Strong Coupling Expansion of Gluodynamics on a Lattice under Rotation

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General Formulation:

Grand Partition Function with a Macroscopic Angular Momentum

$$\mathcal{Q} = \mathrm{Tr}e^{-\frac{1}{T}(H-\omega J_z)} \qquad [H, J_z] = 0$$

Gibbs Free Energy

$$\mathcal{F} = -T \ln \mathcal{Q}$$

Canonical quantization in $A_0 = 0$ gauge

$$H = \frac{1}{2}\int d^3\vec{x}\left(\vec{\Pi}^l \cdot \vec{\Pi}^l + \vec{B}^l \cdot \vec{B}^l\right) = \int d^3\vec{x}\,\mathcal{H}$$

$$\vec{B}^l = \vec{\nabla} \times \vec{A}^l + \frac{1}{2} g f^{lab} \vec{A}^a \times \vec{A}^b$$

$$\left[\Pi_i^a(\vec{x}), A_j^b(\vec{x}')\right] = -i\delta^{ab}\delta_{ij}\delta^3(\vec{x} - \vec{x}')$$

Physical state

$$\left(\vec{\nabla} \cdot \vec{\Pi}^l + g f^{lab} \vec{A}^a \cdot \vec{\Pi}^b\right)|\quad\rangle = 0$$

Angular momentum

$$\vec{J} = \int d^3\vec{x}\, \vec{x} \times \left(\vec{B}^l \times \vec{\Pi}^l\right)$$

Toward Path integral formulation:

$$\dot{\vec{A}}^l = \frac{\partial \mathcal{H}}{\partial \vec{\Pi}^l} = \vec{\Pi}^l + \omega\left[\left(\vec{x} \cdot \vec{B}^l\right)\hat{z} - \left(\hat{z} \cdot \vec{B}^l\right)x\right]$$

Lagrangian density

$$\mathcal{L} = \vec{\Pi}^l \cdot \dot{\vec{A}}^l - \mathcal{H} = -\frac{1}{4}(F^{\mu\nu})^l\left(F_{\mu\nu}\right)^l$$

$$= \frac{1}{2}\left(F_{tx}^l F_{tx}^l + F_{ty}^l F_{ty}^l + F_{tz}^l F_{tz}^l\right) - \frac{1}{2}(1 - \omega^2 r^2)F_{xy}^l F_{xy}^l$$

$$-\frac{1}{2}(1 - \omega^2 x^2)F_{yz}^l F_{yz}^l - \frac{1}{2}(1 - \omega^2 y^2)F_{zx}^l F_{zx}^l$$

$$+\omega\left(xF_{xy}^l F_{tx}^l + yF_{xy}^l F_{ty}^l - xF_{yz}^l F_{tz}^l - yF_{zx}^l F_{tz}^l\right) + \omega^2 xyF_{yz}^l F_{zx}^l$$

$$r^2 = x^2 + y^2$$

$$F_{\mu\nu}^l = \partial_\mu A_\nu^l - \partial_\nu A_\mu^l + gf^{lab}A_\mu^a A_\nu^b \qquad A_0^l = 0$$

Nonzero A_0^l can be restored when transformed to other gauges, e.g. covariant gauge

$\boldsymbol{\mathcal{L}} =$ the Lagrangian density in a global rotating frame with angular velocity $\boldsymbol{\omega}$.

$$ds^2 = (-1 + \omega^2 r^2)dt^2 + \omega(xdy - ydx)dt + dx^2 + dy^2 + dz^2$$

Thermal Ensemble:

$$t \to -it \qquad F^l_{tj} \to iF^l_{tj}$$

$$A^l_\mu\left(t+\frac{1}{T},\vec{x}\right) = A^l_\mu(t,\vec{x})$$

$$\mathcal{S}_G \equiv \int_{S_1\times R^3} d^4x\,\mathcal{L} \to i\mathcal{S}_G$$

$$\mathcal{Q} = \text{const.}\int [dA]\, e^{-\mathcal{S}_G + \text{gauge fixing terms}}$$

