Viscosity of Dense and Cold Nuclear Matter

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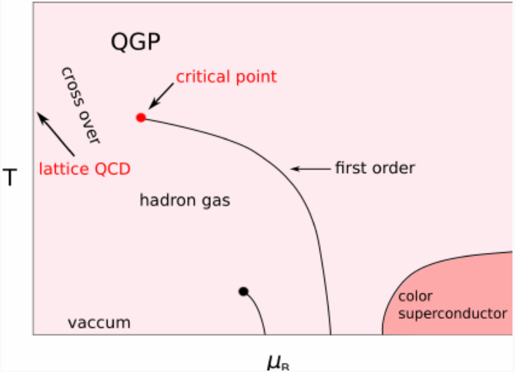
Based on Jianing Li, WK, PRC111(2025)044904,
Jianing Li, WK, Jin Hu, in preparation.

Boundaries of nuclear matter



- The hot boundary: $T \gg \mu$.
  The cold boundary: $T \ll \mu$.
- The system may be simplified in such limits.
- Understanding physics at boundaries can provide constrains to the bulk.

The physics at the two extremes

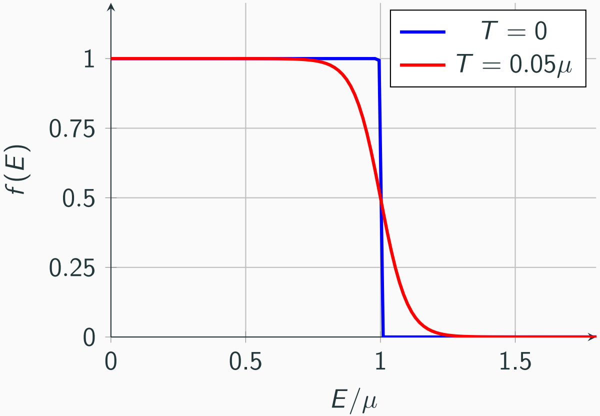
Hot QCD Matter

- Temperature dominated: $T \gg \mu$.
- Hot hadron gas, quark-gluon plasma.
- Well-established frameworks.
- Collective flow, jet quenching

Cold Dense Matter

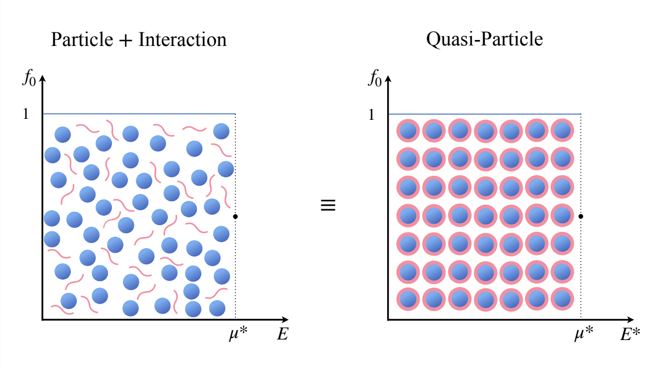
- Chemical potential dominated: $T \ll \mu$
- Degenerate fermions: nucleons, quarks. Effective dof lives on the boundary of the Fermi surface.
- Fermi liquid, non-fermi liquid?
- Phenomena in low/intermediate collisions.

The picture of degenerate fermion system



- The formation of a fermi surface at $T = 0$.
- Finite-temperature effect, introduce the divisiveness of the fermi-surface. Allows collisions and introduce dissipation.

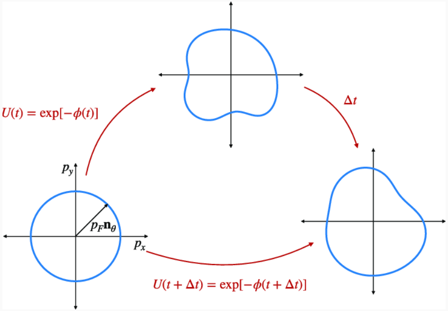
Quasi-particle dispersion relation



- Mean field interactions are absorbed into quasi-particles
- $E_p^2 = p^2 + {m^*}^2$
- $m^* = m^*(\mu^*)$.
- $\mu^* = \mu -$ "mean field effect".

