

Toward constraining QCD phase transitions in neutron star interiors: Bayesian inference with a TOV linear response analysis

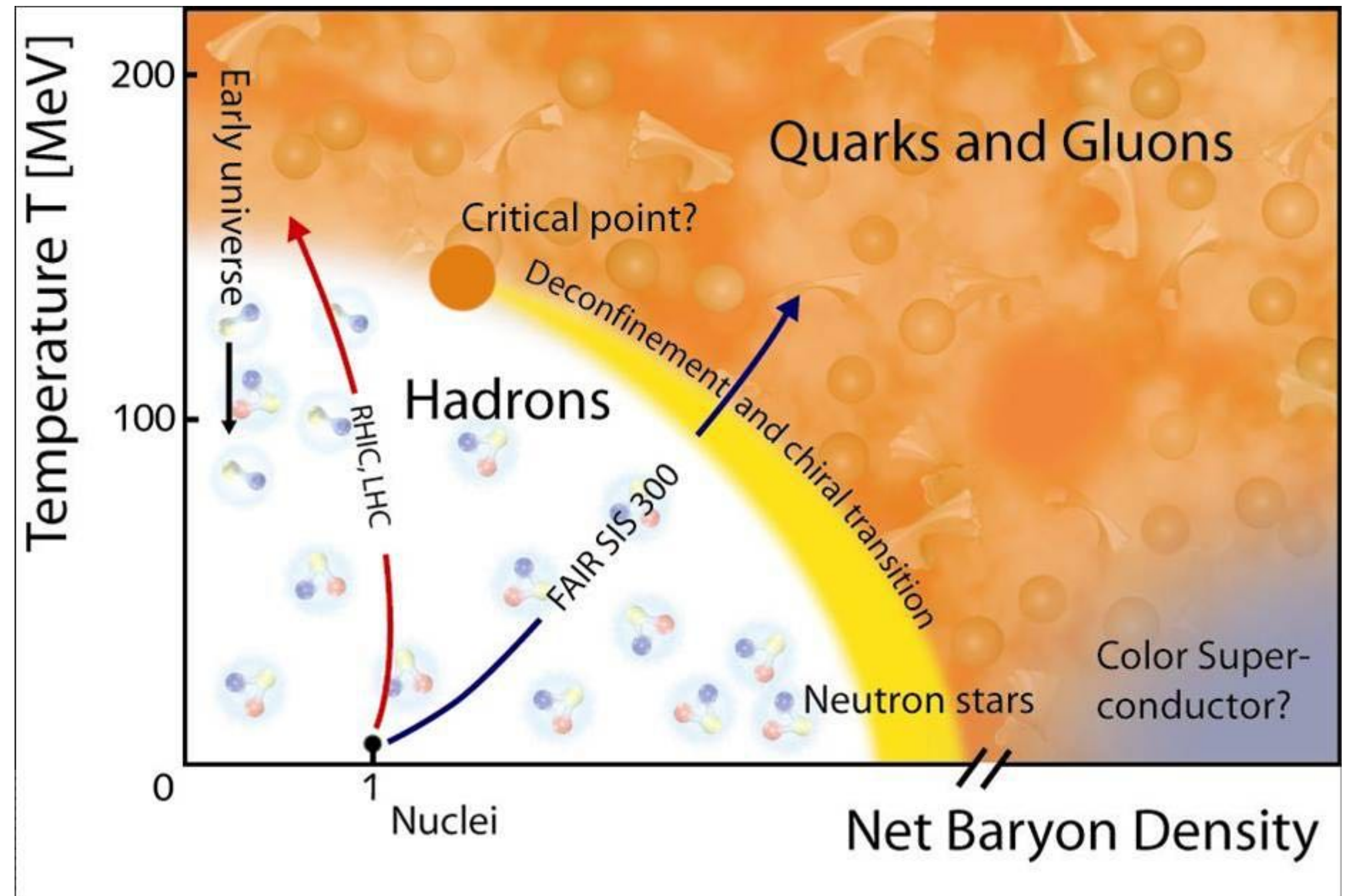
Shuzhe Shi, Tsinghua University

Ronghao Li, Sophia Han, Zidu Lin, Lingxiao Wang, Kai Zhou, **SS**,
Phys. Rev. D, 111, 074026 + work in progress

see also:

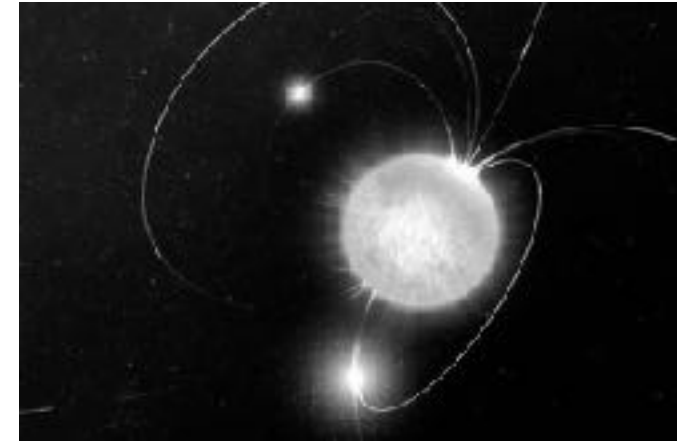
Soma, Wang, **SS**, Stoecker, Zhou, JCAP08 (2022) 071; Phys. Rev. D 107, 083028

from heavy-ion collisions:
crossover @ zero μ_B

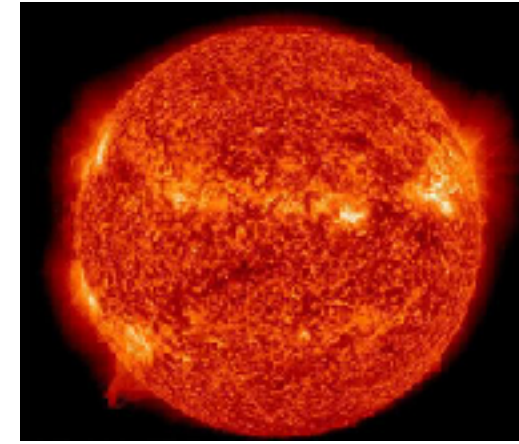


no experimental evidence for a first-order phase transition, yet

Neutron Star



Neutron Star



Sun



Nucleus

Mass
($M_{\odot}$)

0.5 ~ 2

1

$\sim 10^{0\sim 2} \times 10^{-57}$

Radius
(km)

~ 10

7×10^5

$\sim 10^{-18}$

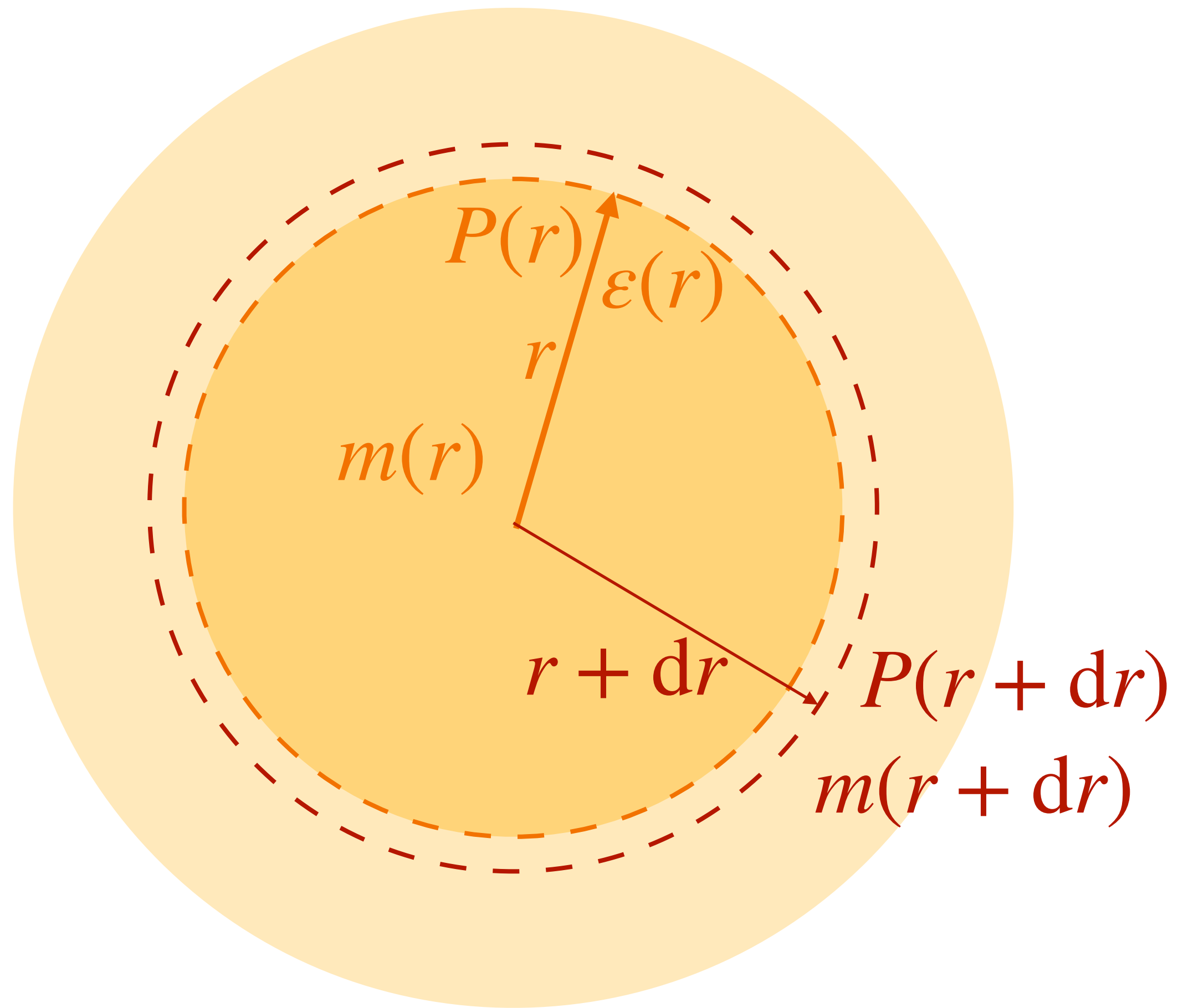
Density
($M_{\odot}/\text{km}^3$)

$\sim 5 \times 10^{-3}$

$\sim 7 \times 10^{-19}$

$\sim 3 \times 10^{-3}$

Tolman-Oppenheimer-Volkoff equations



Pressure at radius r : $P(r)$

Mass density at r : $\epsilon(r)$

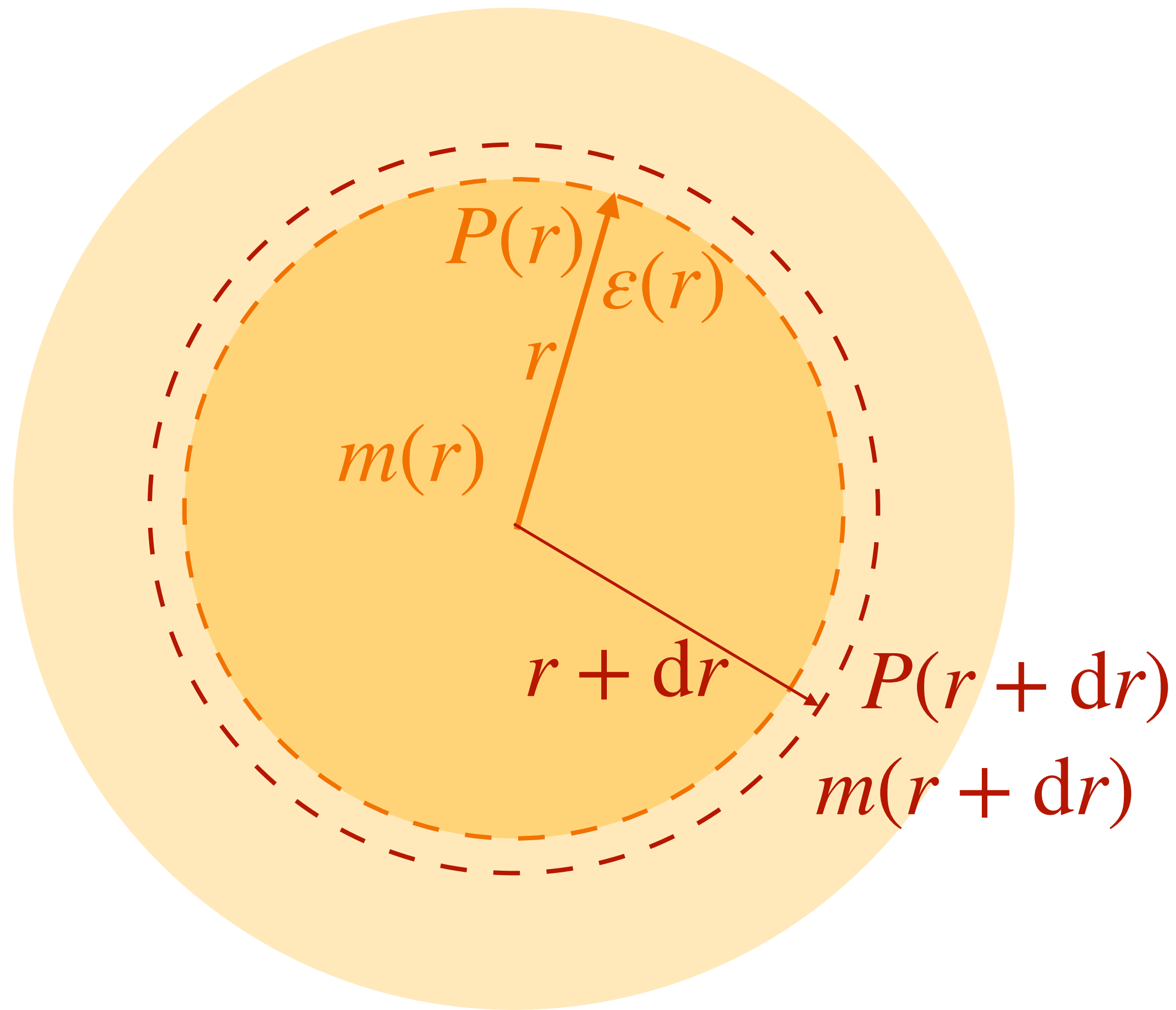
Mass enclosed within shell: $m(r)$

Continuation equation:

$$m(r + dr) = m(r) + \underbrace{4\pi r^2}_{\text{area}} \underbrace{\epsilon(r)}_{\text{density}} \underbrace{dr}_{\text{thickness}},$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon,$$

Tolman-Oppenheimer-Volkoff equations



Pressure at radius r : $P(r)$

Mass density at r : $\epsilon(r)$

Mass enclosed within shell: $m(r)$

Balance of force:

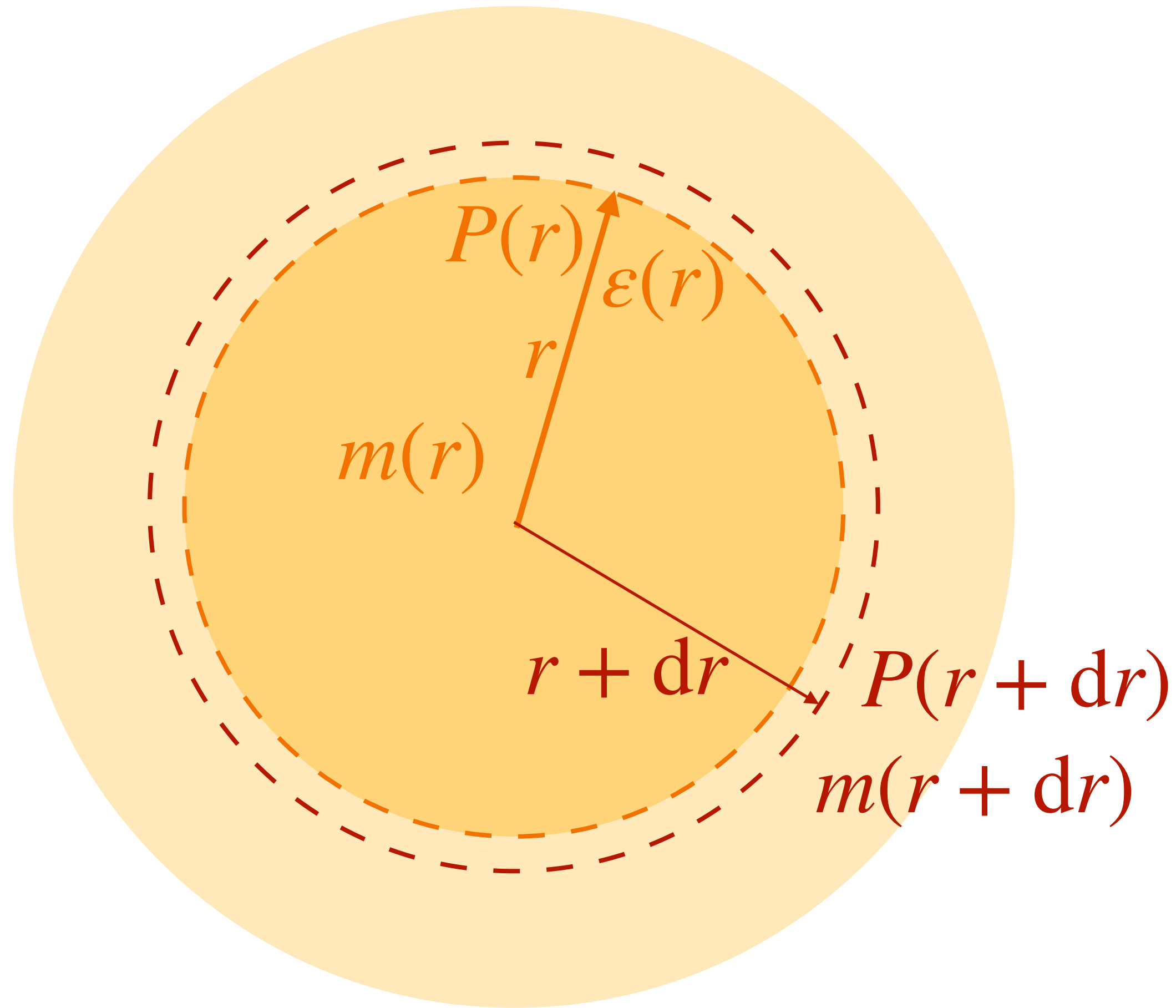
$$4\pi r^2 P(r) = 4\pi (r + dr)^2 P(r + dr) + F_G(r),$$

$$F_G = \frac{G \overset{m_1}{m(r)} \overset{m_2}{4\pi r^2 \epsilon dr}}{\overset{\text{distance-squared}}{r^2}}$$

distance-squared

$$\frac{dm}{dr} = 4\pi r^2 \epsilon,$$

Tolman-Oppenheimer-Volkoff equations



Pressure at radius r : $P(r)$

Mass density at r : $\epsilon(r)$

Mass enclosed within shell: $m(r)$

Balance of force:

$$4\pi r^2 P(r) = 4\pi (r + dr)^2 P(r + dr) + F_G(r),$$

$$F_G = \frac{G \overset{m_1}{m(r)} \overset{m_2}{4\pi r^2 \epsilon dr}}{\overset{\text{distance-squared}}{r^2}}$$

