Parton Shower Algorithm Incorporating Saturation Effects

Jian Zhou

山东大学(青岛)

SHANDONG UNIVERSITY, QINGDAO

Based on the papers: [2211.07174](2211.07174) , [2307.04185](2307.04185), In collaboration with Yu Shi, Shu-yi Wei

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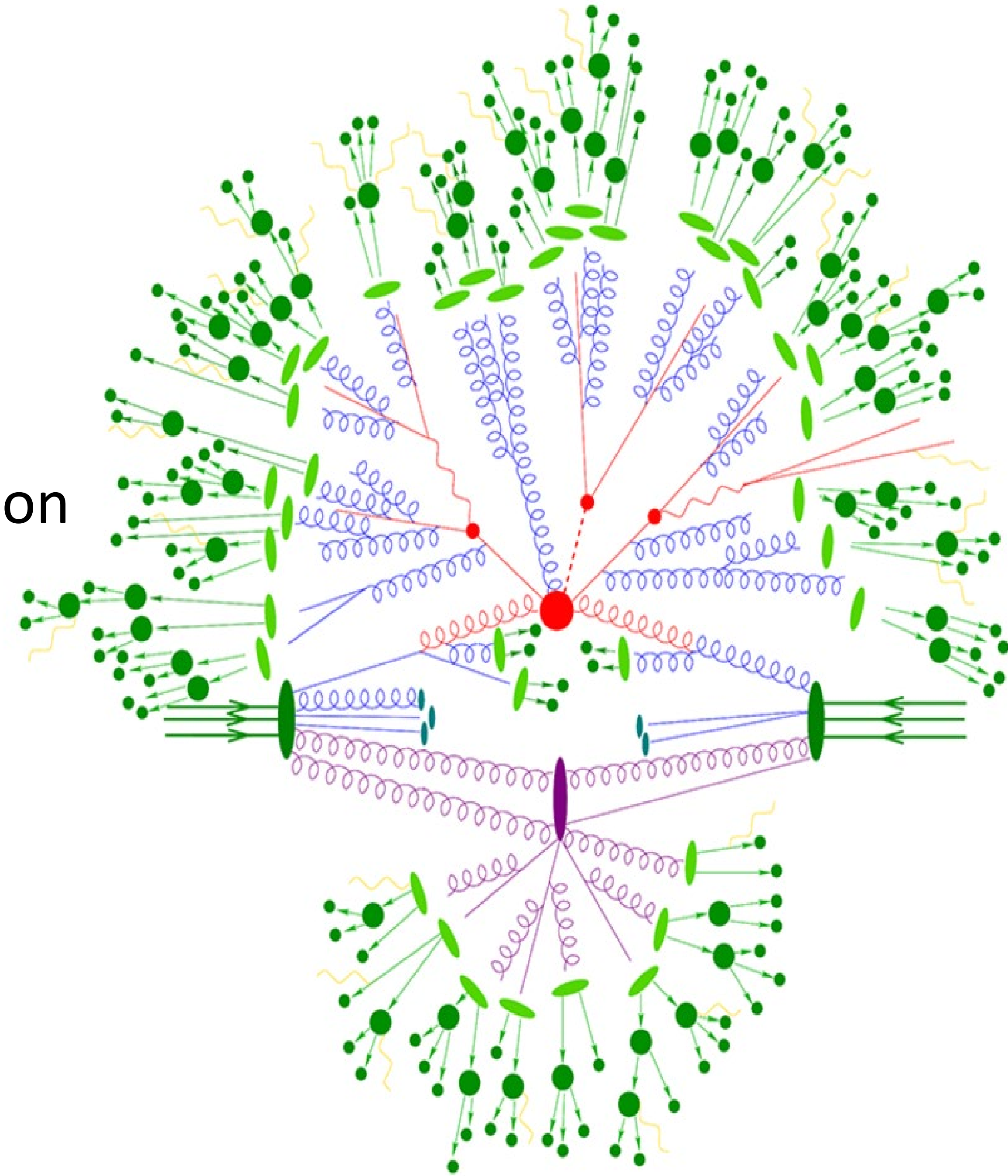
Outline:

- ➢Background
- ➢Forward evolution & backward evolution
- ➢Joint small x and kt resummation
- ➢Summary



Why parton shower generator

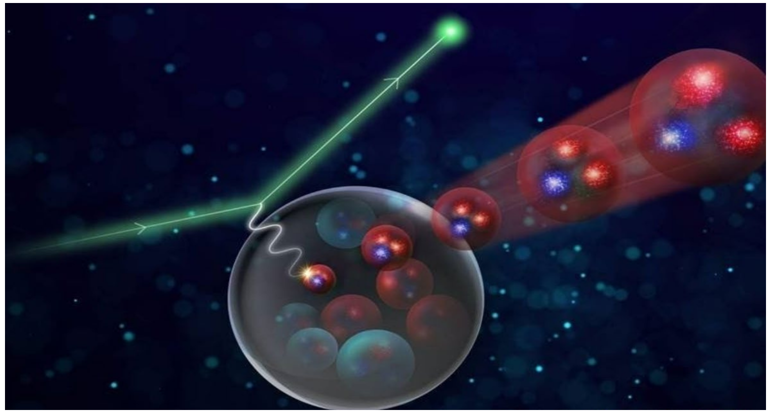
- Describe fully exclusive hadronic state
- Coherent branching
- Keep four momentum conservation in each branching
- Impact studies for future experiments

….

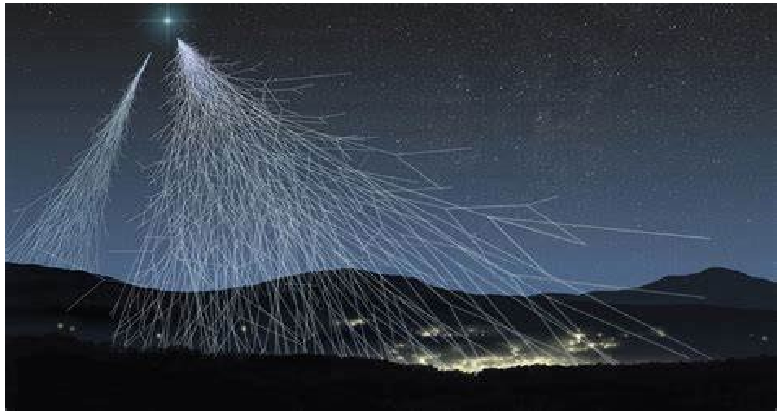
Why small *x* parton shower generator

- ➢ Saturation effect is absent in all existing generators

- ● Aim at developing a PS algorithm to be used:



- ➢ Phenomology in eA collisions @EIC
- ➢ Forward physics in pA collisions @LHC
- ➢ Cosmic ray event generator



Small x evolution equations: $\ln\frac{S}{Q^2}$ or $\ln\frac{1}{x}$

- **BK equation** 2 →1, 3 →1, 4 →1 … gluon fusion & $\ln\frac{1}{x}$ Balitsky, 1996; Kovchegov, 1997

Why not the BK equation:

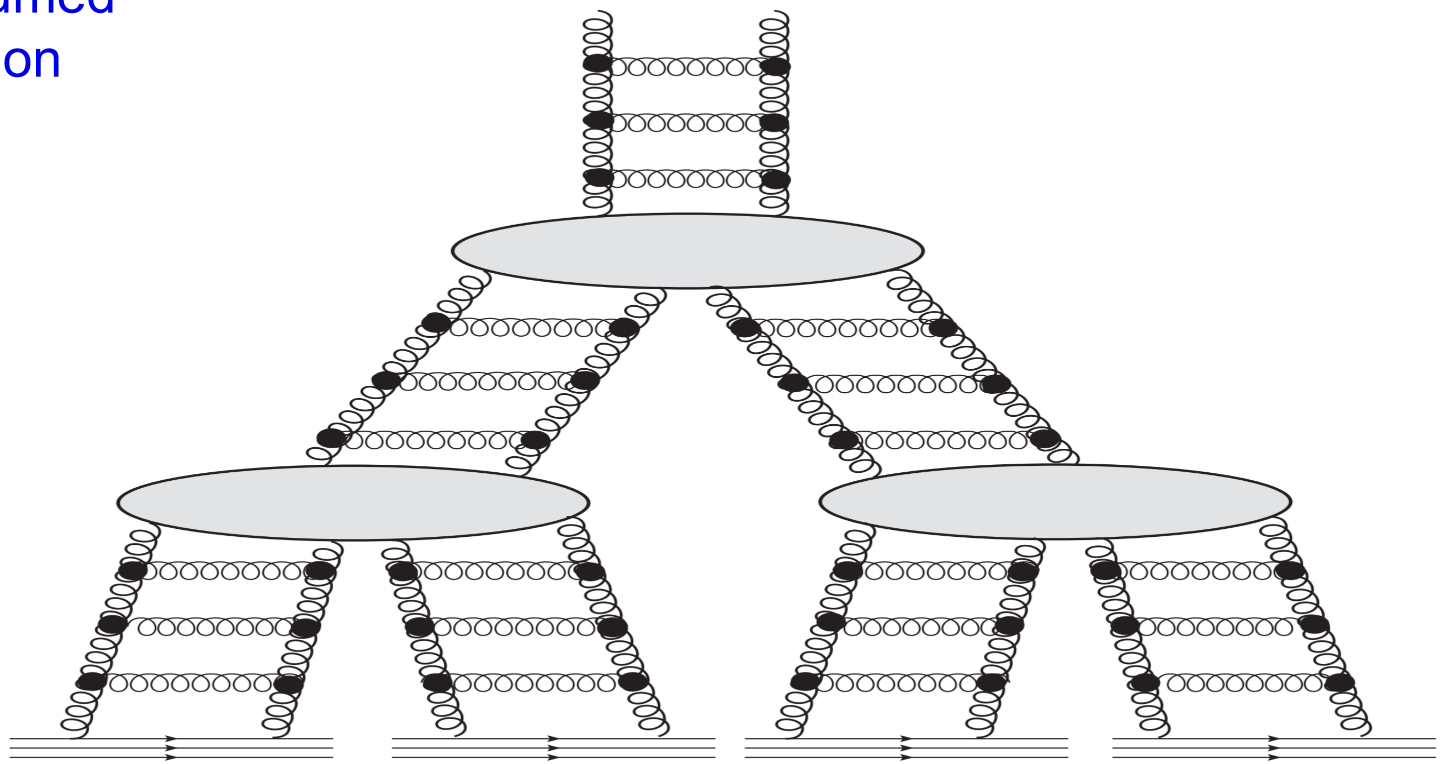
- The Fourier transform of dipole amplitude, lack of probability interpretation.
- Aiming at describing exclusive processes, multiple point correlation functions involved → JIMWLK equation

