Combinatorial Geometry and Feynman Integrals

Tai-Fu Feng, Hai-Bin Zhang

Hebei University, China

微扰量子场论及其应用

ShanDong Univ., China,

July 2025

Contents

I. Embedding of Feynman Integrals in Grassmannians

- Feynman integrals involving several energy scales can be given by some finite linear combinations of generalized hypergeometric functions.
- Any commonly used functions of one indeterminate of analysis can be expressed as the Gauss function

$$
{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\ (c)_n}\, x^n\ , \tag{1.1}
$$

  where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer notation.
- For the given parameters $a,\ b,\ c$, there are 24 hypergeometric series solutions totally of the partial differential equation (PDE) which can be written as the GKZ-system on the Grassmannians $G_{2,4}$.

I. Embedding of Feynman Integrals in Grassmannians

- Some hypergeometric functions are defined on the Grassmannian $G_{k,n}$ in a natural way, where a point of Grassmannian $G_{k,n}$ is a k-dimensional vector subspace C^k of n-dimensional vector space C^n $(k < n)$.
- One can regarded the manifold $Z_{k,n}$ of $k \times n$ matrices of rank k being a bundle space over the base space $G_{k,n}$ whose projection $Z_{k,n} \to G_{k,n}$ assigns to each matrix $Z_{k,n}$ the k-dimensional vector subspace spanned by the row vectors of this matrix.

I. Embedding of Feynman Integrals in Grassmannians

- We consider an arbitrary smooth function on the manifold $Z_{k,n}$ satisfying the homogeneous condition

$$f(x_1 z_1, \cdots, x_n z_n) = \prod_{i=1}^{n} x_i^{\beta_i - 1} f(z_1, \cdots, z_n) \,, x_i \neq 0 \,, \tag{1.2}$$

where z_i, $(i = 1, \cdots, n)$ denotes the i-th column vector of the $k \times n$ matrix $Z_{k,n}$, and

$$\sum_{i=1}^{n} \beta_i = n - k \,. \tag{1.3}$$

- Let $t = (t_1, \cdots, t_k)$ denote the local coordinates of the $(k-1)$-dimensional projective subspace, and the volume element

$$\omega(t) = \sum_{i=1}^{k} (-)^{i-1} dt_1 \wedge \cdots dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_k \tag{1.4}$$

I. Embedding of Feynman Integrals in Grassmannians

- The integral

$$F(z) = \int_S f(t \cdot z)\omega(t) \tag{1.5}$$

satisfies the homogeneous conditions

$$F(g \cdot z) = |\det g|^{-1} F(z), \quad g \in GL(k, C),$$
$$F(z \cdot \chi) = \prod_{i=1}^{n} \chi_i^{\alpha_i - 1} F(z), \quad \chi = \mathrm{diag}(\chi_1, \cdots, \chi_n)\ . \tag{1.6}$$

Here S is an arbitrary $(k-1)$-dimensional hypersurface in the vector space C^k not passing the origin.

- Obviously the first condition is equivalent to the system of partial differential equations (PDEs)

$$\sum_{j=1}^{n} z_{ij} \frac{\partial F}{\partial z_{i'j}} = -\delta_{i,i'} F, \quad i \in [1,k],\ i' \in [1,k], \tag{1.7}$$

with $\delta_{i,i'}$ denoting the Kronecker symbol.

I. Embedding of Feynman Integrals in Grassmannians

- The second condition of Eq.(1.6) is equivalent to the system of PDEs

$$\sum_{i=1}^{k} z_{ij} \frac{\partial F}{\partial z_{ij}} = (\alpha_j - 1)F, \quad j \in [1, n], \tag{1.8}$$

- Using the GKZ-system presented above, we can construct the fundamental solution system which is composed of the hypergeometric functions of local splitting coordinates z_{ij}. In addition, the generalized Gauss inverse relations, the generalized Gauss adjacent relations, and the generalized Gauss-Kummer relations can be derived accordingly.

I. Embedding of Feynman Integrals in Grassmannians

- In α-parameterization, for an example, the Feynman integral of one-loop self-energy is

$$
\begin{aligned}
& iA_{1SE}(p^2, m_1^2, m_2^2) \\
= & -\left(\Lambda_{\rm RE}^2\right)^{2-D/2} \int_0^\infty d\alpha_1 d\alpha_2 \int \frac{d^D q}{(2\pi)^D} \exp\Big\{ i\Big[\alpha_1 (q^2 - m_1^2) \\
& +\alpha_2((q+p)^2 - m_2^2)\Big]\Big\} \\
= & \frac{i^{2-D/2} \exp\left\{\frac{i\pi(2-D)}{4}\right\}\Gamma(2-D/2)\left(\Lambda_{\rm RE}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \\
& \times \int_S \omega_3(t)\delta(t_1 t_2 + t_1 t_3 + t_2 t_3)(t_1 t_2)^{1-D/2} t_3^{D/2-1} \\
& \times \left[t_1 m_1^2 + t_2 m_2^2 + t_3 p^2\right]^{D/2-2} ,
\end{aligned} \tag{1.9}
$$

- The hyperplane S is given by the equation $t_3 + 1 = 0$, and $\omega_3(t) = dt_2 \wedge dt_3 - dt_1 \wedge dt_3 + dt_1 \wedge dt_2$ is the volume element in the projective plane P^2, respectively.

I. Embedding of Feynman Integrals in Grassmannians

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$ where the first row corresponds to the integration variable t_1, the second row corresponds to the integration variable t_2, and the third row corresponds to the integration variable t_3, respectively.
- The first column represents the power function $t_1^{1-D/2}$, the second column represents the power function $t_2^{1-D/2}$, the third column represents the power function $t_3^{D/2-1}$, the sixth column represents the power of the linear polynomial $t_1 m_1^2 + t_2 m_2^2 + t_3 p^2$.
- The polynomial under δ function is taken as the fourth and fifth columns of the subvariety of the Grassmannian $G_{3,6}$.

I. Embedding of Feynman Integrals in Grassmannians

- In order to embed the homogeneous polynomial $(t_1t_2+t_1t_3+t_2t_3)$ under δ function as the fourth and fifth columns of the Grassmannian $G_{3,6}$, we rewrite

$$\begin{aligned}&t_1t_2+t_1t_3+t_2t_3\\&=z_{1,\widehat{\sigma}_1(4)}z_{2,\widehat{\sigma}_1(5)}t_1t_2+z_{1,\widehat{\sigma}_2(4)}z_{3,\widehat{\sigma}_2(5)}t_1t_3+z_{2,\widehat{\sigma}_3(4)}z_{3,\widehat{\sigma}_3(5)}t_2t_3\;,\end{aligned} \tag{1.10}$$

where $\widehat{\sigma}_i,\ (i=1,\ 2,\ 3)$ are elements of the permutation group $S_2=\{\widehat{e},\ \widehat{(45)}\}$ on the column indices $4,\ 5$.

- Taking $\widehat{\sigma}_1=\widehat{\sigma}_3=\widehat{e}$, $\widehat{\sigma}_2=\widehat{(45)}$, we have

$$z_{1,4}z_{2,5}=1\;,\quad z_{1,5}z_{3,4}=1\;,\quad z_{2,4}z_{3,5}=1\;. \tag{1.11}$$

- The solution of Eq.(1.11) in $\mathbf{Z}_2$

$$z_{1,4}=z_{2,4}=z_{3,4}=1\;,\quad z_{1,5}=z_{2,5}=z_{3,5}=1\;. \tag{1.12}$$

I. Embedding of Feynman Integrals in Grassmannians

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$

$$\boldsymbol{\xi}' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & r_1 \\ 0 & 1 & 0 & 1 & 1 & r_2 \\ 0 & 0 & 1 & 1 & 1 & r_3 \end{pmatrix}, \tag{1.13}$$

with $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = p^2$.

- Because the fourth and fifth columns in the matroid Eq.(1.13) coalesce into a same point in projective space P^2, ξ'^{1S} is reduced to the subvariety of the Grassmannian $G_{3,5}$ represented by the matroid ξ of size 3×5

$$\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 & 0 & 1 & r_1 \\ 0 & 1 & 0 & 1 & r_2 \\ 0 & 0 & 1 & 1 & r_3^2 \end{pmatrix}. \tag{1.14}$$

with the exponent vector

$\beta_{(1S)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -1,\ \frac{D}{2} - 1) \in C^5$.

I. Embedding of Feynman Integrals in Grassmannians

- Similarly the Feynman integral of 1-loop massless triangle diagram is embedded in the subvariety of the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.14) with $r_{1,2} = p_{1,2}^2,\ r_3 = p_3^2 = (p_1 + p_2)^2$ and the exponent vector $\beta_{(1T)} = (1,\ 1,\ 1,\ \frac{D}{2} - 2,\ 1 - \frac{D}{2}) \in C^5$.
- The Feynman integral of 2-loop vacuum is embedded in the subvariety of the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.14) with $r_i = m_i^2,\ i = 1, 2, 3$ and the exponent vector $\beta_{(2V)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2} - 2,\ D - 2) \in C^5$.

I. Embedding of Feynman Integrals in Grassmannians

- The Feynman integral of 2-loop sunset diagram can be embedded in the subvariety of the Grassmannian $G_{4,6}$

$$\xi^{2L3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & r_1 \\ 0 & 1 & 0 & 0 & 1 & r_2 \\ 0 & 0 & 1 & 0 & 1 & r_3 \\ 0 & 0 & 0 & 1 & 1 & r_4 \end{pmatrix} , \qquad (1.15)$$

with $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = m_3^2,\ r_4 = p^2$ and the exponent vector $\beta_{(2L3)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -2,\ D-2) \in C^6$.

- The dimension of the solution space of the GKZ system of the Grassmannian $G_{4,6}$ is 4.
- In the above cases, the Feynman integrals are embedded in the general strata of the Grassmannians. For example, the determinant of any 4×4 minor is nonzero in Eq.(1.15).

I. Embedding of Feynman Integrals in Grassmannians

- The Feynman integral of 2-loop self energy with 4 propagators can be embedded in the subvariety of the Grassmannian $G_{5,8}$

$$\xi^{2L4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & r_4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & r_5 \end{pmatrix} . \tag{1.16}$$

  with $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = m_3^2,\ r_4 = m_4^2,\ r_5 = p^2$, and the exponent vector $\beta_{(2L4)} = (1,\ 1,\ 1,\ 1,\ 1,\ 3-D,\ -2,\ D-3) \in C^8$.