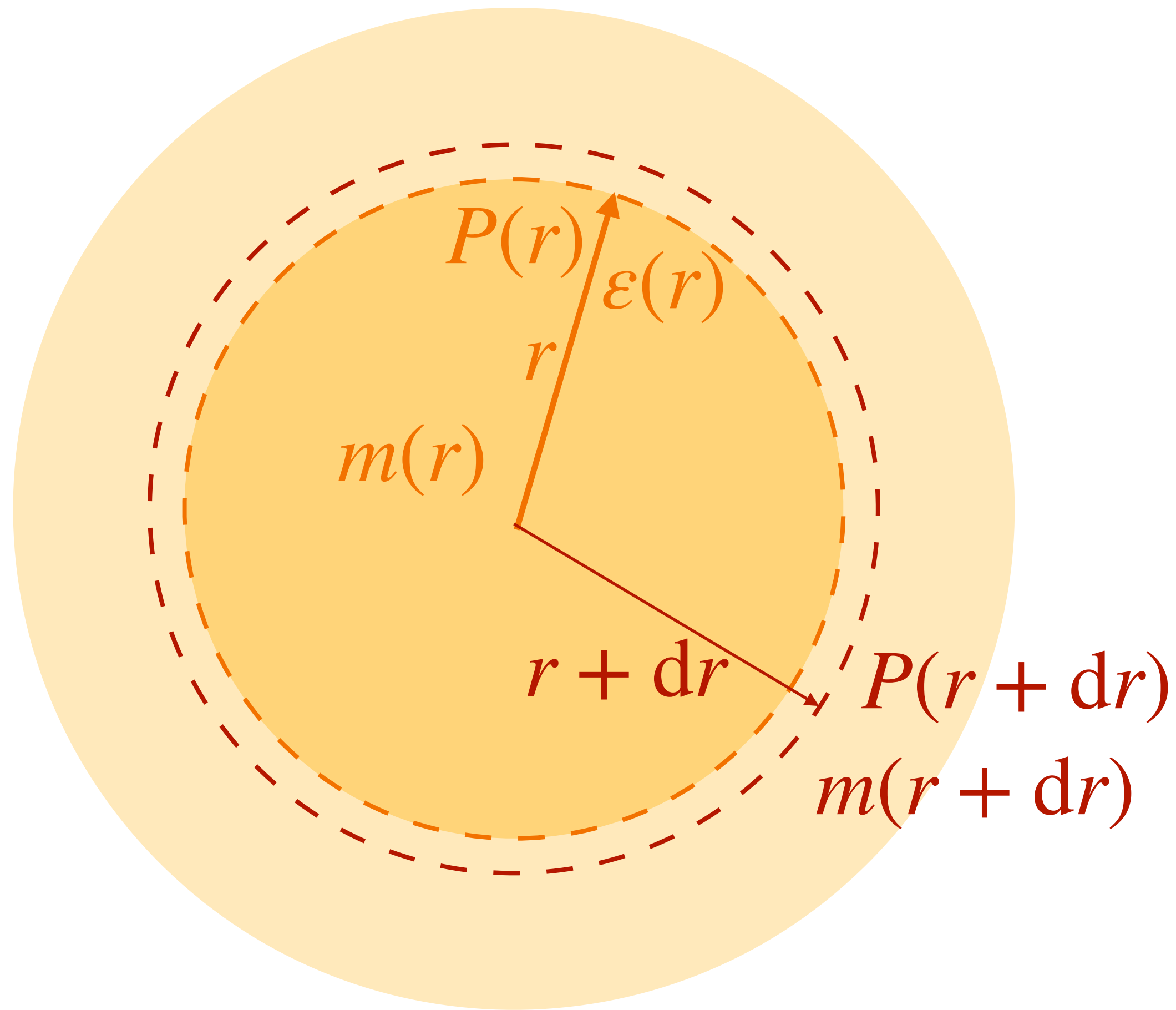
distance-squared

+GR effect

$$\frac{dm}{dr} = 4\pi r^2 \epsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \epsilon)}{r^2 - 2mr},$$

Tolman-Oppenheimer-Volkoff equations



Pressure at radius r : $P(r)$

Mass density at r : $\varepsilon(r)$

Mass enclosed within shell: $m(r)$

Equation of state:

$$\varepsilon = \varepsilon_{\text{EoS}}(P),$$

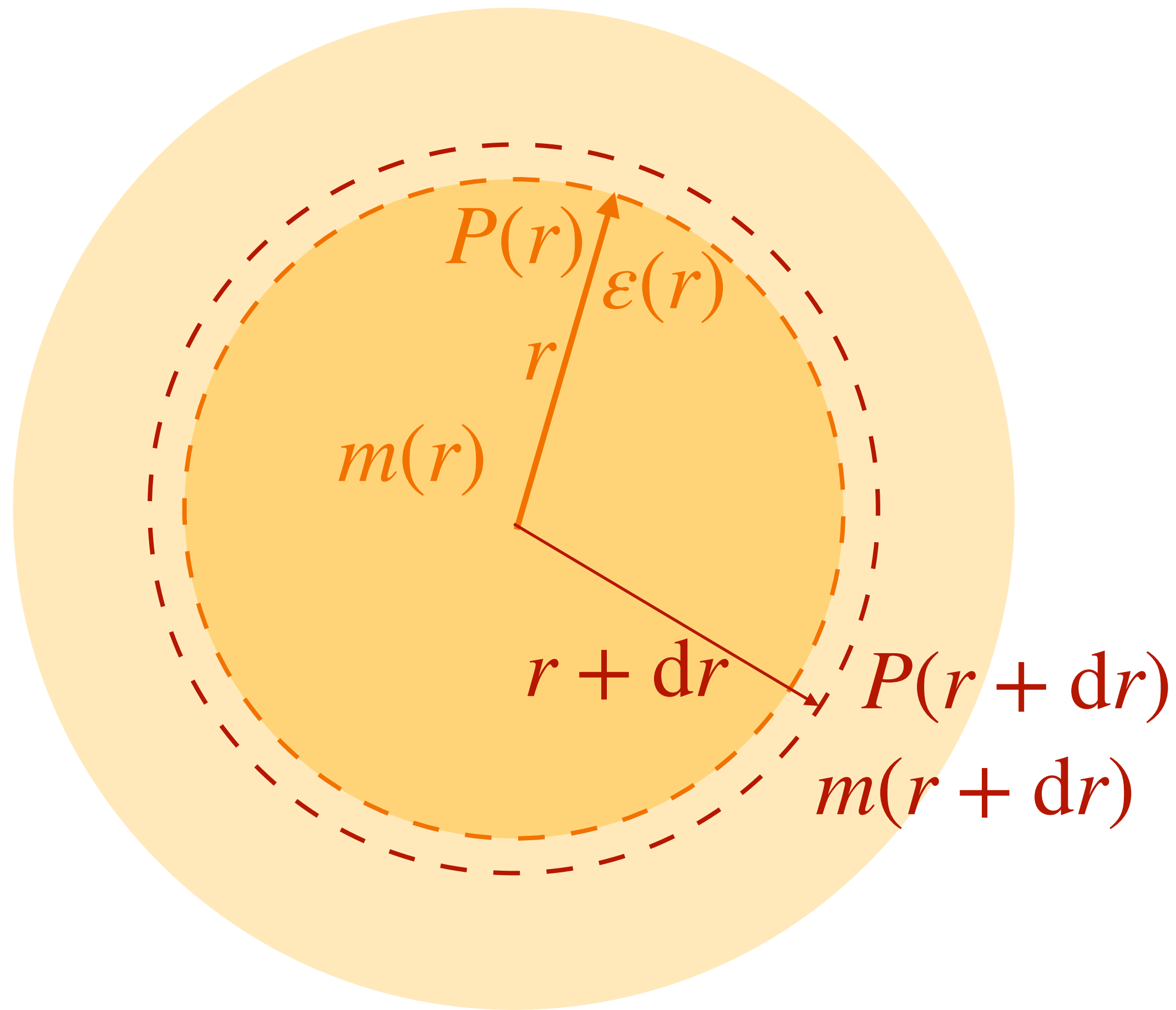
$$\varepsilon(r) = \varepsilon_{\text{EoS}}(P(r)),$$

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr},$$

$$\varepsilon = \varepsilon(P),$$

Tolman-Oppenheimer-Volkoff equations



Pressure at radius r : $P(r)$

Mass density at r : $\epsilon(r)$

Mass enclosed within shell: $m(r)$

boundary conditions:

$$m(r = 0) = 0,$$

$$m(r = R) = M,$$

$$P(r = 0) = P_{\text{cen}},$$

$$P(r = R) = 0.$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \epsilon)}{r^2 - 2mr},$$

$$\epsilon = \epsilon(P),$$

Tolman-Oppenheimer-Volkoff equations

Tolman-Oppenheimer-Volkoff equations:

Tolman, Phys.Rev. 55 (1939) 364-373

Oppenheimer, Volkoff, Phys.Rev. 55 (1939) 374-381

$$m(r = 0) = 0,$$

$$P(r = 0) = P_{\text{cen}},$$

$$m(r = R) = M,$$

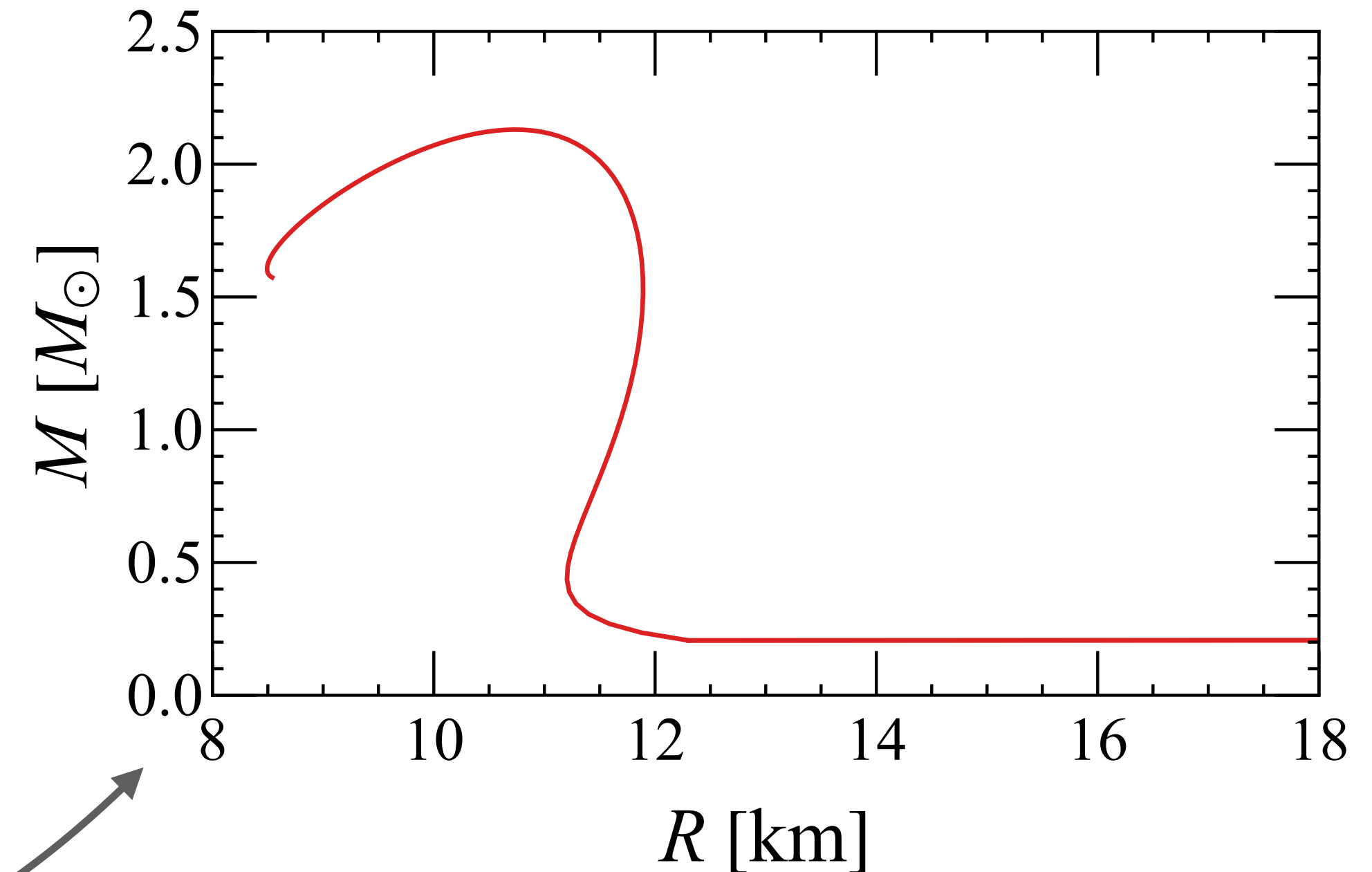
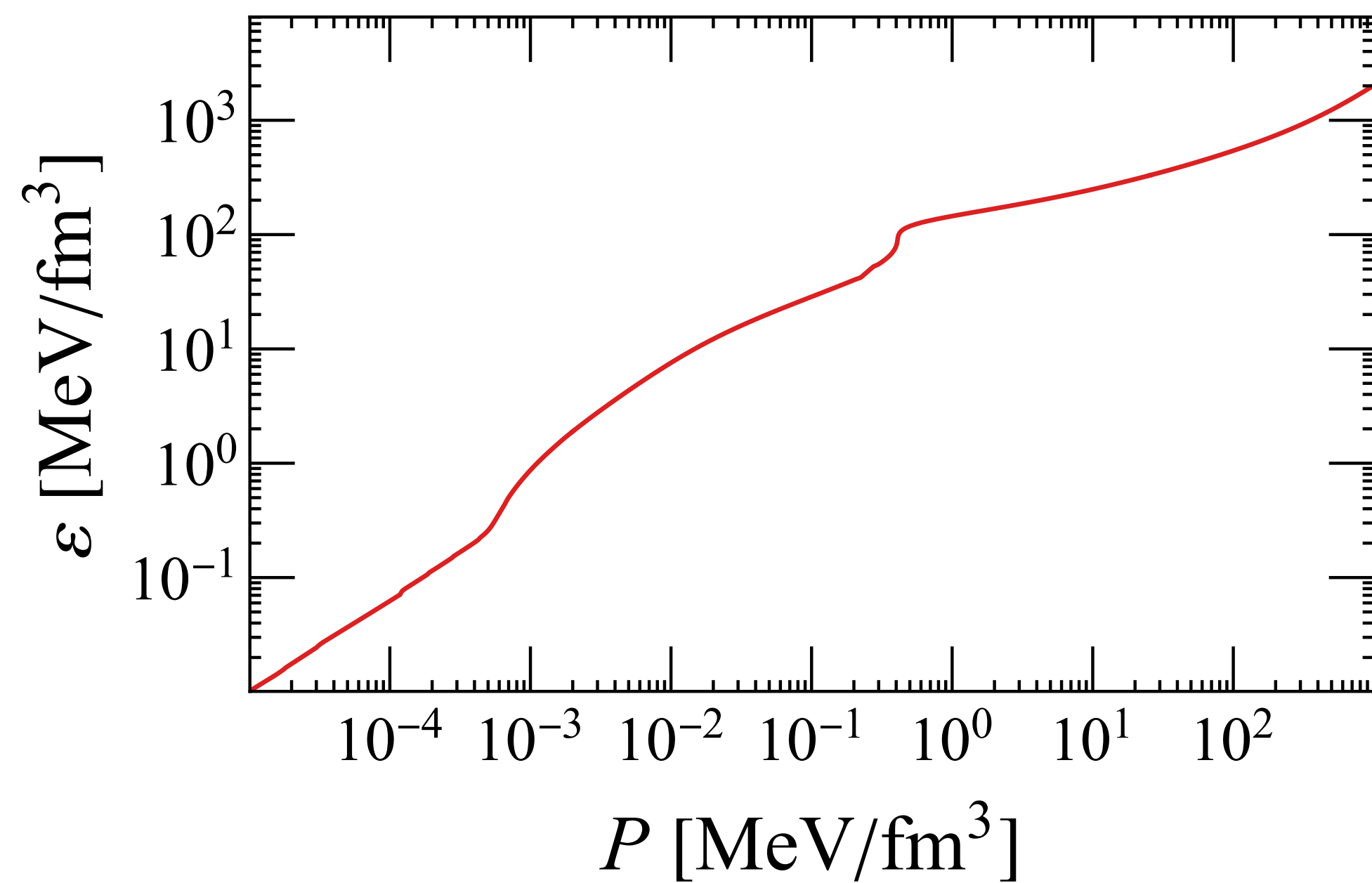
$$P(r = R) = 0.$$

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2 m r},$$

$$\varepsilon = \varepsilon(P),$$

Tolman-Oppenheimer-Volkoff equations



Tolman-Oppenheimer-Volkoff equations:

Tolman, Phys.Rev. 55 (1939) 364-373

Oppenheimer, Volkoff, Phys.Rev. 55 (1939) 374-381

$$m(r = 0) = 0,$$

$$P(r = 0) = P_{\text{cen}},$$

$$m(r = R) = M,$$

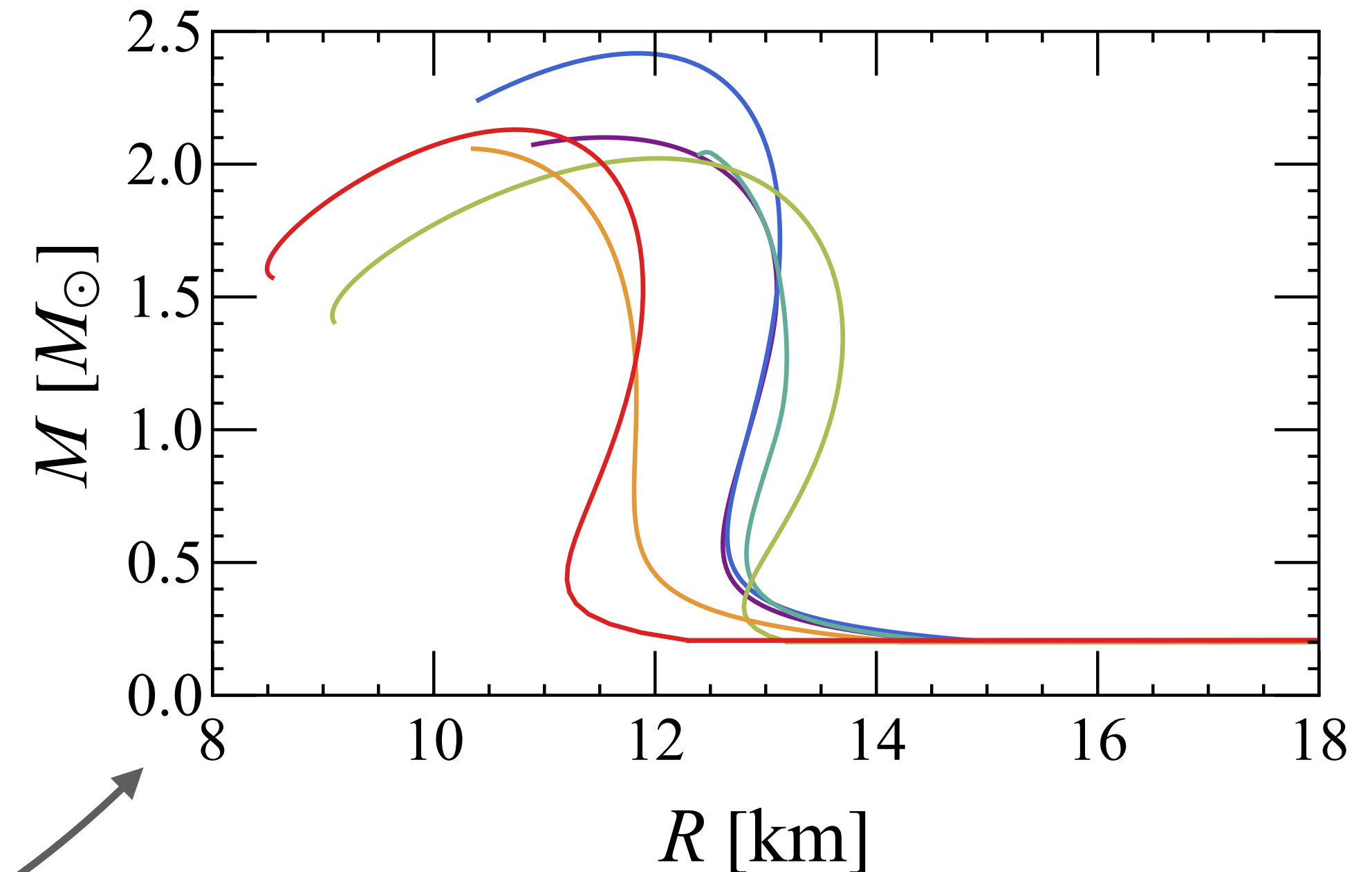
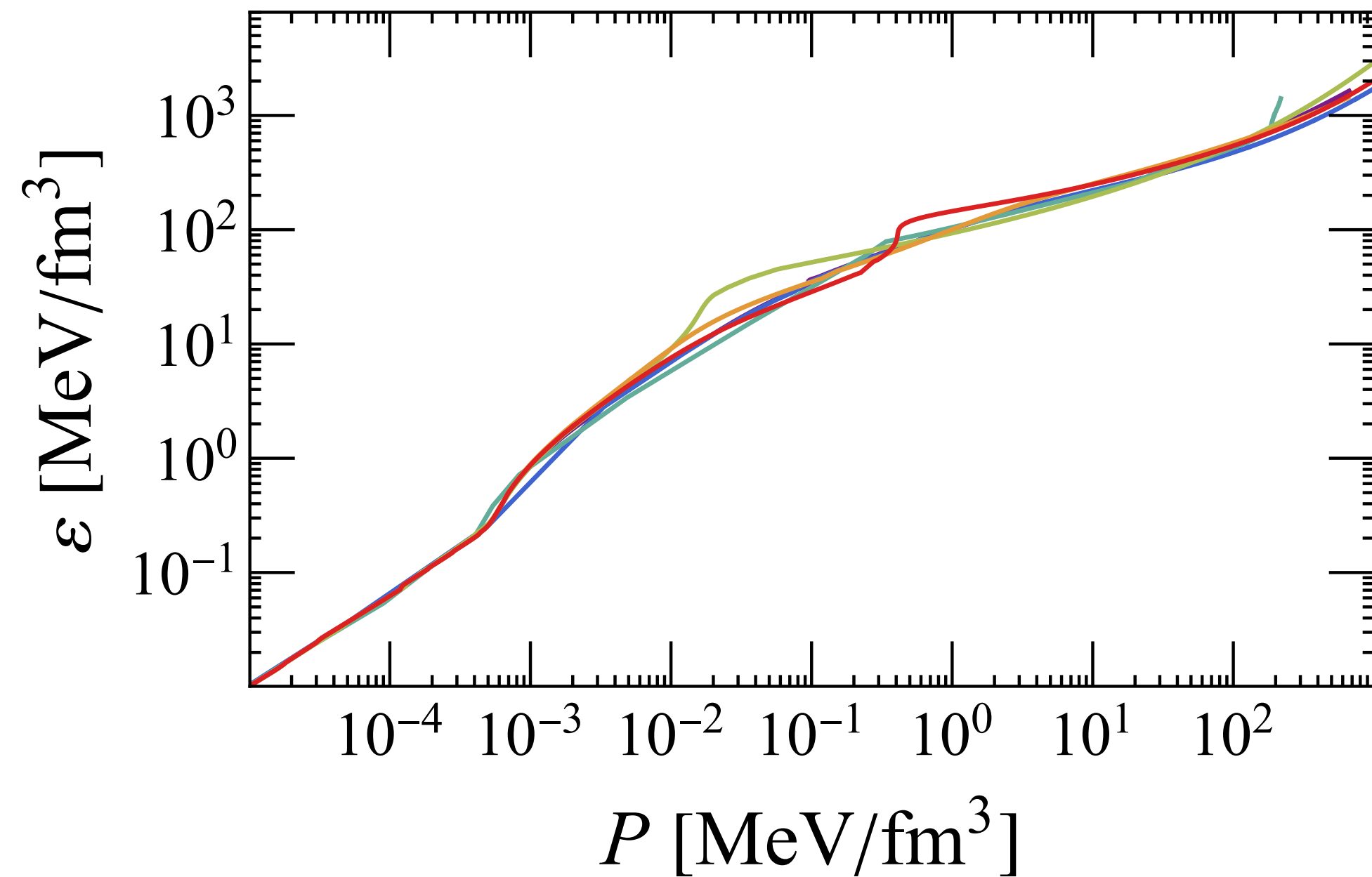
$$P(r = R) = 0.$$

