion and nucleon简单！



重夸克偶素是研究QCD的理想探针, 深化理解微扰和非微扰效应如何interplay

重夸克偶素在高能核物理中也有很多应用。比如， J/ψ suppression 作为QGP的理想探针； J/ψ近阈photoproduction用于探测核子质量起源(QCD trace anomaly)；强磁场和转动介质中的夸克偶素 …

Rule of thumb: 估算重夸克偶素的三个特征能标

第一径向激发和基态的质量劈裂：$M_{\psi(2S)} - M_{J/\psi}$,$M_{\Upsilon(2S)} - M_{\Upsilon} = 600\ MeV$

第一轨道激发和基态的质量劈裂：$M_{\chi_{cJ}(^3P_J)} - M_{J/\psi}$,$M_{\chi_{bJ}(^3P_J)} - M_{\Upsilon} = 400\ MeV$

取中心值作为估计，$mv^2 = 500\ MeV$

因此 $v^2 = \frac{500}{1500} = \frac{1}{3}$ **for charmonium**

$v^2 = \frac{500}{4700} = \frac{1}{9}$ **for bottomonium**

	$c\bar{c}$	$b\bar{b}$	$t\bar{t}$
M	1.5 GeV	4.7 GeV	180 GeV
Mv	0.9 GeV	1.5 GeV	16 GeV
Mv^2	0.5 GeV	0.5 GeV	1.5 GeV

几何平均

$m \gg mv \gg mv^2$

Thus v can acts expansion parameter

Curiously, $mv^2 \sim \Lambda_{QCD}$ for charmonium and bottomnium

Rule of thumb: 估算在三个特征能标的强耦合常数的大小

large M limit (Coulomb-dominant):

$$Mv^2 \sim \frac{4}{3}\frac{\alpha_s(1/r)}{r}\,,\; r \sim \frac{1}{Mv} \;\; \Rightarrow \; v \sim \alpha_s(Mv)$$

“small” M limit (assume linear potential dominance):

$$Mv^2 \sim \kappa^2 r\,, r \sim \frac{1}{Mv} \;\; \Rightarrow \;\; \kappa \sim \Lambda_{QCD} \sim Mv^{3/2}$$

If Coulomb and linear is equally important, plausibly both scaling Rules hold simultaneously

	$c\bar{c}$	$b\bar{b}$	$t\bar{t}$
$\alpha_s(M)$	0.35	0.22	0.11
$\alpha_s(Mv)$	0.52	0.35	0.16
$\alpha_s(Mv^2)$	$\gg 1$	$\gg 1$	0.35

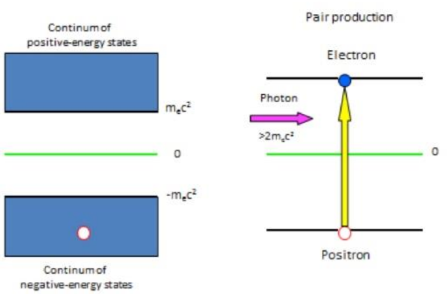
Application of pert. theory at charmonium momentum scale Mv is questionable

构建NRQCD/NRQED背后的物理考量

更加有趣的是将携带色(电)荷的费米子场和规范场耦合，**QED/QCD**作为UV理论:

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm light} + \overline{\Psi}(i\gamma^\mu D_\mu - m_Q)\Psi,$$

根据狄拉克海的图像，粒子和反粒子（空穴）存在能隙>2 mc^2; 在非相对论(低能)极限下，粒子和反粒子阴阳相隔，彼此退耦



非相对论极限下粒子数和反粒子(空穴)数目必须分别守恒

Dirac场的平面波分解

$$\Psi(x) = \int_{|\vec{p}|<\Lambda\sim m_c} \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sum_s (a^s_{\vec{p}}\, u^s_p\, e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}}\, v^s_p\, e^{ip\cdot x})$$

正频部分 **负频部分**

非相对论极限下, 最自然的做法是退耦场的正频和负频部分，即退耦粒子和反粒子(空穴)

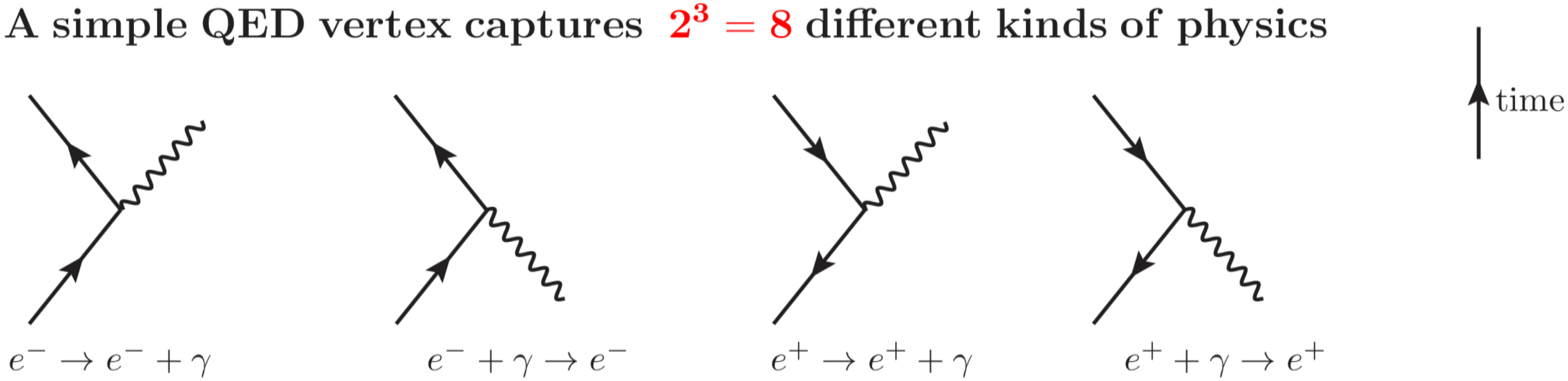
$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \chi(x) \end{pmatrix} \quad \psi(x) = \int_{|\vec{p}|<\Lambda} \frac{d^3\vec{p}}{(2\pi)^3} \sum_{s=1}^{2} a^s_{\vec{p}} \xi^s\, e^{-ip\cdot x} \quad \chi(x) = \int_{|\vec{p}|<\Lambda} \frac{d^3\vec{p}}{(2\pi)^3} \sum_{s=1}^{2} b^{s\dagger}_{\vec{p}} \eta^s\, e^{ip\cdot x})$$

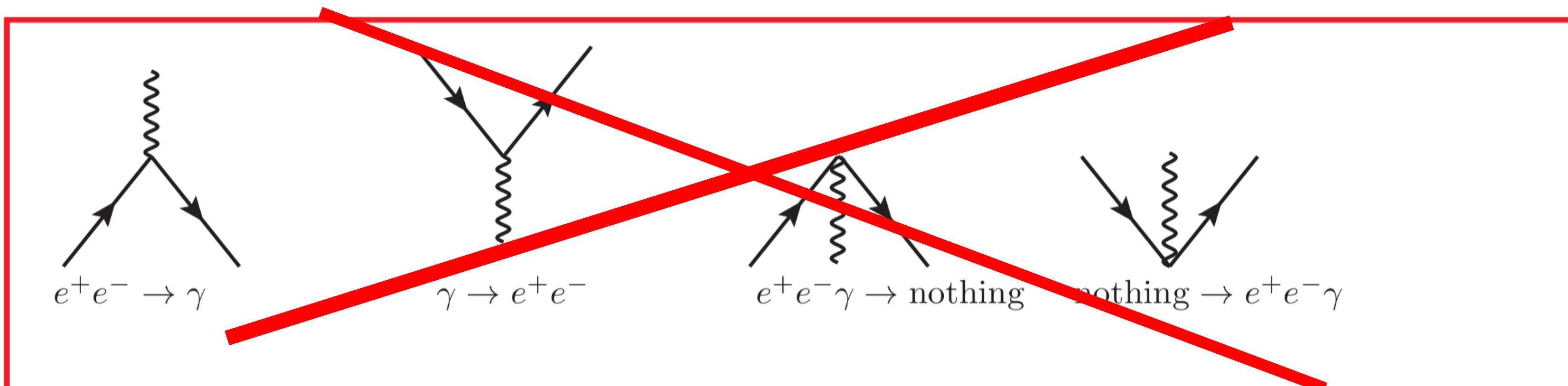
QED/QCD vertex vs. NRQED vertex

By construction, NRQED的Hilbert空间里只能包含低能电子,正电子和低能光子, 仅是QED Hilbert空间的一个子集

Feynman rules in relativistic QFT such as QED is deceptively simple!

A simple QED vertex captures $2^3 = 8$ different kinds of physics

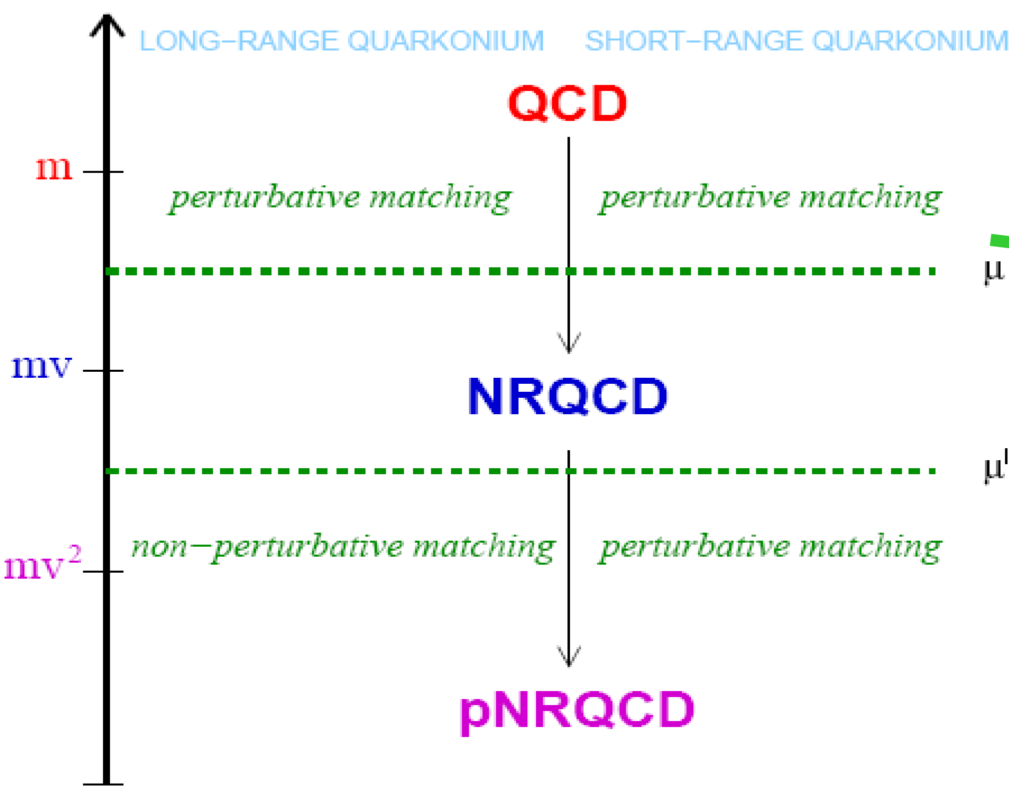




In the 2nd row, photon is too energetic, must be forbidden in NRQED

NRQCD因子化：积掉相对论性(~ mc^2)的量子张落

Caswell, Lepage (1986); Bodwin, Braaten, Lepage (1995)