Lattice version of $\mathcal{S}_G$ (Yamamoto & Hirono, 2013, Braguta et. al., 2021)

$$
\begin{aligned}
\mathcal{S}_G = \frac{2N_c}{g^2}\sum_X \Big[& (1-r^2\omega^2)\Big(1-\frac{1}{N_c}\mathrm{ReTr}\bar{U}_{xy}\Big) + (1-y^2\omega^2)\Big(1-\frac{1}{N_c}\mathrm{ReTr}\bar{U}_{xz}\Big) \\
& + (1-x^2\omega^2)\Big(1-\frac{1}{N_c}\mathrm{ReTr}\bar{U}_{yz}\Big) + 3 - \frac{1}{N_c}\mathrm{ReTr}\big(\bar{U}_{x\tau}+\bar{U}_{y\tau}+\bar{U}_{z\tau}\big) \\
& + \frac{1}{N_c}\mathrm{ReTr}\big(iy\omega(\bar{V}_{xy\tau}+\bar{V}_{xz\tau}) - ix\omega(\bar{V}_{yx\tau}+\bar{V}_{yz\tau}) + xy\omega^2\bar{V}_{xzy}\big)\Big].
\end{aligned}
$$

Formulated on a lattice of $N_t \times N_s^3$ sites

$$\mathcal{Q} = \int \prod_{x,\mu} dU_\mu(x)\, e^{-\mathcal{S}_G}$$

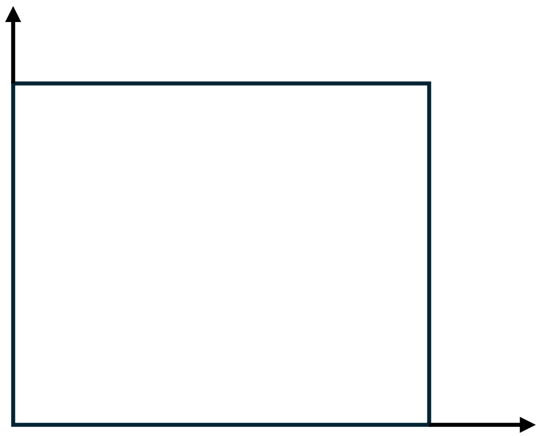
$$\mathcal{F} = -T\ln\mathcal{Q} \qquad\qquad T = \frac{1}{N_t a}$$

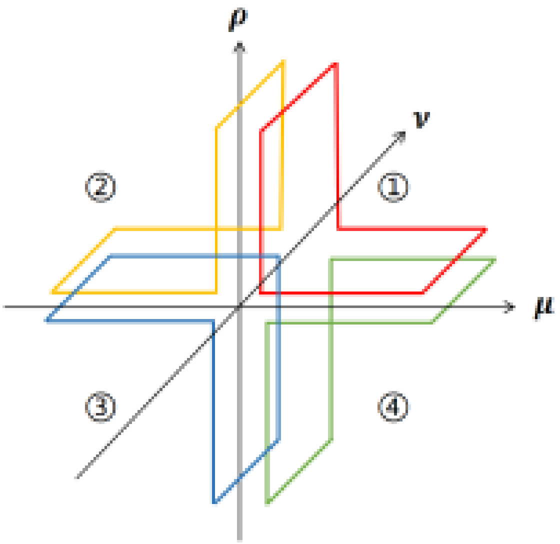
$$\mathcal{S}_G = \mathcal{S}_0 - \frac{1}{g^2}(E_0 + \omega E_1 + \omega^2 E_2),$$

$$E_0 = 2\mathrm{ReTr}\sum_P (U_{xt} + U_{yt} + U_{zt} + U_{xy} + U_{xz} + U_{yz}),$$

$$E_1 = \frac{1}{4}\sum_P \Big\{ \sum_{s=1,2,3,4} (-1)^s 2\mathrm{ReTr}[i(\bar{y}V^s_{xyt} + yV^s_{xzt}) - i(\bar{x}V^s_{yxt} + xV^s_{yzt})] \Big\},$$

$$E_2 = -2\mathrm{ReTr}\sum_P \Big[\bar{\rho}^2 U_{xy} + \bar{y}^2 U_{xz} + \bar{x}^2 U_{yz} - \frac{1}{4}\sum_{s=1,2,3,4} (-1)^s xy V^s_{xzy}\Big],$$