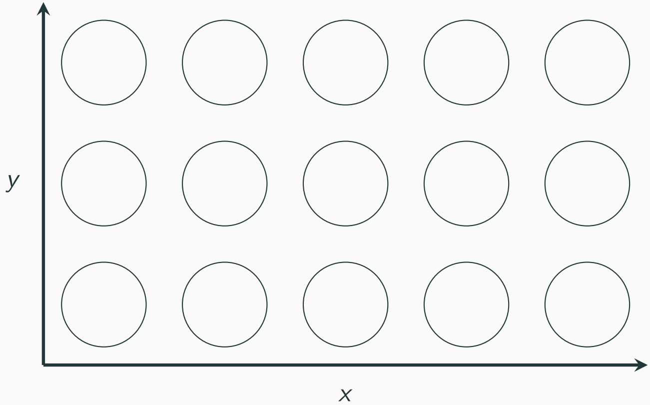
Excitations at zero temperature

A distortion of the shape of the Fermi surface. $\triangledown$ An illustration from L. V. Delacretaz, Y.-H. Du, U. Mehta, D. T. Son, Phys. Rev. Research 4, 033131 (2022)



At $T = 0$, all shape fluctuations are long-lived $\delta n(\Omega) = \sum_{lm} a_{lm} Y_{lm}(\Omega)$. At finite temperature, only energy and momentum $l = 0, 1$ are still conserved.

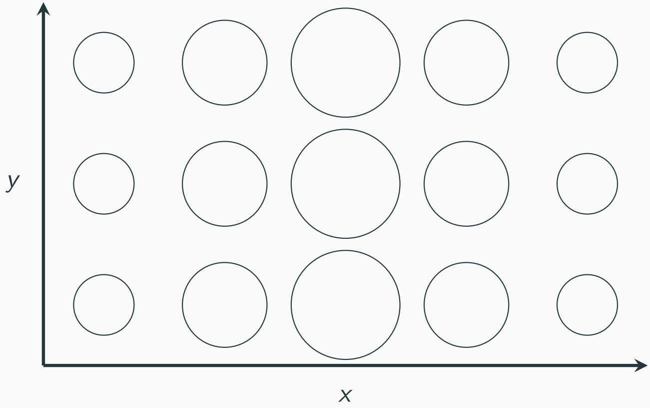
Long wave length excitations at finite temperature



- Ground state: at each location $\vec{x}$, there is a fermi sphere with the given fermi energy, or the chemical potential.

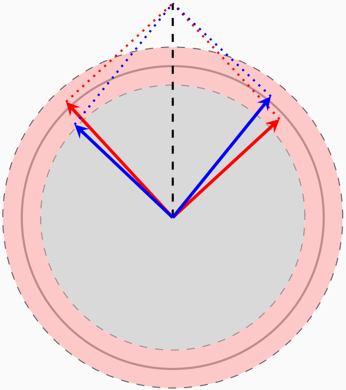
- Localized collisions can only happen near the fermi-surface. For a sharp fermi surface, this is no collision.

Long wave length excitations



- Perturbation type I: a change in the chemical potential over space-time.
- The fermi surface still remains sharp. Such a perturbative does not come with dissipation!

Boltzmann equation and it relaxation time approximation



$$\left(\frac{\partial}{\partial t} + \nabla_p E_p \nabla_x - \nabla_x E_p \nabla_x\right) f(t, x, p) = \mathcal{C}[f]$$

Collision term

$$\mathcal{C}[f] = \frac{1}{2E_1} \int \mathrm{d}\Phi_{2,3,4} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \overline{|M_{12\to 34}|^2}$$
$$\{f_1 f_2 (1 - f_3)(1 - f_4) - f_3 f_4 (1 - f_1)(1 - f_2)\}$$

Collision can only happen near the fermi-surface. The available phase space dies off as T^2.

Linearized collision operator

Linearized collision operator: $f = f_{\rm eq} + \delta f$. Because most of the variation comes from the dof near the fermi surface, δf should be dominated by a delta function near the surface $\delta(E-\mu) = -d\Theta(\mu - E)/d\mu$

$$\delta f = -\frac{\partial f}{\partial \mu}\phi = \beta f_{\rm eq}(1 - f_{\rm eq})\phi$$

The linearized collision operator is integral operator over ϕ

$$\mathcal{C}'[\delta f] = \frac{1}{2E_1}\int \mathrm{d}\Phi_{2,3,4}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\overline{|M_{12\to 34}|^2}$$
$$f_1 f_2(1 - f_3)(1 - f_4)\left\{\phi_1 + \phi_2 - \phi_3 - \phi_4\right\}$$

Note that by symmetry in the local rest frame, if $\phi(E) = a + bE$, the collision operator always vanishes.