GLR equation

- **GLR equation** 2 →1 gluon fusion & $\ln\frac{1}{x}$

Gribov-Levin-Ryskin,1983

“Fan” diagram resumed
by the GLR equation



Folded and unfolded GLR equation

- The standard GLR equation(unfolded one)

$$\frac{\partial N(\eta,k_\perp)}{\partial\eta}=\frac{\bar{\alpha}_s}{\pi}\left[\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp+l_\perp)-\int\frac{\mathrm{d}^2l_\perp}{(k_\perp-l_\perp)^2}\frac{k_\perp^2}{2l_\perp^2}N(\eta,k_\perp)\right]-\bar{\alpha}_sN^2(\eta,k_\perp)$$

- Resolved and unresolved branching:

$$\int\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp+l_\perp)\approx\int_\mu\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp+l_\perp)+\int_0^\mu\frac{\mathrm{d}^2l_\perp}{l_\perp^2}N(\eta,k_\perp)$$

- Folded GLR equation: virtual correction is manifestly resumed to all orders

$$\frac{\partial}{\partial\eta}\frac{N(\eta,k_\perp)}{\Delta(\eta,k_\perp)}=\frac{\bar{\alpha}_s}{\pi}\int_\mu\frac{\mathrm{d}^2l_\perp}{l_\perp^2}\frac{N(\eta,l_\perp+k_\perp)}{\Delta(\eta,k_\perp)}$$

Shi-Wei-ZJ, 2022

- Non-Sudakov form factor

$$\Delta(\eta,k_\perp)=\exp\left\{-\bar{\alpha}_s\int_{\eta_0}^{\eta}d\eta'\left[\ln\frac{k_\perp^2}{\mu^2}+N(\eta',k_\perp)\right]\right\}$$

Parton shower algorithms

GLR v.s. DGLAP/CCFM

$$\Delta(\eta, k_\perp) = \exp\left\{-\bar{\alpha}_s \int_{\eta_0}^{\eta} d\eta' \left[\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right]\right\}$$

$$\Delta_a(t, t') = \exp\left\{-\sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z \, \frac{\alpha_s}{2\pi} \frac{1}{2} P_{ab}(z)\right\}$$

GLR	DGLAP/CCFM
gluon splitting gluon fusion	parton splitting

The evolution variable:

$\eta = \ln(1/x)$	Q

The generated event:

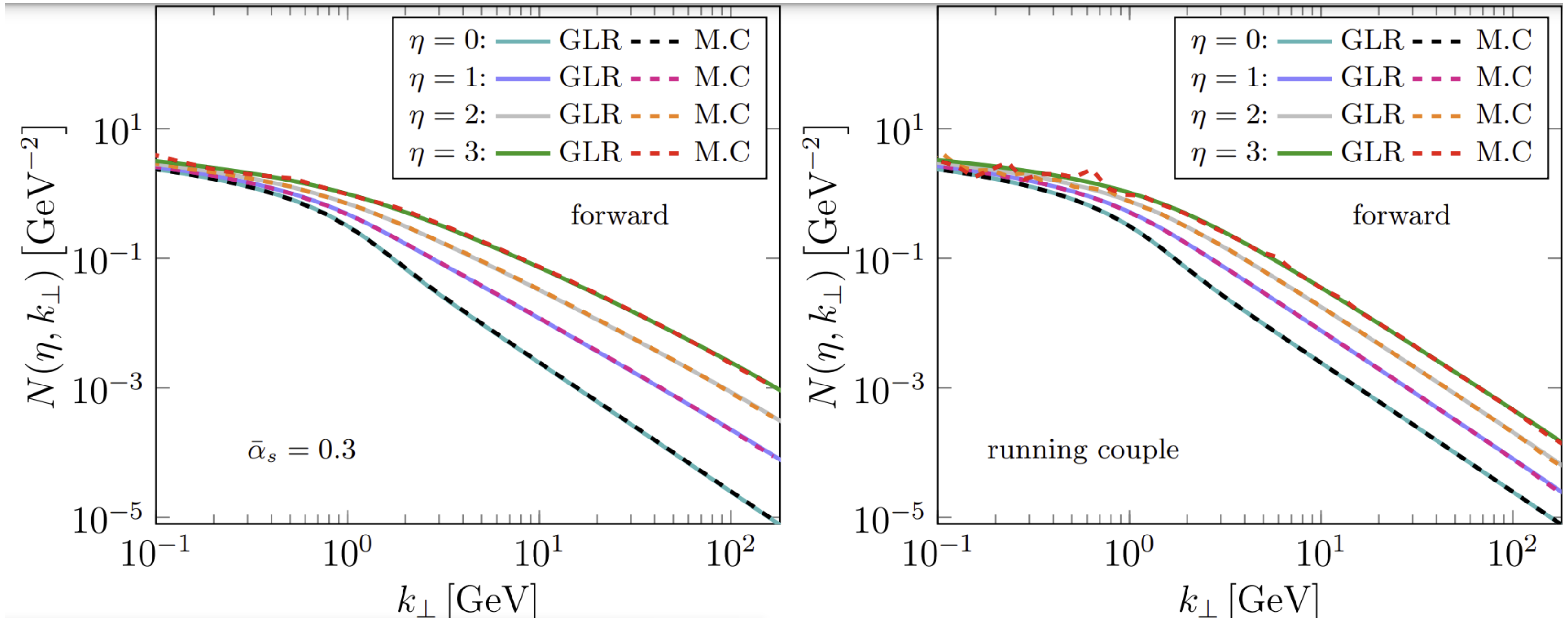
reweight	Unitary

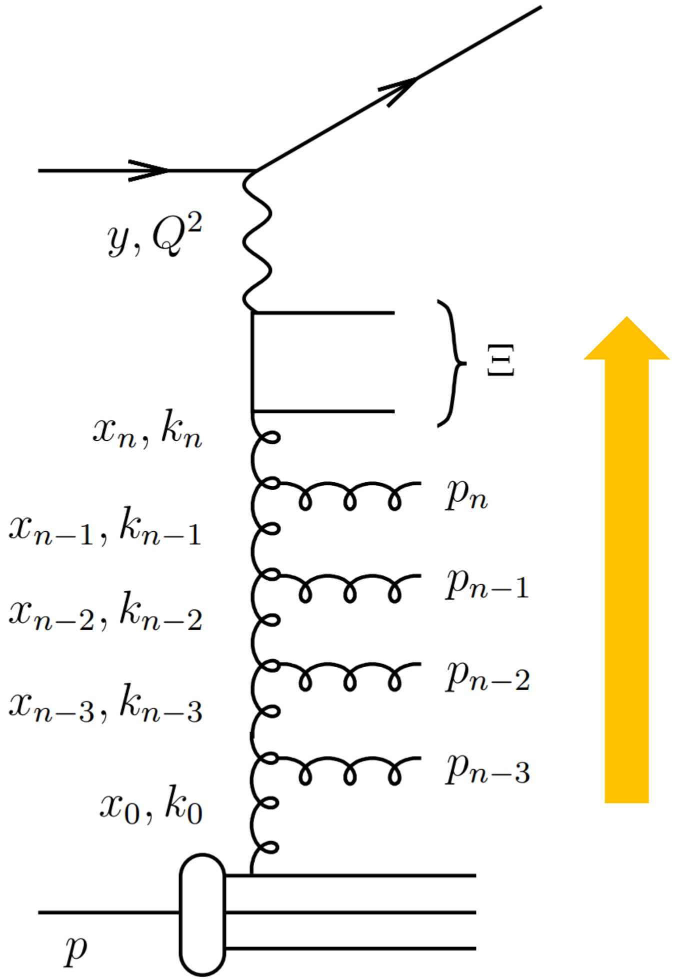
Forward evolution

- MV model as the initial condition at x_0=0.01:

$$N(\eta=0,k_\perp)=\int\frac{d^2r_\perp}{2\pi}e^{-ik_\perp\cdot r_\perp}\frac{1}{r_\perp^2}\left(1-\exp[-\frac{1}{4}Q_{s0}^2r_\perp^2\ln(e+\frac{1}{\Lambda r_\perp})]\right)$$