Integrate out relativistic (hard) quantum fluctuation

根据渐近自由，微扰论可以工作

This scale separation is usually referred to as **NRQCD factorization.**

The NRQCD short-dist. coefficients can be computed in perturbation theory, order by order

Foldy-Wouthuysen-Tani transformation (FWT变换) 1950s

对重夸克Dirac场做FWT transformation:

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$

Dirac-Pauli basis

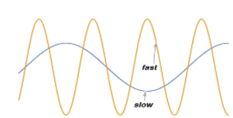
$$\Psi \to exp(-i\boldsymbol{\gamma}\cdot\boldsymbol{D}/2m_Q)\Psi = exp(-i\boldsymbol{\gamma}\cdot\boldsymbol{D}/2m_Q)\begin{pmatrix}\psi\\ \chi\end{pmatrix}$$

FWT变换的目标是逐阶对角化重夸克sector的QCD拉氏量，即逐阶移除形如$\psi^\dagger\chi$，$\chi^\dagger\psi$的off-diagonal二次型

将其带回到QCD/QED拉氏量，重夸克相关项近似变为：

$$\begin{pmatrix}\psi\\ \chi\end{pmatrix}^\dagger \begin{pmatrix} -m_Q + iD_0 + \mathbf{D}^2/2m_Q & 0 \\ 0 & m_Q + iD_0 - \mathbf{D}^2/2m_Q \end{pmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix}$$

质量项可通过对场做rephasing变换exp(-*i m t*) 移除



速度标度率(velocity scaling rule) 【库伦规范】

$$v \sim \alpha_s(Mv)$$

P. Lepage, P. Mackenzie, 1992
E. Braaten, hep-ph/9702225

Operator	Estimate
ψ	$(Mv)^{3/2}$
χ	$(Mv)^{3/2}$
D_0 (acting on ψ or χ)	Mv^2
$\mathbf{D}$	Mv
$g\mathbf{E}$	M^2v^3
$g\mathbf{B}$	M^2v^4
gA_0 (in Coulomb gauge)	Mv^2
$g\mathbf{A}$ (in Coulomb gauge)	Mv^3

A^0 (potential)more important than **A (dynamic)**

|e gamma> higher Fock state in H)

$$\langle H|H\rangle = 1, \int \mathrm{d}^3x \sim \frac{1}{(Mv)^3}, \alpha_s \sim v$$

$$\left\langle H \left| \int \mathrm{d}^3x\, \psi^\dagger\psi \right| H \right\rangle \approx 1 \Rightarrow \psi \sim (Mv)^{3/2}$$

$$\left\langle H \left| \int \mathrm{d}^3x\, \psi^\dagger \frac{\mathbf{D}^2}{2M}\psi \right| H \right\rangle \sim Mv^2$$

$$\Rightarrow \mathbf{D} \sim (Mv)^{3/2}$$

$$\left(\mathrm{i}D_0 - \frac{\mathbf{D}^2}{2M}\right)\psi = 0 \Rightarrow D_0 \sim Mv^2$$

物质场运动方程

矢量场运动方程
（库仑规范）

$$\left(\mathrm{i}\partial_0 - gA_0 + \frac{\nabla^2}{2M}\right)\psi \approx 0 \Rightarrow gA_0 \sim Mv^2$$

M^3v^5

$$(\partial_0^2 - \nabla^2)\, g\mathbf{A} - gA_0\nabla gA_0 - \frac{g^2}{M}\psi^\dagger\nabla\psi \approx 0 \Rightarrow g\mathbf{A} \sim Mv^3$$

$$g\mathbf{E} \sim -g\nabla A_0 \sim M^2v^3, \quad g\mathbf{B} \sim g\nabla \times \mathbf{A} \sim M^2v^4$$

NRQCD Lagrangian

William E. Caswell and Peter Lepage, 1986

$$\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}} + \delta\mathcal{L}.$$

$$\mathcal{L}_{\text{light}} = -\frac{1}{2}\text{tr}\, G_{\mu\nu}G^{\mu\nu} + \sum \bar{q}\, i\not{D} q,$$

$$\mathcal{L}_{\text{heavy}} = \psi^\dagger \left(iD_t + \frac{\mathbf{D}^2}{2M}\right)\psi + \chi^\dagger \left(iD_t - \frac{\mathbf{D}^2}{2M}\right)\chi,$$

$$\int d^3x \sim V \sim \frac{1}{(Mv)^3}$$

$M^4 v^5$

$$\text{H}=\int d^3x\, \mathcal{L} = Mv^2$$

$$\delta\mathcal{L} = \delta\mathcal{L}_{\text{bilinear}} + \delta\mathcal{L}_{4-\text{fermion}}$$

$$\begin{aligned}\delta\mathcal{L}_{\text{bilinear}} = &\frac{c_1}{8M^3}(\psi^\dagger(\boldsymbol{D}^2)^2\psi - \chi^\dagger(\boldsymbol{D}^2)^2\chi) \\ &+\frac{c_D}{8M^2}(\psi^\dagger(\boldsymbol{D}\cdot g\boldsymbol{E} - g\boldsymbol{E}\cdot\boldsymbol{D})\psi + \chi^\dagger(\boldsymbol{D}\cdot g\boldsymbol{E} - g\boldsymbol{E}\cdot\boldsymbol{D})\chi) \\ &+\frac{c_S}{8M^2}(\psi^\dagger(i\boldsymbol{D}\times g\boldsymbol{E} - g\boldsymbol{E}\times i\boldsymbol{D})\cdot\boldsymbol{\sigma}\psi + \chi^\dagger(i\boldsymbol{D}\times g\boldsymbol{E} - g\boldsymbol{E}\times i\boldsymbol{D})\cdot\boldsymbol{\sigma}\chi) \\ &+\frac{c_F}{2M}(\psi^\dagger(g\boldsymbol{B}\cdot\boldsymbol{\sigma})\psi - \chi^\dagger(g\boldsymbol{B}\cdot\boldsymbol{\sigma})\chi)\end{aligned}$$

$M^4 v^7$

$$\text{H}=\int d^3x\, \delta\mathcal{L}_{\text{bilinear}} = Mv^4$$

Bilinear sector形式上和HQET一样, 但power counting rule完全不同

NRQCD Lagrangian：4费米子算符

Such terms are absent in HQET

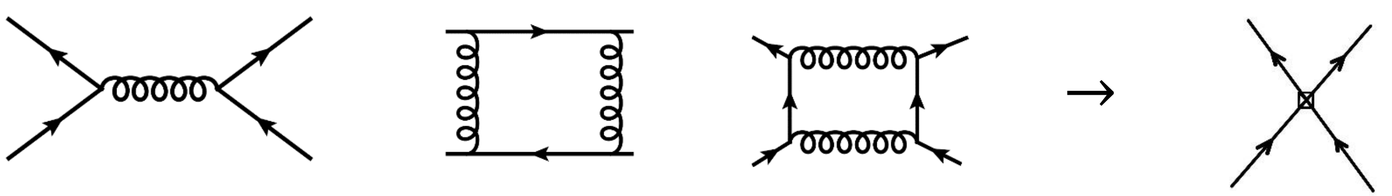
$$\delta\mathcal{L}_{4-\text{fermion}} = \sum_n \frac{f_n(\Lambda)}{M^{d_n-4}} \mathcal{O}_n(\Lambda)$$

$M^4 v^6$

$$(\delta\mathcal{L}_{4-\text{fermion}})_{d=6} = \frac{f_1(^1S_0)}{M^2}\mathcal{O}_1(^1S_0) + \frac{f_1(^3S_1)}{M^2}\mathcal{O}_1(^3S_1) + \frac{f_8(^1S_0)}{M^2}\mathcal{O}_8(^1S_0) + \frac{f_8(^3S_1)}{M^2}\mathcal{O}_8(^3S_1)$$

$$\mathcal{O}_1(^1S_0) = \psi^\dagger\chi\chi^\dagger\psi \qquad \mathcal{O}_1(^3S_1) = \psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi$$

$$\mathcal{O}_8(^1S_0) = \psi^\dagger T^a\chi\cdot\chi^\dagger T^a\psi \qquad \mathcal{O}_8(^3S_1) = \psi^\dagger\boldsymbol{\sigma}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi$$



4费米子算符的Wilson系数可以是复数→NRQCD Hamiltonian不是厄米的

NRQCD遵守的对称性

Braaten, hep-ph/9702225

- SU(3) 色规范对称性
- 转动不变性
- 电荷共轭不变性

$$\psi \rightarrow i\,(\chi^\dagger \sigma_2)^t, \qquad \chi \rightarrow -i\,(\psi^\dagger \sigma_2)^t$$

- 空间反射不变性

$$\psi(t,\mathbf{r}) \rightarrow \psi(t,-\mathbf{r}), \qquad \chi(t,\mathbf{r}) \rightarrow -\chi(t,-\mathbf{r})$$