The generalized relaxation time approximation

Separate the off-equilibrium correction from the local equilibrium $f = f_{\mathrm{eq}} + \delta f$, and solve the linearized Boltzmann equation

$$\begin{aligned} \frac{d}{dt} f_{\mathrm{eq}}(p^*(x), \mu^*(x)) &\equiv \left(\frac{\partial}{\partial t} + \nabla_p E_p^* \nabla_x - \nabla_x E_p^* \nabla_x \right) f_{\mathrm{eq}}(p^*(x), \mu^*(x)) \\ &= \mathcal{C}'[\delta f] \end{aligned}$$

When inverting the linearized collision operator, there are ambiguities from due to conservation laws $\mathcal{C}'[a + bE] = 0$

$$\mathcal{C}'^{-1}\left[\frac{d}{dt} f_{\mathrm{eq}}(t, x, p) \right] = \delta f - \frac{\partial f}{\partial \mu^*} \times (A + BE)$$

The relaxation time approximation states that $\mathcal{C}'^{-1}[...] \approx \tau_{\mathrm{rel}} \times (...)$

Gradient expansion of the off-equilibrium effects

The most general solution to the δf correction under RTA:

$$
\begin{aligned}
\delta f =& \frac{\partial f}{\partial \mu^*}\left\{ p^{*\mu} p^{*\nu} \underbrace{\sigma_{\mu\nu}}_{\text{Shear}} \phi_1 - \underbrace{\partial \cdot u}_{\text{Bulk}}(\phi_2 - a - bE^*) \right\}, \\
\phi_1(p) =& \frac{\tau_{\mathrm{rel}}}{2E_p^*}, \\
\phi_2(p) =& \frac{\tau_{\mathrm{rel}}}{3E_p^*}\left\{ (p^* \cdot u)^2 \left[1 - 3c_s^2 \left(1 - \frac{m^*}{\mu^*}\frac{dm^*}{d\mu^*} \right) \right] - (m^*)^2 \right\}.
\end{aligned}
$$

The mean field effect enters in two ways:

- The mean-field mass and chemical potential: m^*, μ^*.
- The mass dependence on chemical potential $\kappa^* = dm^*/d\mu^*$

Expression for the shear and bulk viscosity

Matching this result to the hydrodynamic description

$$\delta T^{\mu\nu} = \int \frac{p^{*\mu} p^{*\nu}}{E^*} \delta f \equiv \eta \sigma^{\mu\nu} - \zeta (g^{\mu\nu} - u^\mu u^\nu) \partial \cdot u$$

The expressions for shear and bulk viscosity are

$$\eta = \frac{2}{15} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f_{\mathrm{eq}}}{\partial \mu^*} \frac{\mathbf{p}^4}{E_p^*} \phi_1 \equiv \frac{2}{15} \left\langle \frac{\mathbf{p}^4}{E_p^*}, \ \phi_1 \right\rangle ,$$

$$\zeta = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f_{\mathrm{eq}}}{\partial \mu^*} \frac{\mathbf{p}^2}{E_p^*} (\phi_2 - a - bE^*) \equiv \frac{1}{3} \left\langle \frac{\mathbf{p}^2}{E_p^*}, \ \phi_2 - a - bE^* \right\rangle .$$

At this point $\eta > 0$, but ζ is not yet manifestly positive definite.

Landau Matching Condition and Positivity of ζ

The arbitrariness in a and b are fixed by the Landau matching conditions

$$\delta\epsilon \propto \langle E_p^*, \phi_2 - a - bE_p^* \rangle = 0\,,$$
$$\delta n \propto \langle 1, \phi_2 - a - bE_p^* \rangle = 0\,.$$

This way, the bulk viscosity can be shown to be positive definite

$$\zeta = \frac{1}{3}\left\langle \frac{\mathbf{p}^2}{E_p^*},\ \phi_2 - a - bE^* \right\rangle = \frac{1}{\tau_{\rm rel}}\left\langle \frac{\tau_{\rm rel}\mathbf{p}^2}{3E_p^*} - a' - b'E,\ \phi_2 - a - bE^* \right\rangle$$
$$= \frac{1}{\tau_{\rm rel}}\langle \phi_2 - a - bE^*, \phi_2 - a - bE^* \rangle \geq 0\,.$$

