Modified Lüscher formula for long-range interaction

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Based on [JHEP 05 (2024) 168] [arXiv: 2507.18399]

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目录
CONTENTS

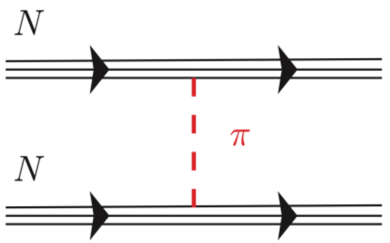
目录

CONTENTS

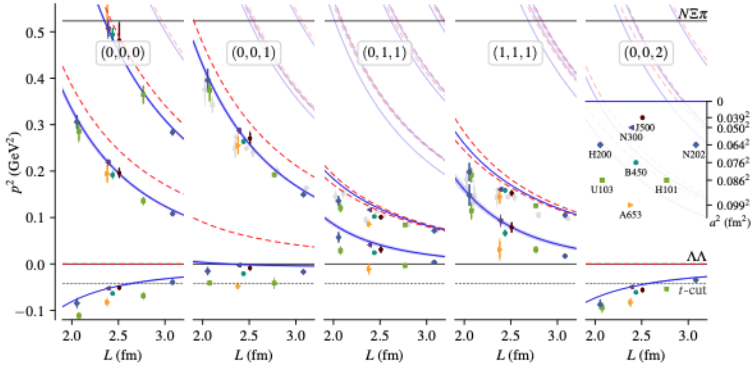
01 Motivation

Inclusion of the long-range forces

- **Finite volume problem of long-range force**
  - Left-hand cut close to threshold: Energy levels below the left-handed branch point cannot be used
  - Slowly converging partial-wave expansion: Expecting strong admixture of higher partial-waves in the quantization condition
  - Exponentially suppressed corrections still sizable

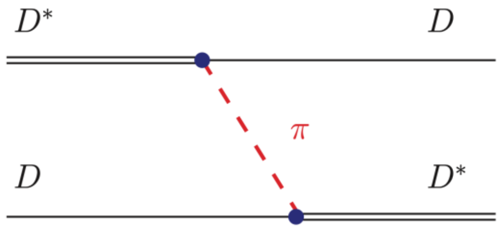


NN scattering



J.R. Green *et al.*, PRL 127(24) (2021) 242003

- **NN scattering**
  - Left-hand cut: $-\infty < s \leq \underbrace{(2m_N)^2 - M_\pi^2}_{(1875\text{MeV})^2}$ right-hand cut: $\underbrace{(2m_N)^2}_{(1880\text{MeV})^2} \leq s < +\infty$
  - Phase shift real below left-handed branch point?



$T_{cc}^+(3875)$

- **Interpretation of $T_{cc}^+(3875)$**
  - Left-hand cut extremely close to threshold
  - Affects the extraction of the phase shifts and the parameters of the $T_{cc}^+(3875)$ from lattice data

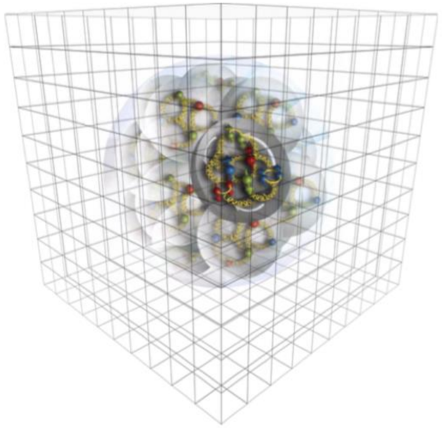
Case of a stable D^*

01 Motivation

Schemes to including long-range force

➢ **Plane-wave basis**

- Describe the system in terms of the parameters of the effective Lagrangian
- Work in the plane-wave basis
- For the NN scattering, it was shown that, at the physical quark masses, the partial-wave mixing is sizable [Meng & Epelbaum, 2021]
- A consistent fit of the DD^* scattering phases to lattice data in the left-hand cut region has been performed [Meng et al. 2023]



➢ **Alternative approaches**

- Splitting long- and short-range interactions [Hansen & Raposo, 2023]
- Using Lüscher equation plus EFT with long-range force in the infinite volume [Collins et al., 2024]
- Other contributions [Lyu et al., 2023] [Du, Guo & Wu, 2024] [Dawid et al., 2025] etc.