- 重夸克相位对称性（粒子数，反粒子数分别守恒）

$$\psi \rightarrow e^{i\alpha}\psi, \qquad \chi \rightarrow e^{i\beta}\chi$$

- （近似的）重夸克自旋对称性(U、V 为独立SU(2) 矩阵)

$$\psi \rightarrow U\psi, \qquad \chi \rightarrow V\chi;$$

NRQCD/NRQED 费曼规则

库伦规范胶子传播子及夸克（反夸克）传播子

库伦规范是没有负模态的物理规范，非常适合研究非相对论性的束缚态。在此规范下，A^0 (potential)要比**A更重要**

$\longrightarrow$ time

temporal gluon

$$\frac{i\delta_{ab}}{\mathbf{l}^2}$$

spatial gluon

$$\frac{i\delta_{ab}\left(\delta^{ij} - l^i l^j/\mathbf{l}^2\right)}{l^2 + i\epsilon}$$

quark

$$\frac{i}{p_0 - \mathbf{p}^2/(2m) + i\epsilon}$$

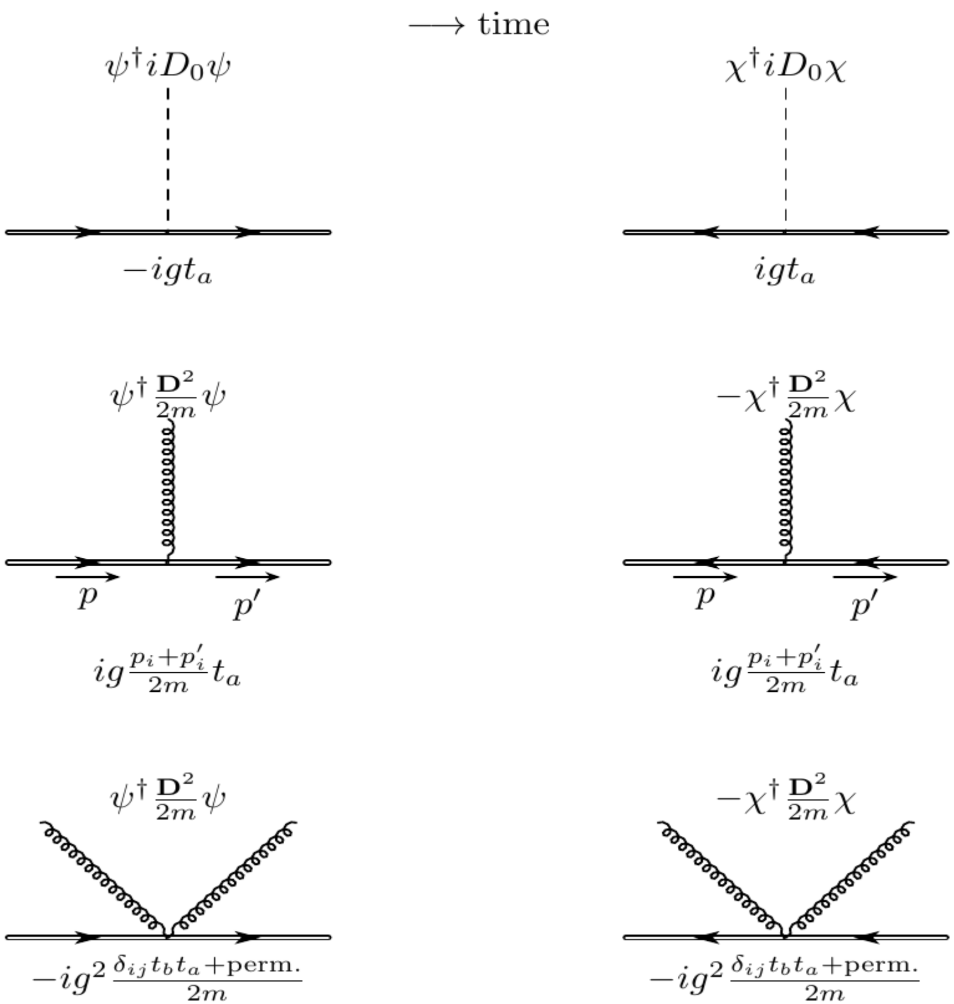
antiquark

$$\frac{i}{-p_0 - \mathbf{p}^2/(2m) + i\epsilon}$$

Single pole only. In contrast to relativistic QFT propagator

NRQCD/NRQED 费曼规则

夸克（反夸克）-胶子顶点



库伦顶点
Leading in v

动能项**p.A**
v压低

动能项$\boldsymbol{A}^2$
海鸥顶角

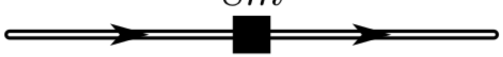
自旋无关

NRQCD/NRQED 费曼规则

夸克（反夸克）-胶子顶点

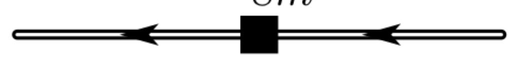
$\longrightarrow$ time

$$\psi^\dagger \frac{\mathbf{D}^4}{8m^3} \psi$$



$$i\frac{\mathbf{p}^4}{8m^3}$$

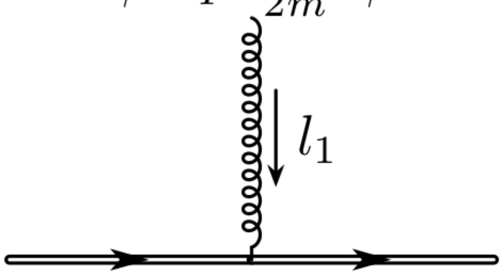
$$-\chi^\dagger \frac{\mathbf{D}^4}{8m^3} \chi$$



$$i\frac{\mathbf{p}^4}{8m^3}$$

色散关系
相对论修正

$$\psi^\dagger c_F \frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2m} \psi$$

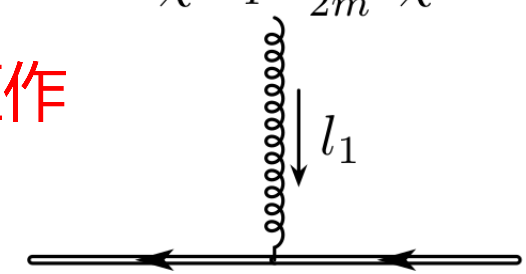


c_F

$$c_F g \frac{\epsilon_{ijk} l_{1j} \sigma_k t_a}{2m}$$

费米项-磁矩作用

$$-\chi^\dagger c_F \frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2m} \chi$$



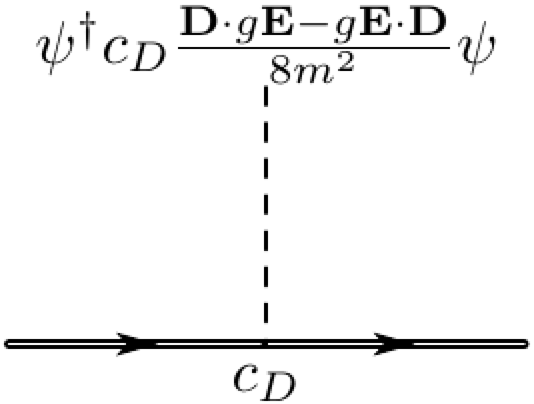
c_F

$$c_F g \frac{\epsilon_{ijk} l_{1j} \sigma_k t_a}{2m}$$

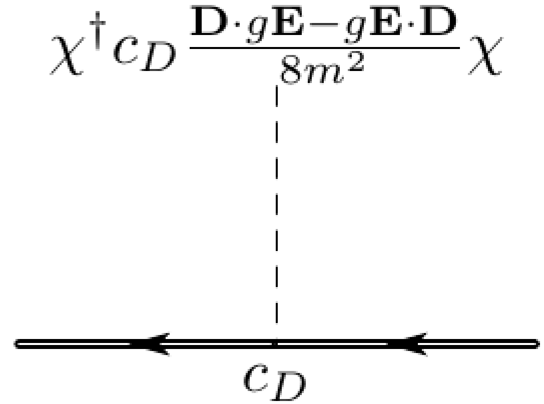
NRQCD/NRQED 费曼规则

夸克（反夸克）-胶子顶点

$\longrightarrow$ time

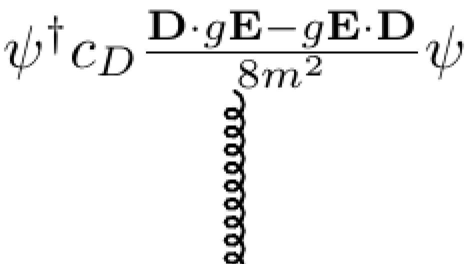


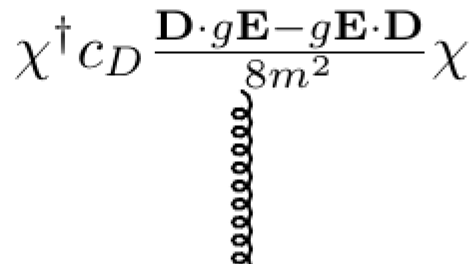
$$ic_D g\frac{\mathbf{l}_1^2 t_a}{8m^2}$$



$$-ic_D g\frac{\mathbf{l}_1^2 t_a}{8m^2}$$

Darwin项





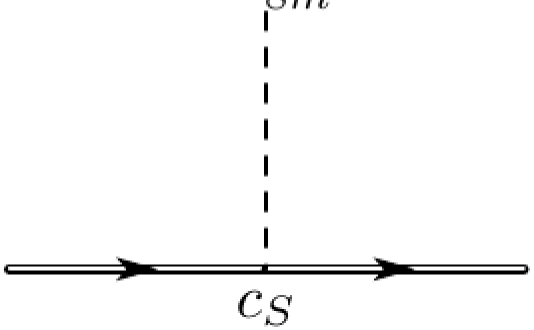
NRQCD/NRQED 费曼规则

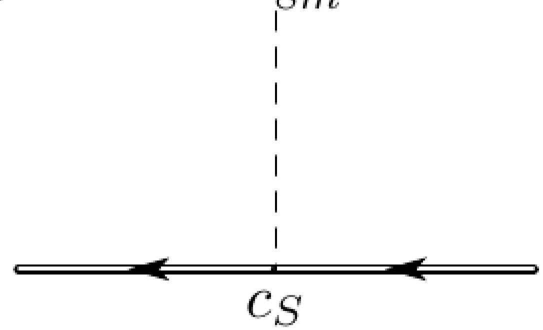
夸克（反夸克）-胶子顶点

spin-orbit耦合

$\longrightarrow$ time

$$\psi^\dagger c_S \frac{(i\mathbf{D}\times g\mathbf{E}-g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}}{8m^2}\psi \qquad\qquad \chi^\dagger c_S \frac{(i\mathbf{D}\times g\mathbf{E}-g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}}{8m^2}\chi$$