- Test the algorithm



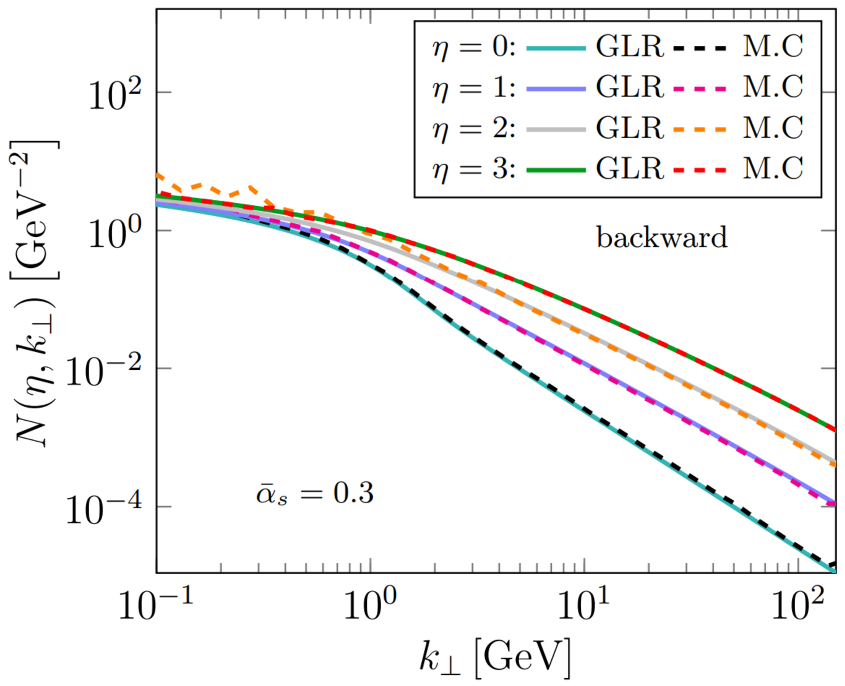


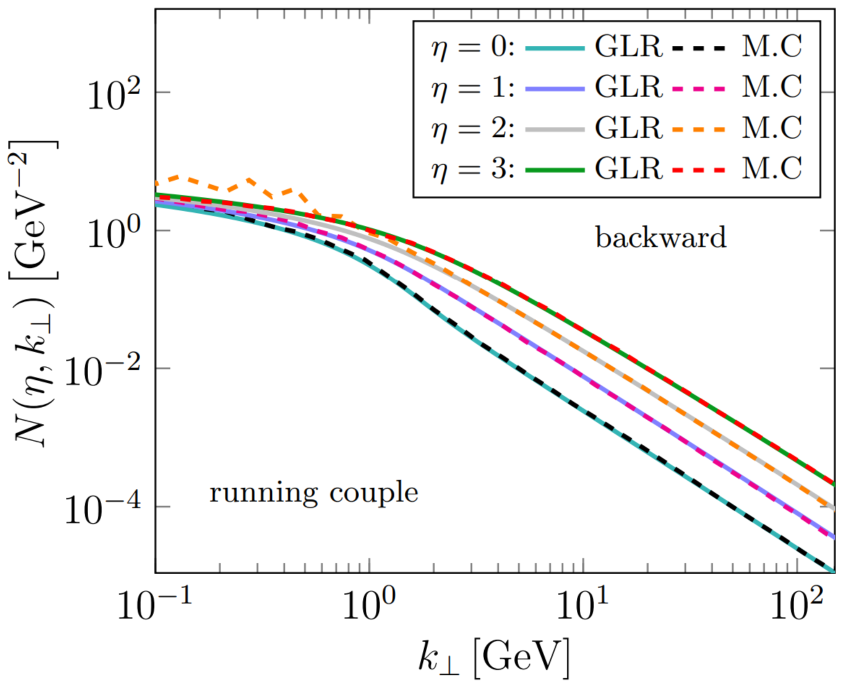
Backward evolution

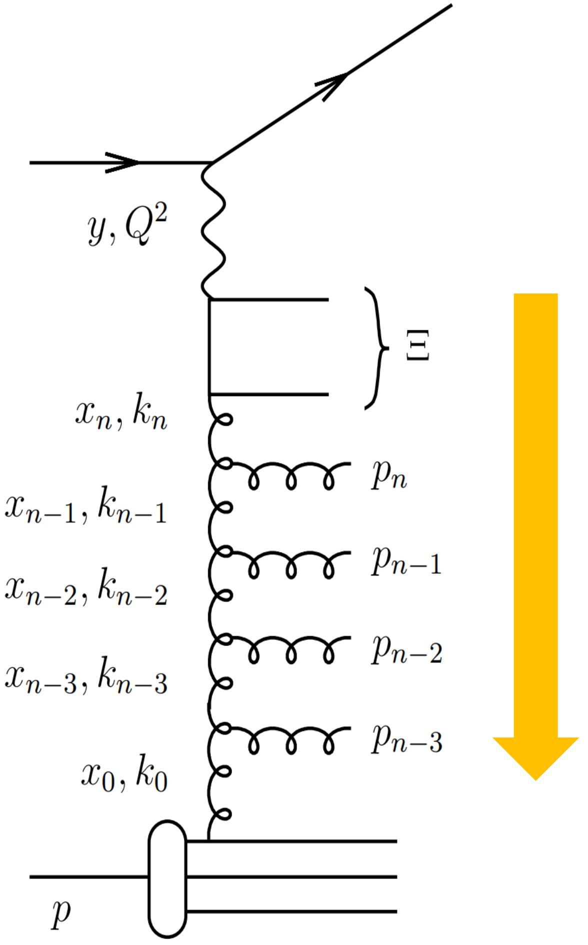
➢ Non-Sudakov form factor:

$$\frac{\Delta(\eta_{i+1},k_{\perp,i+1})N(\eta_i,k_{\perp,i+1})}{\Delta(\eta_i,k_{\perp,i+1})N(\eta_{i+1},k_{\perp,i+1})} = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta \int_{\mu} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\frac{N(\eta,k_{\perp,i+1}+l_\perp)}{N(\eta,k_{\perp,i+1})}\right]$$