- The subvariety presented by the matroid in Eq.(1.16) is a special stratum (not general stratum) of the Grassmannian $G_{5,8}$ because the determinant of the 5×5 minor $\det(\xi^{2L4}_{\{1,2,3,4,7\}}) = 0$.

I. Embedding of Feynman Integrals in Grassmannians

- The subvariety representing a general stratum of the Grassmannian $G_{5,8}$ is given by

$$\xi^{2L4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & u_1 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 1 & u_2 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & u_3 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 1 & u_4 & r_4 \\ 0 & 0 & 0 & 0 & 1 & 1 & u_5 & r_5 \end{pmatrix} . \tag{1.17}$$

  For the general vectors $(u_1, \cdots, u_5) \in \mathbf{C}^5$, $(r_1, \cdots, r_5) \in \mathbf{C}^5$.

- The dimension of the solution space of the GKZ system on the general stratum of the Grassmannian $G_{5,8}$ is 15. We can construct the hypergeometric solutions for the GKZ-system on the general stratum of the Grassmannian, the number of the independent variables in those hypergeometric functions are eight. Certainly, we derive the generalized Gauss inverse relations, the generalized Gauss adjacent relations, and generalized Gauss-Kummer relations among those hypergeometric solutions.

I. Embedding of Feynman Integrals in Grassmannians

- Using the Gauss inverse relations and Gauss-Kummer relations mentioned above, we derive that the dimension of the solution space of the GKZ system constraining on the special stratum of the Grassmannian $G_{5,8}$ is 5, where the stratum is defined by $u_1 = \cdots = u_4 = 1,\ u_5 = 0$ and $(r_1, \cdots, r_5) \in \mathbf{C}^5$ is a general vector.
- The fundamental solution systems of the GKZ system constraining on the special stratum are composed of the canonical hypergeometric functions in which the multiplicity of sums and the number of the independent variables are all equal four.
- The Gauss relations among the canonical hypergeometric functions are induced by those on the general stratum of the Grassmannian $G_{5,8}$ accordingly.

I. Embedding of Feynman Integrals in Grassmannians

- The Feynman integral of 2-loop self energy with 5 propagators can be embedded in the subvariety of the Grassmannian $G_{6,9}$

$$\xi^{2L5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & r_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & r_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & r_6 \end{pmatrix} . \qquad (1.18)$$

  with $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = m_3^2,\ r_4 = m_4^2,\ r_5 = m_5^2,\ r_6 = p^2$, and the exponent vector $\beta_{(2L5)} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 3-D,\ -2,\ D-4) \in C^9$.
- The subvariety presented by the matroid in Eq.(1.18) is a special stratum (not general stratum) of the Grassmannian $G_{6,9}$ because the determinant of the 6×6 minor $\det(\xi^{2L5}_{\{1,2,3,4,5,8\}}) = 0$.

I. Embedding of Feynman Integrals in Grassmannians

- The subvariety representing a general stratum of the Grassmannian $G_{6,9}$ is given by

$$\xi^{2L5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & u_1 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & u_2 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & u_3 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & u_4 & r_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & u_5 & r_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & u_6 & r_6 \end{pmatrix} . \qquad (1.19)$$

  For the general vectors $(u_1, \cdots, u_6) \in \mathbf{C}^6$, $(r_1, \cdots, r_6) \in \mathbf{C}^6$.

- The dimension of the solution space of the GKZ system on the general stratum of the Grassmannian $G_{6,9}$ is 21. We can construct the hypergeometric solutions for the GKZ-system on the general stratum of the Grassmannian, the number of the independent variables in those hypergeometric functions are ten. Certainly, we derive the generalized Gauss inverse relations, the generalized Gauss adjacent relations, and generalized Gauss-Kummer relations among those hypergeometric solutions.

I. Embedding of Feynman Integrals in Grassmannians

- Using the Gauss inverse relations and Gauss-Kummer relations mentioned above, we derive that the dimension of the solution space of the GKZ system constraining on the special stratum of the Grassmannian $G_{6,9}$ is 6, where the stratum is defined by $u_1 = \cdots = u_5 = 1,\ u_6 = 0$ and $(r_1, \cdots, r_6) \in \mathbf{C}^6$ is a general vector.
- The fundamental solution systems of the GKZ system constraining on the special stratum are composed of the canonical hypergeometric functions in which the multiplicity of sums and the number of the independent variables are all equal five.
- The Gauss relations among the canonical hypergeometric functions are induced by those on the general stratum of the Grassmannian $G_{6,9}$ accordingly.

I. Embedding of Feynman Integrals in Grassmannians

- In a similar way, we can embed the Feynman integrals of 2-loop triangle diagrams in some special strata of the Grassmannians. Meanwhile, the Gauss relations are obtained on the general stratum of the Grassmannians.
- The canonical hypergeometric solutions are gotten through constraining that on the general stratum of the Grassmannians. Meanwhile, the Gauss relations are induced by those on the general stratum of the Grassmannians.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

The hypergeometric function on the general stratum of the Grassmannian $G_{3,5}$ with the splitting coordinates in Eq.(1.14) satisfies the GKZ-system as

$$
\begin{aligned}
&\left\{\vartheta_{1,4}+\vartheta_{1,5}\right\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_1\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\left\{\vartheta_{2,4}+\vartheta_{2,5}\right\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_2\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\left\{\vartheta_{3,4}+\vartheta_{3,5}\right\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_3\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\left\{\vartheta_{1,4}+\vartheta_{2,4}+\vartheta_{3,4}\right\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_4-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\left\{\vartheta_{1,5}+\vartheta_{2,5}+\vartheta_{3,5}\right\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_5-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,
\end{aligned}
\tag{2.1}
$$

where the Euler operators $\vartheta_{i,j}=\xi_{i,j}\partial/\partial\xi_{i,j}$, and the exponent vector $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_5)\in C^5$ satisfying $\sum\beta_i=2$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Corresponding to the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.14), the exponent matrix is generally written as

$$\begin{pmatrix} \beta_1 - 1 & 0 & 0 & \alpha_{1,4} & \alpha_{1,5} \\ 0 & \beta_2 - 1 & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & \beta_3 - 1 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix} . \tag{2.2}$$

where

$$\sum_{i=1}^{5} \beta_i = 2, \quad \sum_{j=1}^{3} \alpha_{j,4} = \beta_4 - 1, \quad \sum_{j=1}^{3} \alpha_{j,5} = \beta_5 - 1$$
$$\alpha_{j,4} + \alpha_{j,5} = -\beta_j, \quad j = 1, 2, 3. \tag{2.3}$$

Six indeterminate exponents satisfy four independent linear constraints.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Let $\mathcal{N} = \{1, \cdots, 5\}$ denoting the set of indices of the columns in Eq.(1.14). Choosing the spanning subset $\mathcal{B}$ of the vector subspace C^3 in the vector space C^5 and the integer lattice on the complement $\mathcal{N} \backslash \mathcal{B}$, one gets the hypergeometric function accordingly.
For example as $\mathcal{B} = \{1, 2, 3\}$, there are 12 choices on the matrix of integer lattice whose submatrix composed of the fourth- and fifth columns is formulated as $\pm n_1 E_3^{(i)} \pm n_2 E_3^{(j)}$, where $n_{1,2} \geq 0$, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$, and other elements are all zero.

$$\text{Integer lattice } (0_{3\times 3} \,|\, \pm n_1 E_3^{(i)} \pm n_2 E_3^{(j)}): \quad E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$E_3^{(2)} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}, E_3^{(3)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$
\left(0_{3\times 3}\middle| n_1 E_3^{(1)} + n_2 E_3^{(2)}\right)
= \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & \boxed{n_1} & -n_1 \\ 0 & 0 & 0 & -n_1-n_2 & n_1+n_2 \end{pmatrix}, \qquad (2.4)
$$

- the exponents are given by the matrix

$$
||\boldsymbol{\alpha}||
= \begin{pmatrix} \beta_1-1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2-1 & 0 & \boxed{0} & -\beta_2 \\ 0 & 0 & \beta_3-1 & \beta_4-1 & 1-\beta_3-\beta_4 \end{pmatrix}, \qquad (2.5)
$$

where $\alpha_{1,4}=\alpha_{2,4}=0$ because $n_{1,2}$ are nonnegative.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is

$$\begin{aligned}
\Phi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) &= A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})\frac{(r_1)^{-\beta_1}(r_2)^{-\beta_2}(r_3)^{1-\beta_3-\beta_4}}{\det(\boldsymbol{\xi}_{\{1,2,3\}})} \\
&\times\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_3}{r_1}) \\
&= A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{-\beta_2}(r_3)^{1-\beta_3-\beta_4} \\
&\times\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_3}{r_1})\ , \\
\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) &= \sum_{n_1,n_2} c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2)x_1^{n_1}x_2^{n_2}\ , \qquad (2.6)
\end{aligned}$$

where $\det(\xi_{\{1,2,3\}}) = 1$ denotes the determinant of the 3×3 minor of the matrix in Eq.(1.14) composed of the 1st, 2nd, and 3rd columns.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Where

$$A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)\Gamma(2-\beta_3-\beta_4)} \,,$$

$$c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1, n_2) = \frac{(\beta_2)_{n_1}(\beta_1)_{n_2}(1-\beta_4)_{n_1+n_2}}{n_1!n_2!(2-\beta_3-\beta_4)_{n_1+n_2}} \,. \qquad (2.7)$$

  with the Pochhammer notation $(a)_n = \Gamma(a+n)/\Gamma(a)$.

- The geometric representation of the hypergeometric function is determined by the exponent matrix presented in Eq.(2.5), the determinant of any 2×2 minor of the submatrix consisted of the third and fourth columns is zero. In other words, the two columns coincide as a point on the projective plane P^2.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

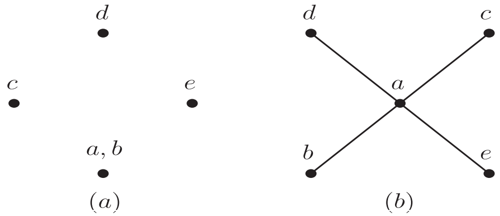

Figure: 1 The geometric configurations of the hypergeometric functions on the projective plane P^2, where the points $a, \cdots, e$ denote the indices of columns of the 3×5 exponent matrix.