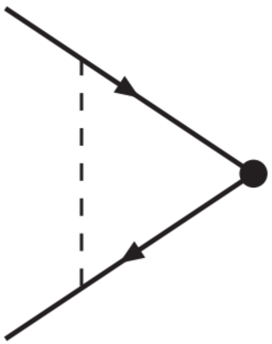


$$c_S g \frac{\epsilon_{ijk} p'_i p_j \sigma_k t_a}{4m^2} \qquad\qquad -c_S g \frac{\epsilon_{ijk} p'_i p_j \sigma_k t_a}{4m^2}$$

$\vdots$

单圈顶角修正: 重夸克-反重夸克对近阈产生

Beneke, Smirnov, NPB 1998



$$C_0\left(m^2,m^2,s,m^2,0,m^2\right)\equiv\frac{\mu^{2\epsilon}}{i\pi^{d/2}r_\Gamma}\int\frac{d^dl}{\left[(l+P/2)^2-m^2\right]\left[(l-P/2)^2-m^2\right](l-q)^2}\equiv\mathcal{I}$$

$$=\frac{\mu^{2\epsilon}}{r_\Gamma}\left(-\frac{1}{m^2v^2}\right)^{1+\epsilon}\frac{\Gamma(\epsilon)}{2}\,{}_2F_1\left(\frac{1}{2},1+\epsilon;\frac{3}{2};1+\frac{1}{v^2}\right)$$

$$v=\frac{|\mathbf{q}|}{m}=\sqrt{\frac{s}{4m^2}-1}$$

夸克对相对速度

$$A_0\left(m^2\right)=m^2\left(\frac{1}{\epsilon_{\mathrm{UV}}}+\ln\frac{\mu^2}{m^2}+1\right)+\mathcal{O}(\epsilon)$$

$$B_0\left(4m^2,m^2,m^2\right)=\frac{1}{\epsilon_{\mathrm{UV}}}+\ln\frac{\mu^2}{m^2}+2+\mathcal{O}(\epsilon)$$

$$C_0\left(m^2,m^2,s,m^2,0,m^2\right)=\frac{1}{4m^2}\left[\frac{1}{\epsilon_{\mathrm{IR}}}\left(\frac{i\pi}{v}-2\right)+\frac{1}{v}\left(i\pi\ln\frac{\mu^2}{4m^2v^2}-\pi^2\right)-2\ln\frac{\mu^2}{m^2}+4\right]$$
$$+\mathcal{O}(\epsilon,v)$$

著名的库伦发散

NRQCD含有三种低能模式：**软，超软，势**

Beneke, Smirnov, NPB 1998

hard region: $l^0 \sim m, \mathbf{l} \sim m$

$$\mathcal{I}^{\mathrm{h}} = \frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma} \int \frac{d^d l}{(l^2 + P\cdot l)(l^2 - P\cdot l)\, l^2} = \frac{\mu^{2\epsilon}}{r_\Gamma}\left(\frac{4}{s}\right)^{1+\epsilon}\left(-\frac{1}{2}\right)\frac{\Gamma(\epsilon)}{1+2\epsilon}$$
$$= \frac{1}{4m^2}\left[-\frac{2}{\epsilon_{IR}} - 2\ln\frac{\mu^2}{m^2} + 4 + \mathcal{O}(\epsilon, v^2)\right]$$

soft region: $l^0 \sim mv, \mathbf{l} \sim mv$

ultrasoft region: $l^0 \sim mv^2, \mathbf{l} \sim mv^2$

$$\mathcal{I}^{\mathrm{s/us}} = \frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma} \int \frac{d^d l}{(P\cdot l)(-P\cdot l)\, l^2} = 0$$

potential region: $l^0 \sim mv^2, \mathbf{l} \sim mv$

$$\mathcal{I}^{\mathrm{p}} = \frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma} \int \frac{d^d l}{(-\mathbf{l}^2 + P^0 l^0 + m^2 v^2)(-\mathbf{l}^2 - P^0 l^0 + m^2 v^2)\left[-(\mathbf{l}-\mathbf{q})^2\right]}$$
$$= \frac{\mu^{2\epsilon}}{\pi^{d/2-1} r_\Gamma}\frac{-1}{\sqrt{s}} \int \frac{d^{d-1} l}{(\mathbf{l}^2 - m^2 v^2)(\mathbf{l}-\mathbf{q})^2} = \frac{\mu^{2\epsilon}}{r_\Gamma}\frac{1}{\sqrt{s}}\left(-\frac{1}{m^2 v^2}\right)^{1/2+\epsilon}\frac{\sqrt{\pi}\,\Gamma(1/2+\epsilon)}{2\epsilon}$$
$$= \frac{1}{4m^2 v}\left[\frac{i\pi}{\epsilon_{IR}} + i\pi\ln\frac{\mu^2}{4m^2 v^2} - \pi^2 + \mathcal{O}(\epsilon, v^2)\right]$$

库伦发散源于“势”region

Sommerfeld增强

小v极限下，需要对$\left(\frac{\alpha_s}{v}\right)^n$做重求和。结果将导致散射截面相比于没有胶子交换有一个增强因子：

Czarnecki, Melnikov, PRL 80, 2531

$$\sigma = \sigma^{(0)}\left\{1 + C_F\left(\frac{\alpha_s}{\pi}\right)\left(\frac{\pi^2}{2v} - 4\right) + C_F\left(\frac{\alpha_s}{\pi}\right)^2\left[C_F\left(\frac{\pi^4}{12v^2} - 2\frac{\pi^2}{v} + \ldots\right) + \ldots\right]\right\}$$

$$\sigma = \sigma^{(0)}|\Psi(0)|^2 = \sigma^{(0)}\frac{C_F\pi\alpha_s/v}{1-\exp\left(-C_F\pi\alpha_s/v\right)} = 1 + \frac{C_F\pi\alpha_s}{2v} + \frac{C_F^2\pi^2\alpha_s^2}{12v^2} + \ldots$$

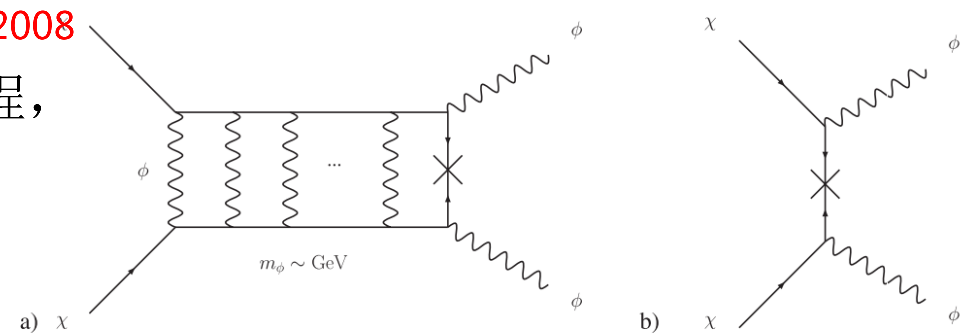
连续态零点波函数

索墨菲因子

在暗物质物理中的应用：

Hisano et.al. PRD(2005); Nima et al., PRD 2008

对于两个重暗物质粒子湮灭过程，如果暗物质粒子之间可以交换轻玻色子，在低相对速度下，散射截面会有索墨菲增强效应。



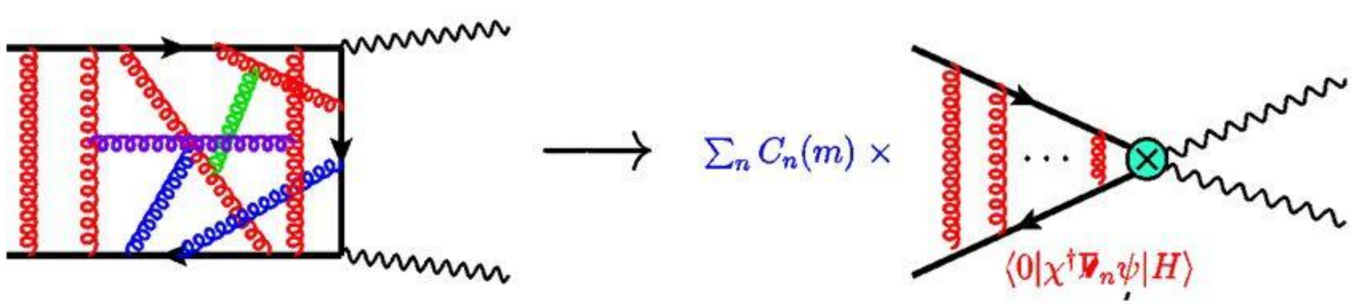
NRQCD因子化的思想

重夸克偶素有三个分得很开的特征能标： $M >> Mv >> Mv^2$

微扰　　非微扰

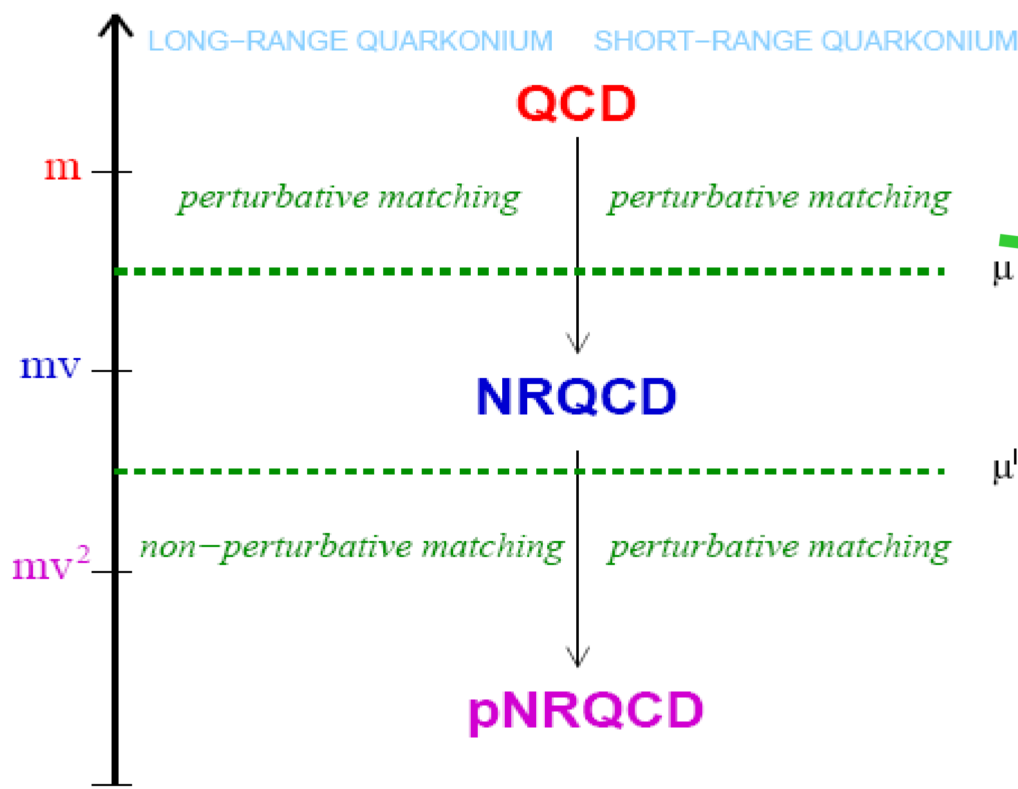
NRQCD系统地**分离短程(l<1/M)、微扰**效应和**长程(l>1/M)、非微扰效应**。**NRQCD因子化**是关于**夸克速度v**和**强作用耦合常数α_s**的**双重展开**。
NRQCD因子化并不区分**M v** and **M v^2**能标的不同modes