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr},$$

$$\varepsilon = \varepsilon(P),$$

Tolman-Oppenheimer-Volkoff equations



Tolman-Oppenheimer-Volkoff equations:

Tolman, Phys.Rev. 55 (1939) 364-373

Oppenheimer, Volkoff, Phys.Rev. 55 (1939) 374-381

$$m(r = 0) = 0,$$

$$P(r = 0) = P_{\text{cen}},$$

$$m(r = R) = M,$$

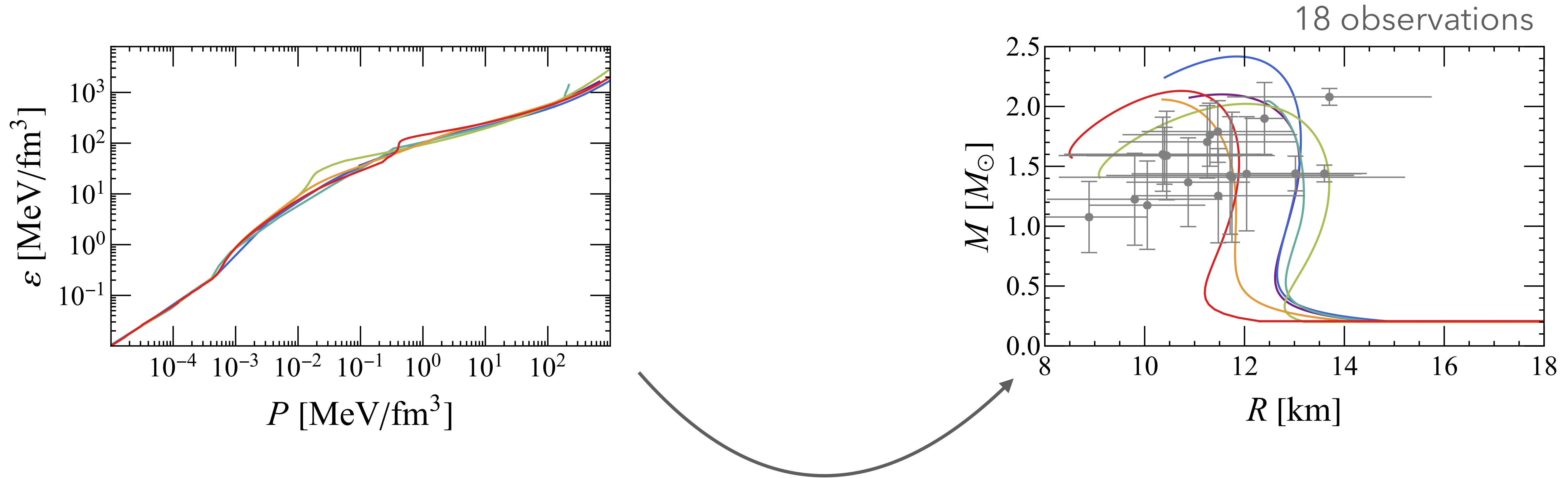
$$P(r = R) = 0.$$

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr},$$

$$\varepsilon = \varepsilon(P),$$

Tolman-Oppenheimer-Volkoff equations



Tolman-Oppenheimer-Volkoff equations:

Tolman, Phys.Rev. 55 (1939) 364-373

Oppenheimer, Volkoff, Phys.Rev. 55 (1939) 374-381

$$m(r = 0) = 0,$$

$$P(r = 0) = P_{\text{cen}},$$

$$m(r = R) = M,$$

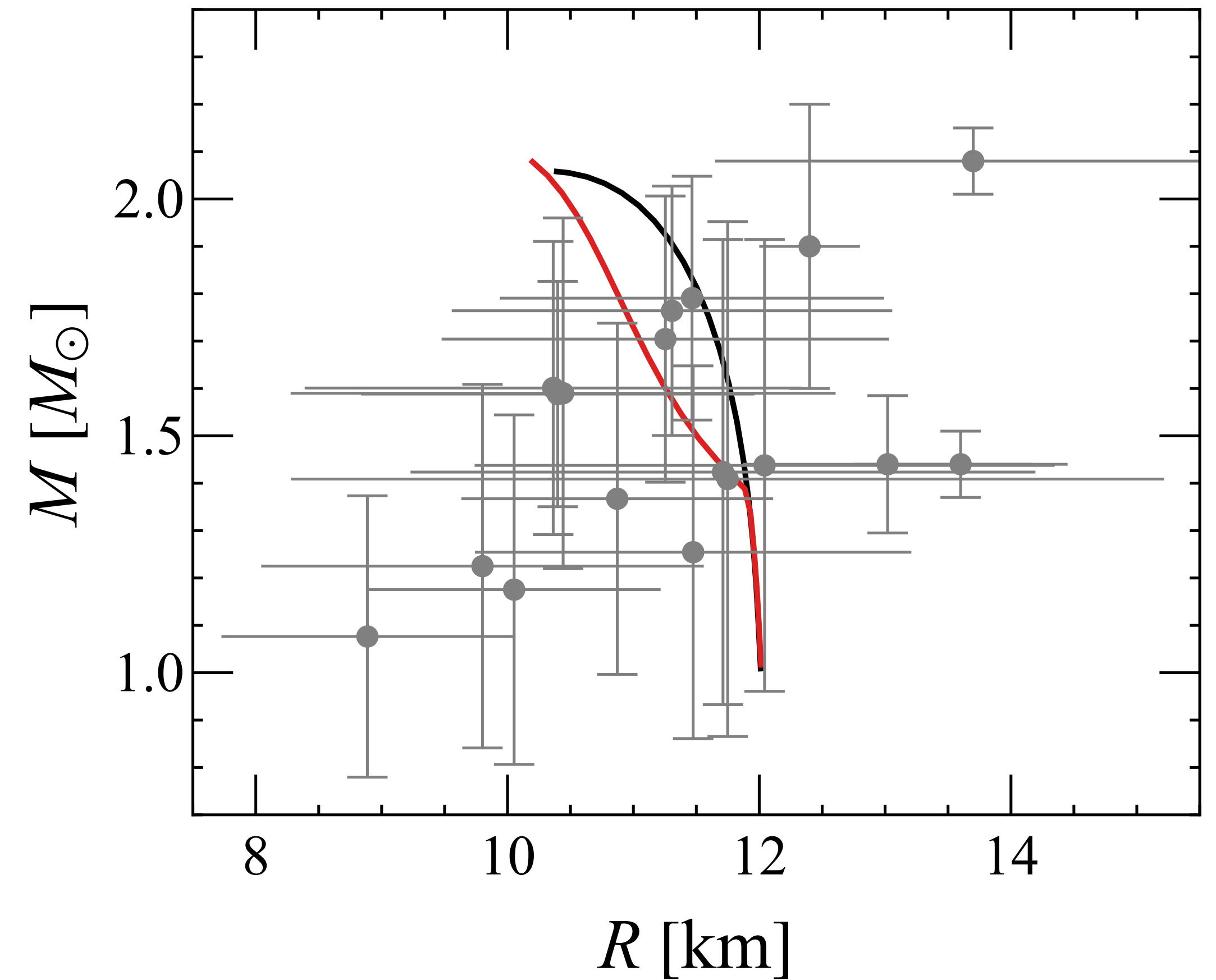
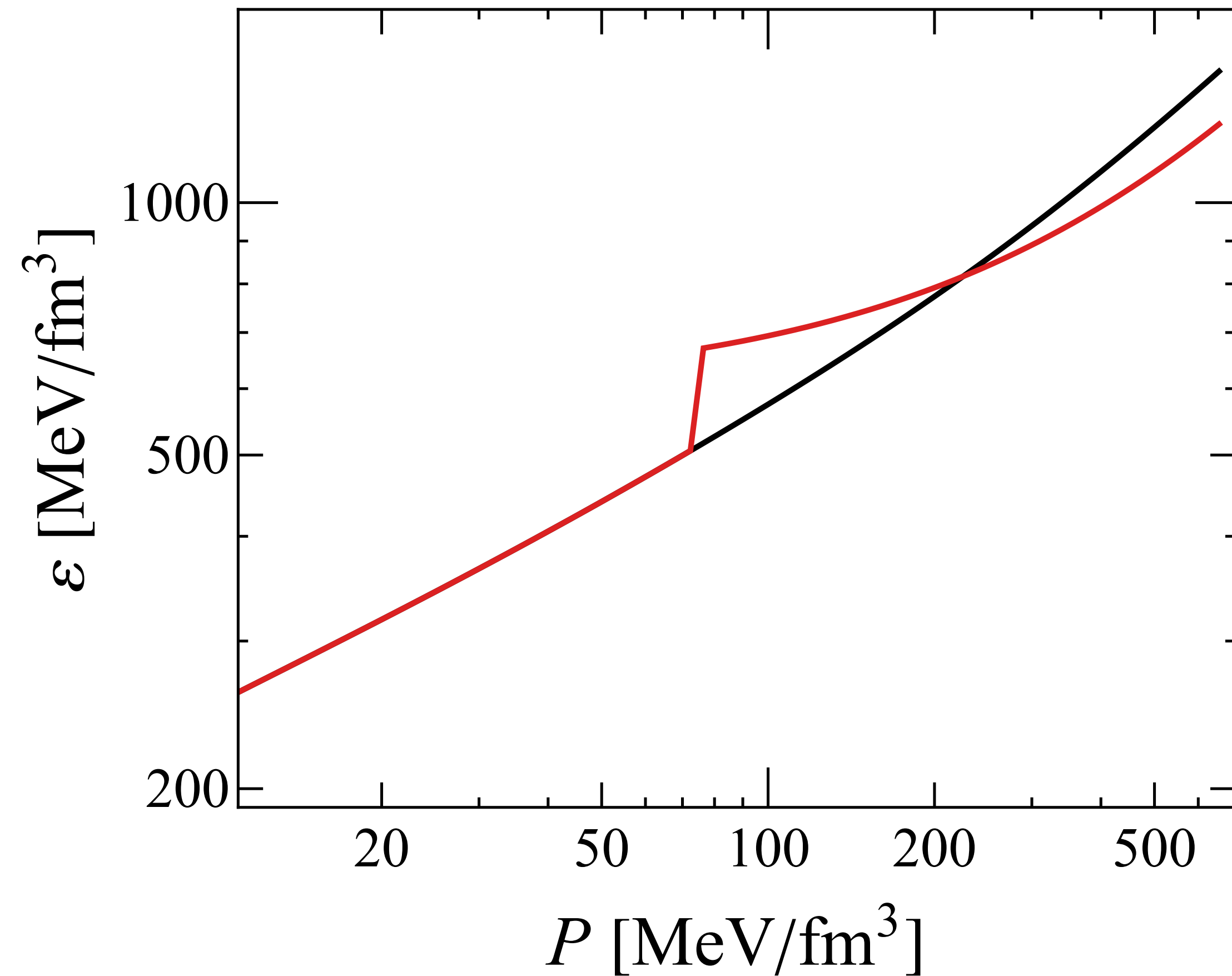
$$P(r = R) = 0.$$

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr},$$

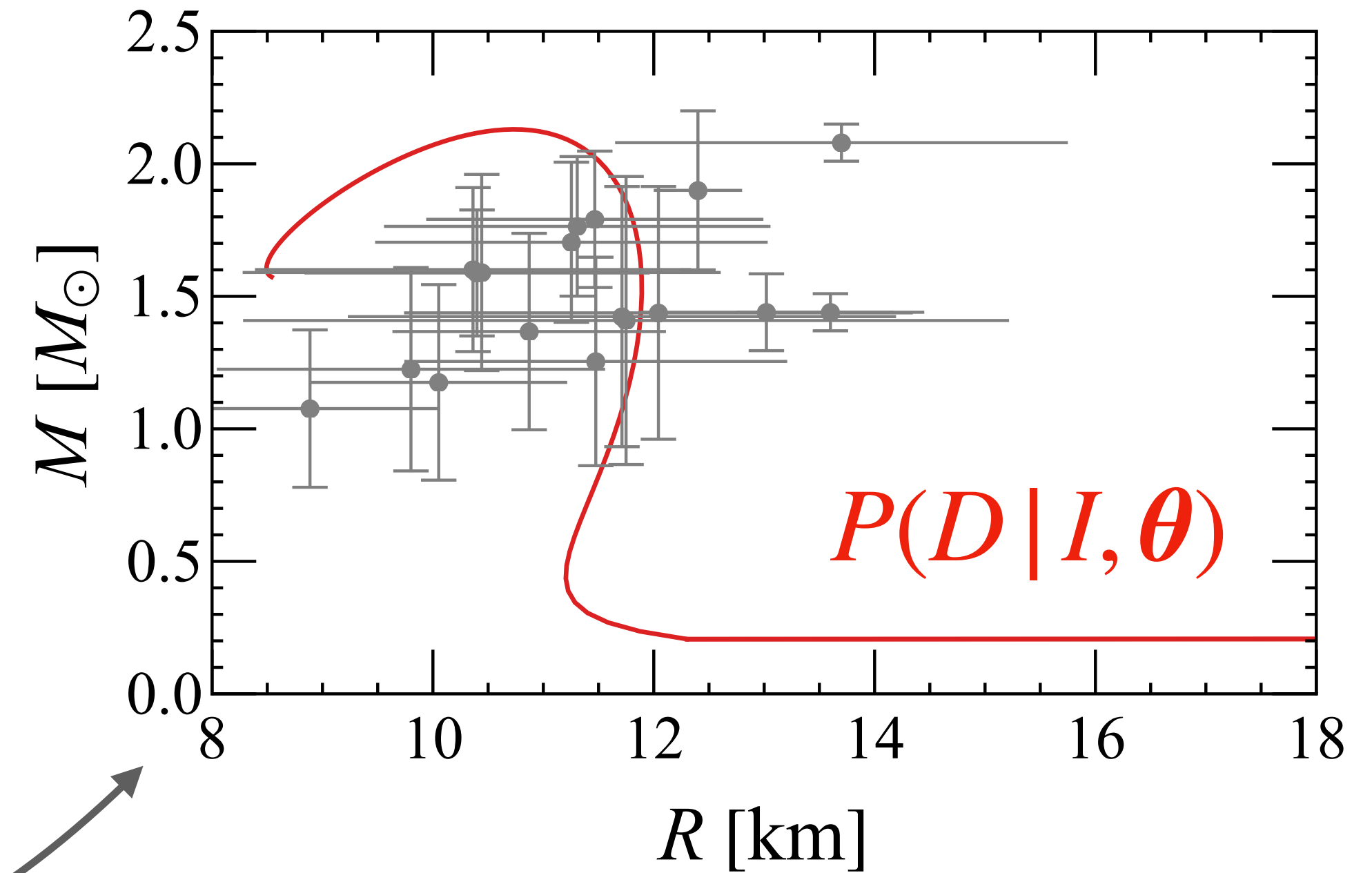
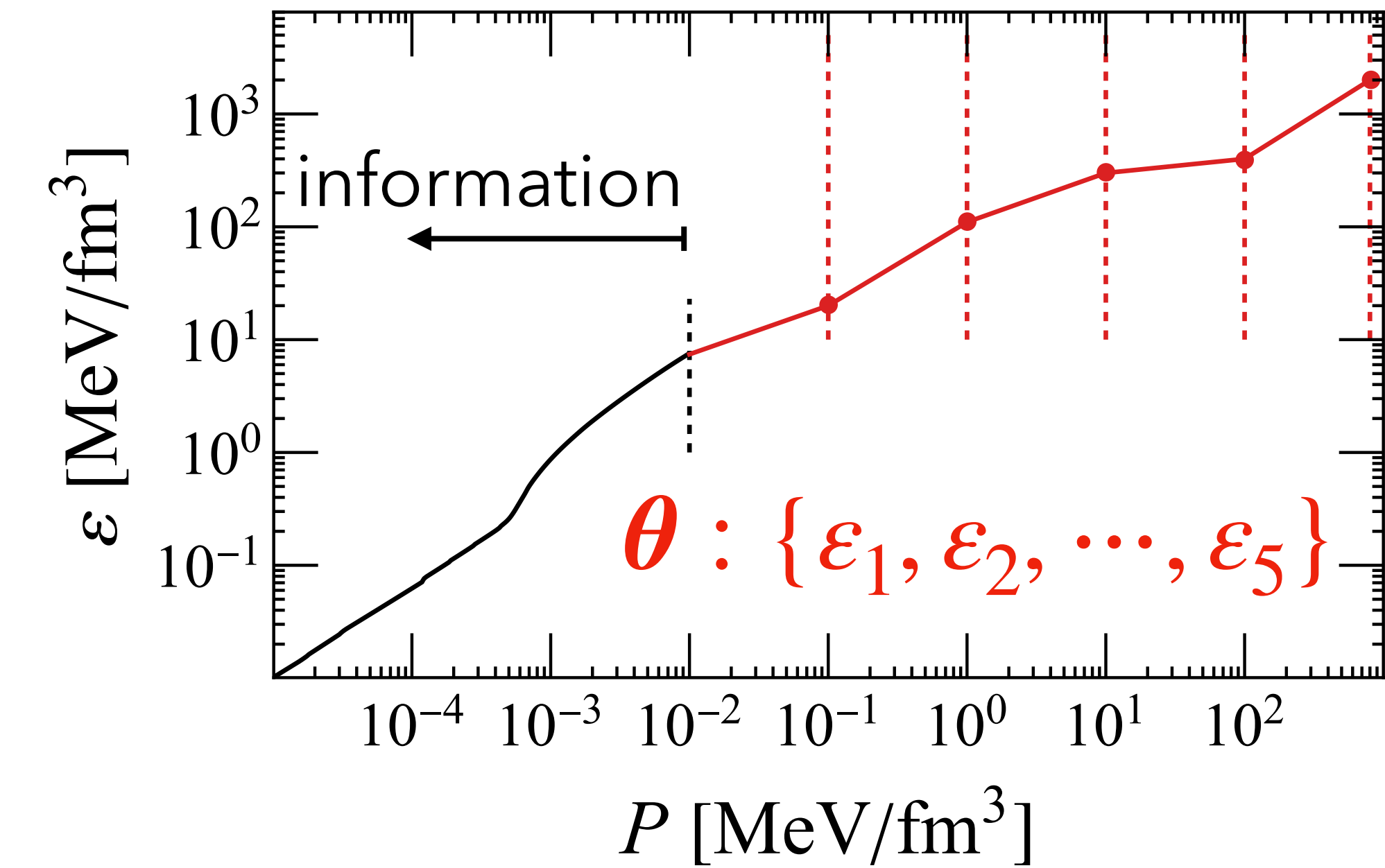
$$\varepsilon = \varepsilon(P),$$

signature of first order phase transition?