The geometric representation of the function $\Phi^{(1)}_{\{1,2,3\}}$ is drown in Fig.1(a) where $\{a, b\} = \{3, 4\}$ and $\{c, d, e\} = \{1, 2, 5\}$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$
\begin{aligned}
&(0_{3\times 3}\,\big|\, n_1 E_3^{(1)} + n_2 E_3^{(3)}) \\
&= \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & n_1 - n_2 & -n_1 + n_2 \\ 0 & 0 & 0 & -n_1 & \boxed{n_1} \end{pmatrix}, \qquad (2.8)
\end{aligned}
$$

- the exponents are given by the matrix

$$
\begin{aligned}
&||\boldsymbol{\alpha}|| \\
&= \begin{pmatrix} \beta_1 - 1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2 - 1 & 0 & \beta_3 + \beta_4 - 1 & \beta_1 + \beta_5 - 1 \\ 0 & 0 & \beta_3 - 1 & -\beta_3 & \boxed{0} \end{pmatrix} \qquad (2.9)
\end{aligned}
$$

where $\alpha_{1,4} = \alpha_{3,5} = 0.$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is formulated as

$$\Phi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) = A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{\beta_1+\beta_5-1} \times \varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_2}{r_1})\ ,$$

$$\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \sum_{n_1,n_2} c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2)x_1^{n_1}x_2^{n_2}\ , \tag{2.10}$$

- Where

$$A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_1+\beta_5)\Gamma(\beta_3+\beta_4)}\ ,$$

$$c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2) = \frac{(-)^{n_1+n_2}(\beta_3)_{n_1}(\beta_1)_{n_2}}{n_1!n_2!(\beta_1+\beta_5)_{-n_1+n_2}(\beta_3+\beta_4)_{n_1-n_2}}\ . \tag{2.11}$$

Note that $1/(a)_{-n} = (-1)^n(1-a)_n$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The geometric representation of the hypergeometric function is determined by the exponent matrix presented in Eq.(2.9), the determinants of the submatrices $\det(||\alpha||_{\{1,2,5\}}) = \det(||\alpha||_{\{2,3,4\}}) = 0$.
- The geometric representation of the function $\Phi^{(2)}_{\{1,2,3\}}$ is drown in Fig.1(b) where $a = 2, \ \{b, c\} = \{1, 5\}$ and $\{d, e\} = \{3, 4\}$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Taking the affine spanning $\mathcal{B} = \{2, 4, 5\}$, one finds $\det(\xi_{\{2,4,5\}}) = r_1 - r_3$ which differs from $\det(\xi_{\{1,2,3\}}) = 1$, where $\det(\xi_{\{2,4,5\}})$ denotes the determinant of the 3×3 minor of the matrix in Eq.(1.14) composed of the 2nd, 4th and 5th columns.
- In addition,

$$\xi^{-1}_{\{2,4,5\}} \cdot \xi = \begin{pmatrix} -\frac{r_3 - r_2}{r_3 - r_1} & 1 & -\frac{r_2 - r_1}{r_3 - r_1} & 0 & 0 \\ \frac{r_3}{r_3 - r_1} & 0 & -\frac{r_1}{r_3 - r_1} & 1 & 0 \\ -\frac{1}{r_3 - r_1} & 0 & \frac{1}{r_3 - r_1} & 0 & 1 \end{pmatrix}. \quad (2.12)$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$(0_{3\times\mathcal{B}} \,|\, (n_1 E_3^{(1)} + n_2 E_3^{(3)})_{\mathcal{N}\setminus\mathcal{B}}) = \begin{pmatrix} \boxed{n_2} & 0 & -n_2 & 0 & 0 \\ n_1 - n_2 & 0 & -n_1 + n_2 & 0 & 0 \\ -n_1 & 0 & \boxed{n_1} & 0 & 0 \end{pmatrix}, \tag{2.13}$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}|| = \begin{pmatrix} \boxed{0} & \beta_2 - 1 & -\beta_2 & 0 & 0 \\ \beta_1 + \beta_5 - 1 & 0 & \beta_2 + \beta_3 - 1 & \beta_4 - 1 & 0 \\ -\beta_5 & 0 & \boxed{0} & 0 & \beta_5 - 1 \end{pmatrix}, \tag{2.14}$$

where $\alpha_{1,1} = \alpha_{3,3} = 0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is

$$\begin{aligned}
\Phi^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) &= A^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta})\frac{(r_3-r_1)^{2-\beta_1-\beta_3}(r_3)^{\beta_1+\beta_5-1}(r_1-r_2)^{-\beta_2}}{\det(\boldsymbol{\xi}_{\{2,4,5\}})} \\
&\quad\times(-r_1)^{\beta_2+\beta_3-1}\varphi^{(2)}_{\{2,4,5\}}\left(\boldsymbol{\beta},\ \frac{r_3}{r_1},\ \frac{r_1(r_3-r_2)}{r_3(r_1-r_2)}\right) \\
&= A^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta})(r_3-r_1)^{1-\beta_1-\beta_3}(r_3)^{\beta_1+\beta_5-1}(r_1-r_2)^{-\beta_2} \\
&\quad\times(-r_1)^{\beta_2+\beta_3-1}\varphi^{(2)}_{\{2,4,5\}}\left(\boldsymbol{\beta},\ \frac{r_3}{r_1},\ \frac{r_1(r_3-r_2)}{r_3(r_1-r_2)}\right), \\
\varphi^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) &= \varphi^{(2)}_{\{1,2,3\}}(\widehat{(124)}\widehat{(35)}\boldsymbol{\beta},\ x_1,\ x_2)\ . \qquad (2.15)
\end{aligned}$$

- The geometric representation of the hypergeometric function is determined by the exponent matrix presented in Eq.(2.14), the determinants of the submatrices $\det(||\boldsymbol{\alpha}||_{\{1,4,5\}}) = \det(||\boldsymbol{\alpha}||_{\{2,3,4\}}) = 0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The geometric representation of the function $\Phi^{(2)}_{\{2,4,5\}}$ is drown in Fig.1(b) where $a=4,\ \{b,c\}=\{1,5\}$ and $\{d,e\}=\{2,3\}$.
- In order to obtain the analytical expressions in the whole domain of definition, we present the fundamental solution systems under all possible affine spanning $\mathcal{B}$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Except the constant factors and power functions, the corresponding hypergeometric functions can be written as

$$\begin{aligned}
&\varphi^{(i)}_{\{1,2,4\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(34)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{1,2,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(345)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{1,3,4\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(234)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{1,3,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(2354)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{2,3,4\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(1234)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{2,3,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(12354)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{1,4,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(24)}\widehat{(35)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{2,4,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(124)}\widehat{(35)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,\\
&\varphi^{(i)}_{\{3,4,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(i)}_{\{1,2,3\}}(\widehat{(13524)}\boldsymbol{\beta},\ x_1,\ x_2)\ ,
\end{aligned} \tag{2.16}$$

where the $\widehat{(34)}$, $\widehat{(345)}$, $\cdots$ are the elements of the permutation group S_5 acting on the components of exponent vector $\beta \in C^5$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- In these hypergeometric functions, $\varphi_{\mathcal{B}}^{(i)}$, $i = 1, 3, 5, 8, 10, 12$ are the first Appell functions, while $\varphi_{\mathcal{B}}^{(j)}$, $j = 2, 4, 6, 7, 9, 11$ are the Horn functions.
- It is easy to find that the convergent regions of $\varphi_{\{1,2,3\}}^{(1)}$, $\varphi_{\{1,2,3\}}^{(2)}$, and $\varphi_{\{1,2,3\}}^{(3)}$ have nonempty intersections in a connected component of definition domain, thus they constitute a fundamental solution system in the proper nonempty subset of the parameter space.
- The linear combinations of hypergeometric functions on the different nonempty proper subsets of the parameter space are regarded as analytic continuations of each other.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

$$\begin{aligned}
\Psi(\boldsymbol{\beta},\ \boldsymbol{\xi}) =& \sum_{i=\{1,2,3\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{1,5,6\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{3,7,8\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{4,5,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{8,9,10\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{10,11,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi})\ . \qquad (2.17)
\end{aligned}$$

Using the Gauss inverse relations below, we can derive the combinatorial coefficients uniquely, then continue the analytic expressions to the whole domain of definition of the Feynman integral by the Gauss-Kummer relations.

III. Generalized Gauss inverse relations

- The Gauss inverse relations include the following analytic continuation together with its various variants

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a} \, {}_2F_1\left(\begin{matrix} a, 1+a-c \\ 1+a-b \end{matrix} \middle| \frac{1}{x} \right) \\
&+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b} \, {}_2F_1\left(\begin{matrix} b, 1+b-c \\ 1-a+b \end{matrix} \middle| \frac{1}{x} \right) \; . \qquad (3.1)
\end{aligned}
$$

- Note that this transformation satisfies the idempotent property. Performing the inverse transformation on the terms of the right side, one finds that the sum of the results after transformation is exactly the term on the left side.

III. Generalized Gauss inverse relations

The Gauss inverse relations, i.e. the analytic continuation formulas from one connected component to another in the domain of definition, are obtained through the Mellin-Barnes's contour on the corresponding complex plane.