$U_{\mu\nu} =$ 

$V_{\mu\rho\nu} =$ 

Deconfinement Phase Transition:

After integrating all spatial links

$$e^{-\frac{1}{T}\mathcal{F}} = \int \prod_{\vec{x}} dW(\vec{x})dW^*(\vec{x})\, e^{-\mathcal{S}_{eff.}[W,T,\omega]}$$

$$---- \text{Polyakov loop} \qquad W(\vec{x}) = \mathrm{tr}\prod_t U_0(\vec{x},t)$$

$----$ $\mathcal{S}_{\mathrm{eff.}}[W,T]$ is invariant under Z_{N_c} rotation of W.

Mean-field theory of the deconfinement transition:

- Saddle point approximation:

$$\mathcal{F} = T\mathcal{S}_{\mathrm{eff.}}[\mathcal{W},T,\omega]$$

where $\mathcal{W}$ = the global minimum of $\mathcal{S}_{\mathrm{eff.}}[\mathcal{W},T,\omega]$

- $\mathcal{S}_{\mathrm{eff.}}[W,T]$ has two local minima: $\mathcal{W}=0$, Z_{N_c} invariant, confinement;
  $\mathcal{W}\neq 0$, Z_{N_c} breaking, deconfinement.
- **Deconfinement transition occurs when the two local minima are degenerate**

$$\boldsymbol{\left(\frac{\partial\mathcal{F}}{\partial\mathcal{W}(\vec{x})}\right)_{T,\omega} = 0 \qquad \mathcal{F}[\mathcal{W},T,\omega] = \mathcal{F}[0,T,\omega]}$$

Shift of the Deconfinement Temperature for Low Angular Velocity ω:

- Following *Braguta et. al.* assuming a cylindrical volume around the rotation axis of radius R.
- Expand the effective action to the order ω^2

$$\mathcal{F}[\mathcal{W},T,\omega] = \mathcal{F}[\mathcal{W},T] - \frac{1}{2}\omega^2 I[\mathcal{W},T]$$

Moment of inertial

- Expand $\mathcal{W}$, and T in the neighborhood of the transition at $\omega = 0$

$$\mathcal{W} = \mathcal{W}_0 + \delta\mathcal{W} \qquad T_d = T_d^{(0)} + \delta T$$

- Re-balancing the deconfinement transition condition

$$\boldsymbol{\frac{T_d - T_d^{(0)}}{T_d^{(0)}} = -\frac{\Delta I}{2L}\omega^2 = Bv^2}$$

where ΔI = jump of the moment of inertial at the transition when $\omega = 0$

L = latent heat at the transition when $\omega = 0$

$$\Delta I = I\left(\mathcal{W}_0, T_d^{(0)}\right) - I\left(0, T_d^{(0)}\right)$$

$$L = T_d^{(0)}\left[S\left(\mathcal{W}_0, T_d^{(0)}\right) - S\left(0, T_d^{(0)}\right)\right]$$

Strong coupling expansion of SU(3) at $\omega = 0$ (*M. Gross, et. al. 1983*)

For a homogeneous W

$$S_{\text{eff.}}^{(0)}[W,T] = -\frac{1}{2}\ln[27 - 18|W|^2 + \text{Re}(W^3) - |W|^4] - 3\lambda|W|^2$$

$$\lambda = 2(3g^2)^{-N_t} = 2e^{-\frac{\sigma a}{T}}$$

String tension in strong coupling

$$\sigma = \frac{1}{a^2}\ln(3g^2)$$

$\sigma = (440\text{MeV})^2$, $a^{-1} = 228\text{MeV}$ (*Braguta, et. al.*)