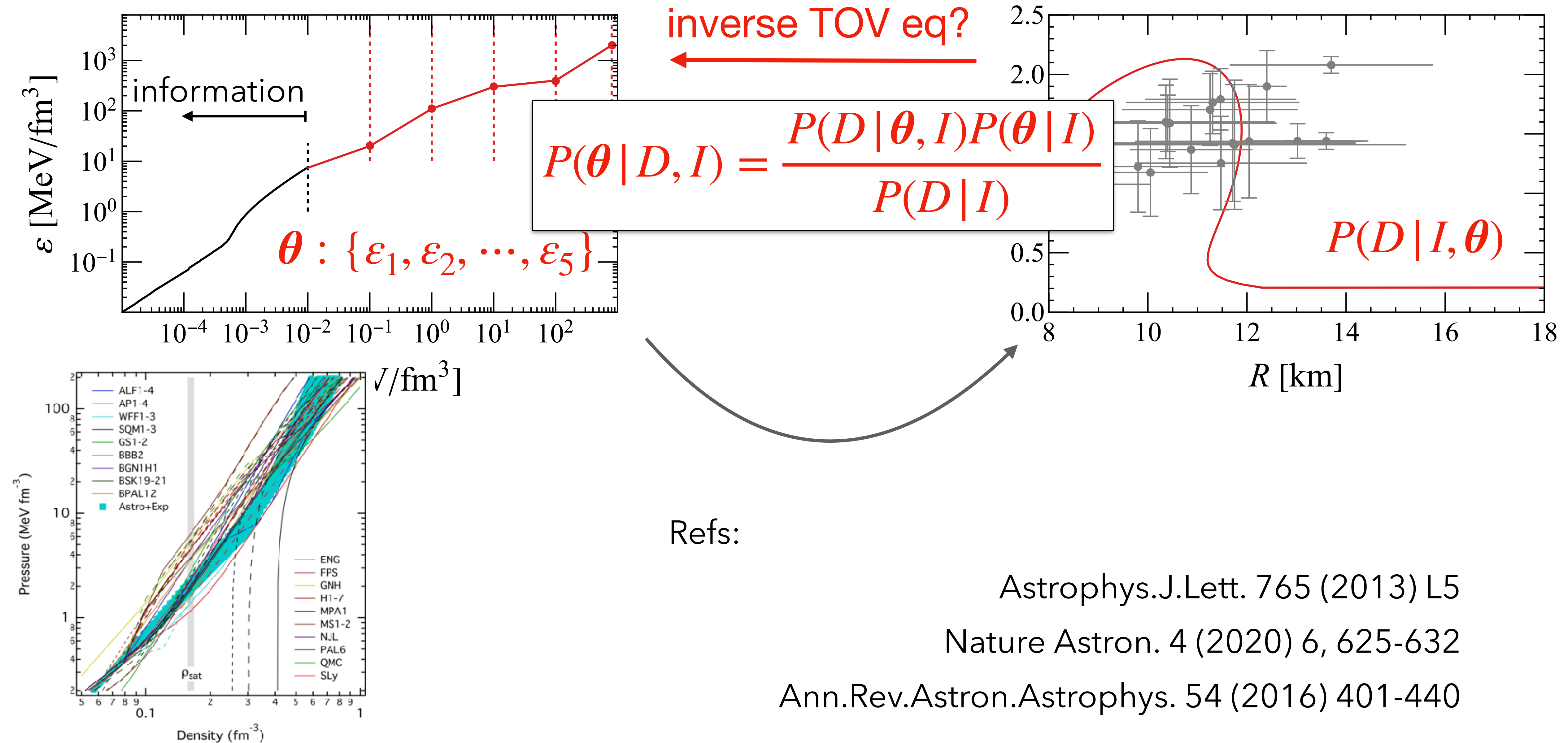


reconstruction given finite number and accuracy?
Bayesian Analysis!

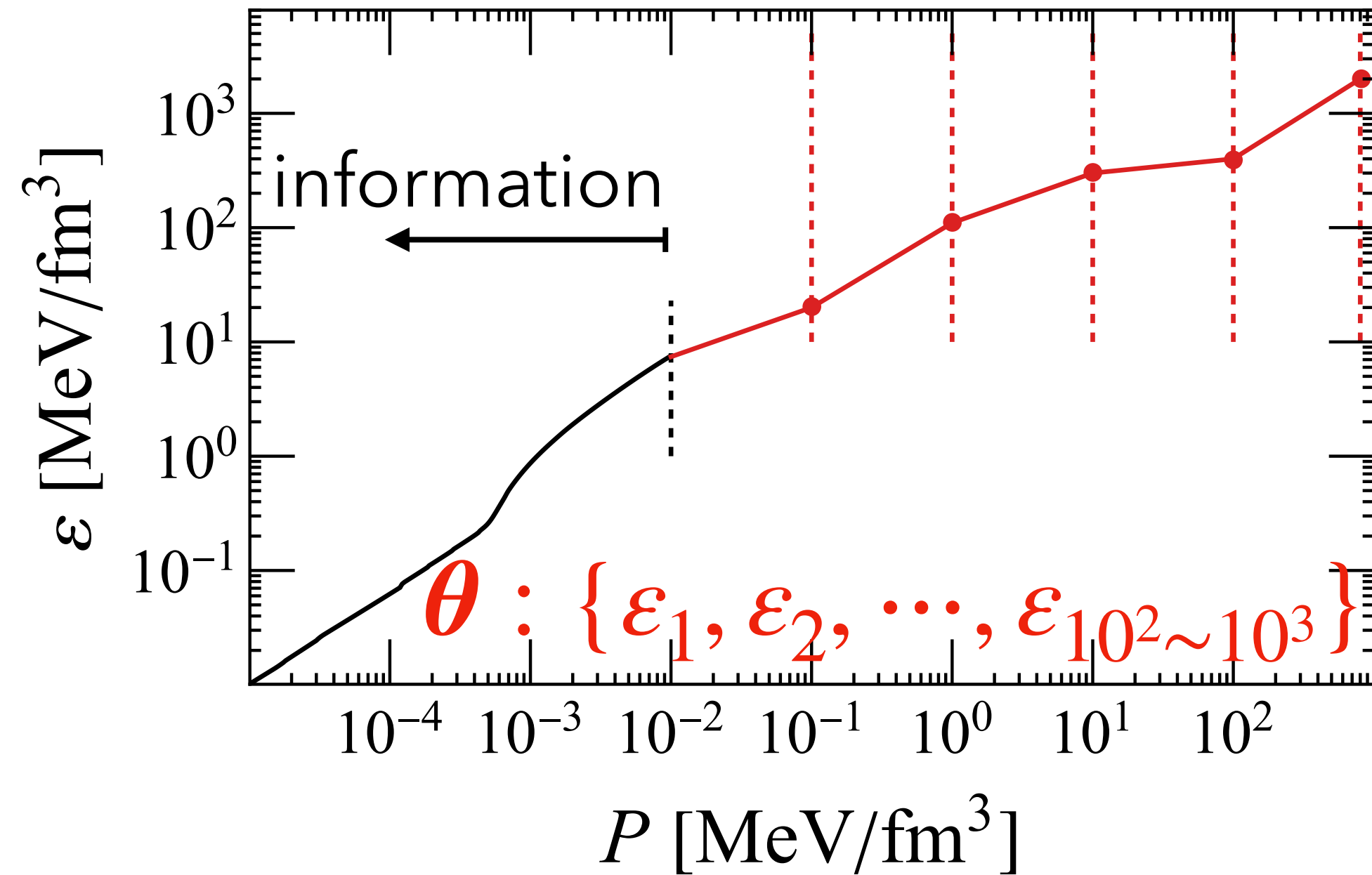
"conventional" Bayesian Analysis



"conventional" Bayesian Analysis



"non-conventional" Bayesian Analysis



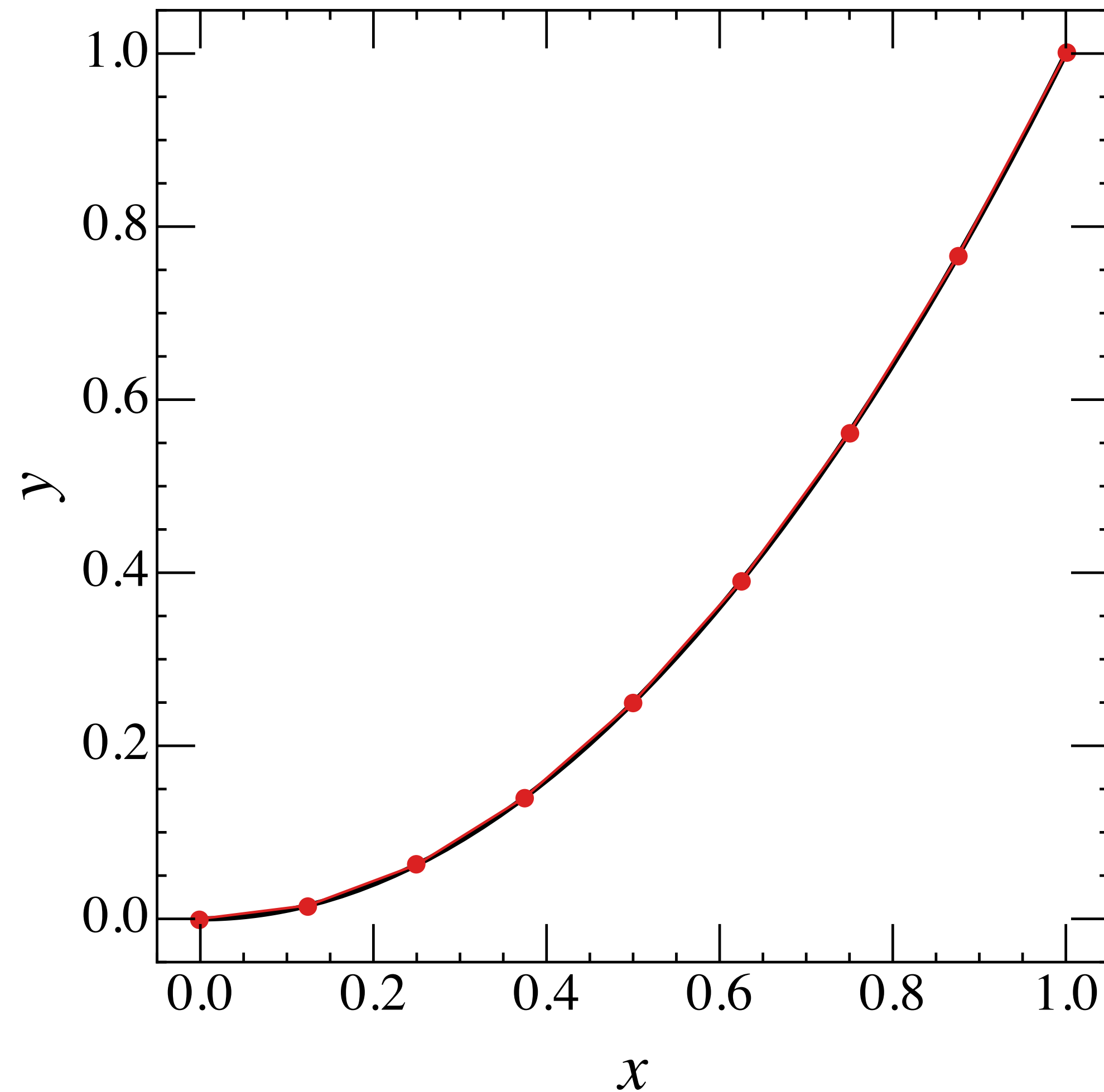
more differential information?

Deep Neural Networks!

What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

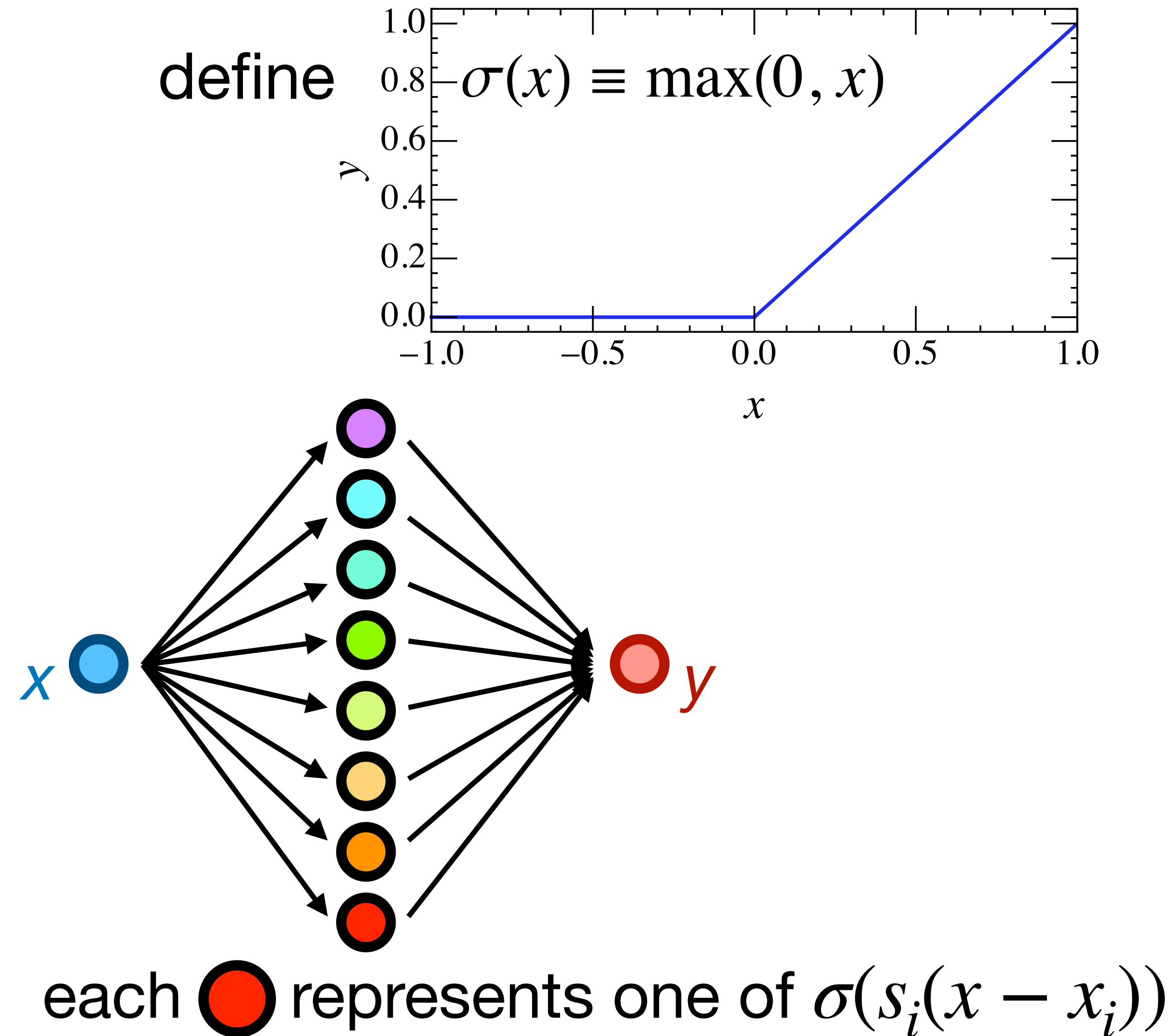
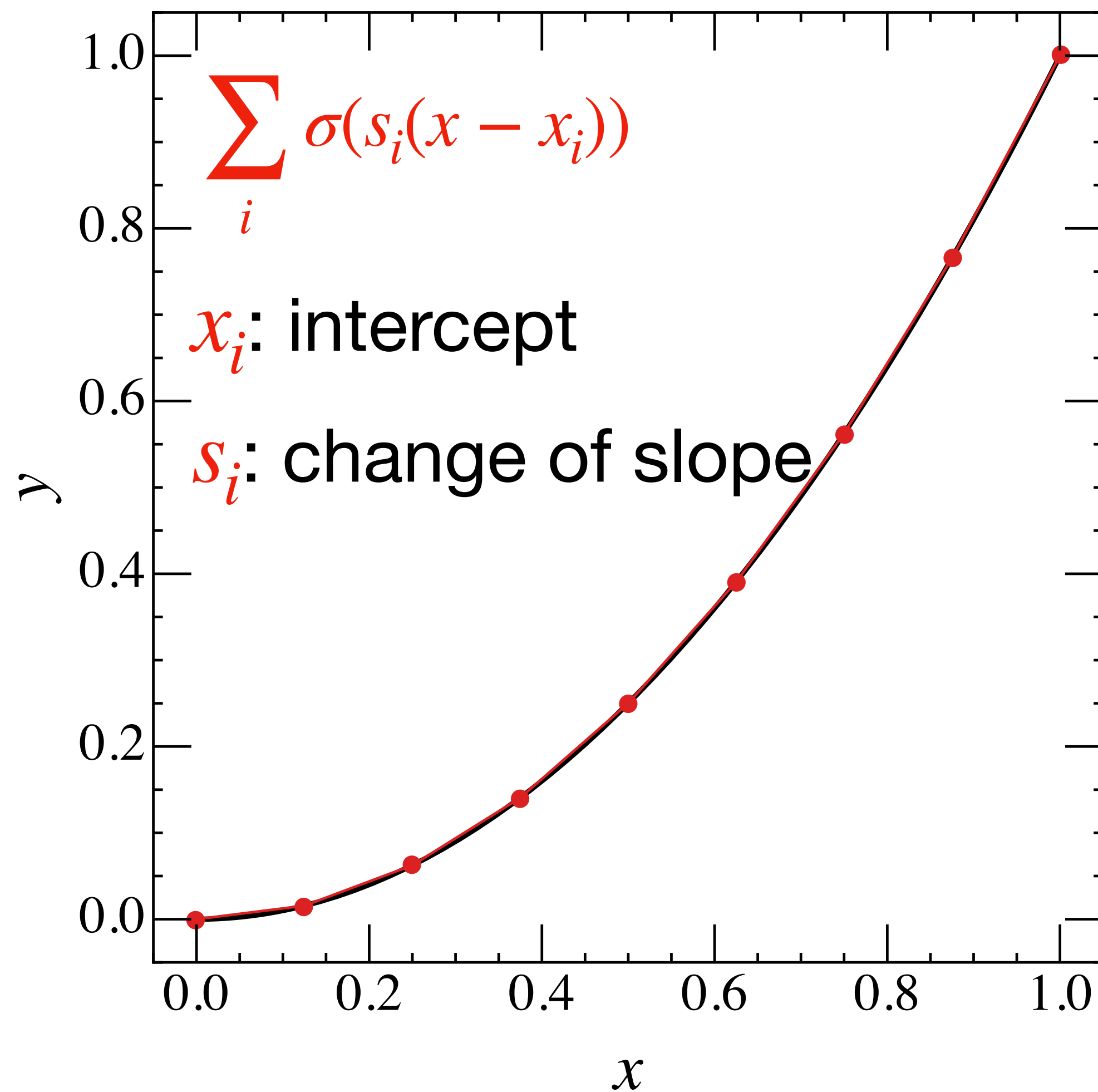
example: approximate $y = x^2$ for $x \in [0,1]$