The Mellin-Barnes representation of the hypergeometric function $\varphi^{(1)}_{\{1,2,3\}}$ is

$$\frac{\Gamma(\beta_2)\Gamma(\beta_1)\Gamma(1-\beta_4)}{\Gamma(2-\beta_3-\beta_4)}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2)$$
$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta_2+s_1)\Gamma(\beta_1+s_2)\Gamma(1-\beta_4+s_1+s_2)}{\Gamma(2-\beta_3-\beta_4+s_1+s_2)}$$
$$\times\Gamma(-s_1)\Gamma(-s_2)(-x_1)^{s_1}(-x_2)^{s_2}ds_1\bigwedge ds_2\ . \tag{3.2}$$

III. Generalized Gauss inverse relations

Performing the transformation $\beta_1 + s_2 = -s'_2$ on the complex plane s_2, we rewrite the Barnes's contour integral in the right-handed of above equation as

$$\frac{(-x_2)^{-\beta_1}}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta_2+s_1)\Gamma(\beta_1+s'_2)\Gamma(1-\beta_1-\beta_4+s_1-s'_2)}{\Gamma(\beta_2+\beta_5+s_1-s'_2)}$$
$$\times\Gamma(-s_1)\Gamma(-s'_2)(-x_1)^{s_1}(-x_2)^{-s'_2}ds_1\bigwedge ds'_2$$
$$=\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(1-\beta_1-\beta_4)}{\Gamma(\beta_2+\beta_5)}(-x_2)^{-\beta_1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2})\ . \tag{3.3}$$

III. Generalized Gauss inverse relations

Under the affine transformation $1-\beta_4+s_1+s_2=-s_2'$ on the complex plane s_2, the Barnes's contour integral in the right-handed of Eq.(3.2) is formulated as

$$\frac{(-x_2)^{\beta_4-1}}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta_2+s_1)\Gamma(\beta_1+\beta_4-1-s_1-s_2')\Gamma(1-\beta_4+s_1+s_2')}{\Gamma(1-\beta_3-s_2')}$$
$$\times\Gamma(-s_1)\Gamma(-s_2')(-x_1)^{s_1}(-x_2)^{-s_1-s_2'}ds_1\bigwedge ds_2'$$
$$=\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(\beta_2)\Gamma(1-\beta_4)}{\Gamma(1-\beta_3)}(-x_2)^{\beta_4-1}\varphi_{\{1,2,3\}}^{(12)}(\boldsymbol{\beta},\ \frac{1}{x_2},\ \frac{x_1}{x_2})\ . \tag{3.4}$$

III. Generalized Gauss inverse relations

Then the residue theorem implies the following equation: Gauss inverse relations

$$\begin{aligned} &\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\ &= \frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_4)}(-x_2)^{-\beta_1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2}) \\ &\quad+\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_4-1}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_2},\ \frac{x_1}{x_2})\ . \end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned} &\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\ &= \frac{\Gamma(1-\beta_2-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_2}\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ x_2) \\ &\quad+\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_3)}(-x_1)^{\beta_4-1}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ \frac{x_2}{x_1})\ . \end{aligned} \tag{3.6}$$

III. Generalized Gauss inverse relations

We derive the generalized Gauss inverse relations for other hypergeometric functions, for example

$$\begin{aligned}
&\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\, x_1,\, x_2) \\
&= \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(\beta_1+\beta_4)}{\Gamma(1-\beta_2-\beta_5)\Gamma(\beta_4)}(-x_2)^{-\beta_1}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\, x_1,\, \frac{1}{x_2}) \\
&\quad + \frac{\Gamma(\beta_1+\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_2+\beta_5-1}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_2},\, x_1x_2) \\
&= \frac{\Gamma(\beta_2+\beta_5)\Gamma(\beta_3+\beta_5-1)}{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)}(-x_1)^{-\beta_2}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_1},\, x_2) \\
&\quad + \frac{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_3-\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_3)}(-x_1)^{\beta_1+\beta_4-1}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_1},\, x_1x_2)\,. \qquad (3.7)
\end{aligned}$$

III. Generalized Gauss inverse relations

$$
\begin{aligned}
&\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta},\, x_1,\, x_2)\\
&= \frac{\Gamma(\beta_3+\beta_5-1)\Gamma(\beta_1+\beta_5)}{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}(-x_2)^{-\beta_1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\, x_1,\, \frac{1}{x_2})\\
&\quad+\frac{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_3-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_2+\beta_4-1}\varphi^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_2},\, x_1x_2)\\
&= \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(\beta_2+\beta_4)}{\Gamma(1-\beta_1-\beta_5)\Gamma(\beta_4)}(-x_1)^{-\beta_2}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_1},\, x_2)\\
&\quad+\frac{\Gamma(\beta_2+\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_3)}(-x_1)^{\beta_1+\beta_5-1}\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\, \frac{1}{x_1},\, x_1x_2)\,. \qquad (3.8)
\end{aligned}
$$

III. Generalized Gauss inverse relations

$$
\begin{aligned}
&\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\
&= \frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_2-\beta_4)}{\Gamma(\beta_3+\beta_5)\Gamma(1-\beta_4)}(-x_2)^{-\beta_1}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2}) \\
&\quad + \frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_2-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_2)}(-x_2)^{\beta_4-1}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{x_1}{x_2},\ \frac{1}{x_2}) \\
&= \frac{\Gamma(1-\beta_3-\beta_4)\Gamma(2-\beta_2-\beta_4)}{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_3}\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ x_2) \\
&\quad + \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(2-\beta_2-\beta_4)}{\Gamma(\beta_3)\Gamma(1-\beta_2)}(-x_1)^{\beta_4-1}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ \frac{x_2}{x_1})\ . \qquad (3.9)
\end{aligned}
$$

III. Generalized Gauss inverse relations

$$
\begin{aligned}
&\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\
&= \frac{\Gamma(2-\beta_1-\beta_4)\Gamma(1-\beta_2-\beta_4)}{\Gamma(\beta_3+\beta_5)\Gamma(1-\beta_4)}(-x_2)^{-\beta_2}\varphi^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2}) \\
&\quad + \frac{\Gamma(\beta_2+\beta_4-1)\Gamma(2-\beta_1-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_1)}(-x_2)^{\beta_4-1}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{x_1}{x_2},\ \frac{1}{x_2}) \\
&= \frac{\Gamma(2-\beta_1-\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_3}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ x_2) \\
&\quad + \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(2-\beta_1-\beta_4)}{\Gamma(\beta_3)\Gamma(1-\beta_1)}(-x_1)^{\beta_4-1}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{x_2}{x_1},\ \frac{1}{x_1})\ .
\end{aligned}
\tag{3.10}
$$

III. Generalized Gauss inverse relations

- Setting certain variable zero and using $1/(a)_{-n} = (-1)^n (1-a)_n$ and $\sum \beta_i = 2$, those relations recover the equation presented in Eq.(3.1). For example, setting $x_1 = 0,\ x_2 = x$ in Eq.(3.5), we have

$$\begin{aligned}
\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ 0,\ x) &= {}_2F_1\left(\begin{matrix} \beta_1, 1-\beta_4 \\ 2-\beta_3-\beta_4 \end{matrix} \middle| x \right) , \\
\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ 0,\ \frac{1}{x}) &= {}_2F_1\left(\begin{matrix} \beta_1, 1-\beta_5 \\ \beta_1+\beta_4 \end{matrix} \middle| \frac{1}{x} \right) , \\
\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ 0,\ \frac{1}{x}) &= {}_2F_1\left(\begin{matrix} \beta_3, 1-\beta_4 \\ 2-\beta_1-\beta_4 \end{matrix} \middle| \frac{1}{x} \right) .
\end{aligned} \tag{3.11}$$

Thus the analytic continuation in Eq.(3.5) recovers the equation presented in Eq.(3.1).

III. Generalized Gauss inverse relations

- Idempotent properties of the Generalized Gauss inverse relations. Substituting the following analytic continuations into the right-handed side of Eq.(3.5)

$$
\begin{aligned}
&\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\
&= \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(\beta_1+\beta_4)}{\Gamma(1-\beta_2-\beta_5)\Gamma(\beta_4)}(-x_2)^{-\beta_1}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ \frac{1}{x_2}) \\
&\quad + \frac{\Gamma(\beta_1+\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_2+\beta_5-1}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_2},\ x_1x_2)\ , \\
&\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) \\
&= \frac{\Gamma(2-\beta_1-\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_3}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_1},\ x_2) \\
&\quad + \frac{\Gamma(\beta_3+\beta_4-1)\Gamma(2-\beta_1-\beta_4)}{\Gamma(\beta_3)\Gamma(1-\beta_1)}(-x_1)^{\beta_4-1}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{x_2}{x_1},\ \frac{1}{x_1})\ ,
\end{aligned}
\tag{3.12}
$$

III. Generalized Gauss inverse relations

- we have

$$\begin{aligned}
&\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2)\\
&=\Big\{\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_1+\beta_4)\Gamma(\beta_3+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_2-\beta_5)\Gamma(\beta_4)\Gamma(1-\beta_4)}\\
&\quad+\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_1-\beta_4)\Gamma(\beta_3+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_1)\Gamma(\beta_3)\Gamma(1-\beta_3)}\Big\}\\
&\quad\times\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2)\\
&\quad+\Big\{\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_1+\beta_4)\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_3)\Gamma(1-\beta_3)}\\
&\quad+\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_1-\beta_4)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_3+\beta_4)}{\Gamma(\beta_3)\Gamma(1-\beta_3)}\Big\}\\
&\quad\times(-x_2)^{\beta_3+\beta_4-1}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_2,\ \frac{x_1}{x_2})\\
&=\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2)\ . \qquad (3.13)
\end{aligned}$$

Because $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ and $\sum\beta_i=2$.

III. Generalized Gauss inverse relations

- Note that

$$\Big\{\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(2-\beta_2-\beta_4)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_3+\beta_4)}{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_1-\beta_5)\Gamma(\beta_4)\Gamma(1-\beta_4)} + \frac{\Gamma(\beta_2+\beta_4)\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_3-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_2)\Gamma(\beta_3)\Gamma(1-\beta_3)}\Big\} = 1$$

- The combinations of Gauss inverse relations and Gauss-Kummer relations below generalize the analytic continuations of the Gauss functions presented in (1.8.1.11)-(1.8.1.19) of *Generalized Hypergeometric Functions* by L. J. Slater, (Cambridge University Press 1966). Those generalizations can be used to continue the analytic expressions of Feynman integral to its whole domain of definition.

III. Generalized Gauss inverse relations

- The images of the generalized hypergeometric functions under the map of inverse transformation of certain variable are the linear combinations of the generalized hypergeometric solutions of the GKZ-system in the same affine spanning.
- The method presented here generalizes the approach adopted in the work *A new development of the theory of hypergeometric functions* by E. W. Barnes, published in Proc. London. Math. Soc. **6**(1907)141-177, and can be used to derive the analytic continuations of any generalized hypergeometric functions. For example, we can derive the analytic continuations of the Pochhammer functions ${}_{p+1}F_p$, and verify those continuations satisfying the idempotent property accordingly.

III. Generalized Gauss inverse relations

- Note that the results presented here coincide with the analytic continuations in *Integration of the Partial Differential Equations for the Hypergeometric Functions* F_1 *and* F_D *of Two and More Variables* by P. O. M. Olsson, published in J. Math. Phys. **5**(1964)420-430, which are derived through the transformation theory of low variable hypergeometric functions to find the transformation formulas of higher variable hypergeometric functions.
- The application of the above method presented there has limitations: the transformation formulas of higher variable hypergeometric functions depend on the transformation theory of low variable hypergeometric functions.