◆ $l_\perp$ **is also sampled in a different way!**

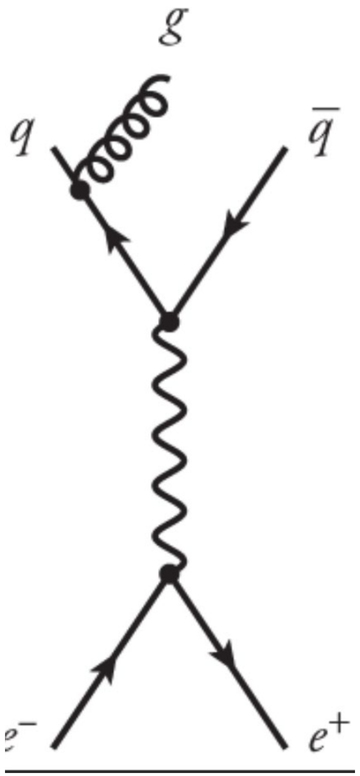




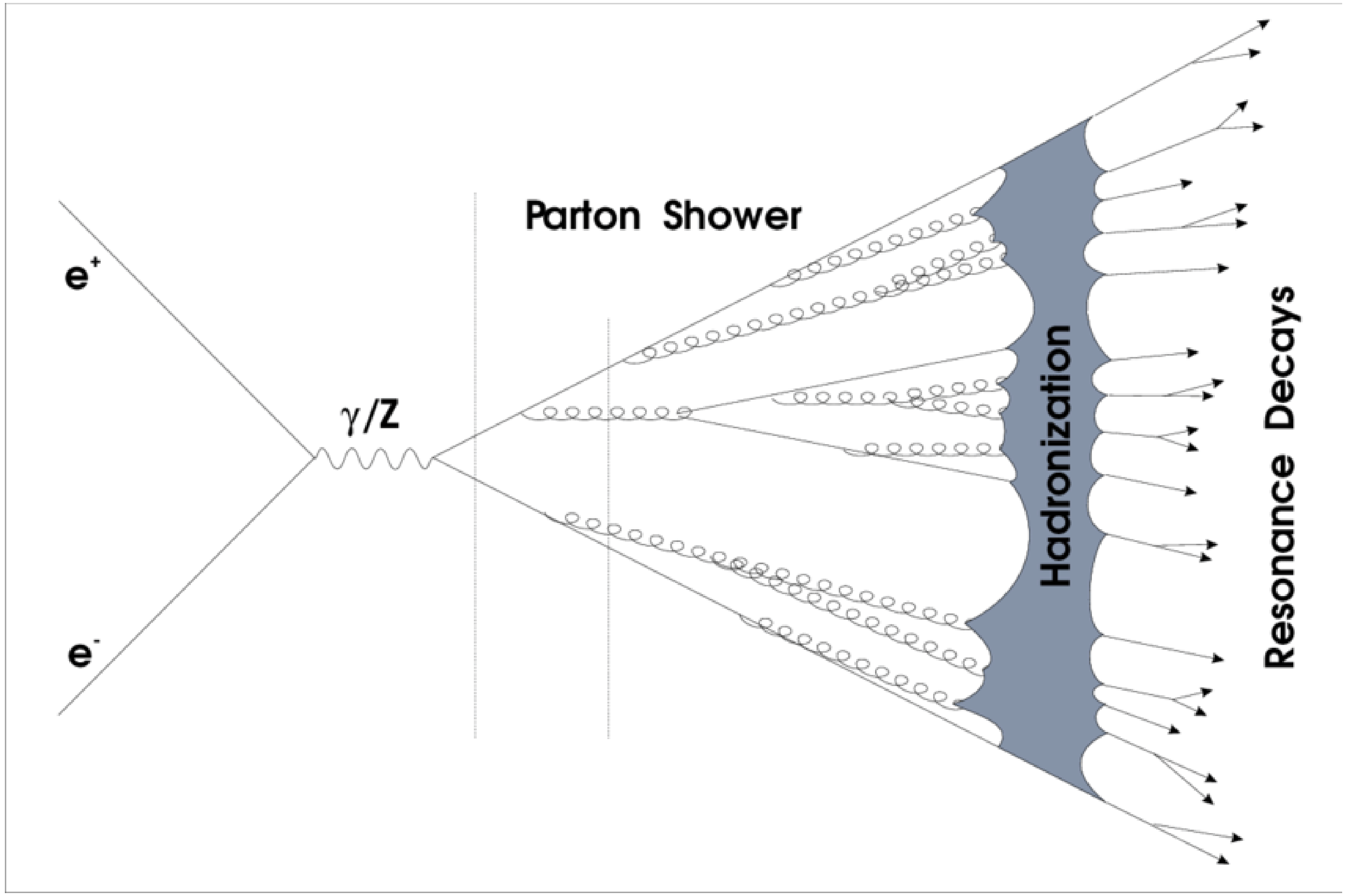


Kinematic constraint & Coherent branching

- The Chudakov effect



- Angular ordering



➢ Wide angle radiation suppressed!

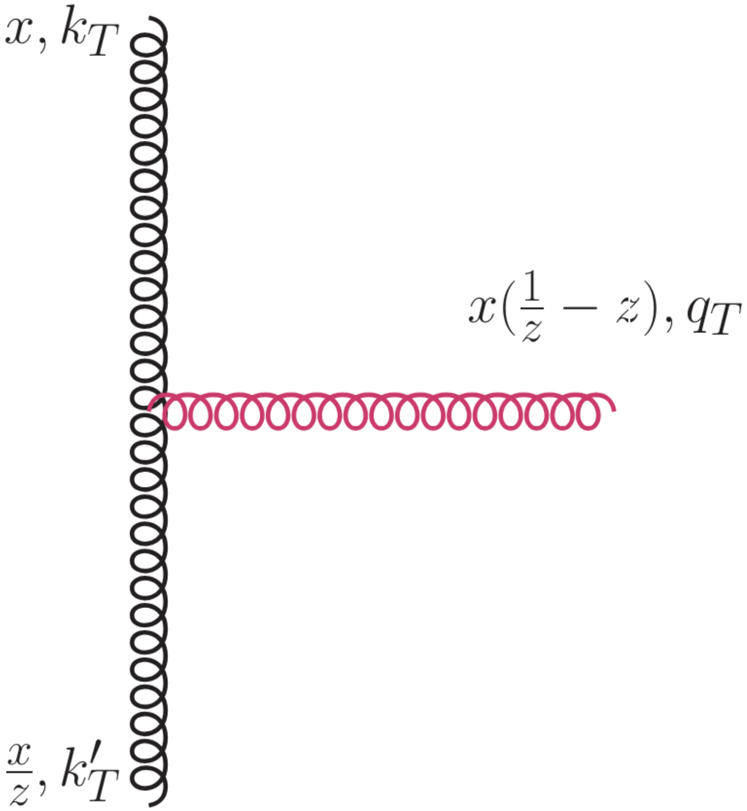
Kinematical constraint in the GLR evolution equation

- **The virtuality of t-channel gluon should be dominated by transverse components**

[Kwiecinski, Martin, Sutton, Z. Phys. C, 96; Deak, Kutak, Li, Stasto, EPJC, 19]

$$k_T^2 > |k^+ k^-| \qquad k^- = k'^- - q^- \simeq -q^- = -q_T^2/q^+$$

$$k^+ k^- \simeq -\frac{k^+}{q^+} q_T^2 = -\frac{k^+}{k'^+ - k^+} q_T^2 = -\frac{z}{1-z} q_T^2$$



- **The on shell condition for s-channel gluon leads to the constraint**

$$q_T^2 < \frac{1-z}{z} k_T^2 \qquad \eta \longrightarrow \eta + \ln \frac{k_\perp^2}{k_\perp^2 + l_\perp^2}$$

- **Results in a non-local GLR equation**

$$\frac{\partial N(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi} \int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N\left(\eta + \ln \frac{k_\perp^2}{k_\perp^2 + l_\perp^2}, l_\perp + k_\perp\right) - \frac{\bar{\alpha}_s}{\pi} \int_0^{k_\perp} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp) - \bar{\alpha}_s N^2(\eta, k_\perp)$$

Coherent branching in the GLR evolution

- Kinematic constrained GLR equation

$$\frac{\partial}{\partial \eta}\frac{N(x,k_\perp)}{\Delta(\eta,k_\perp)}=\frac{\bar{\alpha}_s}{\pi}\int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\frac{N\left(\eta+\ln\left[\frac{k_\perp^2}{k_\perp^2+l_\perp^2}\right],l_\perp+k_\perp\right)}{\Delta(\eta,k_\perp)}$$

Shi-Wei-ZJ, 2022

- Weighting factor for forward evolution:

$$\mathcal{W}_{kc}(\eta_i,\eta_{i+1};k_\perp)=\frac{(\eta_{i+1}-\eta_i)\int_\mu^{\min\left[P_\perp,\sqrt{(k_\perp-l_\perp)^2\frac{1-z}{z}}\right]}\frac{\mathrm{d}^2 l_\perp}{l_\perp^2}e^{-\bar{\alpha}_s\int_{\eta_{i+1}}^{\eta_{i+1}+\ln\frac{(k_\perp-l_\perp)^2}{(k_\perp-l_\perp)^2+l_\perp^2}}\mathrm{d}\eta\left[\ln\frac{k_\perp^2}{\mu^2}+N(\eta,k_\perp)\right]}}{(\eta_{i+1}-\eta_i)\ln\frac{k_\perp^2}{\mu^2}+\int_{\eta_i}^{\eta_{i+1}}d\eta N(\eta,k_\perp)}$$

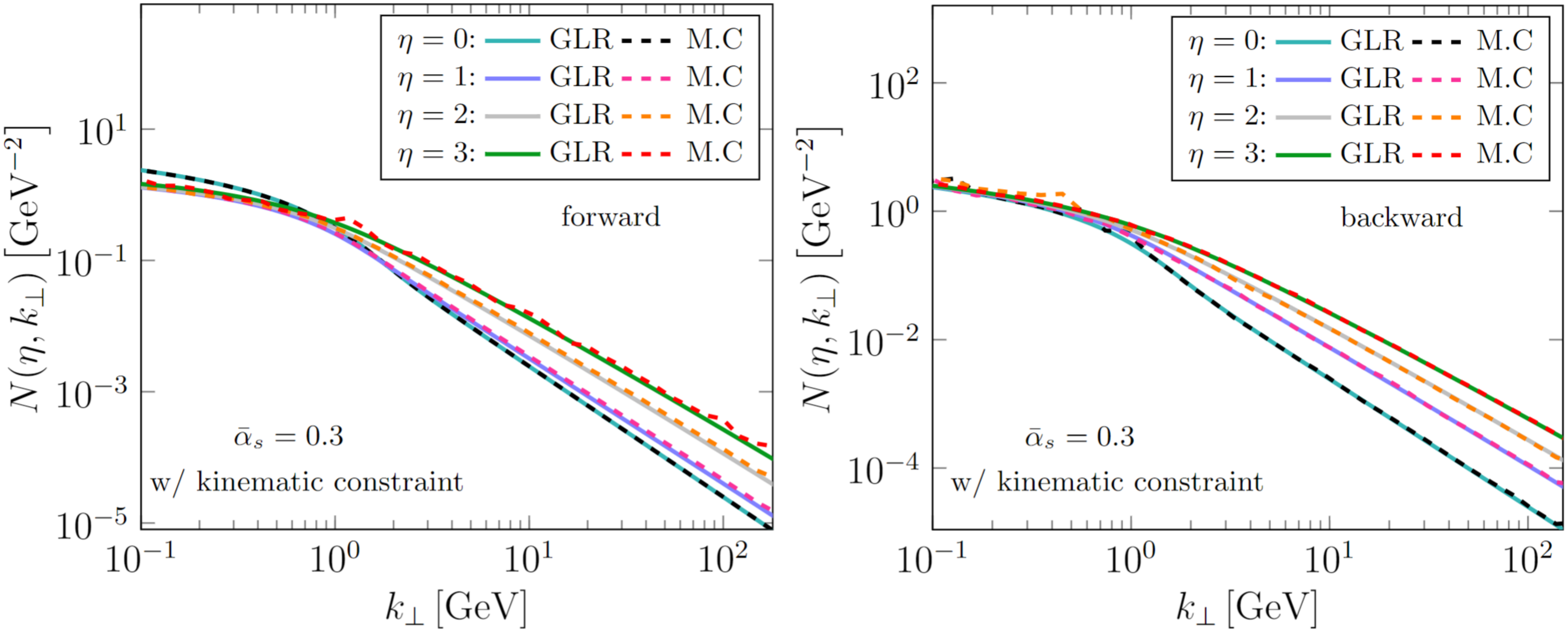
- Weighting factor for backward evolution:

$$\mathcal{W}(\eta_{i+1},\eta_i;k_{\perp,i+1})=\frac{(\eta_{i+1}-\eta_i)\ln\frac{k_{\perp,i}^2}{\mu^2}+\int_{\eta_i}^{\eta_{i+1}}d\eta N(\eta,k_{\perp,i})}{(\eta_{i+1}-\eta_i)\ln\frac{P_\perp^2}{\mu^2}}\frac{N(\eta_i,k_{\perp,i})}{N\left(\eta_i+\ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2+l_\perp^2}\right],k_{\perp,i}\right)}$$

- Non-Sudakov factor for backward evolution

$$\frac{\Delta(\eta_{i+1},k_{\perp,i+1})N(\eta_i,k_{\perp,i+1})}{\Delta(\eta_i,k_{\perp,i+1})N(\eta_{i+1},k_{\perp,i+1})}=\exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}}\mathrm{d}\eta\int_\mu^{\min\left[P_\perp,\sqrt{\frac{1-z}{z}k_{\perp,i+1}^2}\right]}\frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\frac{N\left(\eta+\ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2+l_\perp^2}\right],k_{\perp,i+1}+l_\perp\right)}{N(\eta,k_{\perp,i+1})}\right]$$

Coherent branching: test against the numerical results



Shi-Wei-ZJ, 2022

Joint small x and kt resummation

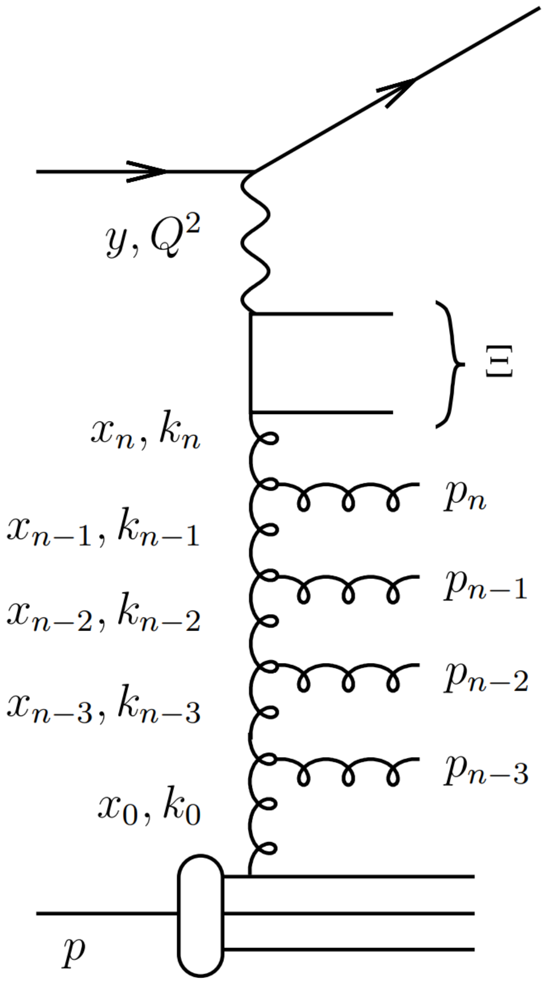
- When $S >> Q^2 >> k_T^2$, resume two type large logs

$$\ln \frac{S}{Q^2} \qquad\qquad \ln^2 \frac{Q^2}{k_\perp^2}$$

- Starting point:

$$xG(x, l_\perp, x\zeta) = \int \frac{dy^- d^2 y_\perp}{(2\pi)^3 P^+} e^{-ixP^+ y^- + il_\perp \cdot y_\perp} \langle P | F_\mu^+(y^-, y_\perp) \mathcal{L}_{\tilde{n}}^\dagger(y^-, y_\perp) \mathcal{L}_{\tilde{n}}(0, 0_\perp) F^{\mu+}(0) | P \rangle$$

Collins, Soper, 81; Collins, Soper, Sterman, 85



Gloun---->gloun splitting kernel; $\mathcal{P}_{gg}(z) = 2C_A \frac{(z^2 - z + 1)^2}{z(1-z)}$

When z(or x) --->0, large logarithm $\ln \frac{1}{x}$ summed by BFKL

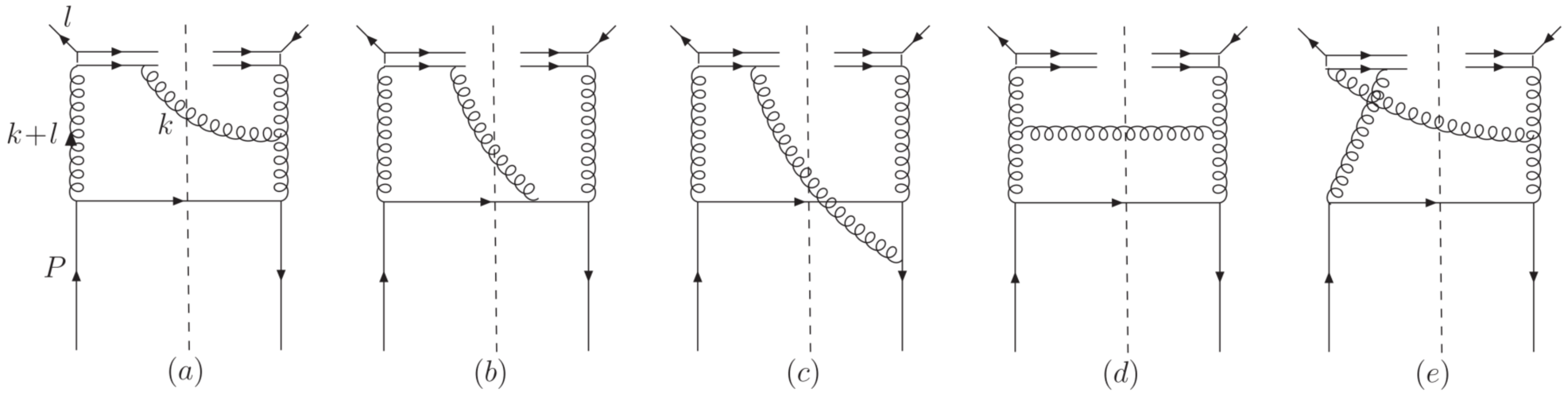
When z--->1, light cone divergence, introduce ζ to regularize

large logarithm $\ln \frac{x^2 \varsigma^2}{k_T^2}$ summed by CS

Mueller, Xiao, Yuan 2012; Xiao, Yuan, Zhou, 2017; Caucal, Salazar, Schenke, Venugopalan, 2022-2023; Taels, Altinoluk, Beuf, Marquet, 2022; Mukherjee, Skokov, Tarasov, Tiwari, 2023….