$Z_3 -$ breaking minimum:

$$W = \mathcal{W}_0 = 1 + \sqrt{4 - \frac{1}{3\lambda}}$$

becomes degenerate with the symmetric minimum, $W = 0$ at

$$\lambda \cong 0.086 \qquad \Longrightarrow \qquad T_d^{(0)} = 270\text{MeV}$$

Technical background:

- Strong coupling expansion

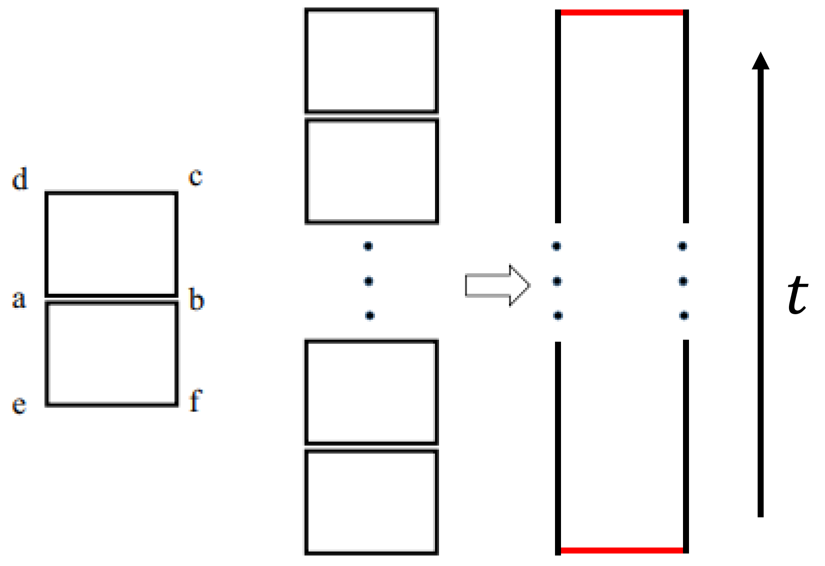
---- N_t – th power of the lattice action $\mathcal{S}_G^{N_t}$

---- The terms $\mathcal{S}_G^{N_t}$ covering each of $U_0(\vec{x}, t)$ once

---- Integration by theorem of orthogonality

$$\int dU_{ab}(\mathrm{tr}U_A U_{ab})(\mathrm{tr}U_{ab}^{\dagger} U_B) = \frac{1}{3}\mathrm{tr}U_A U_B,$$

$$\int dU\, \mathrm{tr}UU_C U^{\dagger} U_D = \frac{1}{3}\mathrm{tr}U_C \mathrm{tr}U_D$$



- Haar measure transformation

$$VUV^{\dagger} = \mathrm{diag.}\left(e^{i\alpha}, e^{i\beta}, e^{i\gamma}\right) \qquad W = e^{i\alpha} + e^{i\beta} + e^{i\gamma}$$

$$\int dU\, f(W, W^*) = \mathrm{const.} \int d\alpha d\beta d\gamma\, \mathcal{J}(\alpha, \beta, \gamma)\delta(\alpha + \beta + \gamma) f(W, W^*)$$

$$= \mathrm{const.} \int dW\, dW^* \left[27 - 18|W|^2 + 8\mathrm{Re}(W^3) - |W|^4\right]^{1/2} f(W, W^*)$$

Order ω^2 correction:

$$\mathcal{S}_G = \mathcal{S}_0 - \frac{1}{g^2}(E_0 + \omega E_1 + \omega^2 E_2),$$

$$E_0 = 2\mathrm{ReTr}\sum_P (U_{xt} + U_{yt} + U_{zt} + U_{xy} + U_{xz} + U_{yz}),$$

$$E_1 = \frac{1}{4}\sum_P \Big\{ \sum_{s=1,2,3,4} (-1)^s 2\mathrm{ReTr}[i(\bar{y}V^s_{xyt} + yV^s_{xzt}) - i(\bar{x}V^s_{yxt} + xV^s_{yzt})]\Big\},$$