Quarkonium is a QCD bound state involving several distinct scales



Nonrelativistic QCD (NRQCD) factorization: for hard process: quarkonium annihilation decay and production

Caswell, Lepage (1986); Bodwin, Braaten, Lepage (1995)



Integrate out relativistic (hard) quantum fluctuation

根据渐近自由，微扰论可以信任

This scale separation is usually referred to as **NRQCD factorization.**

The NRQCD short-dist. coefficients can be computed in perturbation theory, order by order

EFT for Coulomb-Schrödinger atoms

Huang, Jia and Yu 2018

- **Coulomb-Schrödinger EFT** for atoms: (baby version of NRQED+HNET)

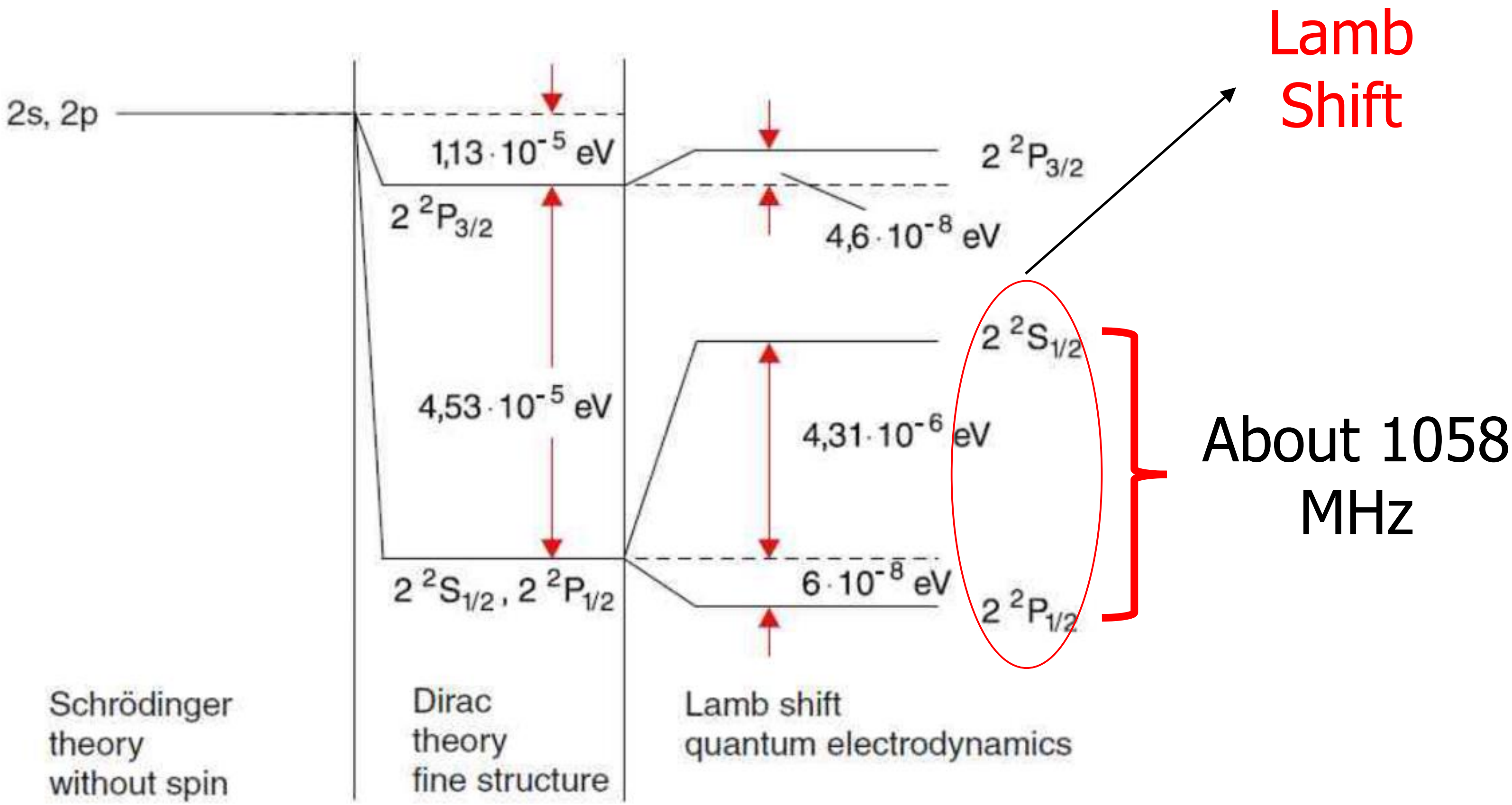
$$\mathcal{L}_{\text{Coul-Schr}} = \psi^\dagger \left\{ iD_0 + \frac{\nabla^2}{2m} \right\} \psi + N^\dagger iD_0 N + \frac{1}{2} \left(\nabla A^0\right)^2 .$$

Field theoretical realization of Schrodinger eq.

$$H_{\text{Coul}} = -\sum_{i=1}^{N} \frac{\nabla_i^2}{2m} - \sum_{i=1}^{N} \frac{Z\alpha}{r_i} + \sum_{j>i=1}^{N} \frac{\alpha}{r_{ij}} \qquad H_{\text{Coul}}\Psi = E\Psi.$$

- Coulomb gauge (only retain instantaneous Coulomb potential)
- No dynamic photons (set **A**=0): so will not see Lamb shift
- No relativistic corrections included

A celebrated example: Lamb Shift



首先考察原子的自发辐射(电偶极矩跃迁)

Barry Holstein, topics in advanced quantum mechanics

We calculate the decay rate of an atom in an excited state $|B\rangle$ and spontaneously decay to the ground state $|A\rangle$. For simplicity we deal with a one electron atom. Following Fermi's golden rule

$$\Gamma = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} 2\pi\delta\left(E_A^{(0)} + \omega_k - E_B^{(0)}\right) \left|\langle f|\hat{V}|i\rangle\right|^2$$

where the initial state $|i\rangle$ is

$$|i\rangle = |B\rangle|0\rangle$$

while the final state $|f\rangle$ is

$$|f\rangle = |A\rangle\left|1_{\boldsymbol{k},\lambda}\right\rangle$$

The Hamiltonian is

$$\hat{H} = \frac{(\boldsymbol{p} - e\boldsymbol{A})}{2m} + e\phi + \hat{H}_{RAD}^{(r)} \equiv \hat{H}_0 + \hat{V}$$

原子电偶极矩跃迁

We need now to evaluate the interaction matrix element

$$\langle A|\langle 1_{\boldsymbol{k},\lambda}|\hat{V}|0\rangle|B\rangle$$

Note that the piece of the potential $\hat{V}$ involving $\boldsymbol{A}\cdot\boldsymbol{A}$ connects only states having the same number of photons or which differ by a pair of photons. Thus $\boldsymbol{A}\cdot\boldsymbol{A}$ cannot connect the vacuum to a single photon state and we shall discard it, keeping instead only the $-\frac{e}{m}\boldsymbol{A}\cdot\boldsymbol{p}$ term for which

$$\left|\langle f|\hat{V}|i\rangle\right| = -\frac{e}{m}\langle A|\langle 1_{\boldsymbol{k},\lambda}|\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p}|0\rangle|B\rangle$$

$$= -\frac{e}{m}\frac{1}{\sqrt{2\omega_k}}\langle A|\varepsilon^*_{\boldsymbol{k},\lambda}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\cdot\boldsymbol{p}|B\rangle$$

The spontaneous decay rate is then given by

$$\frac{d\Gamma}{d\Omega} = \frac{\omega_k^2}{(2\pi)^2}\sum_\lambda\left|\langle f|\hat{V}|i\rangle\right|^2 = \frac{\alpha\omega_k}{2\pi m^2}\sum_\lambda\left|\langle A|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\varepsilon^*_{\boldsymbol{k},\lambda}\cdot\boldsymbol{p}|B\rangle\right|^2$$

or

$$\Gamma = \frac{\alpha}{2\pi m^2}\omega_k\int d\Omega\sum_\lambda\left|\langle A|\varepsilon^*_{\boldsymbol{k},\lambda}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\cdot\boldsymbol{p}|B\rangle\right|^2$$

电偶极矩跃迁矩阵元

$$\left\langle A \left| \sum_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \varepsilon^*_{\boldsymbol{k},\lambda} \cdot \boldsymbol{p} \right| B \right\rangle$$

典型的原子能级差是eV范围
典型的原子尺寸是Angstroms：

$$\boldsymbol{k}\cdot\boldsymbol{r} \sim 1eV \times 1\text{Å}\frac{1}{\hbar c} \sim \frac{1}{2000} \ll 1$$

合理的近似是多极展开(multipole expansion):

$$e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \approx 1$$

称为电偶极近似。

我们的任务是计算如下矩阵元

$$\langle A|\boldsymbol{p}|B\rangle$$

电偶极矩跃迁公式

利用量子力学的对易关系 $\left[\hat{H}_{ATOMIC},\boldsymbol{r}\right]=-i\frac{1}{m}\boldsymbol{p}$

我们则有

$$\begin{aligned}\langle A|\boldsymbol{p}|B\rangle &= im\langle A|\left[\hat{H}_{ATOMIC},\boldsymbol{r}\right]|B\rangle \\ &= im(E_B^{(0)}-E_A^{(0)})\langle A|\boldsymbol{r}|B\rangle\end{aligned}$$