What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

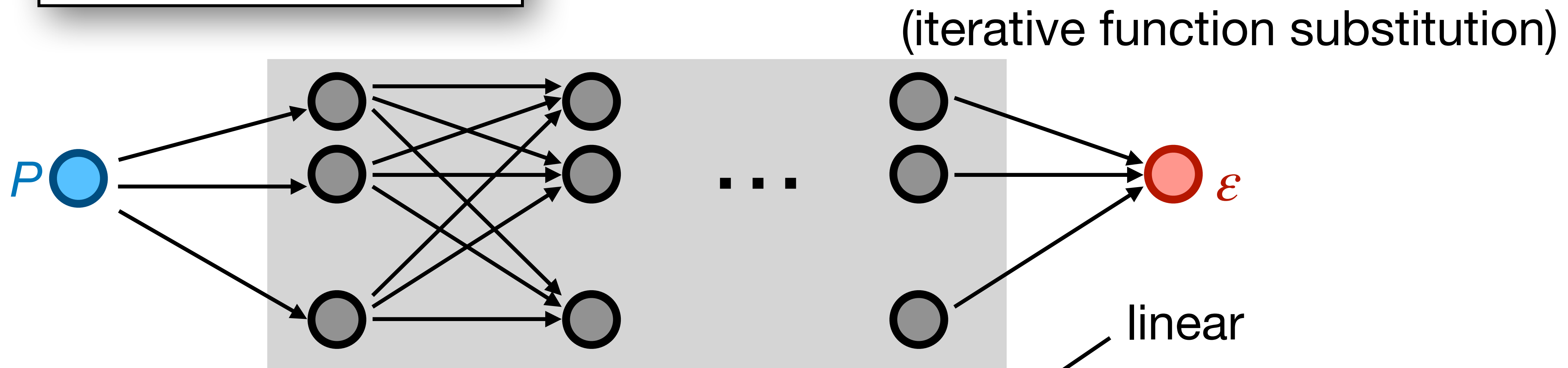
example: approximate $y = x^2$ for $x \in [0,1]$

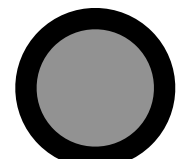


What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

$$\varepsilon(P) \approx \varepsilon_{\text{DNN}}(P | \theta)$$



Each  is an intermediate function ($a_i^{(l)}$):

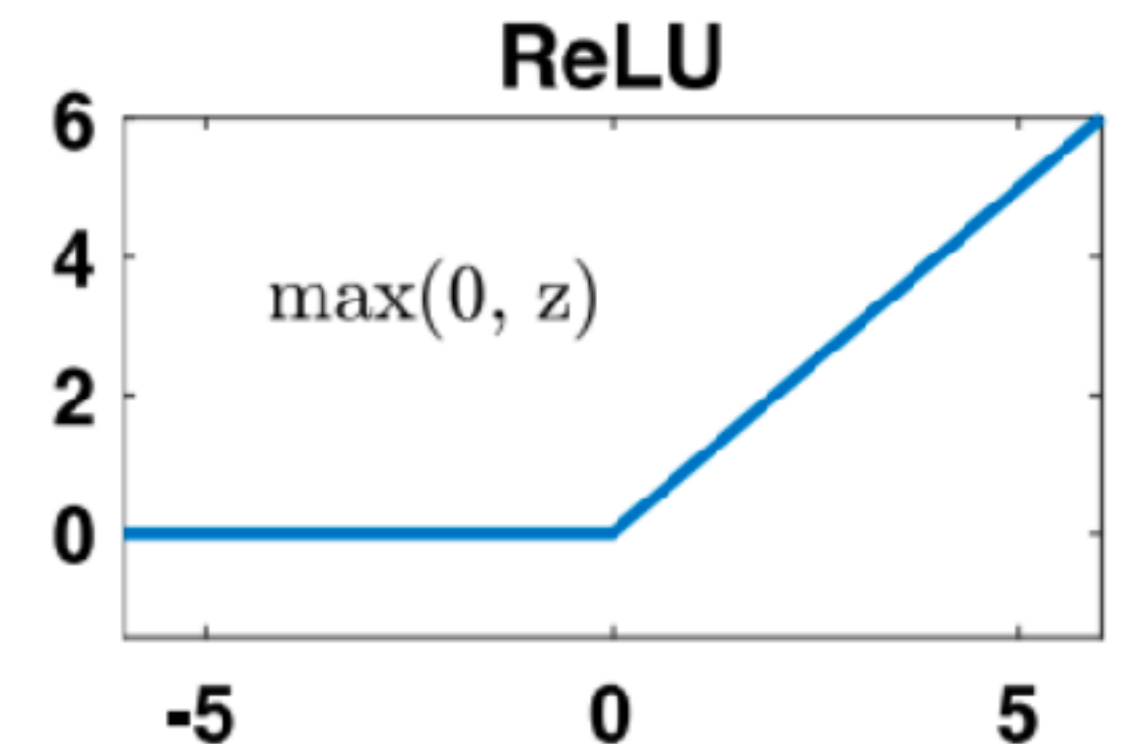
- At the first layer:

$$z_i^{(1)} = b_i^{(1)} + W_{i,1}^{(1)} r, \quad a_i^{(1)} = \sigma(z_i^{(1)})$$

- At later layers:

$$z_i^{(l)} = b_i^{(l)} + \sum_j W_{i,j}^{(l)} a_j^{(l-1)}, \quad a_i^{(l)} = \sigma(z_i^{(l)})$$

nonlinear

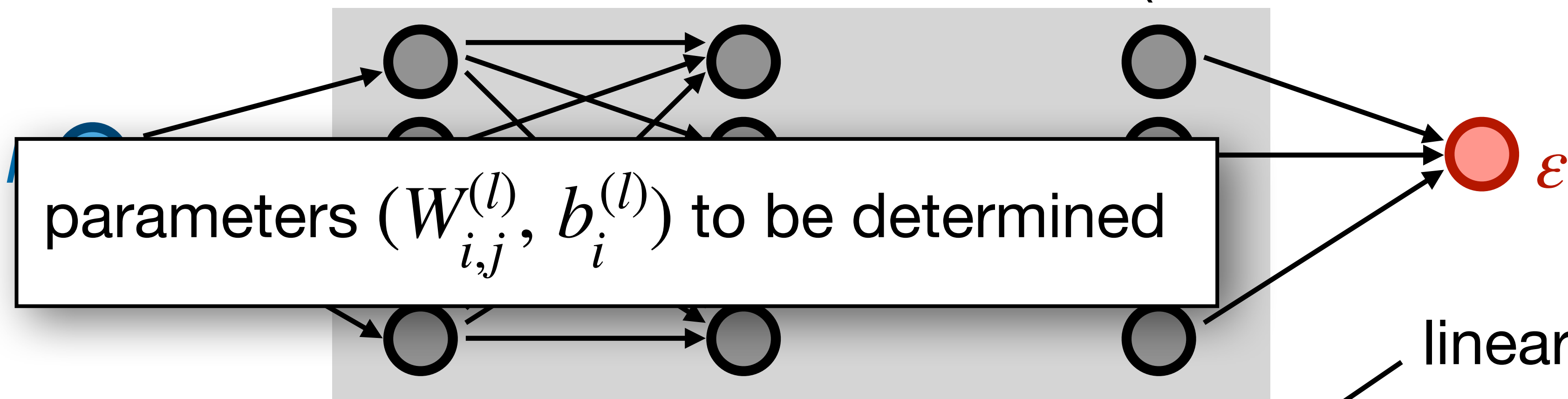


What are Deep Neural Networks?

--- a general parameterization scheme to approximate continuous functions.

$$\varepsilon(P) \approx \varepsilon_{\text{DNN}}(P | \theta)$$

(iterative function substitution)



Each $\bullet$ is an intermediate function $(a_i^{(l)})$:

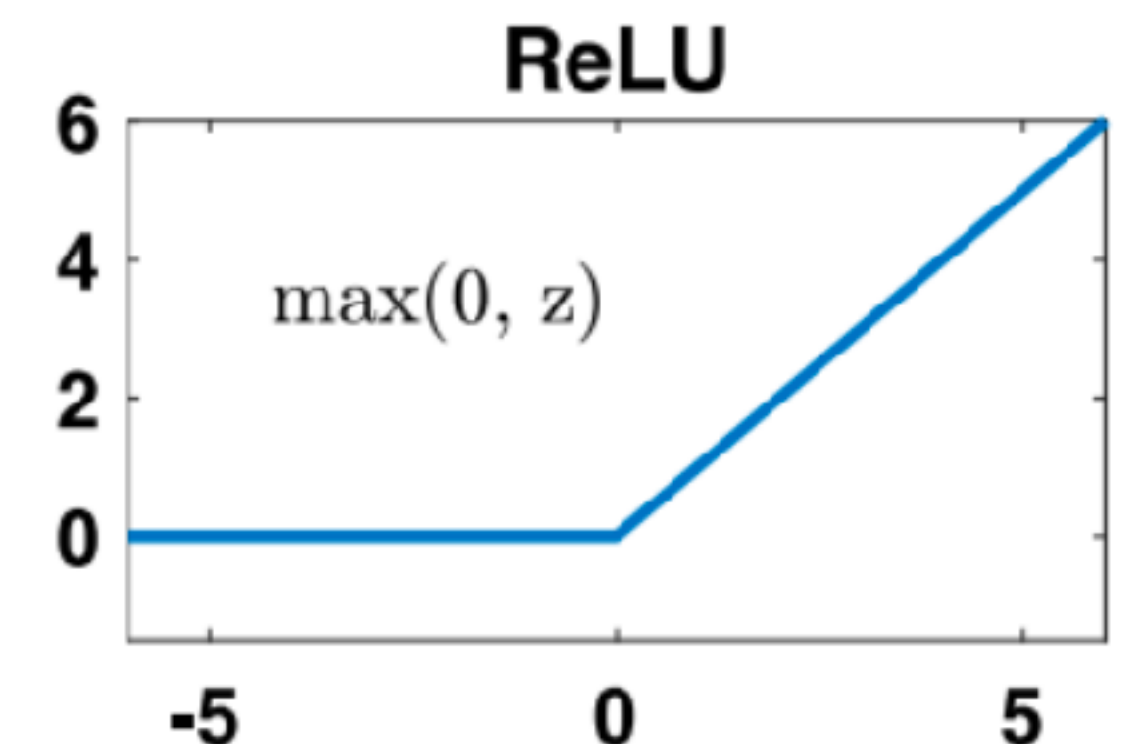
- At the first layer:

$$z_i^{(1)} = b_i^{(1)} + W_{i,1}^{(1)} r, \quad a_i^{(1)} = \sigma(z_i^{(1)})$$

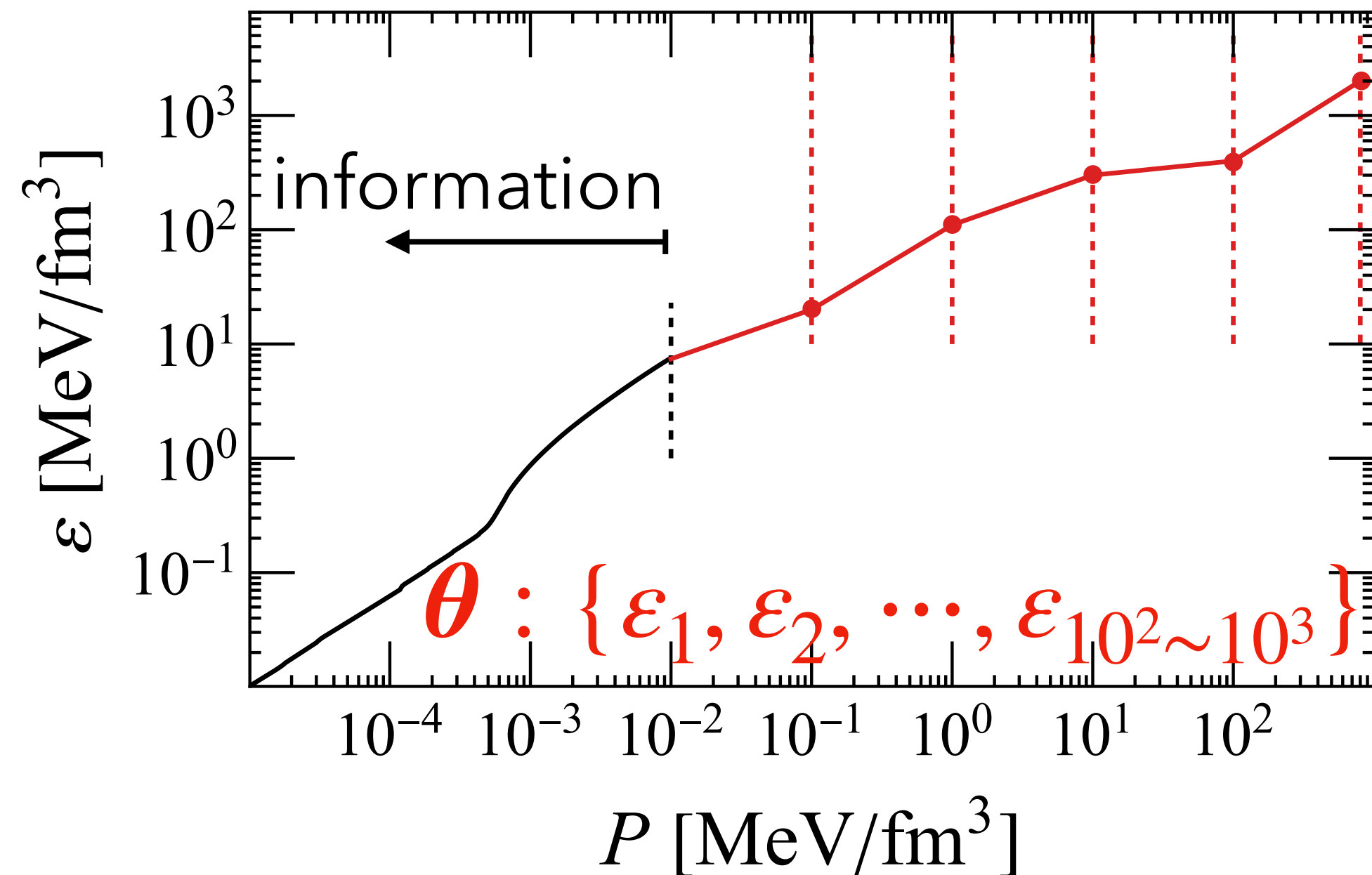
- At later layers:

$$z_i^{(l)} = b_i^{(l)} + \sum_j W_{i,j}^{(l)} a_j^{(l-1)}, \quad a_i^{(l)} = \sigma(z_i^{(l)})$$

nonlinear



“non-conventional” Bayesian Analysis



more differential information?

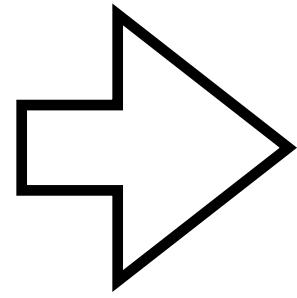
Deep Neural Networks!

BA with $\sim 10^2 - 10^3$ parameters?

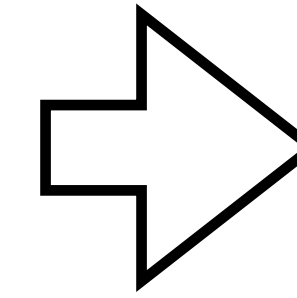
$\nabla_{\theta} P(\theta)$?

gradient: linear response of TOV

input:
EoS



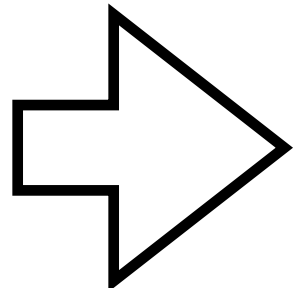
$$\begin{aligned}\frac{dP}{dr} &= -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr}, \\ \frac{dm}{dr} &= 4\pi r^2 \varepsilon, \\ \varepsilon &= \varepsilon(P),\end{aligned}$$



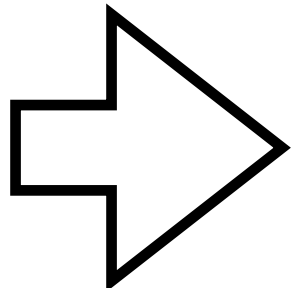
output:
 M - R curve

gradient: linear response of TOV

input:
EoS

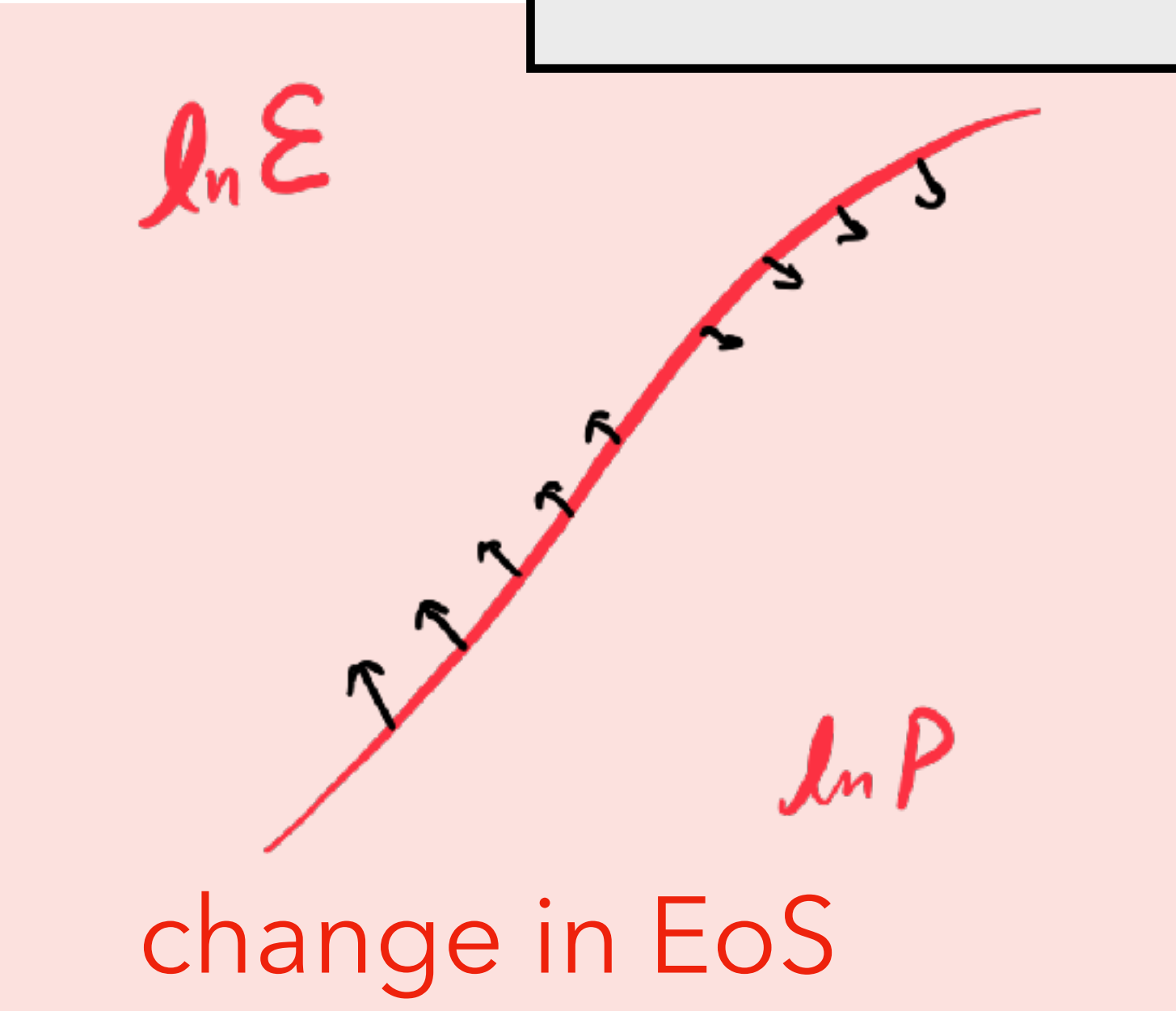


$$\frac{dP}{dr} = -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr},$$
$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

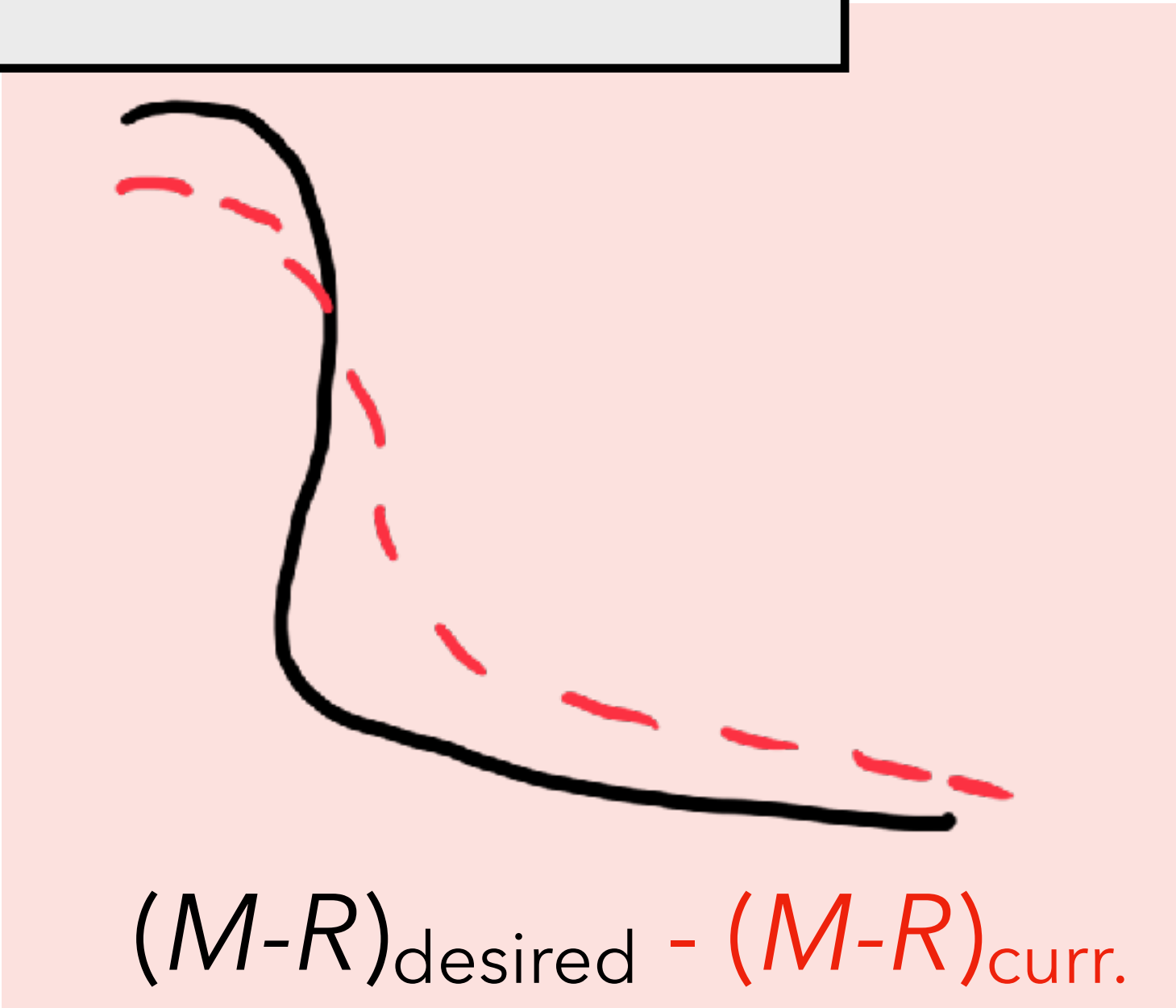


output:
M-R curve

TOV equation: functional mapping EoS to M-R;
Inverse TOV: functional derivatives!