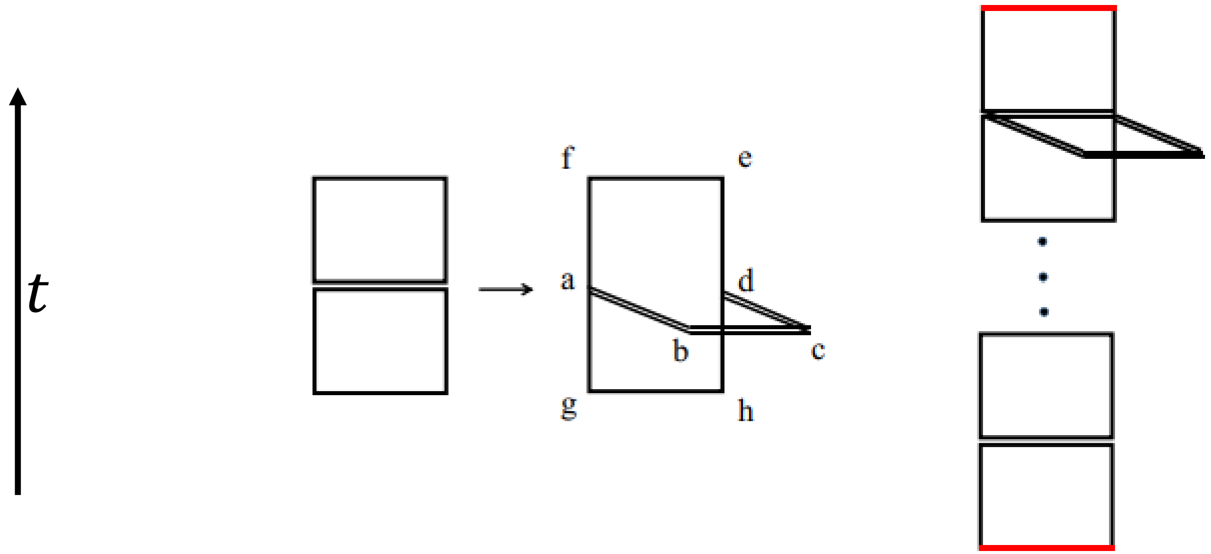
$$E_2 = -2\mathrm{ReTr}\sum_P \Big[\bar{\rho^2}U_{xy} + \bar{y^2}U_{xz} + \bar{x^2}U_{yz} - \frac{1}{4}\sum_{s=1,2,3,4}(-1)^s xyV^s_{xzy}\Big],$$

- $E_0^{N_t}$ already contributed to leading order. $E_0^{N_t-2}E_2^2$ contributes to the leading order of corrections.
- An adjacent pair of U's is replaced by a pair of V's.

$$\mathcal{S}_{\text{eff.}} = \mathcal{S}^{(0)}_{\text{eff.}} + \mathcal{S}^{(1)}_{\text{eff.}} + \cdots \qquad \mathcal{S}^{(1)}_{\text{eff.}} = O(\omega^2)$$

- Inhomogeneity does not contribute to this order.

Technical details



(a) (b)

$$\Longrightarrow \quad \mathcal{S}^{(1)}_{\text{eff.}}$$

$$\int dU_{ab}dU_{bc}dU_{cd}(\mathrm{tr}U_A U_{ab} U_{bc} U_{cd})\left(\mathrm{tr}U^{\dagger}_{cd}U^{\dagger}_{bc}U^{\dagger}_{ab}U_B\right) = \frac{1}{3}\mathrm{tr}U_A U_B$$

Result of order ω^2 correction:

Assuming the open boundary condition of *Braguta et. al*.

$$\mathcal{S}^{(1)}_{\text{eff.}} \cong -\frac{1}{4}\lambda v^2 \mathcal{N} N_t W^* W$$

where $\mathcal{N}$ = number of lattice sites within the cylinder

The jump of the moment of inertia at the transition $T = T_d^{(0)}$

$$\Delta I = I\left[\mathcal{W}_0\left(T_d^{(0)}\right), T^{(0)}\right] - I\left[0, T^{(0)}\right] = \frac{1}{2}\mathcal{N} N_t \lambda_0 R^2 \left|\mathcal{W}_0\left(T_0^{(0)}\right)\right|^2$$