注意$|A\rangle$ and $|B\rangle$ 是$\hat{H}_{ATOMIC}$ 的能量本征态. 利用$E_B^{(0)}-E_A^{(0)}=\omega_k$， 我们得到著名的原子的电偶极矩跃迁公式:

$$\frac{d\Gamma}{d\Omega_k}\approx\frac{\alpha\omega_k^3}{2\pi}\sum_\lambda\left|\langle A|\boldsymbol{r}\cdot\varepsilon_{\boldsymbol{k},\lambda}^*|B\rangle\right|^2$$

这个公式也被广泛应用在核物理和强子物理中。

兰姆位移：量子力学二阶微扰论推导 originally by H. Bethe

Imagine attempting an actual calculation of the energy shift due to coupling of the radiation field

$$\Delta E_B = P\sum_{I,\rho}\frac{\left|\langle I,\rho|\hat{V}|B,0\rangle\right|^2}{E_B^{(0)}-E_I^{(0)}-\omega_\rho}$$

长波近似下：

$$\begin{aligned}\Delta E_B &= \sum_I\int\frac{k^2dk}{(2\pi)^3}d\Omega_{\hat{k}}\sum_\lambda\left(\frac{e}{m}\right)^2\frac{\left|\hat{\varepsilon}_{\boldsymbol{k},\lambda}\cdot\boldsymbol{p}_{IB}\right|^2}{E_B^{(0)}-E_I^{(0)}-k}\frac{1}{2k}\\&=\sum_I\int\frac{k\,dk}{(2\pi)^3}d\Omega_{\hat{k}}\frac{e^2}{2m^2}\frac{\boldsymbol{p}_{BI}\cdot\boldsymbol{p}_{IB}-\boldsymbol{p}_{BI}\cdot\hat{k}\boldsymbol{p}_{IB}\cdot\hat{k}}{E_B^{(0)}-E_I^{(0)}-k}\\&=\sum_I\int\frac{k\,dk}{(2\pi)^3}\frac{e^2}{2m^2}\cdot4\pi\cdot\frac{2}{3}\frac{|\boldsymbol{p}_{BI}|^2}{E_B^{(0)}-E_I^{(0)}-k}\\&=-\frac{2}{3\pi}\frac{\alpha}{m^2}\int_0^\infty dk\,k\sum_I\frac{|\boldsymbol{p}_{BI}|^2}{k-E_B^{(0)}+E_I^{(0)}}\end{aligned}$$

Linearly UV divergent

自由电子能级的二阶微扰修正

Since plane waves are eigenstates of the momentum operator, off-diagonal matrix elements vanish and the sum over I reduces to a single term with $\boldsymbol{p}_{BB}$ representing the free electron momentum $\boldsymbol{p}$.

$$\Delta E_p = -\frac{2}{3\pi}\frac{\alpha}{m^2}\boldsymbol{p}^2\int_0^\infty dk$$

The theory is, of course, incorrect for very large photon momenta. However, we may suppose that due to additional effects the integral over k has an effective cutoff $K \sim m$, since for $k \gg K$ e.g. relativistic effects become important.

$$\frac{\Delta E_p}{E_p} \approx -\frac{4}{3\pi}\alpha \ll 1$$

so that this represents only a tiny correction to the electron kinetic energy.

Since $\Delta E_p \propto p^2$ we may consider the energy shift to be associated with a change δm in the electron rest mass

$$\Delta E_p = \frac{\boldsymbol{p}^2}{2(m+\delta m)} - \frac{\boldsymbol{p}^2}{2m} \approx -\frac{\boldsymbol{p}^2}{2m^2}\delta m$$

$$\delta m = \frac{4\alpha}{3\pi}\int_0^K dk\,, \qquad m_{exp} = m + \delta m$$

自由电子能级的二阶微扰修正：电子质量重整化

Rewrite Hamiltonian:

$$\hat{H} = \frac{\boldsymbol{p}^2}{2m} + e\phi(\boldsymbol{r}) + \hat{H}_{RAD}^{(r)} + \hat{V}$$

$$= \frac{\boldsymbol{p}^2}{2m_{exp}} + e\phi(\boldsymbol{r}) + \hat{H}_{RAD}^{(r)} + \hat{V} + \left(\frac{\boldsymbol{p}^2}{2m} - \frac{\boldsymbol{p}^2}{2(m+\delta m)}\right)$$

$$\equiv \hat{H}_0' + \hat{V}'$$

Here $\hat{H}_0'$ is simply the usual Hamiltonian $\hat{H}_0$ but with m_{exp} substituted for the electron mass. However, $\hat{V}'$ now consists of two pieces

$$\hat{V}' = \hat{V} + \frac{\boldsymbol{p}^2}{2m}\delta m \equiv \hat{V}_1 + \hat{V}_2$$

and both must be included in the energy shift calculation, as first done by Bethe. From $\hat{V}_1$ we find as before

$$\Delta E_1 = -\frac{2}{3\pi}\frac{\alpha}{m^2}\int_0^K dk\, k \sum_I \frac{\langle B|\boldsymbol{p}|I\rangle \cdot \langle I|\boldsymbol{p}|B\rangle}{k - E_B^{(0)} + E_I^{(0)}}$$

while for the piece $\hat{V}_2$

$$\Delta E_2 = \frac{1}{2m^2}\delta m \langle B|\boldsymbol{p}\cdot\boldsymbol{p}|B\rangle$$

电子质量重整化后的原子能级修正

$$\Delta E_2 = \frac{1}{2m^2}\delta m\langle B|\boldsymbol{p}\cdot\boldsymbol{p}|B\rangle$$

$$(using\ completeness\ of\ atomic\ states)$$

$$= \frac{4\alpha}{3\pi}\int_0^K dk\,\frac{1}{2m^2}\sum_I \langle B|\boldsymbol{p}|I\rangle\cdot\langle I|\boldsymbol{p}|B\rangle$$

$$= \frac{2\alpha}{3\pi m^2}\int_0^K dk\sum_I (k+E_I^{(0)}-E_B^{(0)})\frac{\langle B|\boldsymbol{p}|I\rangle\cdot\langle I|\boldsymbol{p}|B\rangle}{k+E_I^{(0)}-E_B^{(0)}}$$

Adding the two contributions , we find

$$\Delta E_{tot} = \Delta E_1 + \Delta E_2$$

$$= \frac{2\alpha}{3\pi m^2}\int_0^K dk\sum_I \frac{(E_I^{(0)}-E_B^{(0)})\langle B|\boldsymbol{p}|I\rangle\cdot\langle I|\boldsymbol{p}|B\rangle}{k+E_I^{(0)}-E_B^{(0)}}$$

$$P\int_0^K \frac{dk}{k-a} \simeq \ln\frac{K}{|a|}$$

so that

$$\Delta E = \frac{2\alpha}{3\pi m^2}\sum_I |\langle I|\boldsymbol{p}|B\rangle|^2\left(E_I^{(0)}-E_B^{(0)}\right)\ln\frac{K}{|E_I^{(0)}-E_B^{(0)}|}$$

Lamb shift

Since the logarithm is slowly varying with energy, it makes sense to define

$$\sum_I \left(E_I^{(0)} - E_B^{(0)}\right) |\langle I|\boldsymbol{p}|B\rangle|^2 ln|E_I^{(0)} - E_B^{(0)}| \equiv ln|E - E_B|_{AV} \sum_I \left(E_I^{(0)} - E_B^{(0)}\right) |\langle I|\boldsymbol{p}|B\rangle|^2$$

Now use the completeness property to write

$$\sum_I \left(E_I^{(0)} - E_B^{(0)}\right) \langle B|\boldsymbol{p}|I\rangle \cdot \langle I|\boldsymbol{p}|B\rangle = \sum_I \langle B|\boldsymbol{p}|I\rangle \cdot \langle I\big|[\hat{H}_{ATOMIC}, \boldsymbol{p}]\big|B\rangle$$

$$= \langle B\big|\boldsymbol{p} \cdot [\hat{H}_{ATOMIC}, \boldsymbol{p}]\big|B\rangle$$

Here $\hat{H}_{ATOMIC} = \frac{1}{2m}\boldsymbol{p}^2 - \frac{\alpha}{r}$ is the atomic Hamiltonian.

Then

$$[\hat{H}_{ATOMIC}, \boldsymbol{p}] = -i\nabla\frac{\alpha}{r}$$

$$\langle B\big|\boldsymbol{p} \cdot [\hat{H}_{ATOMIC}, \boldsymbol{p}]\big|B\rangle = -\int d^3r\psi_B^*(r)\nabla \cdot (\psi_B(r)\nabla\frac{\alpha}{r})$$

Since the integral is a real number, we can write

$$\langle B\big|\boldsymbol{p} \cdot [\hat{H}_{ATOMIC}, \boldsymbol{p}]\big|B\rangle = -\frac{1}{2}\int d^3r\psi_B^*(r)\nabla \cdot \left(\psi_B(r)\nabla\frac{\alpha}{r}\right) - \frac{1}{2}\int d^3r\psi_B(r)\nabla \cdot \left(\psi_B^*(r)\nabla\frac{\alpha}{r}\right)$$

Lamb shift

integrating by parts

$$\langle B|\boldsymbol{p}\cdot[\hat{H}_{ATOMIC},\boldsymbol{p}]|B\rangle = -\frac{1}{2}\int d^3r\psi_B^*(r)\left[\nabla\cdot\left(\psi_B(r)\nabla\frac{\alpha}{r}\right) - \nabla\psi_B\,(r)\cdot\nabla\frac{\alpha}{r}\right]$$

$$= -\frac{\alpha}{2}\int d^3r|\psi_B(r)|^2\nabla^2\left(\frac{1}{r}\right)$$

However, we know that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)$$

so

$$\langle B|\boldsymbol{p}\cdot[\hat{H}_{ATOMIC},\boldsymbol{p}]|B\rangle = 2\pi\alpha|\psi_B(0)|^2$$

Since the result is proportional to the value of the atomic wavefunction at the origin so that only S-waves are shifted . Using

$$\left|\psi_{nS_{\frac{1}{2}}}(0)\right|^2 = \frac{1}{\pi n^3 a_0^3}$$

where $a_0 = \frac{1}{\alpha m}$ is the Bohr radius.