IV. Generalized Gauss adjacent relations

- The independent Gauss adjacent relations are the following two equations

$$c\ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}\bigg|\, x\right) = a\, x\ {}_2F_1\left(\begin{array}{c} a+1, b \\ c+1 \end{array}\bigg|\, x\right) + c\ {}_2F_1\left(\begin{array}{c} a, b-1 \\ c \end{array}\bigg|\, x\right),$$

$$(a-c+1)\ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}\bigg|\, x\right) = a\ {}_2F_1\left(\begin{array}{c} a+1, b \\ c \end{array}\bigg|\, x\right) - (c-1)\ {}_2F_1\left(\begin{array}{c} a, b \\ c-1 \end{array}\bigg|\, x\right), \quad (4.1)$$

  together with two equations obtained by the interchanging $a \leftrightarrow b$ in the above equations.

- The adjacent relations of the hypergeometric functions in the (n-1)-dimensional projective space are determined from the quotient module of the free module C^n by the submodule generated by the coefficient vectors of the corresponding GKZ-system.

IV. Generalized Gauss adjacent relations

- For the GKZ-system on the Grassmannian, the adjacent relations of the hypergeometric functions are determined by $G_{k,n}$ and its dual $G^{\perp}_{k,n}$
- If the exponent vector $\beta \in C^n$ satisfies $\sum_{i=1}^{n} \beta_i = n - k$, the adjacent relations are written respectively as followings.
- $\gamma = \beta + e_i \in C^n,\ i \in \{1, \cdots, n\}$, and $\sum_{i=1}^{n} a_i e_i \in G_{k,n}$, then

$$\sum_{j=1}^{n} a_j (\gamma_j - 1) \Phi_{\mathcal{B}}(\boldsymbol{\gamma} - e_j,\ \boldsymbol{\xi}) = 0\ . \tag{4.2}$$

Where $\mathcal{B} \subset \{1, \cdots, n\}$ is the proper subset of the columns which spans the k-dimensional vector subspace in the C^n, and $e_i,\ i \in \{1, \cdots, n\}$ denotes the i-th standard vector in the standard basis of C^n.

IV. Generalized Gauss adjacent relations

- $\gamma = \beta - e_i \in C^n,\ i \in \{1, \cdots, n\}$, and $\sum_{i=1}^{n} a_i e_i \in G_{k,n}^{\perp}$, then

$$\sum_{j=1}^{n} a_j \Phi_{\mathcal{B}}(\gamma + e_j,\ \xi) = 0\ . \tag{4.3}$$

- There are totally n independent adjacent relations from above two equations.
- Since the Gauss functions can be regarded as functions on the Grassmannian $G_{2,4}$, there are four independent adjacent relations presented in Eq.(4.1) for the Gauss functions.
- For the Grassmannian $G_{3,5}$ presented in Eq.(1.14), the exponent vector $\beta = (\beta_1, \cdots, \beta_5) \in C^5$ satisfying $\sum_{i=1}^{5} \beta_i = 2$, and $\mathcal{B} = \{1,\ 2,\ 3\}$.

IV. Generalized Gauss adjacent relations

- The dual variety of the Grassmannian ξ in Eq.(1.14) is given by the matroid

$$\boldsymbol{\xi}_\perp = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ -r_1 & -r_2 & -r_3 & 0 & 1 \end{pmatrix} . \tag{4.4}$$

- Corresponding to $\boldsymbol{\beta} + e_1 = (1+\beta_1,\ \beta_2,\ \cdots,\beta_5)$, we obtain three independent adjacent relations among $\Phi^{(i)}_{\{1,2,3\}},\ i \in \{1,\cdots,12\}$ from Eq.(4.2).

$$\begin{aligned}
&\beta_1 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1)r_1 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_5,\ \boldsymbol{\xi}) \equiv 0\ , \\
&\beta_2 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_2 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1)r_2 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_2 - e_5,\ \boldsymbol{\xi}) \equiv 0\ , \\
&\beta_3 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_3 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1)r_3 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_3 - e_5,\ \boldsymbol{\xi}) \equiv 0\ .
\end{aligned} \tag{4.5}$$

IV. Generalized Gauss adjacent relations

- Corresponding to $\boldsymbol{\beta} - e_1 = (\beta_1 - 1, \beta_2, \cdots, \beta_5)$, we obtain two independent relations among $\Phi^{(i)}_{\{1,2,3\}}$ with contiguous parameters from Eq.(4.3) as

$$\begin{aligned}
&\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta}, \boldsymbol{\xi}) + \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_2, \boldsymbol{\xi}) \\
&+\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_3, \boldsymbol{\xi}) - \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_4, \boldsymbol{\xi}) \equiv 0 , \\
&r_1 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta}, \boldsymbol{\xi}) + r_2 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_2, \boldsymbol{\xi}) \\
&+r_3 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_3, \boldsymbol{\xi}) - \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} - e_1 + e_5, \boldsymbol{\xi}) \equiv 0 .
\end{aligned} \tag{4.6}$$

- The independent adjacent relations of $\Phi^{(1)}_{\{1,2,3\}}$ are concretely written as:

$$\begin{aligned}
&(2 - \beta_3 - \beta_4)\left[\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) - \varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_5)\right] \\
&+(1 - \beta_4)x_2 \varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_4) \equiv 0 ,
\end{aligned} \tag{4.7}$$

IV. Generalized Gauss adjacent relations

$$
\begin{aligned}
&(2-\beta_3-\beta_4)\Big[\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})-\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}+e_2-e_5)\Big]\\
&+(1-\beta_4)x_1\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}+e_2-e_4)\equiv 0\;,\\
&\beta_3\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})+(\beta_4-1)\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}+e_3-e_4)\\
&+(1-\beta_3-\beta_4)\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}+e_3-e_5)\equiv 0\;,\\
&(\beta_1-1)x_2\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})+\beta_2x_1\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_2)\\
&+(\beta_3+\beta_4-1)\Big[\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_3)-\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_4)\Big]\equiv 0\;,\\
&(\beta_1-1)\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})+\beta_2\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_2)\\
&+(\beta_3+\beta_4-1)\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_3)+\beta_5\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}-e_1+e_5)\equiv 0\;,
\end{aligned}
\tag{4.8}
$$

here the variables $x_{1,2}$ in the hypergeometric function $\varphi^{(1)}_{\{1,2,3\}}$ is omitted for concise.

IV. Generalized Gauss adjacent relations

- The generalized hypergeometric function $\varphi^{(1)}_{\{1,2,3\}}$ can be written as

$$\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) = F_1\left(\begin{array}{c} 1-\beta_4,\ \beta_2,\ \beta_1 \\ 2-\beta_3-\beta_4 \end{array}\middle|\ x_1,\ x_2\right)\ , \tag{4.9}$$

where F_1 is the first type Appell function.

- The first adjacent relation is reduced as:

$$\begin{aligned} &F_1\left(\begin{array}{c} 1-\beta_4,\ \beta_2,\ \beta_1 \\ 2-\beta_3-\beta_4 \end{array}\right) + \frac{(1-\beta_4)x_2}{2-\beta_3-\beta_4} F_1\left(\begin{array}{c} 1-\beta_4,\ \beta_2,\ 1+\beta_1 \\ 3-\beta_3-\beta_4 \end{array}\right) \\ &-F_1\left(\begin{array}{c} 1-\beta_4,\ \beta_2,\ 1+\beta_1 \\ 2-\beta_3-\beta_4 \end{array}\right) \equiv 0\ , \end{aligned} \tag{4.10}$$

we also suppress the variables $x_{1,2}$ in the above equation for concise.

IV. Generalized Gauss adjacent relations

- As $x_2 = 0$, this relation is simplified as the trivial equation

$$-\,{}_2F_1\left(\begin{matrix}1-\beta_4,\ \beta_2\\ 2-\beta_3-\beta_4\end{matrix}\,\middle|\,x_1\right) + {}_2F_1\left(\begin{matrix}1-\beta_4,\ \beta_2\\ 2-\beta_3-\beta_4\end{matrix}\,\middle|\,x_1\right) = 0\ . \qquad (4.11)$$

- As $x_1 = 0$, the adjacent relation in Eq.(4.10) is simplified as

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}1-\beta_4,\ \beta_1\\ 2-\beta_3-\beta_4\end{matrix}\,\middle|\,x_2\right) + \frac{(1-\beta_4)x_2}{2-\beta_3-\beta_4}\,{}_2F_1\left(\begin{matrix}1-\beta_4,\ 1+\beta_1\\ 3-\beta_3-\beta_4\end{matrix}\,\middle|\,x_2\right)\\ &-\,{}_2F_1\left(\begin{matrix}1-\beta_4,\ 1+\beta_1\\ 2-\beta_3-\beta_4\end{matrix}\,\middle|\,x_2\right) \equiv 0\ , \qquad (4.12)\end{aligned}$$

which coincides with the first equation in Eq.(4.1).

IV. Generalized Gauss adjacent relations

- Similarly we can verify that other adjacent relations presented in Eq.(4.7) recover the well-known adjacent relations of the Gauss functions as $x_1 = 0$ or $x_2 = 0$, respectively.
- The adjacent relations of other affine spanning $\mathcal{B}$ are obtained from the adjacent relations in $\varphi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta})$ through some permutation on components of the exponent vector $\boldsymbol{\beta} \in C^5$.
- The adjacent relations are adopted to formulae the coefficients of powers of $\varepsilon = 2 - D/2$ in the Laurent series of Feynman integrals at dimension of time-space $D = 4$ as linear combinations of generalized hypergeometric functions with integer parameters.

V. Generalized Gauss-Kummer relations

- The third type Gauss relations are derived through Kummer's classification, which can be written as

$$
\begin{aligned}
{}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, x\right) &= (1-x)^{c-a-b}\,{}_2F_1\left(\begin{array}{c} c-a,\ c-b \\ c \end{array}\middle|\, x\right) \\
&= (1-x)^{-a}\,{}_2F_1\left(\begin{array}{c} a,\ c-b \\ c \end{array}\middle|\, \frac{x}{x-1}\right) \\
&= (1-x)^{-b}\,{}_2F_1\left(\begin{array}{c} c-a,\ b \\ c \end{array}\middle|\, \frac{x}{x-1}\right)
\end{aligned}
\tag{5.1}
$$

  and its various variants.
- For the GKZ-system on the Grassmannian, the generalized hypergeometric solutions corresponding to the same geometric representation are proportional to each other in the intersection of their convergent regions.