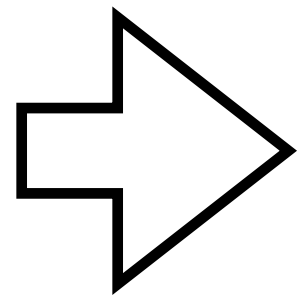


$$\frac{\delta (M-R)}{\delta (\text{EoS})}$$

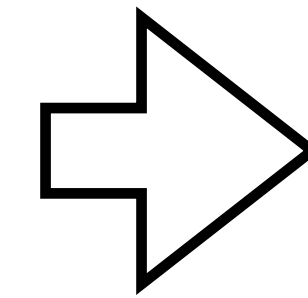


gradient: linear response of TOV

input:
EoS



$$\begin{aligned}\frac{dP}{dr} &= -\frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr}, \\ \frac{dm}{dr} &= 4\pi r^2 \varepsilon, \\ \varepsilon &= \varepsilon(P),\end{aligned}$$



output:
 M - R curve

central
pressure

radius

$$P_c = \int_0^R \frac{(m + 4\pi r^3 P)(P + \varepsilon)}{r^2 - 2mr} dr,$$

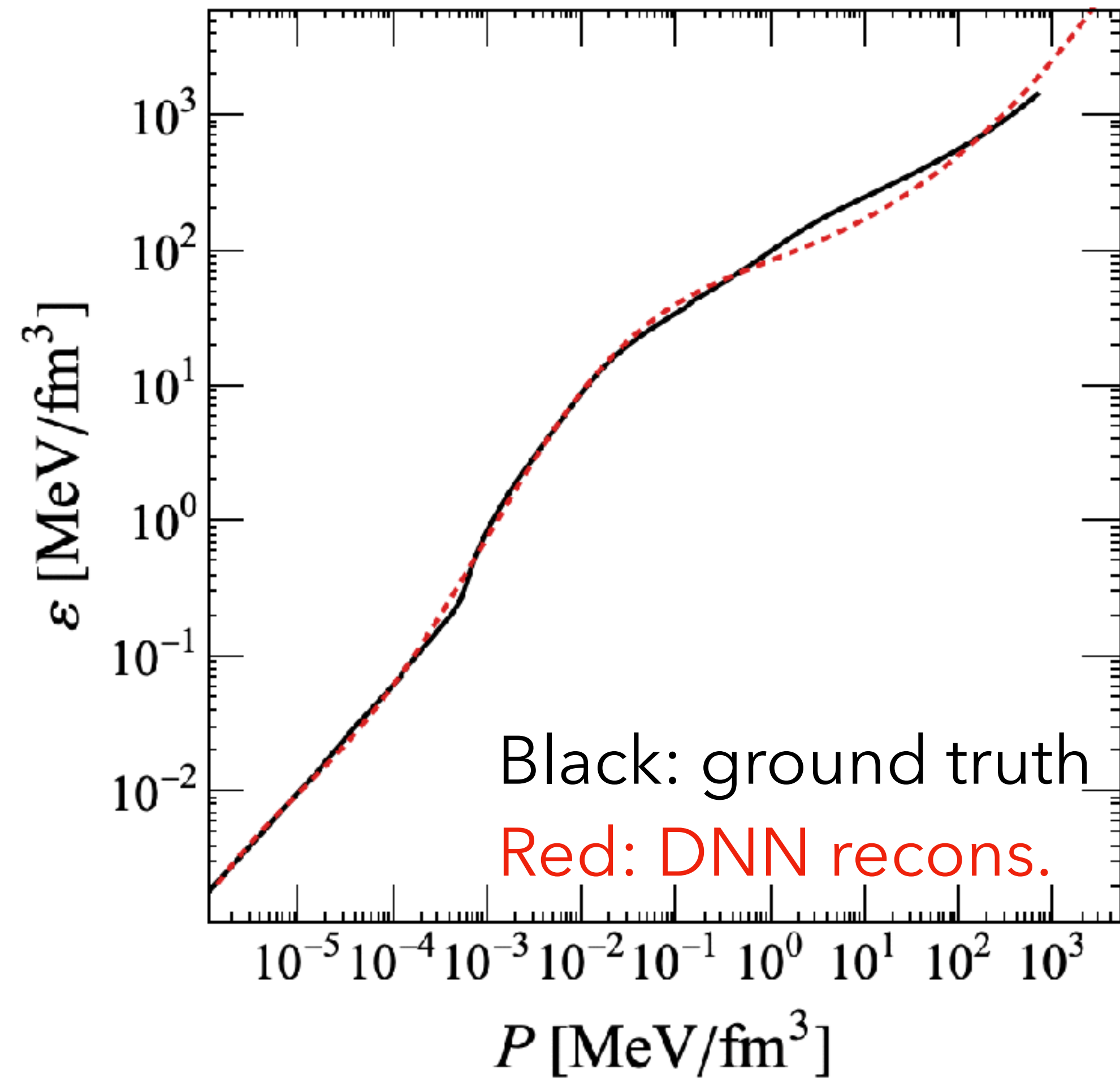
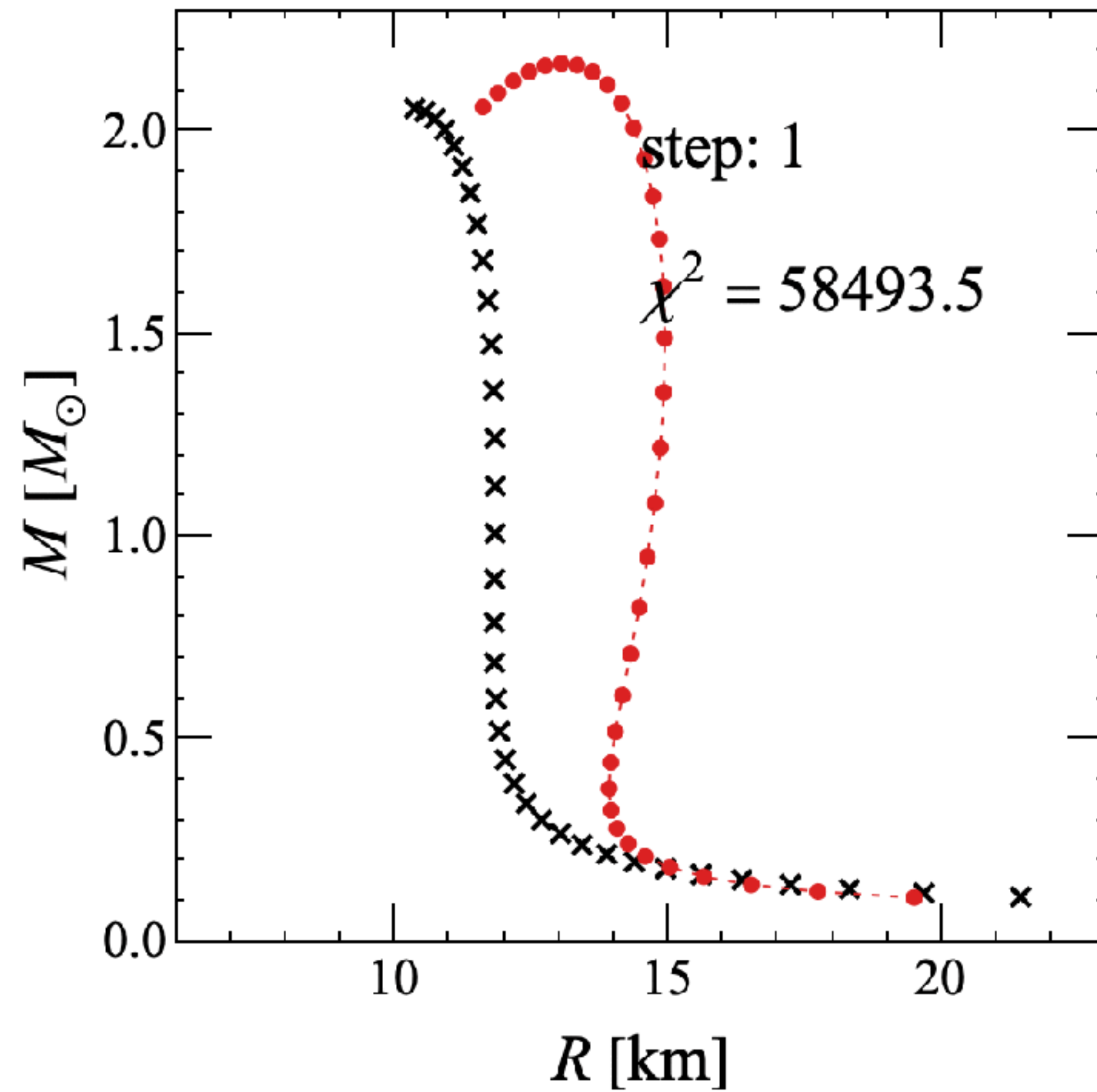
mass

$$M = 4\pi \int_0^R r^2 \varepsilon dr,$$

$$\varepsilon(P) \rightarrow \varepsilon(P) + \delta\varepsilon \delta(P - P')$$

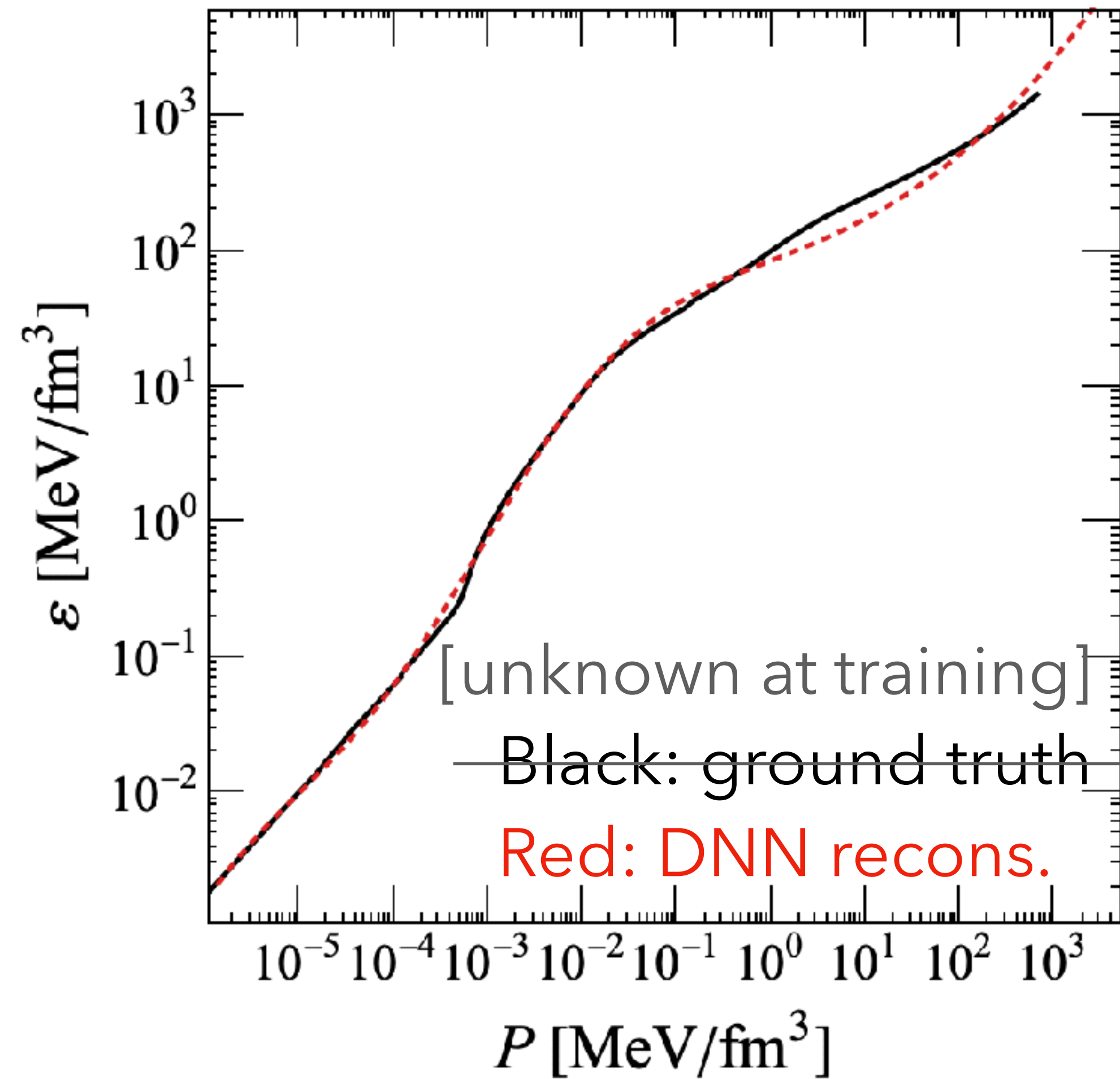
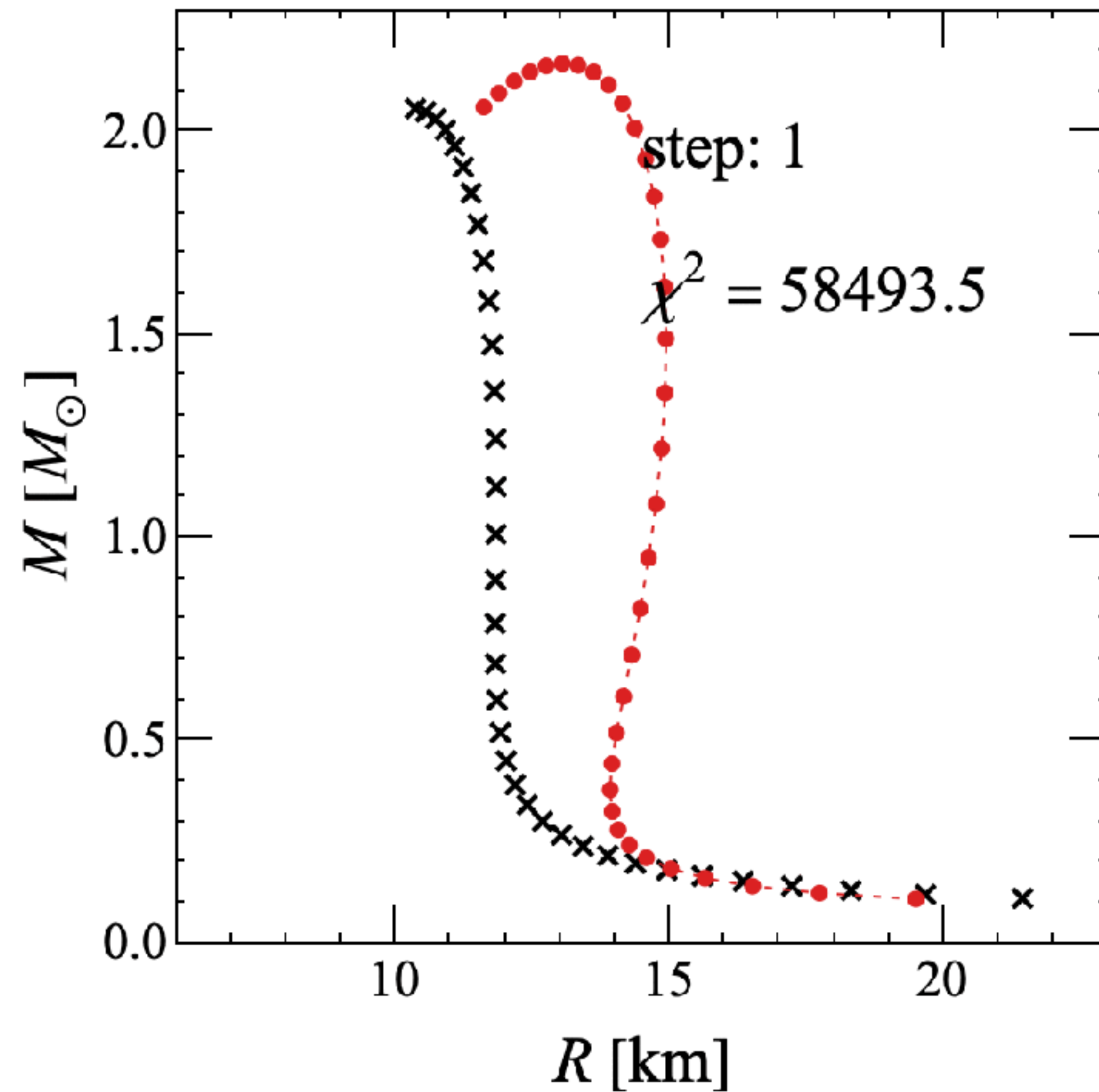
$$R \rightarrow R + \delta R, \quad M \rightarrow M + \delta M$$