Together with the latent heat at the transition

$$L = 3\lambda_0 \mathcal{N} \sigma a \left|\mathcal{W}_0\left(T_0^{(0)}\right)\right|^2 > 0$$

$$B = \frac{T_d - T_d^{(0)}}{T_d^{(0)} v^2} = -\frac{1}{12\sigma a^2} \cong -0.02235$$

In contrast to the lattice simulation by *Braguta et. al*. $B = 0.7$ (arXiv:2102.05084)

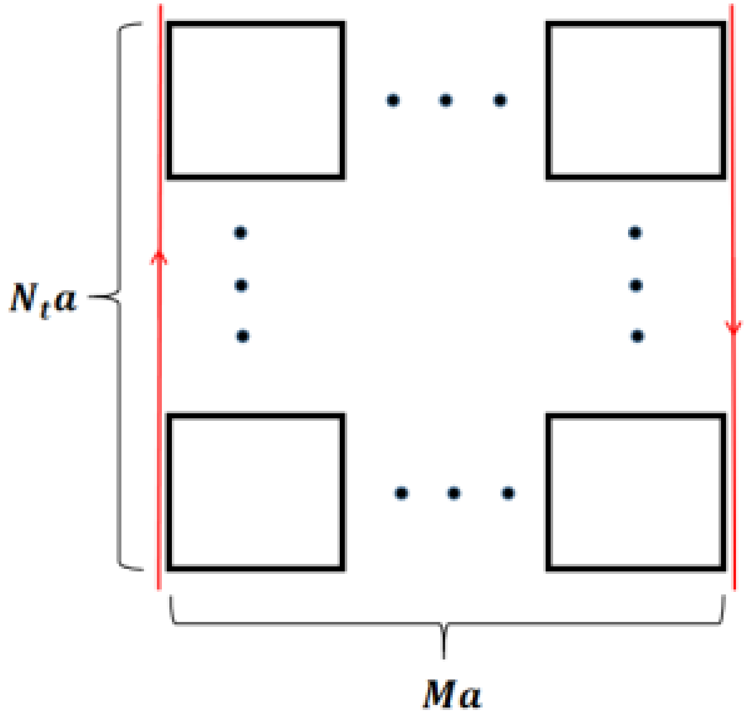
The same behavior was also obtained by *Fukushima & Shimada* (arXiv:2506.03560).

ω^2-Correction to string tension

$$e^{-\beta U(L)} = \frac{1}{Q} \int \prod_{x,\mu} dU_\mu(x) e^{-S_G[U]} W(\vec{x}_1) W^*(\vec{x}_2)$$

$$e^{-\beta U(L)} = \left(\frac{1}{3g^2}\right)^{N_t M} (1 + N_t M \omega^2 d^2)$$

$$\sigma = \frac{\ln(3g^2) + \omega^2 d^2}{a^2}$$



Summary and Outlook:

- A systematic approximation from the first principle.
- If lattice result is robust, the discrepancy maybe caused by

  ---- The order of strong coupling expansion is not enough to capture the physics within the critical window.

  ---- While the physics in the critical window is universal, the lattice formulation can be different and there maybe a lattice formulation that capture critical behavior better.

- From large N_c perspective, the extensive thermodynamic quantities carry N_c^2.

  ---- If the moment of inertia is positive, $\Delta I > 0$ at the transition and the deconfinement temperature **decreases** with the angular velocity.

  ---- The moment of inertia

$$I = \frac{1}{T}\left\langle J_z - \langle J_z \rangle \right\rangle^2 > 0 \qquad \langle O \rangle = \frac{\mathrm{Tr} e^{-\frac{1}{T}H} O}{\mathrm{Tr} e^{-\frac{1}{T}H}}$$

  ---- Our result is consistent with most of holographic calculations (*X. Chen, et. al.* arXiv: 2010.14478).

  and AdS/QCD assumes large N_c.

  ---- Is $N_c = 3$ sufficiently large?