Lamb shift

We find

$$\Delta E = \frac{4}{3\pi}\alpha^5 m \ln \frac{K}{\left|E - E_B^{(0)}\right|_{AV}} \times \begin{cases} \frac{1}{n^3} & for\ S - states \\ 0 & otherwise \end{cases}$$

For the $2S_{\frac{1}{2}}$ state of hydrogen $\left|E - E_B^{(0)}\right|_{AV}$ is calculated by Bethe, Brown and Stehn as

$$\left|E - E_B^{(0)}\right|_{AV} = 16.6\text{Rydberg} = 8.3\alpha^2 m$$

The Dirac theory of the hydrogen atom predicts an exact degeneracy for the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen. However, because of the coupling to the radiation field, we expect the degeneracy to be lifted, with

$$E_{2S_{1/2}} - E_{2P_{\frac{1}{2}}} = -\frac{4}{3\pi} m\alpha^5 \left(\ln 8.3\alpha^2\right) \cdot \frac{1}{8} \approx 1048\ MHz$$

This splitting was first measured by Lamb and Retherford, who found

$$E_{2S_{1/2}} - E_{2P_{1/2}} = 1057.8 \pm 0.1\ MHz$$

Lamb shift：更加直觉的推导

Welton 1948

Consider an electron in some state of the hydrogen atom $\psi_n(\boldsymbol{r})$ with no photons present. That is, the state vector is

$$|\psi\rangle|0\rangle$$

The meaning of $|0\rangle$ is that the radiation field is in its ground state. As we have previously argued in this configuration the expectation value of the electric and magnetic fields must vanish

$$\langle 0|\boldsymbol{E}|0\rangle = \langle 0|\boldsymbol{B}|0\rangle = 0$$

at each point in spacetime since the vector potential $\boldsymbol{A}$ either creates or destroys photons and therefore has a vanishing vacuum expectation value

$$\langle 0|\boldsymbol{A}|0\rangle = 0$$

On the other hand , the field energy is non-zero , namely

$$\left\langle 0\left|\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)\right|0\right\rangle = \sum_{modes}\frac{1}{2}\omega_k = \int\frac{d^3k}{(2\pi)^3}\omega_k$$

$$= \frac{1}{2\pi^2}\int_0^\infty d\omega_k\,\omega_k^3$$

Lamb shift : Intuitive Derivation 源于电磁场在真空中的量子涨落

Since for the radiation field

$$\langle 0|\boldsymbol{E}^2|0\rangle = \langle 0|\boldsymbol{B}^2|0\rangle$$

we have

$$\langle 0|\boldsymbol{E}^2|0\rangle = \frac{1}{2\pi^2}\int_0^{\infty} d\omega_k\, \omega_k^3$$

For the high frequency components of the field fluctuations we shall ignore the binding energy and will treat the electron motion classically. Let $\triangle \boldsymbol{r}$ be the displacement of the electron from its equilibrium " orbit " in response to the rapidly fluctuating electric field $\boldsymbol{E}(t)$. Then

$$m\frac{d^2}{dt^2}\triangle \boldsymbol{r} = e\boldsymbol{E}$$

Now perform a Fourier analysis. Consider the x-component of the motion and write

$$\triangle x(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\ \triangle x(\omega)$$

where

$$\triangle x(\omega) = \triangle x^*(-\omega), \qquad \triangle x\ \ is\ real$$

Lamb shift : Intuitive Derivation

$$E_x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, E_x(\omega)$$

$$E_x(\omega) = E_x^*(-\omega), \qquad E_x(t)\ \ is\ real$$

From the equation of motion, $m\frac{d^2}{dt^2}\triangle \boldsymbol{r} = e\boldsymbol{E}$, we have

$$\triangle\, x(\omega) = -\frac{e}{m\omega^2}E_x(\omega)$$

Consider the average value of $(\triangle\, x)^2$ over some long time interval T

$$\langle(\triangle\, x)^2\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\ \ \triangle\, x(t)\ \triangle\, x^*(t)$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\ \triangle\, x(\omega)\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\, e^{i\omega' t}\ \triangle\, x^*\,(\omega')$$

$$\simeq \frac{1}{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi} 2\pi\,\delta(\omega' - \omega\,)\ \triangle\, x(\omega)\ \triangle\, x^*\,(\omega')$$

$$= \frac{1}{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ \triangle\, x(\omega)\ \triangle\, x^*\,(\omega)$$

Lamb shift : Intuitive Derivation

i.e.,

$$\langle(\Delta\, x)^2\rangle = \frac{1}{T}\frac{e^2}{m^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{1}{\omega^4}E_x(\omega)E_x^*(\omega)$$

However

$$\langle {E_x}^2\rangle = \frac{1}{3}\langle E^{\,2}\rangle = \frac{1}{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;E_x(\omega)E_x^*(\omega)$$

$$= \frac{2}{T}\int_0^{\infty}\frac{d\omega}{2\pi}\;E_x(\omega)E_x^*(\omega) = \frac{1}{6\pi^2}\int_0^{\infty}d\omega\;\;\omega^3$$

or

$$\frac{1}{T}E_x^*(\omega)E_x(\omega) = \frac{1}{6\pi}\omega^3$$

Thus we find

$$\langle(\Delta\, \boldsymbol{r})^2\rangle = 3\langle(\Delta\, x)^2\rangle = \frac{2\alpha}{\pi m^2}\int_0^{\infty}\frac{d\omega}{\omega}$$

Lamb shift : Intuitive Derivation

The frequency integral diverges logarithmically at both limits. We have argued however, that the electron cannot respond to the fluctuating field if the frequency is much less than the atomic binding frequency. Thus take

$$\omega_{min} \sim E_{Rydberg} = \frac{\alpha^2 m}{2}$$

$$\omega_{max} \sim m$$

since we have neglected relativistic effects. We thus have

$$\langle (\Delta\, \boldsymbol{r})^2 \rangle = \frac{2\alpha}{\pi m^2} \ln \frac{\omega_{max}}{\omega_{min}}$$

In order finally to compute the effect of these fluctuations on the potential energy, we write

$$V\big(\boldsymbol{r}(t) + \Delta\, \boldsymbol{r}(t)\big) = V(\boldsymbol{r}) + \Delta\, \boldsymbol{r} \cdot \nabla V(\boldsymbol{r}) + \frac{1}{2}(\Delta\, \boldsymbol{r} \cdot \nabla)^2 V(\boldsymbol{r}) + \cdots$$

Where $\boldsymbol{r}(t)$ is the (smooth) classical trajectory and

$$V(r) = -\frac{\alpha}{r}$$

is the Coulomb potential energy.

Lamb shift : Intuitive Derivation

Since

$$\langle \triangle \boldsymbol{r} \rangle = 0$$

the second term in the expansion of the potential drops out. For the third , when we take the angular average we have

$$\int \frac{d\Omega}{4\pi} \triangle r_i \triangle r_j \left(\nabla_i \nabla_j \right) = \frac{1}{3} (\triangle \boldsymbol{r})^2 \nabla^2$$

Then

$$\triangle V = \langle V(\boldsymbol{r} + \triangle \boldsymbol{r}) - V(\boldsymbol{r}) \rangle = \frac{1}{6} \langle (\triangle \boldsymbol{r})^2 \rangle \nabla^2 V$$

$$= 4\pi \delta^3(r) \alpha \frac{1}{6} \langle (\triangle \boldsymbol{r})^2 \rangle$$

This may be regarded as a perturbation in the Hamiltonian for the atom.

Lamb shift : Intuitive Derivation

For a state with wave function $\psi(\boldsymbol{r})$ this leads to an energy shift

$$\triangle E = \frac{4\pi\alpha}{6}\langle(\triangle \boldsymbol{r})^2\rangle \int d^3r\, \psi(\boldsymbol{r})\delta^3(\boldsymbol{r})\psi(\boldsymbol{r})$$

$$= \frac{4\pi\alpha}{6}\langle(\triangle \boldsymbol{r})^2\rangle|\psi(0)|^2$$

$$= \frac{4\alpha^2}{3m^2}|\psi(0)|^2 \ln\frac{\omega_{max}}{\omega_{min}}$$

$$= \frac{4\alpha^5}{3\pi}\begin{Bmatrix} \frac{1}{n^3} & for\ S-states \\ 0 & otherwise \end{Bmatrix} ln\frac{\omega_{max}}{\omega_{min}}$$

which is very similar to Bethe's result. For a numerical estimate, we have

$$\triangle E_{2S_{\frac{1}{2}}-2P_{\frac{1}{2}}} = m\frac{\alpha^5}{6\pi}\ln\frac{8}{\alpha^2} = \frac{\alpha^3}{3\pi}\left(\frac{\alpha}{2a_0}\right)\ln\frac{8}{\alpha^2}$$

$$= \frac{\alpha^3}{3\pi} \times 13.6eV \times 11.9 \approx 1.6\times 10^9 Hz$$

Which is the correct order of magnitude and sign!

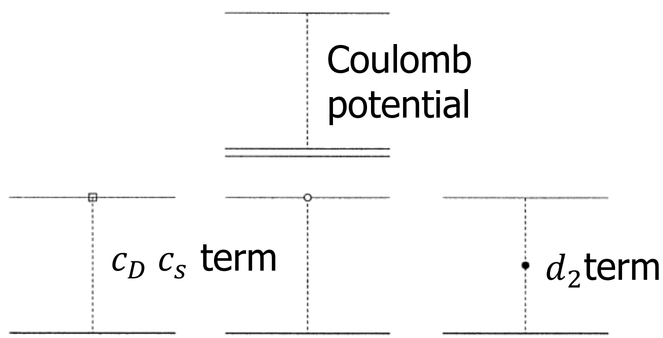
Lamb Shift——Systematical treatment: Potential NRQED （modern EFT framework）

$$\mathscr{L}_{\rm NRQED} = \psi^\dagger \Big\{ iD^0 + \frac{\boldsymbol{D}^2}{2m} + \frac{\boldsymbol{D}^4}{8m^3} + c_F\, g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m} + c_D\, g \frac{[\boldsymbol{\nabla}\cdot\boldsymbol{E}]}{8m^2} + ic_S\, g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{8m^2} \Big\} \times \psi + N^\dagger iD^0 N - \frac{1}{4} d_1 F_{\mu\nu}F^{\mu\nu} + \frac{d_2}{m^2} F_{\mu\nu} D^2 F^{\mu\nu}$$

Treat nucleon to be a static heavy field

Integrate out potential photon exchanged between nucleon and electron

$$L_{\rm pNRQED} = \int d^3\boldsymbol{x} \Big(\psi^\dagger \Big\{ iD^0 + \frac{\boldsymbol{D}^2}{2m} + \frac{\boldsymbol{D}^4}{8m^3} \Big\} \psi + N^\dagger iD^0 N - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \Big) + \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2 N^\dagger N(t,\boldsymbol{x}_2) \times \Big(\frac{Z\alpha}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \frac{Ze^2}{m^2}\Big(-\frac{c_D}{8} + 4d_2\Big) \times \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2) + ic_S \frac{Z\alpha}{4m^2}\boldsymbol{\sigma} \cdot \Big(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3} \times \boldsymbol{\nabla} \Big) \Big) \psi^\dagger \psi(t,\boldsymbol{x}_1).$$



Project this Lagrangian to one nucleon-one electron state. Since there is no space derivative acting on nucleon we can treat nucleon to be static at origin.