Formulation in terms of gluon TMD(dilute limit)

◆ Sample real diagrams



➢ The resulting gluon TMD indeed simultaneously satisfies the both

$$\int_0^\infty \frac{dk^+}{k^+} = \int_{l^+}^\infty \frac{dk^+}{k^+} + \int_0^{l^+} \frac{dk^+}{k^+}$$

2016, ZJ

BFKL equation:

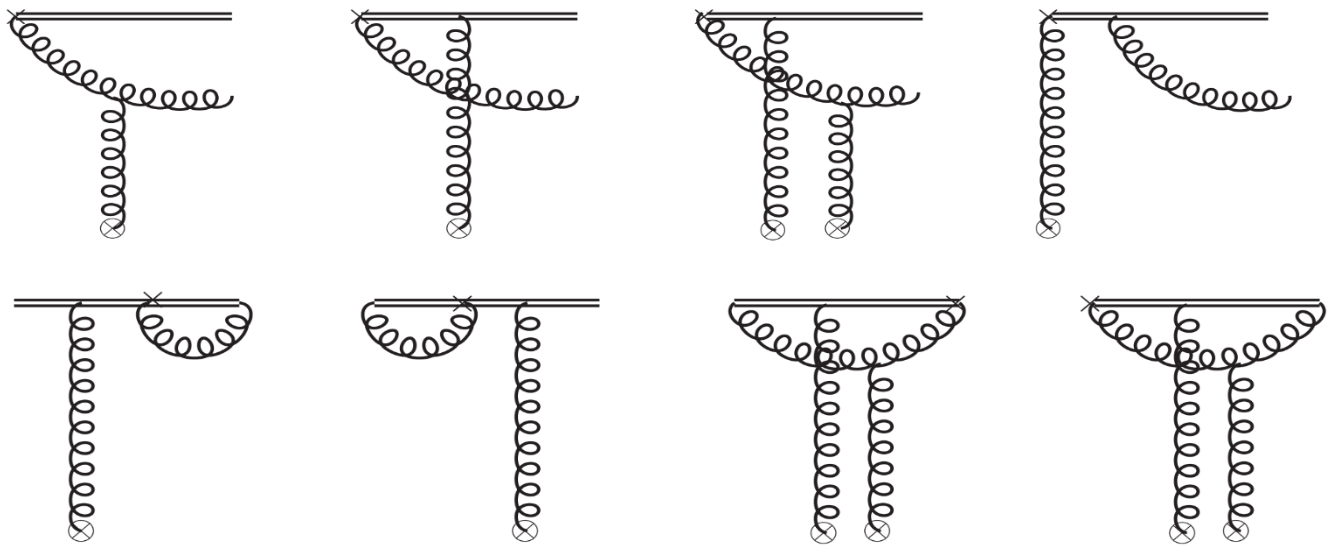
$$\frac{\partial\left[xG(x,l_\perp,x\zeta)\right]}{\partial\ln(1/x)} = \frac{\alpha_s N_c}{\pi^2}\int\frac{d^2k_\perp}{k_\perp^2}\left\{xG(x,k_\perp+l_\perp,x\zeta)-\frac{l_\perp^2}{2(l_\perp+k_\perp)^2}xG(x,l_\perp,x\zeta)\right\}$$

CS equation:

$$\frac{\partial\left[G(x,b_\perp,x\zeta)\right]}{\partial\ln\zeta} = -\frac{\alpha_s N_c}{\pi}\ln\left[\frac{x^2\zeta^2 b_\perp^2}{4}e^{2\gamma_E-\frac{1}{2}}\right]G(x,b_\perp,x\zeta)$$

Small x TMDs in CGC at NLO(double log)

Sample diagrams (**Collins-2011 scheme**)



Xiao, Yuan and ZJ, 2017.

The basic nonperturbative ingredient:

$$U(x_\perp) = \langle \mathcal{P} e^{-ig\int_{-\infty}^{+\infty} dx^- A^+(x^-,\, x_\perp)} \rangle$$

Multiple gluon rescattering is treated in an equal way.

Collinear approach v.s. CGC

- Collinear factorization:

$$\tilde{f}_g^{(sub.)}(x, r_\perp, \zeta_c{}') = e^{-S^g_{pert}(Q, r_\perp)} \sum_i C_{g/i}(\mu_r/\mu) \otimes f_i(x, \mu)$$

Sudakov factor

Hard coefficient

Colliner PDF

- CGC **(Colliner divergence absent)**

$$xG^{(1)}(x, k_\perp, \zeta) = -\frac{2}{\alpha_S} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^4} e^{ik_\perp \cdot r_\perp} \mathcal{H}^{WW}(\alpha_s(Q)) e^{-S_{sud}(Q^2, r_\perp^2)} \mathcal{F}^{WW}_{Y=\ln 1/x}(x_\perp, y_\perp)$$

Hard coefficient

Sudakov factor

Two point function

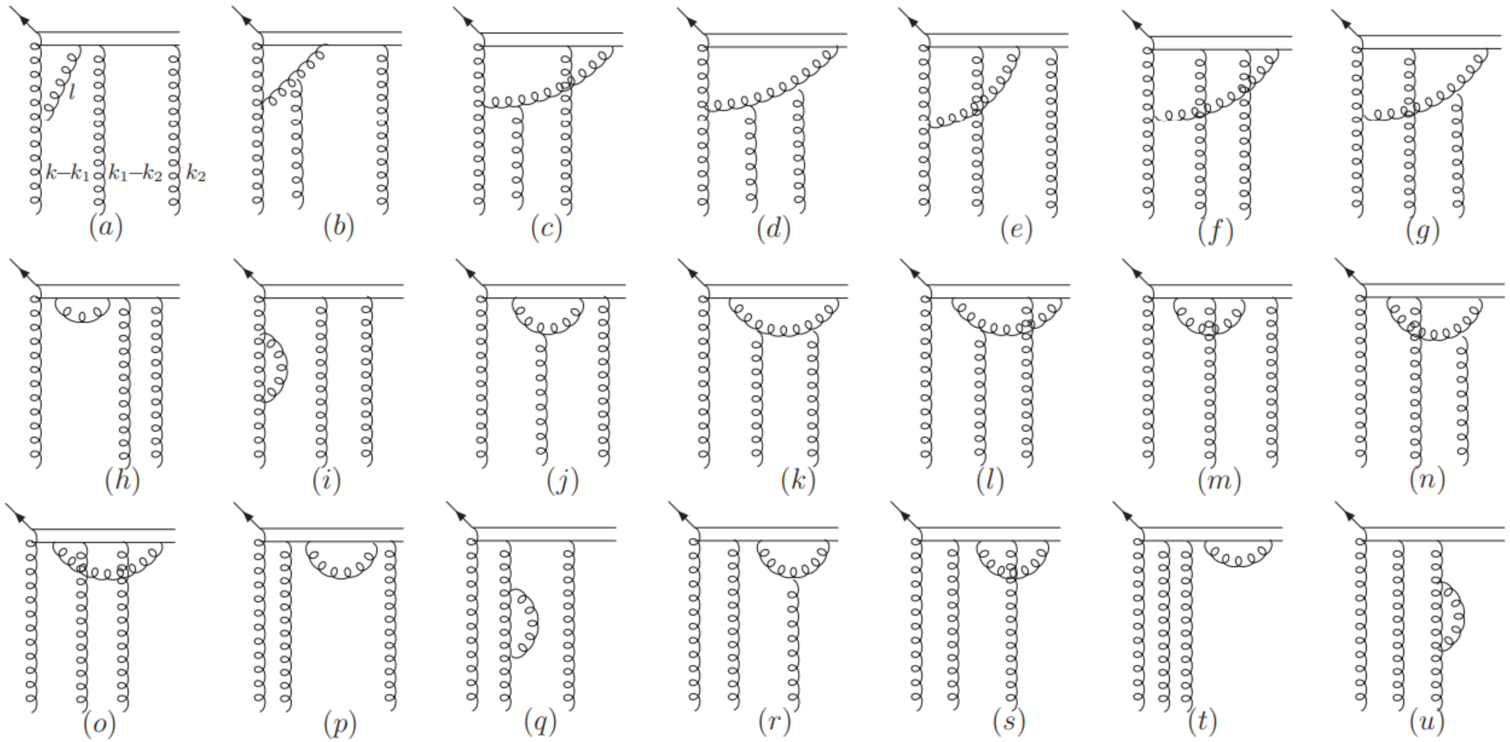
Two step evolution: $x_0 \longrightarrow x$ $\quad k_t \longrightarrow Q$

Sudakov Single log at small x

The anomalous dimension at small x?

$$\frac{\mathrm{d}\ \ln G(x, b_\perp, \mu^2, \zeta_c^2)}{\mathrm{d}\ \ln\mu} = \gamma_G\left(g(\mu), \zeta_c^2/\mu^2\right)$$