V. Generalized Gauss-Kummer relations

In order to obtain the generalized Gauss-Kummer relations properly, we assume

- The variables are nonnegative real numbers
- The bases of all power factors are nonnegative real numbers.

Corresponding to the geometric representation shown in Fig.1(a) with $\{a,b\}=\{3,5\}$, $\{c,d,e\}=\{1,2,4\}$, we derive the following six solutions of the GKZ-system presented in Eq.(2.1) which are proportional to each other in the intersection of their convergent regions,

$$\Phi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) \sim \Phi^{(1)}_{\{1,2,5\}}(\boldsymbol{\beta}) \sim \Phi^{(5)}_{\{1,3,4\}}(\boldsymbol{\beta}) \sim \Phi^{(10)}_{\{1,4,5\}}(\boldsymbol{\beta}) \sim \Phi^{(10)}_{\{2,4,5\}}(\boldsymbol{\beta}) \sim \Phi^{(5)}_{\{2,3,4\}}(\boldsymbol{\beta}) \,. \quad (5.2)$$

V. Generalized Gauss-Kummer relations

Dividing each function by a common power factor and requiring equality to each other on the concrete principal value plane, we obtain the generalized Gauss-Kummer relations as

$$\begin{aligned}
&\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ y)\\
&= (1-y)^{-\beta_1}(1-x)^{-\beta_2}\varphi^{(1)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{y}{y-1})\\
&= (1-x)^{\beta_5-1}\varphi^{(5)}_{\{1,3,4\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{x-y}{x-1})\\
&= (1-y)^{\beta_5-1}\varphi^{(5)}_{\{2,3,4\}}(\boldsymbol{\beta},\ \frac{y}{y-1},\ \frac{y-x}{y-1})\\
&= (1-x)^{1-\beta_2-\beta_3}(1-y)^{-\beta_1}\varphi^{(10)}_{\{1,4,5\}}(\boldsymbol{\beta},\ x,\ \frac{x-y}{1-y})\\
&= (1-y)^{1-\beta_1-\beta_3}(1-x)^{-\beta_2}\varphi^{(10)}_{\{2,4,5\}}(\boldsymbol{\beta},\ y,\ \frac{y-x}{1-x})\ ,
\end{aligned} \tag{5.3}$$

with $x = r_2/r_3,\ y = r_1/r_3$.

V. Generalized Gauss-Kummer relations

- The relations coincide with those equations presented in Eq.(1),(2),(3),(4),(5) in the section 9.4 of *Generalized Hypergeometric Series* by W. N. Bailey, published by Stechert-Hafner Service Agency 1964. Where those relations are derived through the Euler integral expression of the first type Appell function.
- We provide an explanation on those relations in point of view of combinatorial geometry.
- There are totally 60 solutions $\Phi_{\mathcal{B}}^{(i)},\ i = 1, 3, 5, 8, 10, 12$ which are formulated as the first type Appell functions with different parameters. Here $\mathcal{B}$ denotes a possible affine spanning of the vector subspace C^3.

V. Generalized Gauss-Kummer relations

- In another group of the first type Appell functions which correspond to same geometric representation, the obtained Gauss-Kummer relations are various variants of the equations presented in Eq.(5.3).

- Corresponding to the geometric representation shown in Fig.1(b) with $a = 2$, $\{b, c\} = \{1, 4\}$, and $\{d, e\} = \{3, 5\}$, we have the following four solutions of the GKZ-system presented in Eq.(2.1) which are proportional to each other in the intersection of their convergent regions,

$$\Phi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}) \sim \Phi^{(2)}_{\{1,2,5\}}(\boldsymbol{\beta}) \sim \Phi^{(6)}_{\{2,3,4\}}(\boldsymbol{\beta}) \sim \Phi^{(9)}_{\{2,4,5\}}(\boldsymbol{\beta}) \,. \tag{5.4}$$

V. Generalized Gauss-Kummer relations

- If the products of power factors and the hypergeometric functions in these solutions are equal to each other on the principal value plane, we derive the generalized Gauss-Kummer relations as

$$
\begin{aligned}
& x^{\beta_3+\beta_5-1}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ \frac{y}{x}) \\
&= x^{\beta_3+\beta_5-1}(1-x)^{\beta_1+\beta_4-1}(1-y)^{-\beta_1}\varphi^{(2)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{y(x-1)}{x(y-1)}) \\
&= (x-y)^{\beta_3+\beta_5-1}(1-y)^{-\beta_3}\varphi^{(6)}_{\{2,3,4\}}(\boldsymbol{\beta},\ \frac{y}{y-x},\ \frac{x-y}{1-y}) \\
&= (1-x)^{\beta_1+\beta_4-1}(x-y)^{\beta_3+\beta_5-1}(1-y)^{1-\beta_1-\beta_3}\varphi^{(9)}_{\{2,4,5\}}(\boldsymbol{\beta},\ \frac{y-x}{1-x},\ \frac{y(1-x)}{y-x})\ .
\end{aligned}
\tag{5.5}
$$

- There is no the Euler integral representation for the Horn series here so far, the above Gauss-Kummer relations cannot be obtained through the method to derive that of the first type Appell functions. The relations presented here recover the well-known relations of the Gauss functions in some special case.

V. Generalized Gauss-Kummer relations

- As $y = 0$, the above equations are reduced as

$$\begin{aligned}
&\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ 0) \\
&= (1-x)^{\beta_1+\beta_4-1}\varphi^{(2)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ 0) \\
&= \varphi^{(6)}_{\{2,3,4\}}(\boldsymbol{\beta},\ 0,\ x) \\
&= (1-x)^{\beta_1+\beta_4-1}\varphi^{(9)}_{\{2,4,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ 0)\ .
\end{aligned} \tag{5.6}$$

- Using $1/(a)_{-n} = (-1)^n(1-a)_n$,

$$\begin{aligned}
&\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ 0) = {}_2F_1\left(\begin{array}{c}\beta_3,\ 1-\beta_1-\beta_4 \\ \beta_3+\beta_5\end{array}\middle|\ x\right)\ , \\
&(1-x)^{\beta_1+\beta_4-1}\varphi^{(2)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ 0) \\
&= (1-x)^{\beta_1+\beta_4-1}\ {}_2F_1\left(\begin{array}{c}\beta_5,\ 1-\beta_1-\beta_4 \\ \beta_3+\beta_5\end{array}\middle|\ \frac{x}{x-1}\right)\ , \\
&\varphi^{(6)}_{\{2,3,4\}}(\boldsymbol{\beta},\ 0,\ x) = {}_2F_1\left(\begin{array}{c}\beta_3,\ 1-\beta_1-\beta_4 \\ \beta_3+\beta_5\end{array}\middle|\ x\right)\ , \\
&(1-x)^{\beta_1+\beta_4-1}\varphi^{(9)}_{\{2,4,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ 0) \\
&= (1-x)^{\beta_1+\beta_4-1}\ {}_2F_1\left(\begin{array}{c}\beta_5,\ 1-\beta_1-\beta_4 \\ \beta_3+\beta_5\end{array}\middle|\ \frac{x}{x-1}\right)\ ,
\end{aligned} \tag{5.7}$$

V. Generalized Gauss-Kummer relations

- The reduced relations in Eq.(5.6) coinciding with that presented in Eq.(5.1).
- There are totally 60 solutions $\Phi_{\mathcal{B}}^{(i)},\ i=2,4,6,7,9,11$ which are written as the Horn series with different parameters. In another group of the Horn series which correspond to same geometric representation, the obtained Gauss-Kummer relations are various variants of the equations presented in Eq.(5.5).
- The solutions of GKZ-system in projective spaces can obtain adjacent and inverse relations, but Gauss-Kummer relations cannot be obtained.

VI. The analytic expressions for 1-loop self energy

In this scheme, we obtain the analytic expressions of a Feynman integral in its whole domain of definition through the following steps.

- After embedding the Feynman integral on a variety (a special stratum) of the Grassmannian $G_{k,n}$ $(k < n)$, we construct all hypergeometric solutions for the general stratum of the Grassmannian $G_{k,n}$ under all possible affine spanning.
- We derive the inverse and adjacent relations among hypergeometric solutions under the same affine spanning, and the Gauss-Kummer relations among hypergeometric solutions from different affine spanning.

VI. The analytic expressions for 1-loop self energy

- Constraining those hypergeometric solutions for the general stratum on the special stratum through inverse and Kummer relations, we derive the canonical series solutions for our special stratum.
- The various Gauss relations for the canonical series solutions are induced by those of the general stratum.
- In the neighborhood of the regular singularities, we write the Feynman integral as a finite linear combinations of the canonical series solutions for our special stratum under same affine spanning.
- The combination coefficients are obtained by the reduced Gauss inverse relations among the canonical series solutions, then the analytic expressions of the Feynman integral are continued to its whole domain of definition.

VI. The analytic expressions for 1-loop self energy

- The adjacent relations of generalized hypergeometric functions can be used to demonstrate the equivalence of fundamental solution systems constructed based on different integral representations, and those relations can rewrite the coefficients of the Laurent expansion of Feynman integrals around $D = 4$ as generalized hypergeometric functions with integer parameters.

In the example of 1-loop self energy, its Feynman integral is embedded in the general stratum of $G_{3,5}$, i.e. the determinant of any 3×3 minor is non zero for the general vector $(r_1,\ r_2,\ r_3) \in C^3$.

VI. The analytic expressions for 1-loop self energy

- The exponent vector

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{(1S)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -1,\ \frac{D}{2} - 1) \in C^5\ . \tag{6.1}$$

  Where D is the time-space dimension in dimensional regularization.

- The boundary conditions:

$$\begin{aligned} iA_{1SE}(p^2, 0, 0) &= \frac{i\Gamma(2 - \frac{D}{2})\Gamma^2(\frac{D}{2} - 1)}{(4\pi)^{D/2}\Gamma(D-2)} \Big(\frac{-p^2}{\Lambda_{\rm RE}^2}\Big)^{\frac{D}{2}-1}\ , \\ iA_{1SE}(0, m^2, 0) = iA_{1SE}(0, 0, m^2) &= \frac{i\Gamma(2 - \frac{D}{2})\Gamma(\frac{D}{2} - 1)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})} \Big(\frac{m^2}{\Lambda_{\rm RE}^2}\Big)^{\frac{D}{2}-1}\ , \end{aligned} \tag{6.2}$$

  which are used to obtain the combinatorial coefficients. Here $\Lambda_{\rm RE}$ is the renormalization scale.