Low temperature expansion

At low temperature, use the Sommerfeld expansion

$$f_{eq}(T \ll \mu^*) = \Theta(\mu^* - E^*) - \sum_{n=1}^{\infty} \left[1 - \frac{1}{2^{2n-1}}\right] 2\zeta_{2n} \delta^{(2n-1)}(E^* - \mu^*) + \mathcal{O}(e^{-\frac{\mu^*}{T}})$$

This result

$$\frac{\eta}{\tau_{\rm rel}} = \frac{1}{30\pi^2} \frac{(p_F^*)^5}{\mu^*}(1 + \mathcal{O}(T^2/\mu^2))$$

$$\frac{3\zeta}{\tau_{\rm rel}} = \frac{8\pi^2}{135} \frac{{m^*}^4 p_F^* T^4}{{\mu^*}^5}(1 + \mathcal{O}(T^2/\mu^2))$$

- At leading order in T/μ, they only depends on m^* and μ^*, but not $\kappa^* = dm^*/d\mu^*$.
- $\frac{\zeta}{\eta} \propto \left(\frac{T}{p_F^*}\right)^4 \times \left(\frac{m^*}{\mu^*}\right)^4$

Compare to results from R. Hakim and L. Mornas PRC47(1993)2846, expanded at $T \ll \mu$ and $p_F \gg M$.

$$\frac{\eta}{\tau_{\rm rel}} = \frac{1}{30\pi^2} \frac{(\mu_F^*)^5}{p_F^*}$$

$$\frac{3\zeta}{\tau_{\rm rel}} = \frac{8\pi^2}{135} \frac{{m^*}^4 \mu^* T^4}{{p_F^*}^5}$$

Note that they do agree since $p_F \approx \mu$ in this limit.

Put in actually physics: the Walecka model

$$
\begin{aligned}
\mathcal{L}_{\mathrm{W}} = \sum_{\mathrm{N=n,p}} & \underbrace{\bar{\psi}_{\mathrm{N}} \left(i\partial\!\!\!/ - m_{\mathrm{N}} + g_\sigma \sigma - g_\omega \omega\!\!\!/ + \mu_B \gamma_0 \right) \psi_{\mathrm{N}}}_{\text{Nucleon fields}} \\
& \underbrace{+ \frac{1}{2} \left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 \right) + \frac{1}{3} b m_{\mathrm{N}} \left(g_\sigma \sigma \right)^3 + \frac{1}{4} c \left(g_\sigma \sigma \right)^4}_{\sigma \text{ meson field with self interaction}} \\
& \underbrace{- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu}_{\omega \text{ meson fields}}
\end{aligned}
$$

- σ meson: long-range attraction between nucleons. σ meson condensation corrects the quasi-particle mass in the mean field level.
- ω meson: short-range repulsion. ω meson condensation corrects the chemical potential.

Walecka model: thermal dynamics

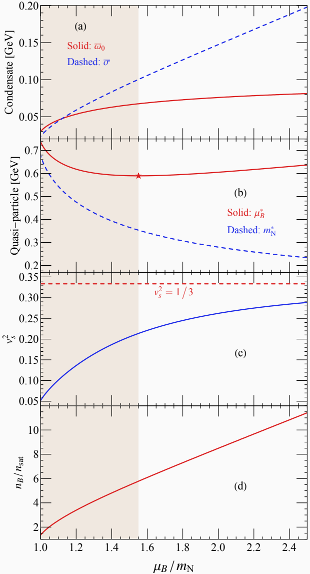
Thermal dynamics at the mean-field level $\sigma = \bar{\sigma} + (\sigma - \bar{\sigma})$, $\omega = \bar{\omega} + (\omega - \bar{\omega})$.