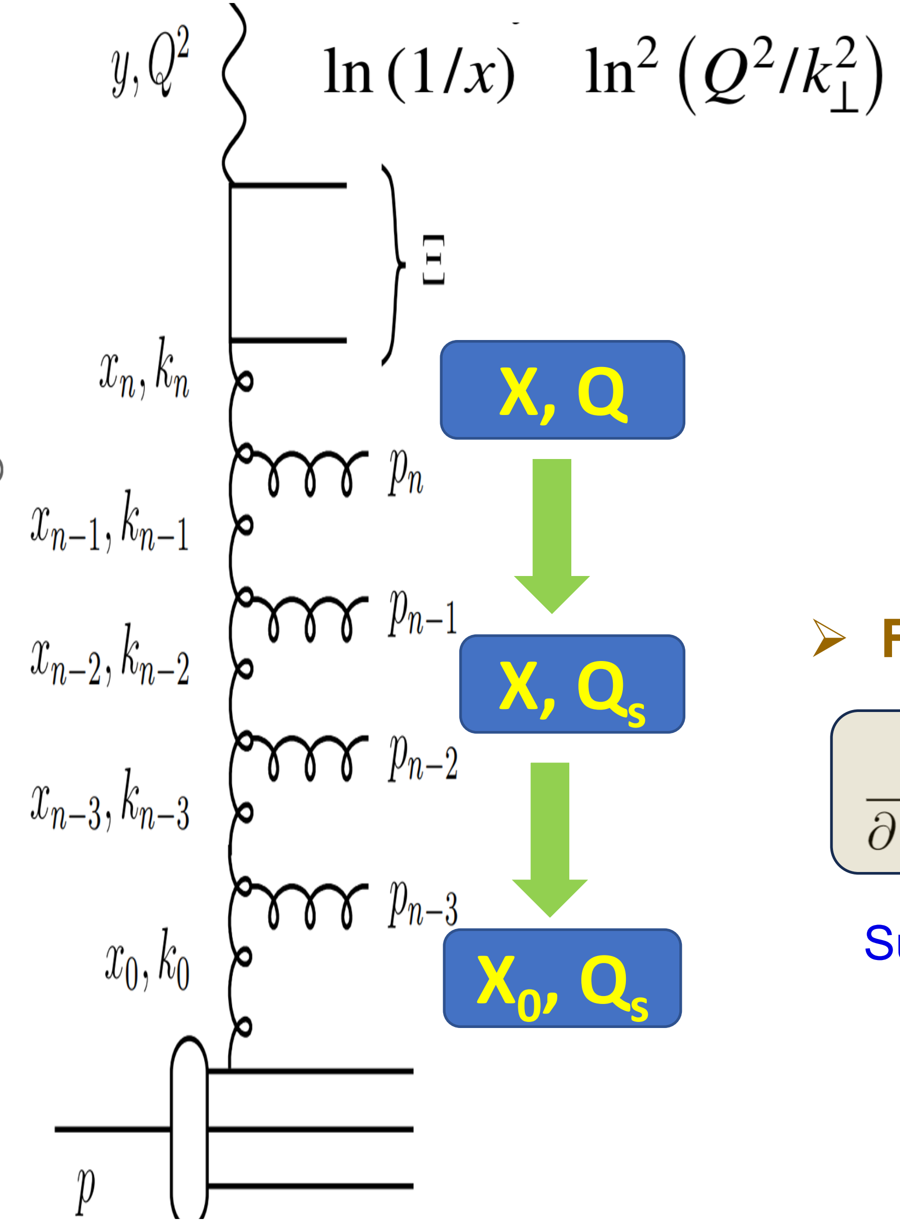
Compute UV part first; employ the Eikonal approximation next!



2019, ZJ

- Single log is not affected by saturation effect.

Monte Carlo implementation of the joint resummation



- Combing Collins-Soper equation

$$\frac{\partial N(\mu^2,\zeta^2,x,k_\perp)}{\partial \ln \zeta^2} = \frac{2\alpha_s N_c}{\pi^2}\int_0^\zeta \frac{d^2 l_\perp}{l_\perp^2}\left[N(\mu^2,\zeta^2,x,k_\perp+l_\perp) - N(\mu^2,\zeta^2,x,k_\perp)\right]$$

with renormalization group equation

$$\frac{\partial N(\mu^2,\zeta^2,x,k_\perp)}{\partial \ln \mu^2} = \frac{\alpha_s N_c}{\pi}\left[\frac{\beta_0}{6} - \ln\frac{\zeta^2}{\mu^2}\right] N(\mu^2,\zeta^2,x,k_\perp)$$

- **Folded CS+RG:** $N(Q^2,x,k_\perp) \equiv N(\mu^2 = Q^2, \zeta^2 = Q^2, x, k_\perp)$

$$\frac{\partial}{\partial \ln Q^2}\frac{N(Q^2,x,k_\perp)}{\Delta_s(Q^2,k_\perp)} = \frac{2\bar{\alpha}_s}{\pi}\int_{Q_0}^{Q}\frac{d^2 l_\perp}{l_\perp^2}\frac{N(Q^2,x,k_\perp+l_\perp)}{\Delta_s(Q^2,k_\perp)}$$

Shi-Wei-ZJ, 2023

Sudakov form factor

$$\Delta_s(Q^2,k_\perp) = \exp\left[-\bar{\alpha}_s\int_{Q_0^2}^{Q^2}\frac{dt}{t}\left(2\ln\frac{t}{Q_0^2} - \frac{\beta_0}{6}\right)\right]$$

Monte Carlo implementation of kt resummation

- The modified Sudakov factor in the backward evolution:

$$\frac{\Delta_s(Q_n^2, k_{\perp,n}) N(Q_{n-1}^2, x_n, k_{\perp,n})}{\Delta_s(Q_{n-1}^2, k_{\perp,n}) N(Q_n^2, x_n, k_{\perp,n})} = \exp\left[-\int_{Q_{n-1}^2}^{Q_n^2} \frac{dt}{t} \int_{Q_0}^{t} \frac{d^2 l_\perp}{l_\perp^2} \frac{2\bar{\alpha}_s(l_\perp^2)}{\pi} \frac{N(t, x_n, k_{\perp,n} + l_\perp)}{N(t, x_n, k_{\perp,n})}\right] = \mathcal{R}$$

- Sample lt and reconstruct z

Shi-Wei-ZJ, 2023

$$\int \frac{d^2 l_\perp}{l_\perp^2} \frac{\bar{\alpha}_s(l_\perp^2)}{\pi} N(Q_{n-1}^2, x_n, k_{\perp,n} + l_\perp) \qquad l_{\perp,n}^2 \approx Q_{n-1}^2 (1 - z_n)$$

- Weight factor

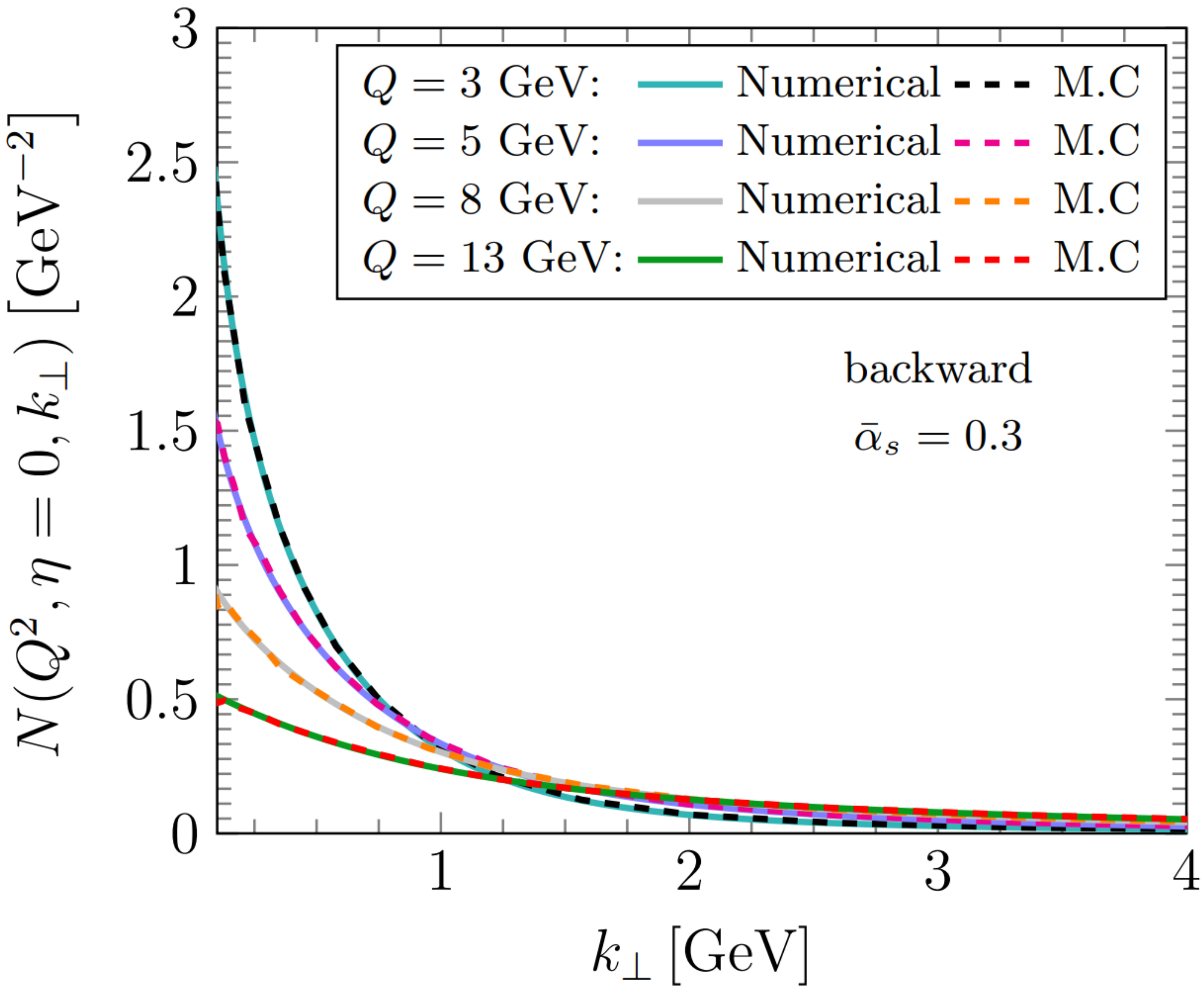
$$\mathcal{W} = \frac{\int_{Q_{n-1}^2}^{Q_n^2} \frac{dt}{t} \ln \frac{t^2}{Q_0^2}}{\int_{Q_{n-1}^2}^{Q_n^2} \frac{dt}{t} \left[\ln \frac{t^2}{Q_0^2} - \frac{\beta_0}{12}\right]}$$