VI. The analytic expressions for 1-loop self energy

- $|p^2| < m_1^2 < m_2^2$

$$
\begin{aligned}
&A_{1SE}(p^2, m_1^2, m_2^2) \\
&= C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{-\beta_2}(p^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{p^2}{m_1^2}) \\
&+C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\
&+C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_2+\beta_5-1}(m_2^2)^{-\beta_2}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_1^2}{m_2^2})
\end{aligned} \tag{6.3}
$$

- $m_2^2 < |p^2| < m_1^2$

$$
\begin{aligned}
&A_{1SE}(p^2, m_1^2, m_2^2) \\
&= C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_5-1}\varphi^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_2^2}{m_1^2}) \\
&+C^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(p^2)^{\beta_1+\beta_5-1}\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{p^2}{m_1^2}) \\
&+C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_2^2}{m_1^2})
\end{aligned} \tag{6.4}
$$

VI. The analytic expressions for 1-loop self energy

- $m_2^2 < m_1^2 < |p^2|$

$$
\begin{aligned}
& A_{1SE}(p^2, m_1^2, m_2^2) \\
= & C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_2^2}{m_1^2}) \\
& +C^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_2^2}{m_1^2}) \\
& +C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{p^2})
\end{aligned} \tag{6.5}
$$

- $m_1^2 < m_2^2 < |p^2|$

$$
\begin{aligned}
& A_{1SE}(p^2, m_1^2, m_2^2) \\
= & C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{p^2}) \\
& +C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{m_2^2}) \\
& +C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2})
\end{aligned} \tag{6.6}
$$

VI. The analytic expressions for 1-loop self energy

- $m_1^2 < |p^2| < m_2^2$

$$
\begin{aligned}
& A_{1SE}(p^2, m_1^2, m_2^2) \\
= & C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{-\beta_2}(p^2)^{\beta_2+\beta_5-1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{p^2}) \\
& + C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\
& + C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2})
\end{aligned} \tag{6.7}
$$

- Using the boundary conditions in Eq.(6.2), we have

$$
\begin{aligned}
& C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) = C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}, \\
& C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{(-1)^{D/2-2}\Gamma^2(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(D-2)}.
\end{aligned} \tag{6.8}
$$

VI. The analytic expressions for 1-loop self energy

- Other coefficients are linear combinations of the above coefficients through the Gauss inverse relations.
- In order to derive other combinatorial coefficients, we apply the Gauss-inverse relations.
- Performing the inverse transformation of suitable variables in Eq.(6.3) and Eq.(6.4), for example, one gets

$$\begin{aligned}
C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) &= (-1)^{\beta_5-1}\frac{\Gamma(\beta_1+\beta_5-1)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&\quad +(-1)^{-\beta_2}\frac{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_2+\beta_5)}{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) &= (-1)^{-\beta_1}\frac{\Gamma(1-\beta_1-\beta_5)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_5)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&\quad +(-1)^{\beta_3+\beta_4-1}\frac{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_1-\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_1)}C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})\;,
\end{aligned} \tag{6.9}$$

VI. The analytic expressions for 1-loop self energy

- $$C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) = (-1)^{-\beta_1}\frac{\Gamma(\beta_1+\beta_5)\Gamma(\beta_2+\beta_5-1)}{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) +(-1)^{\beta_5-1}\frac{\Gamma(\beta_2+\beta_5-1)\Gamma(2-\beta_1-\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_1)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})\;,$$
  $$C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}) = (-1)^{\beta_3+\beta_4-1}\frac{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_2-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)}C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) +(-1)^{-\beta_2}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_2-\beta_5)}{\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \tag{6.10}$$
- thus
  $$C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}) = (-1)^{\beta_2}\frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_2+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) +(-1)^{\beta_2+\beta_5}\frac{\Gamma(2-\beta_2-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta_5)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})\;,$$
  $$C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) = (-1)^{\beta_1+\beta_5}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) +(-1)^{\beta_1}\frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})\;. \tag{6.11}$$

VI. The analytic expressions for 1-loop self energy

- In a similar way, the combinatorial coefficients of $\phi^{(i)}_{\{1,2,3\}}(\beta)$, $i=4,7,9,11$ are respectively written as

$$
\begin{aligned}
C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_2+\beta_5}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_2+\beta_5)\Gamma(\beta_3)\Gamma(1-\beta_2)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_2}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2+\beta_5)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_3)\Gamma(1-\beta_1)\Gamma(\beta_1+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_3}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_3+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)\Gamma(2-\beta_3-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_3}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2+\beta_5-1)\Gamma(\beta_3+\beta_5)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_3-\beta_5)\Gamma(\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) \quad (6.12)
\end{aligned}
$$

VI. The analytic expressions for 1-loop self energy

- The combinatorial coefficients of $\phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta}),\ i=1,8,12$ are respectively written as

$$\begin{aligned}
C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5-1}\frac{\Gamma(\beta_3+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_3)\Gamma(\beta_1+\beta_5-1)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{1-\beta_2-\beta_5}\frac{\Gamma(\beta_3+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_2)\Gamma(\beta_3)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1+\beta_2}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(2-\beta_3-\beta_4)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ , \\
C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5-1}\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_5-1)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1+\beta_3}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(2-\beta_2-\beta_4)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{1-\beta_3-\beta_5}\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ , \\
C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_2+\beta_3}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(2-\beta_1-\beta_4)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_2+\beta_5-1}\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5-1}\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ . \qquad (6.13)
\end{aligned}$$

VI. The analytic expressions for 1-loop self energy

- Taking the time-space dimension $D=4-2\varepsilon$ in dimensional regularization, one finds that the ultraviolet divergence in Eq(6.3)$\sim$Eq.(6.8) is $1/\varepsilon$.
- It is popularly believed that the physical thresholds obtained from Cutkosky cuts are proper subsets of the singular locus of the Feynman integral. The dominant contributions of the Feynman integral at the threshold are derived by the expansion of its analytic expressions in the limit of heavy masses and large momenta. Actually the singular locus of a Feynman integral depends on determinant of the resultant complex in an affine algebra, we will release our relevant calculations elsewhere.

VII. The 2-loop Massive Dune diagram

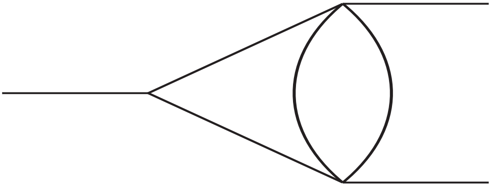

Figure: 1 The 2-loop Massive Dune diagram.

The Feynman integral of the 2-loop Dune is embedded in a special stratum of the Grassmannian $G_{5,8}$.

VII. The 2-loop Massive Dune diagram

The splitting coordinates are reduced to the matroid ξ_{DU} of size 5×8

$$\xi_{DU} = \left(\begin{array}{c|c} I_5 & Z_{DU} \end{array} \right) \tag{7.1}$$

with the exponent vector $\beta_{DU} = (0, 0, 0, 0, 0, -D, D-4, -1) \in \mathbf{C}^8$, and

$$Z_{DU} = \begin{pmatrix} 1 & p_1^2 & m_1^2 \\ 1 & p_1^2 & m_2^2 \\ 1 & p_2^2 & m_3^2 \\ 1 & p_3^2 & m_4^2 \\ \zeta & \Lambda^2 & \Lambda^2 \end{pmatrix} . \tag{7.2}$$

For a generic stratum of the Grassmannian $G_{5,8}$ there are 56 affine spanning. In each affine spanning there are 1905 linearly independent hypergeometric functions. In total there are 106680 hypergeometric functions in our analysis.

VII. The 2-loop Massive Dune diagram

The matroid ξ_{DU} represent a collection of eight points in the projective space P^4 which has nine geometric configurations. Those 106680 hypergeometric functions above are attributed to nine types of hypergeometric functions which are transformed into each other by the various Gauss relations. In addition, those functions depend on 8 variables, the Feynman integral depends on 6 variables! Those analytical continuations from the inverse transformations and Kummer relations corresponding to geometric representations can be used to calculate the restrictions of those hypergeometric functions on the special stratum (a hypergeometric series with 6 independent variable and a six-fold summation).

VII. The 2-loop Massive Dune diagram

The indices of columns $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ belong to the set $\{1, \cdots, 8\}$ and are all distinct from each another. The geometric configurations of the exponent matrices of hypergeometric functions are divided into the following types.

- The configuration I: any 2×2 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are zero, and any 5×5 minors of the 5×6 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$, or $\mathcal{C}$), $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are nonzero.

$$H_{\rm I}\left(\begin{array}{c} \boldsymbol{a} \\ c \end{array}\middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n}\in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}}(a_2)_{|n|_{34}}(a_3)_{|n|_{56}}(a_4)_{|n|_{78}}(a_5)_{|n|_{1357}}(a_6)_{|n|_{2468}}}{\Big[\prod\limits_{i=1}^{8} n_i!\Big]\,(c)_{|n|}}\,\boldsymbol{x}^{\boldsymbol{n}}\;, \qquad (7.3)$$

here the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^6$ in the nominator, and the parameter $c \in \boldsymbol{C}$ in the denominator.