closure test



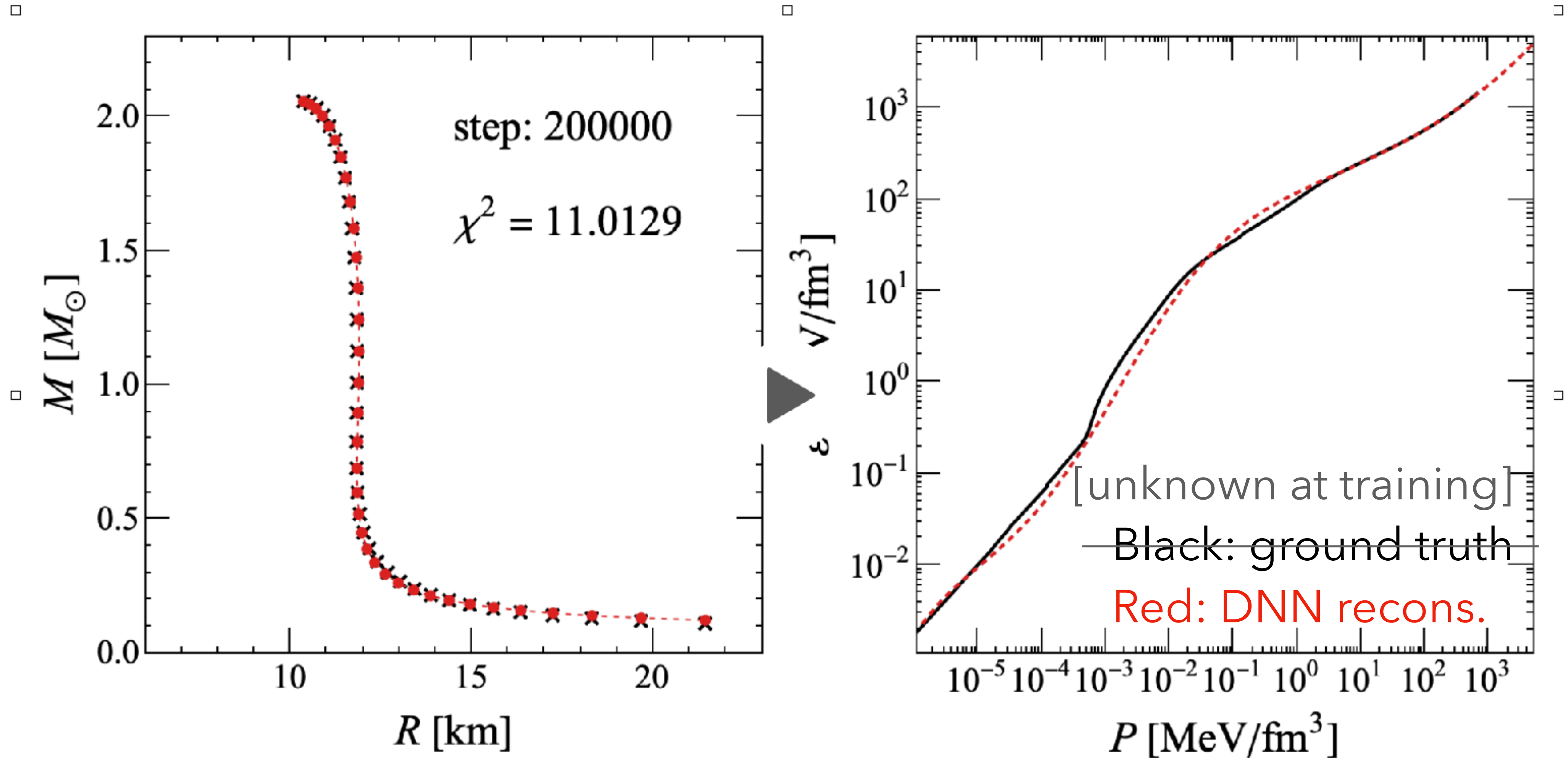
$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{\text{obs}}}{\Delta m_i} \right)^2 + \left(\frac{R_i - R_i^{\text{obs}}}{\Delta R_i} \right)^2$$

closure test



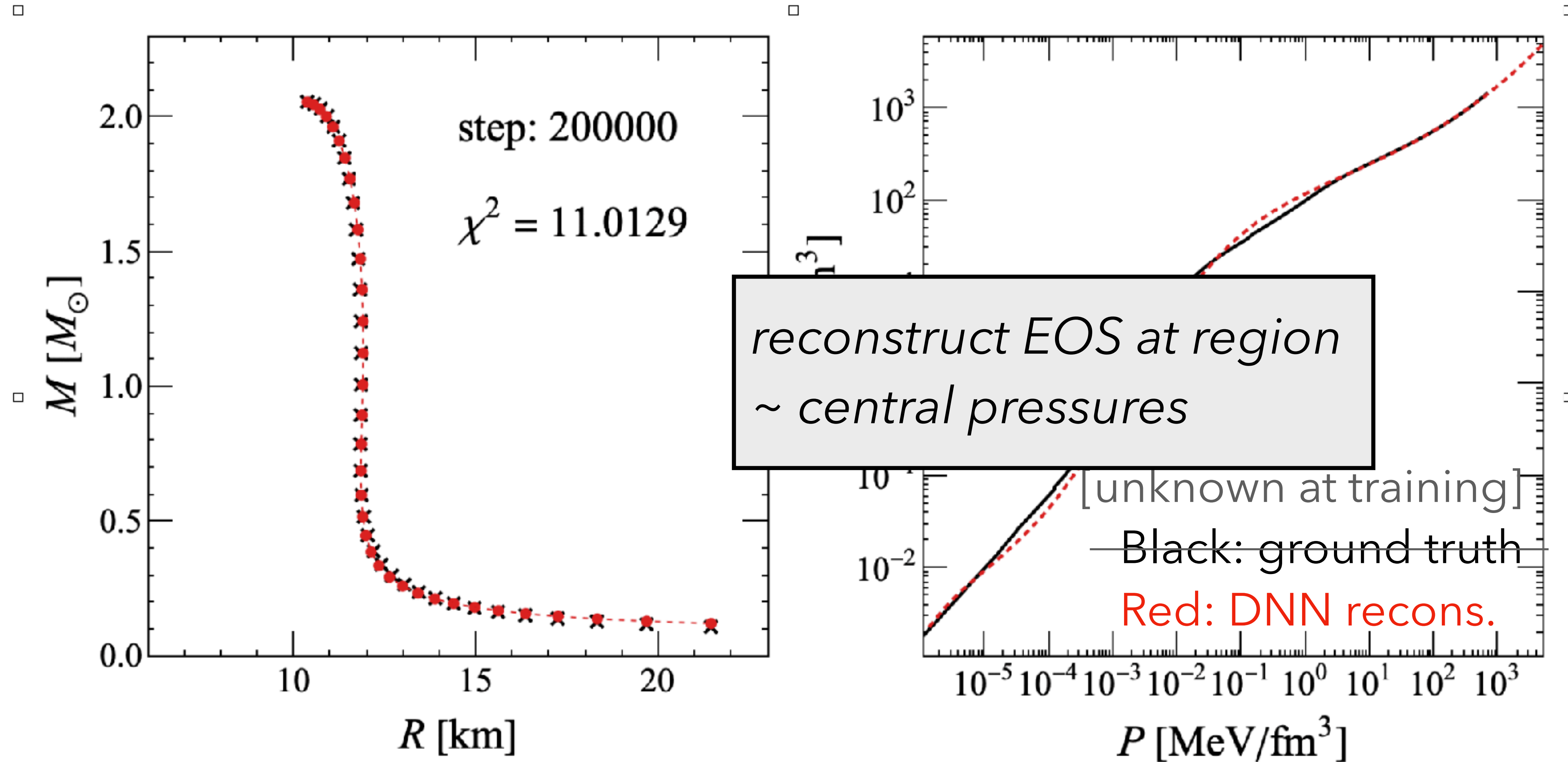
$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{\text{obs}}}{\Delta m_i} \right)^2 + \left(\frac{R_i - R_i^{\text{obs}}}{\Delta R_i} \right)^2$$

closure test



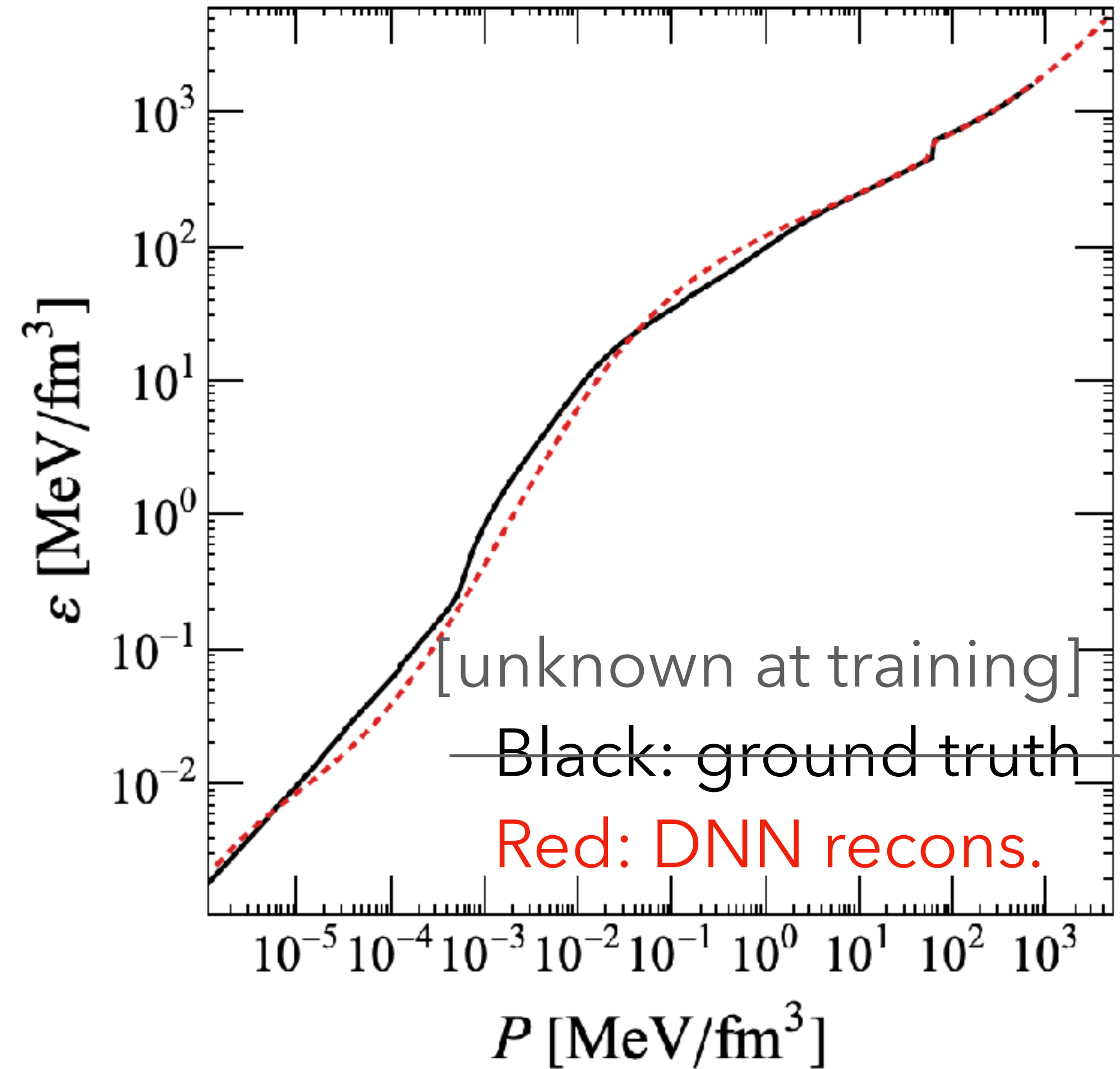
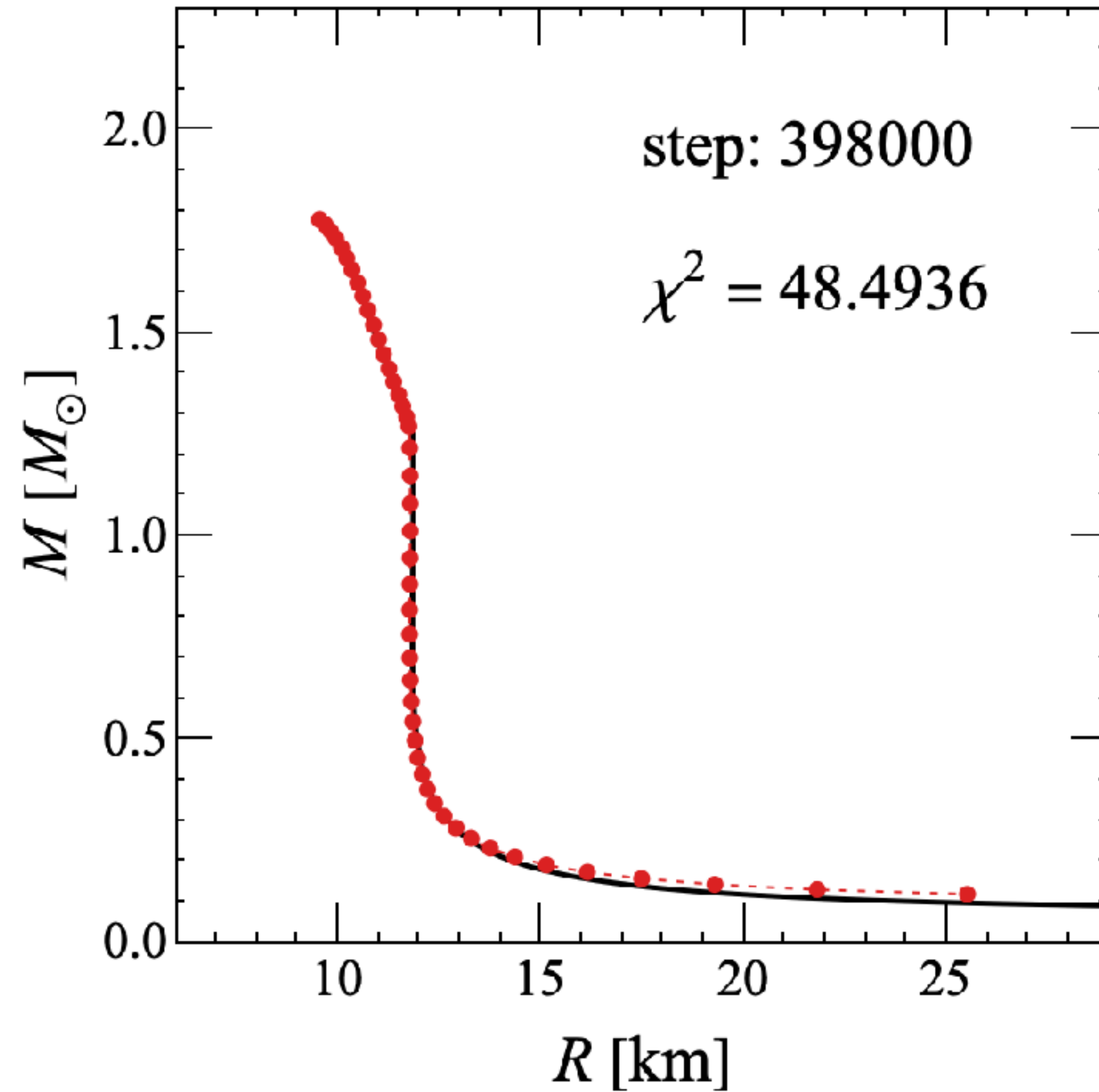
$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{\text{obs}}}{\Delta m_i} \right)^2 + \left(\frac{R_i - R_i^{\text{obs}}}{\Delta R_i} \right)^2$$

closure test

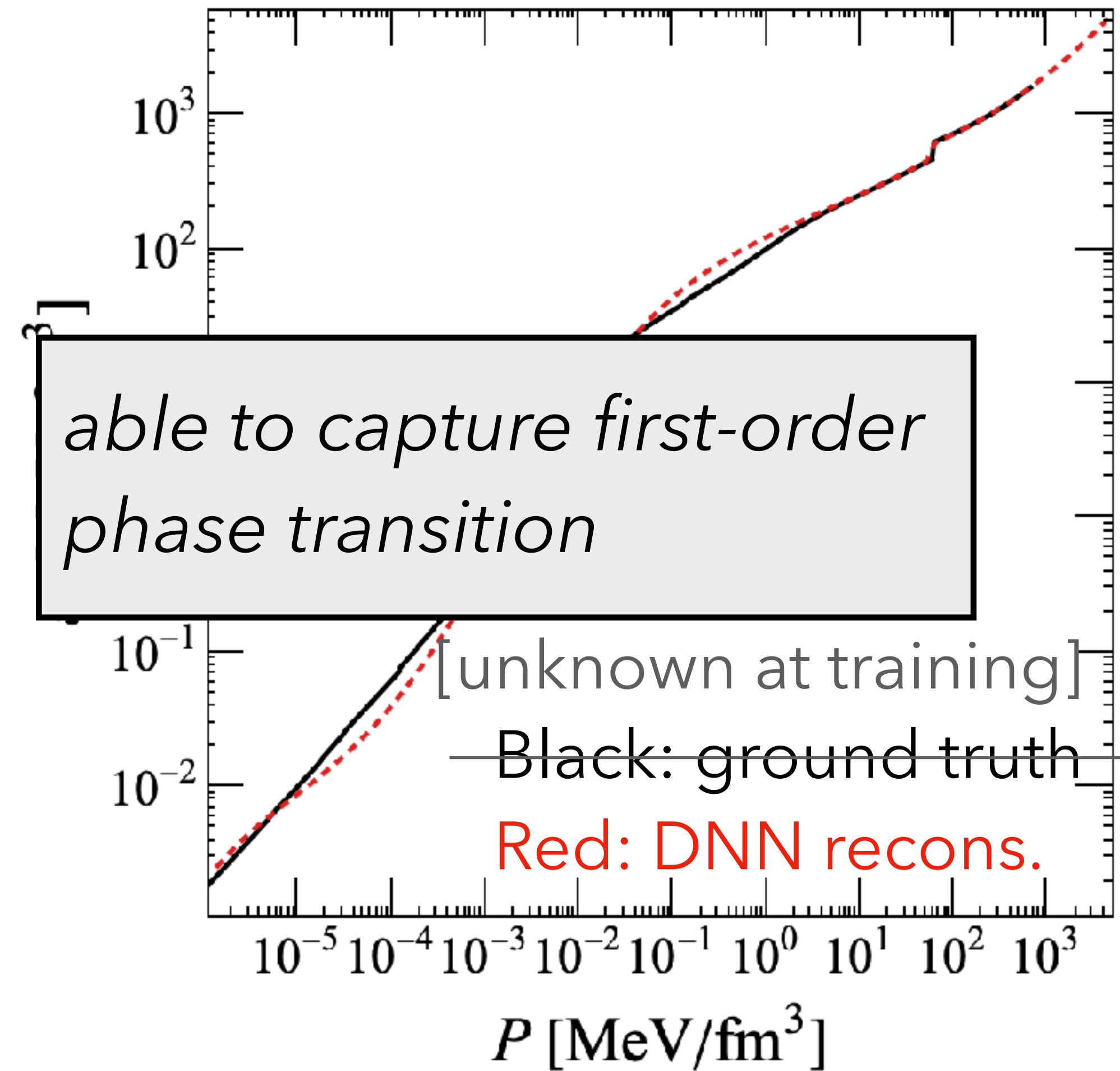
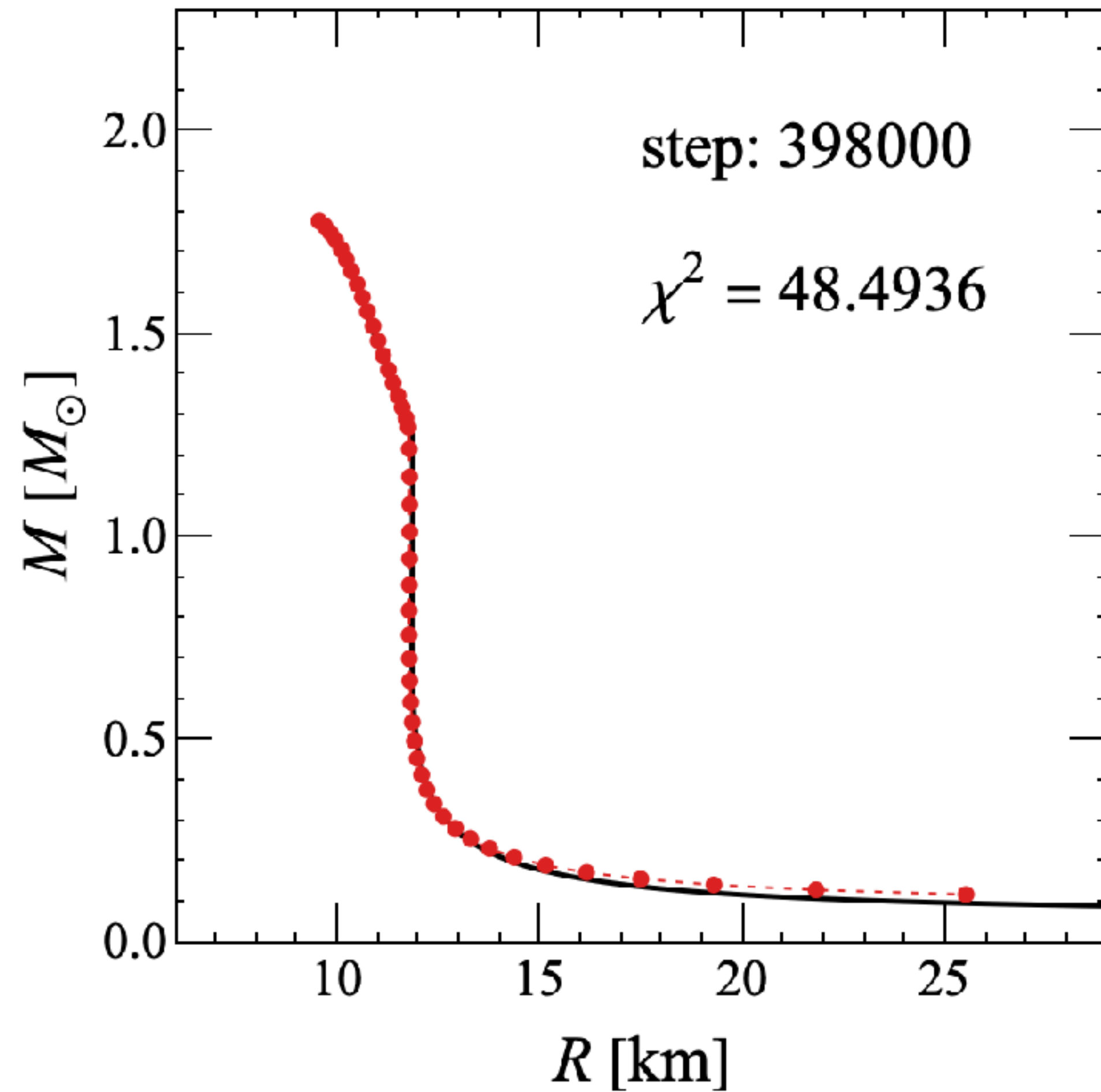


$$\chi^2 = \sum_i \left(\frac{m_i - m_i^{\text{obs}}}{\Delta m_i} \right)^2 + \left(\frac{R_i - R_i^{\text{obs}}}{\Delta R_i} \right)^2$$