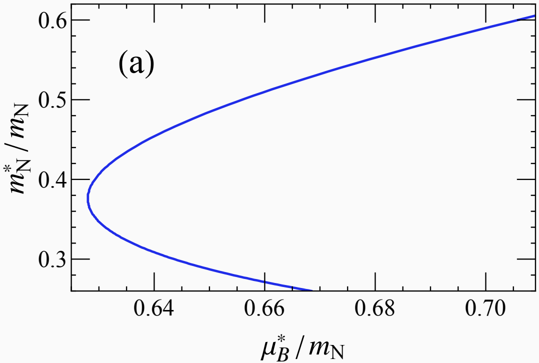
$$\mathcal{V}(T, \mu_N; \bar{\sigma}, \bar{\omega}) = -\ln \mathrm{Tr} e^{-\int_0^\beta \mathcal{L}_E d\tau}$$

- Extrema conditions (Gap equations)

$$\frac{\partial \mathcal{V}}{\partial \bar{\sigma}} = m_\sigma^2 \bar{\sigma} + \left.\frac{\partial U}{\partial \sigma}\right|_{\sigma=\bar{\sigma}} - 2 g_\sigma n_0(T, \mu_B^*, m_N^*) = 0$$
$$\frac{\partial \mathcal{V}}{\partial \bar{\omega}} = m_\omega^2 \bar{\sigma} - 2 g_\omega n_s(T, \mu_B^*, m_N^*) = 0$$

Static properties

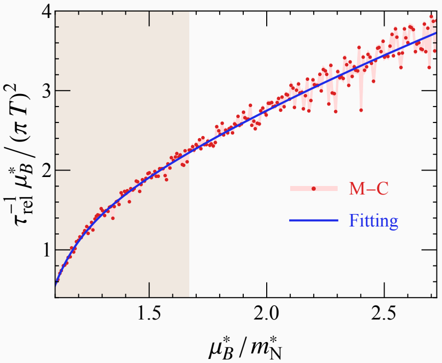


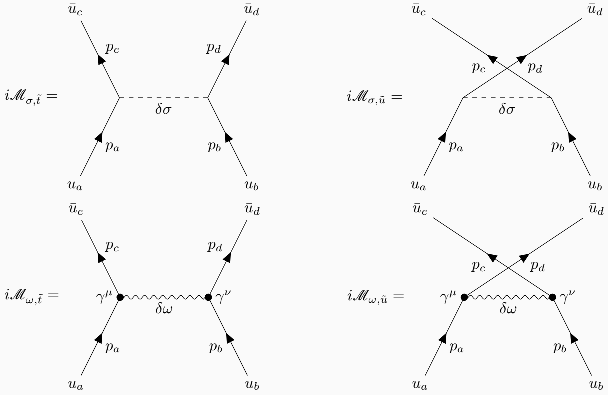


Note the turning point in the relation between m_N^* and μ^*. We use this as the point of the region of validity of the theory.

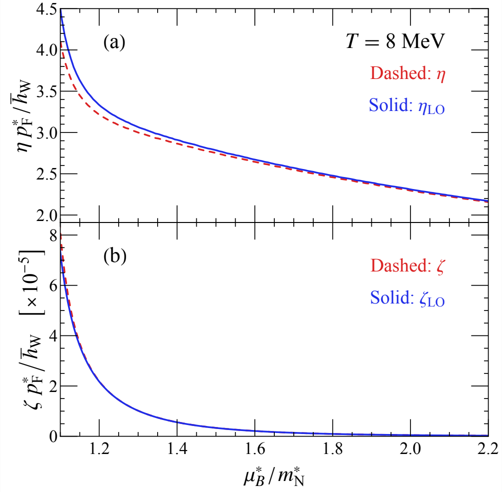
Walecka model: estimating the relaxation time

Scatterings are mediated by the residue quantum fluctuations $\delta\omega$ and $\delta\sigma$ around the mean field. Relaxation time scales as μ_B^*/T^2.





Results for shear and bulk viscosity



Use the dimensionless ratio to estimate the ideal-ness of hydrodynamic: $\frac{\eta p_F^*}{e+P}$ and $\frac{\zeta p_F^*}{e+P}$

$$\frac{\zeta}{\eta} \sim \left(\frac{T}{p_F^*}\right)^4 \times \left(\frac{m^*}{\mu^*}\right)^4 \ll 1$$

Conclusion and prospective

- Revisit the derivation of shear and bulk viscosity of cold nuclear matter with mean field effects under the RTA approximation of Boltzmann equation.
- Results suitable for $T \ll \mu$ and $p_F \lesssim m_N$.
- $\zeta/\eta \sim \left(\frac{T}{p_F^*}\right)^4 \times \left(\frac{m^*}{\mu^*}\right)^4$
- When using the Walecka model:
    - Range of validity also limited by the non-monotonic relation between μ^* and m^*.
    - Future: improve with non-linear ω potential and ω-σ coupling.

Questions?