VII. The 2-loop Massive Dune diagram

- The configuration II: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{A}$ and $\mathcal{B}$ are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{C}$, and $\mathcal{D}$ are zero, and the determinant of the 5×5 matrix composed by the columns with the indices $\mathcal{C}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is zero.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm II}\left(\begin{array}{c}\boldsymbol{a},b\\ \boldsymbol{c}\end{array}\Big|\,\boldsymbol{x}\right)=\sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8}\frac{(a_1)_{|n|_{34}}(a_2)_{|n|_{56}}(a_3)_{|n|_{78}}(a_4)_{|n|_{2357}}(b)_{-n_1+|n|_{2\cdots8}}}{\Big[\prod_{i=1}^{8}n_i!\Big](c_1)_{-n_1+|n|_{3\cdots8}}(c_2)_{-n_1+|n|_{2357}}}\times(-1)^{|n|_{1357}}\boldsymbol{x}^{\boldsymbol{n}}\;, \tag{7.4}$$

where the parameter vector $\boldsymbol{a}\in\boldsymbol{C}^4$, the parameter $b\in\boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c}\in\boldsymbol{C}^2$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration III: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{A}$ and $\mathcal{B}$ are zero, any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are zero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm III}\left(\begin{array}{c}\boldsymbol{a},b\\ \boldsymbol{c}\end{array}\Big|\boldsymbol{x}\right)=\sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8}\frac{(a_1)_{|n|_{23}}(a_2)_{|n|_{56}}(a_3)_{|n|_{78}}(a_4)_{|n|_{2457}}(b)_{-|n|_{13}+|n|_{4\cdots8}}}{\left[\prod\limits_{i=1}^{8}n_i!\right](c_1)_{-n_1+|n|_{2457}}(c_2)_{-|n|_{13}+|n|_{5678}}}\times(-1)^{|n|_{157}}\boldsymbol{x}^{\boldsymbol{n}}\;,\qquad(7.5)$$

where the parameter vector $\boldsymbol{a}\in\boldsymbol{C}^4$, the parameter $b\in\boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c}\in\boldsymbol{C}^2$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration IV: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{A}$ and $\mathcal{B}$ are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are zero, and the determinant of the 5×5 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is zero.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm IV}\left(\begin{array}{c} \boldsymbol{a}, b \\ \boldsymbol{c} \end{array}\middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{23}}(a_2)_{|n|_{45}}(a_3)_{|n|_{78}}(a_4)_{|n|_{2467}}(b)_{-|n|_{135}+|n|_{678}}}{\Big[\prod_{i=1}^{8} n_i!\Big](c_1)_{-n_1+|n|_{2467}}(c_2)_{-|n|_{135}+|n|_{78}}} \times(-1)^{|n|_{17}}\boldsymbol{x}^{\boldsymbol{n}}\ , \tag{7.6}$$

where the parameter vector $\boldsymbol{a}\in\boldsymbol{C}^4$, the parameter $b\in\boldsymbol{C}$ in the nominator, and the parameter vector $\boldsymbol{c}\in\boldsymbol{C}^2$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration V: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{A}$ and $\mathcal{B}$ are zero, any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{C}$ and $\mathcal{D}$ are zero, and any 5×5 minors of the 5×6 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{C}$ (or $\mathcal{D}$), $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are nonzero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm V}\left(\begin{array}{c}\boldsymbol{a},b\\ c\end{array}\Big|\boldsymbol{x}\right)=\sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8}\frac{(a_1)_{|n|_{23}}(a_2)_{|n|_{45}}(a_3)_{|n|_{67}}(a_4)_{|n|_{1357}}(a_5)_{|n|_{2468}}(b)_{-|n|_{1357}+n_8}}{\left[\prod\limits_{i=1}^{8}n_i!\right](c)_{-n_1+|n|_{2468}}} \times(-1)^{|n|_{357}}\boldsymbol{x}^{\boldsymbol{n}}\;, \tag{7.7}$$

where the parameter vector $\boldsymbol{a}\in\boldsymbol{C}^5$, the parameter $b\in\boldsymbol{C}$ in the nominator, and the parameter $\boldsymbol{c}\in\boldsymbol{C}$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration VI: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{A}$ and $\mathcal{B}$ are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{C}$ and $\mathcal{D}$ are zero, and the determinant of the 5×5 matrix composed by the columns with the indices $\mathcal{A}$ (or $\mathcal{B}$), $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is zero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm VI}\left(\begin{array}{c} \boldsymbol{a}, b \\ c \end{array}\middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n}\in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}}(a_2)_{|n|_{34}}(a_3)_{|n|_{56}}(a_4)_{|n|_{78}}(a_5)_{|n|_{1468}}(b)_{-|n|_{12}+|n|_{357}}}{\left[\prod_{i=1}^{8} n_i!\right](c)_{-n_2+|n|_{3\cdots 8}}} \times(-1)^{n_1}\boldsymbol{x}^{\boldsymbol{n}}\;, \tag{7.8}$$

where the parameter vector $\boldsymbol{a} \in \boldsymbol{C}^5$ in the nominator, and the parameter $c \in \boldsymbol{C}$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration VII: any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{D}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are zero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\text{VII}}\left(\begin{array}{c} \boldsymbol{a}, \boldsymbol{b} \\ \boldsymbol{c} \end{array} \middle| \boldsymbol{x}\right) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}_+^8} \frac{(a_1)_{|n|_{12}} (a_2)_{|n|_{56}} (a_3)_{|n|_{78}} (b_1)_{-|n|_{12}+|n|_{457}} (b_2)_{-|n|_{23}+|n|_{4\ldots8}}}{\left[\prod_{i=1}^{8} n_i!\right] (c_1)_{-|n|_{123}+|n|_{457}} (c_2)_{-|n|_{23}+|n|_{5678}}} \times (-1)^{|n|_{2357}} \boldsymbol{x}^{\boldsymbol{n}} , \tag{7.9}$$

where the parameter vectors $\boldsymbol{a} \in \boldsymbol{C}^3$, $\boldsymbol{b} \in \boldsymbol{C}^2$ in the nominator, and the parameter $\boldsymbol{c} \in \boldsymbol{C}^2$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration VIII: any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are zero, any 3×3 minors of the 5×3 matrix composed by the columns with the indices $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{C}$, $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{H}$ are zero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm VIII}\left(\begin{array}{c}\boldsymbol{a},\boldsymbol{b}\\ \boldsymbol{c}\end{array}\Big|\boldsymbol{x}\right)=\sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8}\frac{(a_1)_{|n|_{12}}(a_2)_{|n|_{45}}(a_3)_{|n|_{78}}(b_1)_{-|n|_{12}+|n|_{467}}(b_2)_{-|n|_{235}+|n|_{678}}}{\Big[\prod\limits_{i=1}^{8}n_i!\Big](c_1)_{-|n|_{123}+|n|_{467}}(c_2)_{-|n|_{235}+|n|_{78}}}\times(-1)^{|n|_{237}}\boldsymbol{x}^{\boldsymbol{n}}\;, \tag{7.10}$$

where the parameter vectors $\boldsymbol{a}\in\boldsymbol{C}^3$, $\boldsymbol{b}\in\boldsymbol{C}^2$ in the nominator, and the parameter $\boldsymbol{c}\in\boldsymbol{C}^2$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- The geometric configuration IX: any 2×2 minors of the 5×2 matrix composed by the columns with the indices $\mathcal{D}$, and $\mathcal{E}$ are zero, any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ (or $\mathcal{E}$) are zero, and any 4×4 minors of the 5×4 matrix composed by the columns with the indices $\mathcal{D}$ (or $\mathcal{E}$), $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are zero, respectively.

The geometric configuration corresponds to the following generalized hypergeometric function

$$H_{\rm IX}\left(\begin{array}{c}\boldsymbol{a},b\\ c\end{array}\middle|\boldsymbol{x}\right)=\sum_{\boldsymbol{n}\in\boldsymbol{Z}_+^8}\frac{(a_1)_{|n|_{12}}(a_2)_{|n|_{34}}(a_3)_{|n|_{56}}(a_4)_{|n|_{78}}(a_5)_{|n|_{1368}}(b)_{-|n|_{1234}+|n|_{57}}}{\left[\prod\limits_{i=1}^{8}n_i!\right](c)_{-|n|_{24}+|n|_{5678}}}\times(-1)^{|n|_{13}}\boldsymbol{x}^{\boldsymbol{n}}\;, \tag{7.11}$$

where the parameter vector $\boldsymbol{a}\in\boldsymbol{C}^5$, the parameter $b\in\boldsymbol{C}$ in the nominator, and the parameter $c\in\boldsymbol{C}$ in the denominator, respectively.

VII. The 2-loop Massive Dune diagram

- Under the inverse transformation, the image function is some linear combination of these nine hypergeometric functions.

$$
\begin{aligned}
& H_{\rm II}\left(\begin{array}{c} \boldsymbol{a}, b \\ \boldsymbol{c} \end{array} \middle| \boldsymbol{x}\right) \\
&= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a_3-b)\Gamma(a_4-b)}{\Gamma(a_3)\Gamma(a_4)\Gamma(c_1-b)\Gamma(c_2-b)} x_7^{-b} \\
&\quad \times H_{\rm II}\left(\begin{array}{c} (a_1, a_2, 1-c_1+b, 1-c_2+b), b \\ (1-a_3+b, 1-a_4+b) \end{array} \middle| (x_1 x_7, \frac{x_8}{x_7}, \frac{x_4}{x_7}, \frac{x_3}{x_7}, \frac{x_6}{x_7}, \frac{x_5}{x_7}, \frac{1}{x_7}, \frac{x_2}{x_7})\right) \\
&\quad + \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a_3-a_4)\Gamma(b-a_4)}{\Gamma(a_3)\Gamma(b)\Gamma(c_1-a_4)\Gamma(c_2-a_4)} x_7^{-a_4} \\
&\quad \times H_{\rm IV}\left(\begin{array}{c} (a_2, a_1, 1-c_2+a_4, a_4), 1-c_1+a_4 \\ (1-a_3+a_4, 1-b+a_4) \end{array} \middle| (x_8, \frac{x_5}{x_7}, x_6, \frac{x_3}{x_7}, x_4, \frac{x_2}{x_7}, \frac{1}{x_7}, x_1)\right) \\
&\quad + \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a_4-a_3)\Gamma(b-a_3)}{\Gamma(a_4)\Gamma(b)\Gamma(c_1-a_3)\Gamma(c_2-a_3)} x_7^{-a_3} \\
&\quad \times H_{\rm VII}\left(\begin{array}{c} (a_3, a_2, a_1), (1+a_3-c_2, a_4-a_3, b-a_3) \\ c_1-a_3 \end{array} \middle| (\frac{x_8}{x_7}, \frac{1}{x_7}, x_1, x_2, x_5, x_6, x_3, x_4)\right) \quad (7.12)
\end{aligned}
$$

VII. The 2-loop Massive Dune diagram

- With the geometric representations of those hypergeometric functions, we obtain the generalized Gauss-Kummer relations.
- With those Gauss relations, we can reduce these eight-fold summation hypergeometric series to six-fold summation hypergeometric series for the 2-loop Dune diagram, and reduce these eight-fold summation hypergeometric series to four-fold summation hypergeometric series for the 2-loop self-energy with 4 propagators, respectively.

VIII. Summary

- In this approach, one topological diagram corresponds to one set of hypergeometric solutions. We make the classification among those hypergeometric solutions by the geometric configurations at first, make the classification further by the geometric representations, then make the classification by the affine spanning finally.
- GKZ-systems of Grassmannians give the analytic expressions of Feynman integrals in whole domain of definition. For example, 2-loop 4-propagator and 2-loop 5-propagator self-energies.

河北大学
HEBEI UNIVERSITY