- Unitary is preserved only within the double leading log approximation

- Terminate kt PS and turn on small x PS

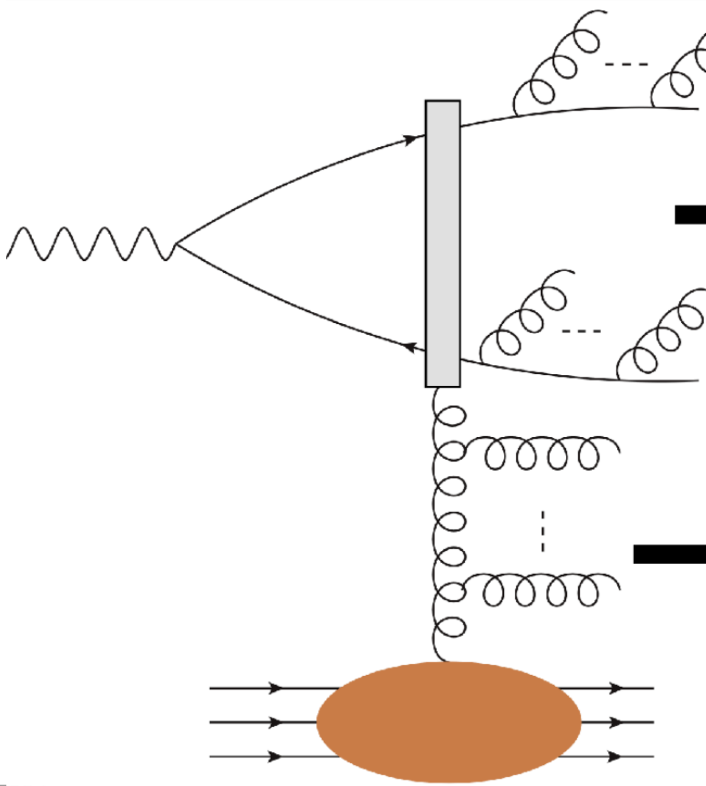
Z_i>0.5 Q_i>Q_s

Numerical V.S. MC:



Initial condition: $$N(Q_0^2 = 1\ \mathrm{GeV}^2, x = 0.01, k_\perp) = \int \frac{d^2 r_\perp}{2\pi} e^{ik_\perp \cdot r_\perp} \frac{1}{r_\perp^2} \left[1 - e^{-\frac{Q_s^2 r_\perp^2}{4} \log\left(\frac{1}{r_\perp \Lambda_{\mathrm{MV}}} + e\right)}\right]$$

Hadronization and final state radiations



FSR, Pythia

ISR, our algorithm

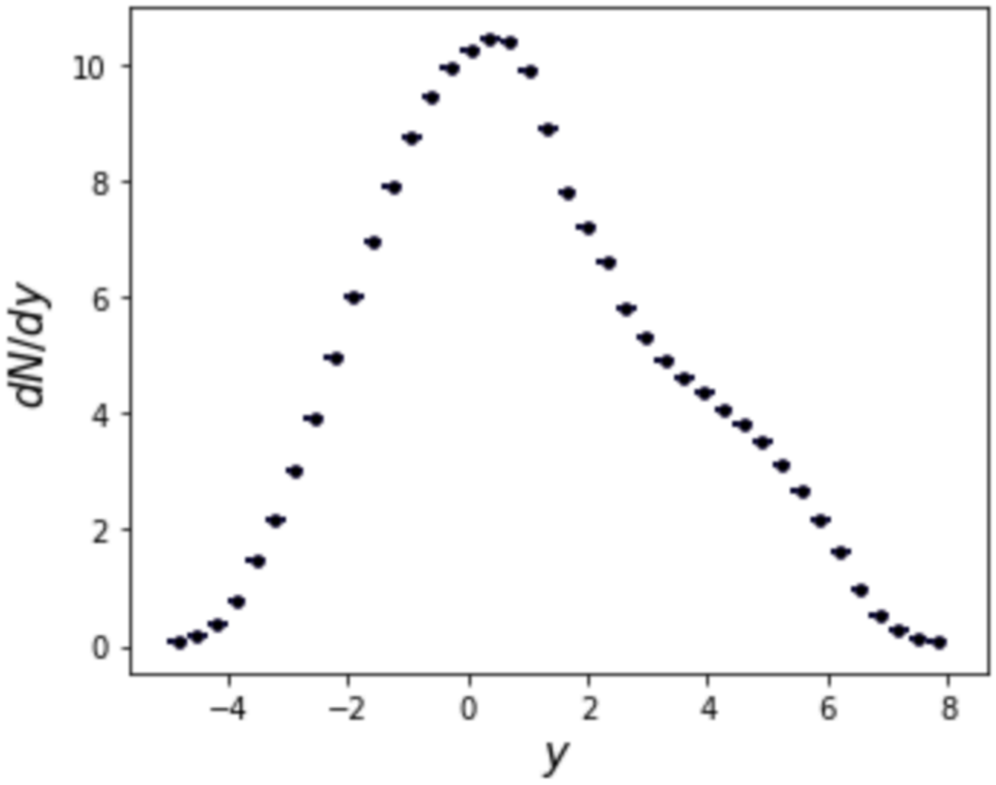
Hadronization Pythia

Di-jet/di-hadron production in the DIS

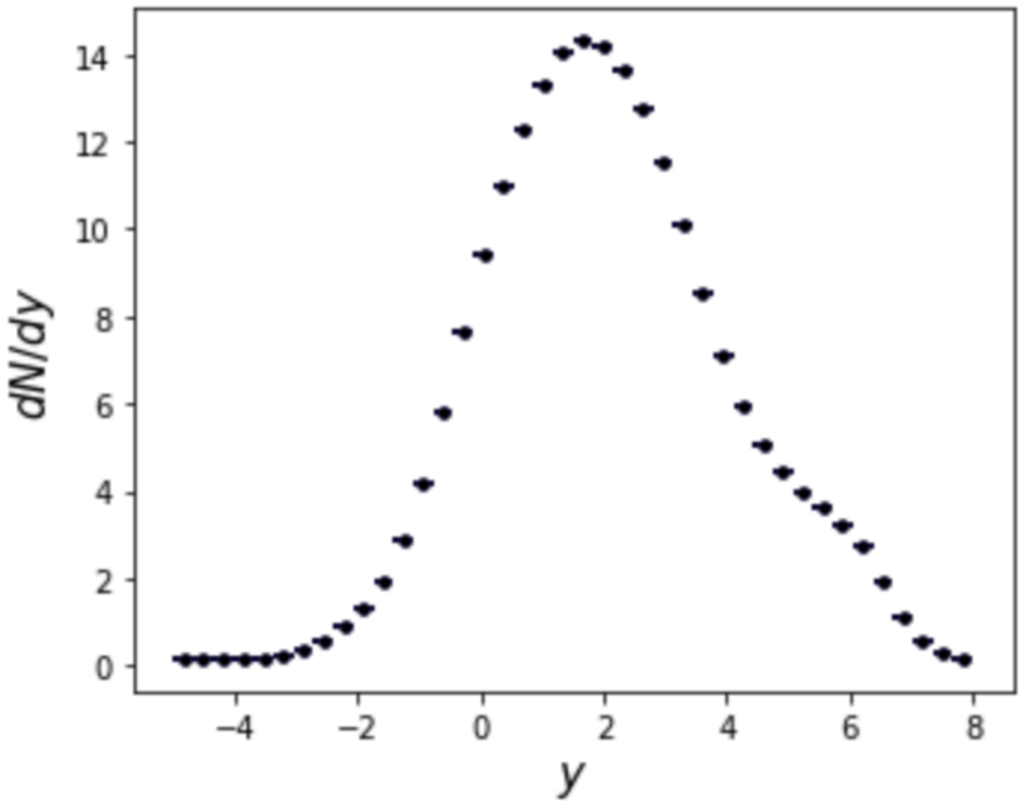
Lepton-proton collider at HERA (Photon is quasi-real photon.)

Working in progress

Preliminary results



Small-x Cascade
Pythia + hadronization



Pythia

Summary and Outlook

- The first PS algorithm incorporating saturation effect
- The implementation of the joint resummation is sketched

- Recoiled effect; linear polarization effect; multiple gluon fusion effect
- Smooth transition to non-linear QCD regime: compare with HERA data
- Integrate into eHIJING, working with Yu Shi, Wei-yao Ke and Xin-nian Wang

Thank you for your attention!

山东大学（青岛）
SHANDONG UNIVERSITY, QINGDAO