Lamb Shift——Systematical treatment: Potential NRQED

$$L_{\text{pNRQED}} = \int d^3\mathbf{x}\,\varphi^\dagger(t,\mathbf{x})\Bigg(i\partial_0 - e(1-Z)A_0(t,\mathbf{0}) - e x^i \partial_i A_0(t,\mathbf{0}) + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{Z\alpha}{|\mathbf{x}|} - ie\frac{\mathbf{A}(t,\mathbf{0})\cdot\boldsymbol{\nabla}}{m}$$

$$+\frac{\boldsymbol{\nabla}^4}{8m^3} + \frac{Ze^2}{m^2}\left(-\frac{c_D}{8} + 4d_2\right)\delta^3(\mathbf{x})$$

$$+ic_S\frac{Z\alpha}{4m^2}\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\times\boldsymbol{\nabla}\right)\Bigg)\varphi(t,\mathbf{x})$$

$$-\int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$\int d^4x\,\varphi^\dagger i\frac{e}{m}\mathbf{A}(t,\mathbf{0})\cdot\boldsymbol{\nabla}\varphi$$

$$= \int d^4x\,\varphi^\dagger ie\mathbf{A}(t,\mathbf{0})\cdot\left[\mathbf{x},\hat{h}_0\right]\varphi$$

$$= \int d^4x\,\varphi^\dagger e\partial_0\mathbf{A}(t,\mathbf{0})\cdot\mathbf{x}\,\varphi$$

where

$$\hat{h}_0 = -\frac{\boldsymbol{\nabla}^2}{2m} - \frac{Z\alpha}{|\mathbf{x}|}.$$

$$L_{\text{pNRQED}} = \int d^3\mathbf{x}\,S^\dagger(t,\mathbf{x})\Bigg(i\partial_0 - e(1-Z)A_0(t,\mathbf{0}) + e\mathbf{x}\cdot\mathbf{E}(t,\mathbf{0}) + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{Z\alpha}{|\mathbf{x}|} + \frac{\boldsymbol{\nabla}^4}{8m^3}$$

$$+\frac{Ze^2}{m^2}\left(-\frac{c_D}{8} + 4d_2\right)\delta^3(\mathbf{x})$$

$$+ic_S\frac{Z\alpha}{4m^2}\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\times\boldsymbol{\nabla}\right)\Bigg)S(t,\mathbf{x})$$

$$-\int d^3\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Explicit gauge invariant so that we can calculate at any gauge

Note that we have done muti-pole expansion because ultrasoft photon($m\alpha^2$) can't detect the inner detail of nucleon-electron system($1/mZ\alpha$)

Lamb Shift——Systematical treatment: Potential NRQED

$$\delta^S E_n = \delta^{S,K} E_n + \delta^{S,\delta} E_n + \delta^{S,S} E_n \,,$$

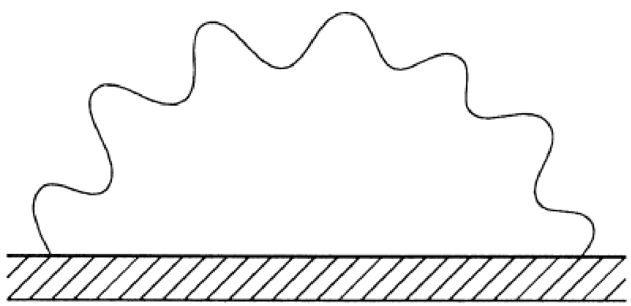
$$\delta^{S,K} E_n = -\frac{1}{8m^3}\langle nlj|\boldsymbol{\nabla}^4|nlj\rangle, \quad \longleftarrow \quad \frac{\boldsymbol{\nabla}^4}{8m^3}$$

$$\delta^{S,\delta} E_n = \frac{Ze^2}{m^2}\left(\frac{c_D}{8} - 4d_2\right)|\phi_n(\mathbf{0})|^2, \quad \longleftarrow \quad \frac{Ze^2}{m^2}\left(-\frac{c_D}{8} + 4d_2\right)\delta^3(\boldsymbol{x})$$

$$\delta^{S,S} E_n = c_S \frac{Z\alpha}{4m^2}\left(j(j+1) - l(l+1) - \frac{3}{4}\right) \times \langle nlj|\frac{1}{\boldsymbol{x}^3}|nlj\rangle \quad \longleftarrow \quad ic_S\frac{Z\alpha}{4m^2}\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}\times\boldsymbol{\nabla}\right)$$

A direct contribution is potential terms in pNRQED Lagrangian

Lamb Shift——Systematical treatment: Potential NRQED



$$I_{ij}(p) = \int \frac{d^D k}{(2\pi)^D} \frac{i}{k^2} \left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2} \right) \frac{i}{p - k^0 + i\eta}$$
$$= -ip \frac{1}{6\pi^2} \delta^{ij} \left(\frac{1}{\epsilon} + \frac{1}{2} \log 4\pi + \log \frac{\mu}{-p - i\eta} + \frac{5}{6} - \frac{\gamma}{2} - \log 2 \right). \qquad (13)$$

$$\Pi(q, \boldsymbol{x}) := \int dx^0 e^{iqx^0} < T\{\varphi(0)\varphi^\dagger(x)\} >$$

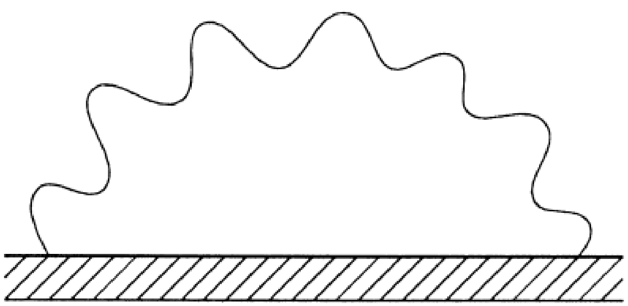
$$\Pi(q, \boldsymbol{x}) \sim \frac{i\phi_n(0)\phi_n^\dagger(\boldsymbol{x})}{q - E_n} - e^2 \frac{i\phi_n(0)}{q - E_n} \times \sum_m \langle n|\boldsymbol{v}^i|m\rangle I_{ij}(q - E_m) \times \langle m|\boldsymbol{v}^j|n\rangle \frac{i\phi_n^\dagger(\boldsymbol{x})}{q - E_n},$$

LSZ

$$\boldsymbol{v} = -i\boldsymbol{\nabla}/m.$$

Note there is a UV pole, next we confirm this pole is proportional to δ-function so that it can be absorbed into corresponding potential terms

Lamb Shift——Systematical treatment: Potential NRQED



$$I_{ij}(p) = \int \frac{d^D k}{(2\pi)^D} \frac{i}{k^2} \left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2} \right) \frac{i}{p - k^0 + i\eta}$$
$$= -ip \frac{1}{6\pi^2} \delta^{ij} \left(\frac{1}{\epsilon} + \frac{1}{2}\log 4\pi + \log \frac{\mu}{-p - i\eta} + \frac{5}{6} - \frac{\gamma}{2} - \log 2 \right). \quad (13)$$

$$\Pi(q, \boldsymbol{x}) := \int dx^0 e^{iqx^0} < T\{\varphi(0)\varphi^\dagger(x)\} >$$

$$\Pi(q, \boldsymbol{x}) \sim \frac{i\phi_n(0)\phi_n^\dagger(\boldsymbol{x})}{q - E_n} - e^2 \frac{i\phi_n(0)}{q - E_n} \times \sum_m \langle n|\boldsymbol{v}^i|m\rangle I_{ij}(q - E_m) \times \langle m|\boldsymbol{v}^j|n\rangle \frac{i\phi_n^\dagger(\boldsymbol{x})}{q - E_n},$$

LSZ

$$\boldsymbol{v} = -i\boldsymbol{\nabla}/m.$$

Note there is a UV pole, next we confirm this pole is proportional to δ-function so that it can be absorbed into corresponding potential terms

Lamb Shift——Systematical treatment: Potential NRQED

Then the remaining finite term is :

$$\delta^{US}E_n = \frac{2}{3}\frac{\alpha}{\pi}\left(\left(\log\frac{\mu}{m} + \frac{5}{6} - \log 2\right)\right.$$

$$\times\left(\frac{Ze^2}{2}\right)\frac{|\phi_n(\mathbf{0})|^2}{m^2} - \sum_{m\neq n}|\langle n|\boldsymbol{v}|m\rangle|^2$$

$$\left.\times(E_n - E_m)\log\frac{m}{|E_n - E_m|}\right)$$

Bethe Log

$$\delta E_n = \delta^S E_n + \delta^{US} E_n$$

$$\delta^S E_n = \delta^{S,K}E_n + \delta^{S,\delta}E_n + \delta^{S,S}E_n\,,$$

$$\delta^{S,K}E_n = -\frac{1}{8m^3}\langle nlj|\boldsymbol{\nabla}^4|nlj\rangle\,,$$

$$\delta^{S,\delta}E_n = \frac{Ze^2}{m^2}\left(\frac{c_D}{8} - 4d_2\right)|\phi_n(\mathbf{0})|^2\,,$$

$$\delta^{S,S}E_n = c_S\frac{Z\alpha}{4m^2}\left(j(j+1) - l(l+1) - \frac{3}{4}\right)$$

$$\times\langle nlj|\frac{1}{\boldsymbol{x}^3}|nlj\rangle$$

$\log\mu$ dependence in self energy correction and in c_D cancel. The final result is independent of $\log\mu$ and agrees with the well known result

$$c_D = 1 + \frac{\alpha}{\pi}\left(\frac{8}{3}\log\frac{m}{\mu}\right) \qquad c_S = 1 + \frac{\alpha}{\pi}$$

Thanks for your attention