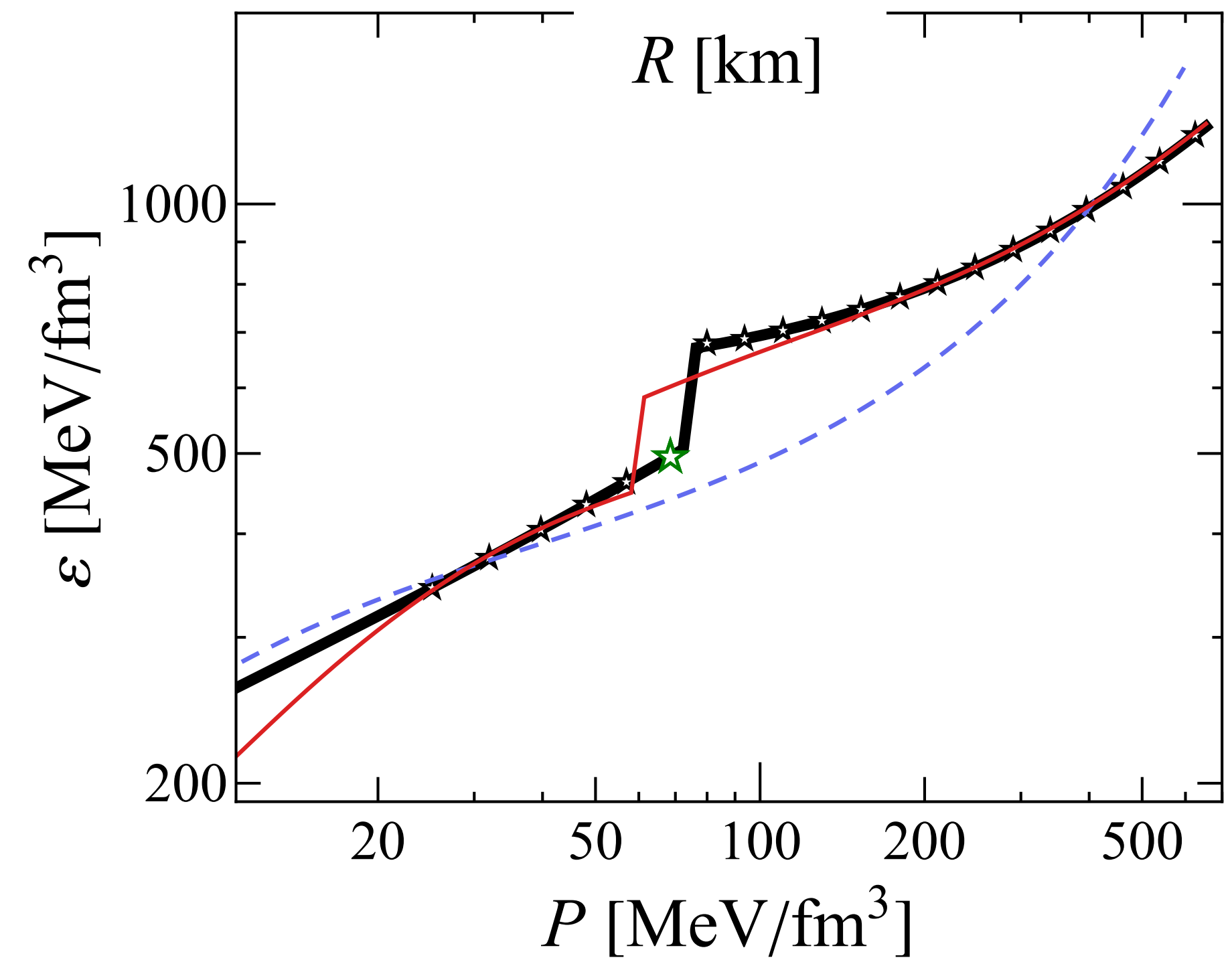
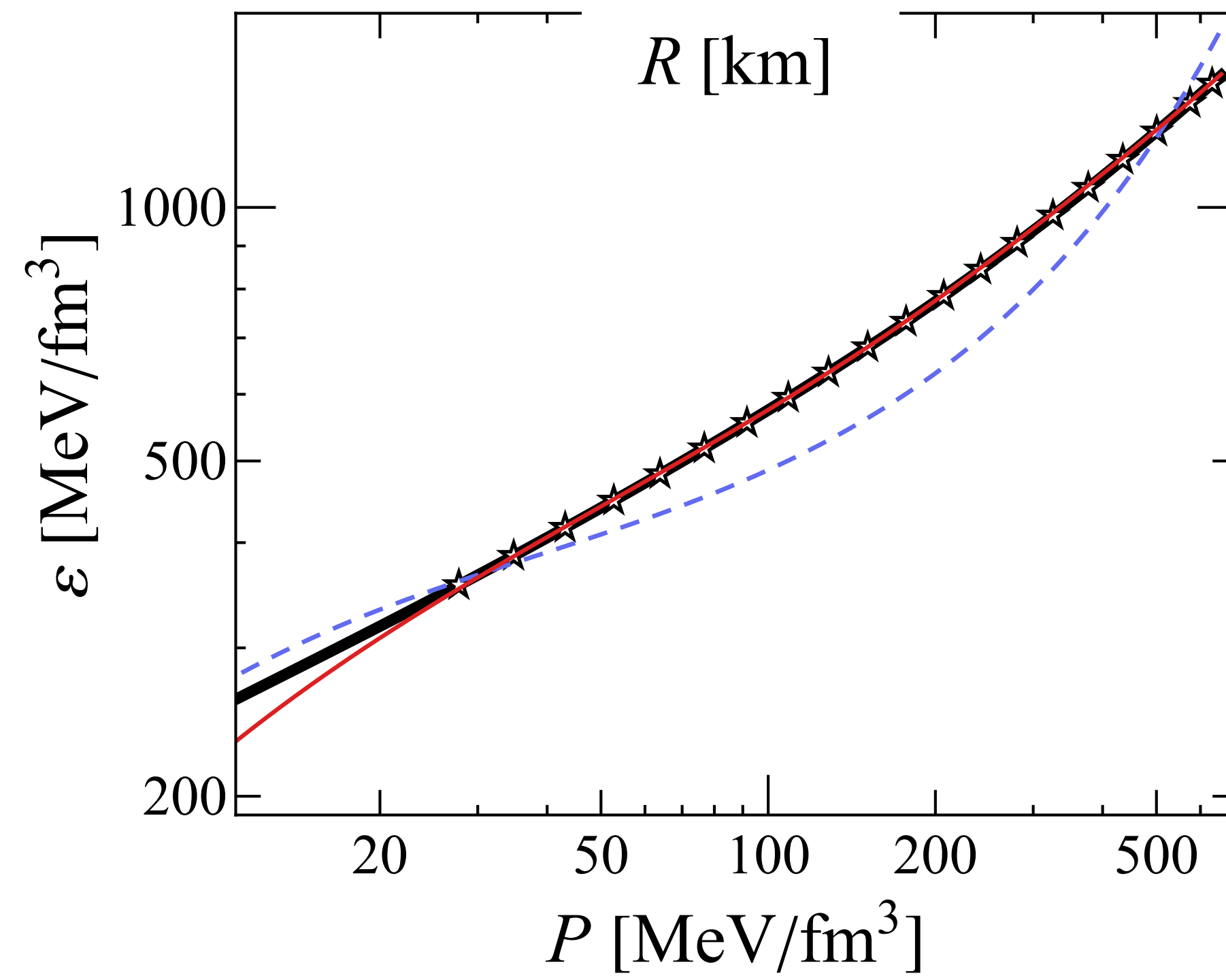
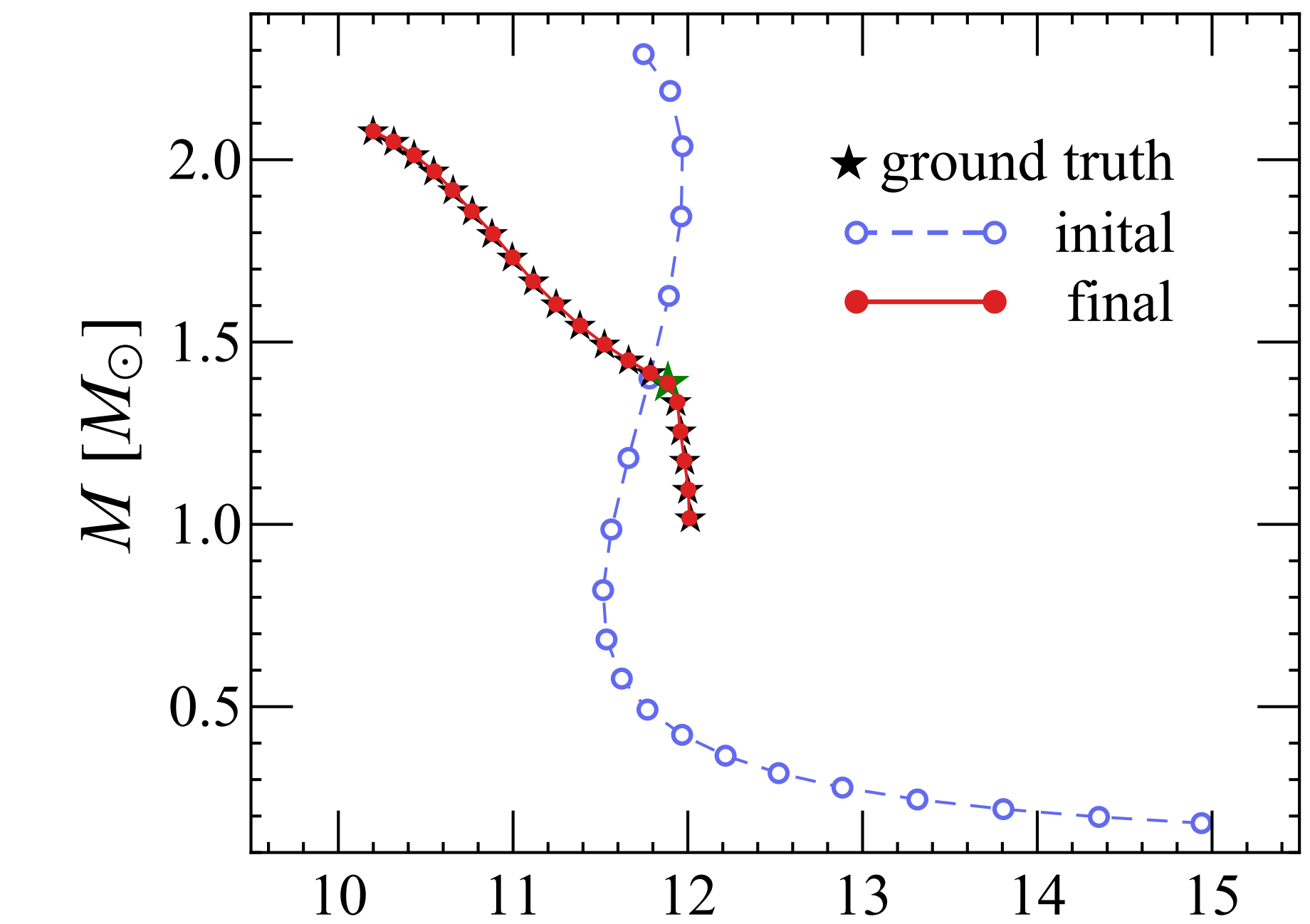
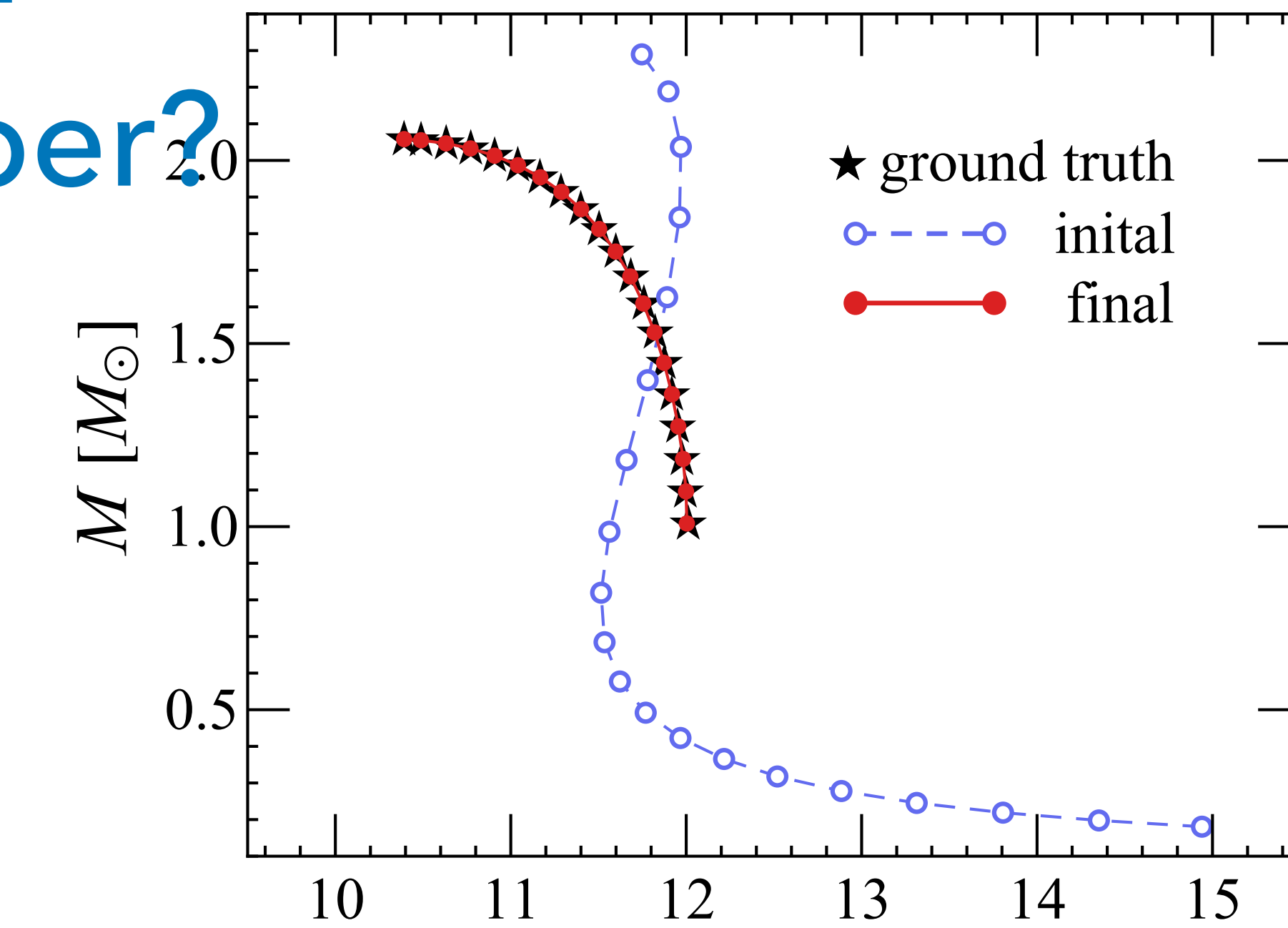
closure test: with phase transition



closure test: with phase transition

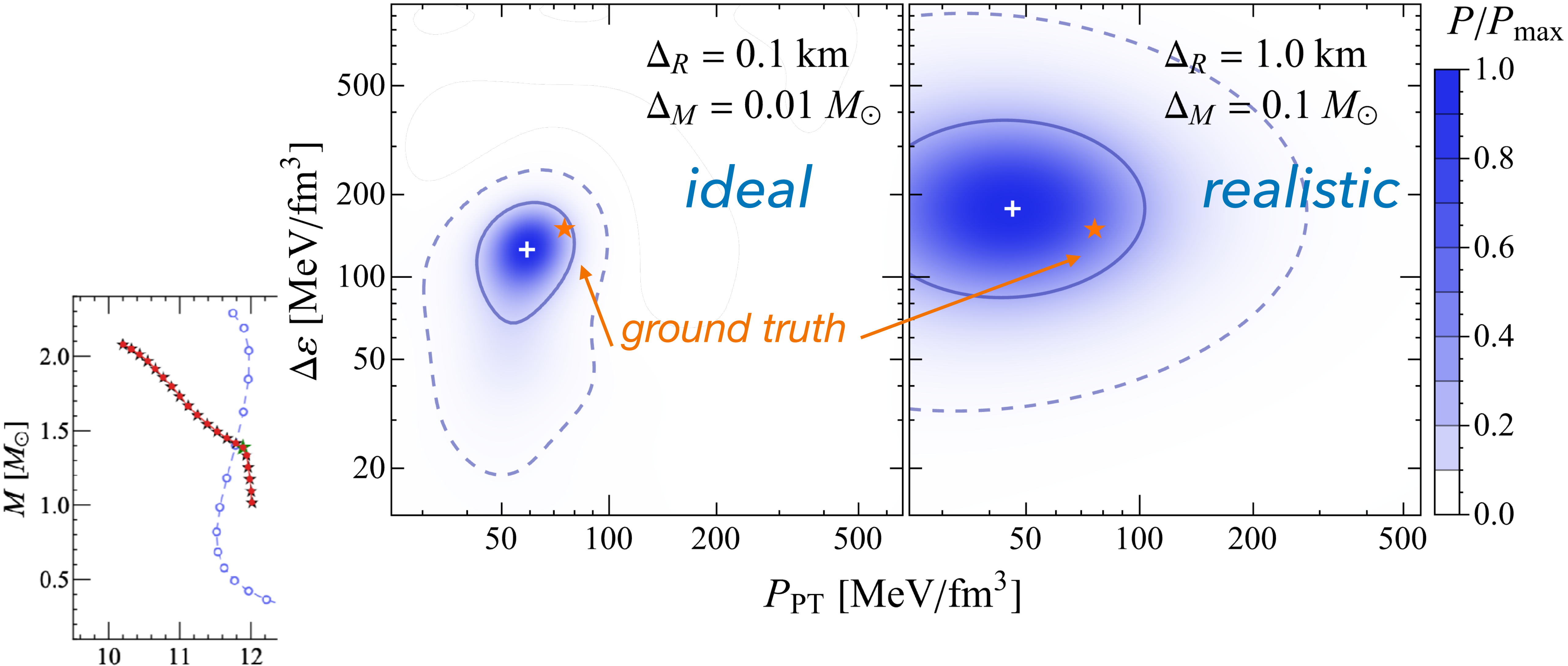


reconstruction
w/ finite number?



reconstruction power for different uncertainties

marginal posterior distribution of phase transition pressure and latent heat



assign different level of uncertainties and perform BA.

summary and outlook

- Derived *analytical* differential equations for linear responses of the TOV eq.
 - enables gradient driven BA;
 - suitable for any parameterization of the EoS;
- Studied the *reconstruction power* of first-order phase transition based on NS observations with *finite number and accuracy*.
- On going: Apply to *tidal deformability* observables;
- On going: BA based on real NS observations