➢ **Our scheme**

- Modified Lüscher formula [Bubna, Hammer, Müller, JYP, Rusetsky & Jia-Jun Wu, JHEP 05 (2024) 168] [Bubna, Hammer, Boid, JYP, Rusetsky & Jia-Jun Wu, arXiv:2507.18399]
- Finite volume version of the modified effective range expansion [van Haeringen & Kok, 1982]

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2}\left(\cot(\delta_\ell(q) - \sigma_\ell(q)) - i\right) = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\tilde{r}_\ell q^2 + O(q^4)$$

目录

CONTENTS

02 Modified Lüscher formula (infinite volume part)

Decomposition of interaction

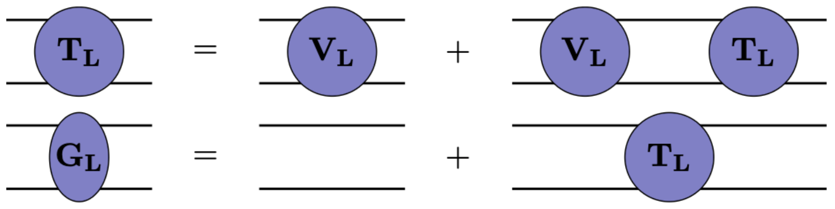
➢ **Decomposition of potential**

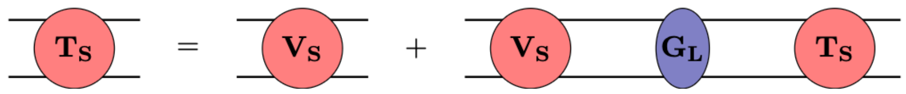
$$V = V_L + V_S$$

- V_L: long-range potential which is known from, e.g., χPT
- V_S: short-range potential which is unknown, but local

➢ **Amplitudes and full propragators**

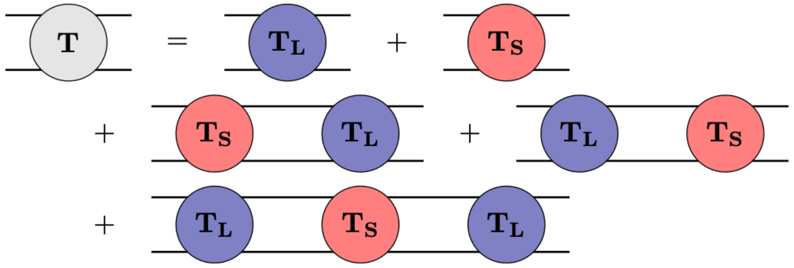
- T_L: long-range amplitude which is established only by V_L
- G_L: long-range propagator
- T_S: effective short-range amplitude which is established by both long- and short-range interaction





➢ **Decomposition of T-matrix**

$$T = T_L + (1 + T_L G)T_S(GT_L + 1)$$



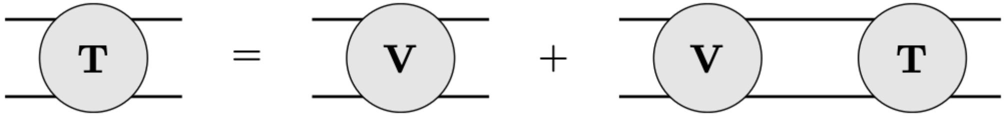
02 Modified Lüscher formula (infinite volume part)

Pole structure

➢ **Pole position**

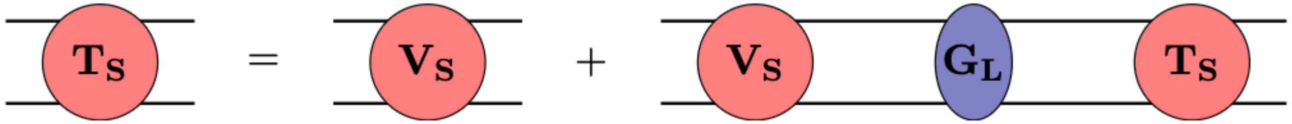
$$T = T_L + (1 + T_L G) T_S (G T_L + 1)$$

- Poles of T are equivalent to T_S
- Free energy (pole of G)
- Long-range eigen-energy (pole of T_L)
- Eigen-energy (pole of T)



➢ **Effective short-range LS equation**

$$T_S = V_S + V_S G_L T_S$$



- Solve effective equation for T_S instead of total LS equation for T

02 Modified Lüscher formula (infinite volume part)

Effective short-range amplitude

➢ **Effective short-range LS equation**

$$T_S = V_S + V_S G_L T_S$$

➢ **Effective short-range amplitude**

$$\boxed{\langle \psi^{(+)}(q_0^2), \ell m | T_S | \psi^{(+)}(q_0^2), \ell m \rangle = \frac{1}{q_0 \cot(\delta_\ell - \sigma_\ell) - i q_0}.}$$

$$\langle q_0, \ell m | T | q_0, \ell m \rangle = \frac{1}{q_0 \cot \delta_\ell - i q_0},$$

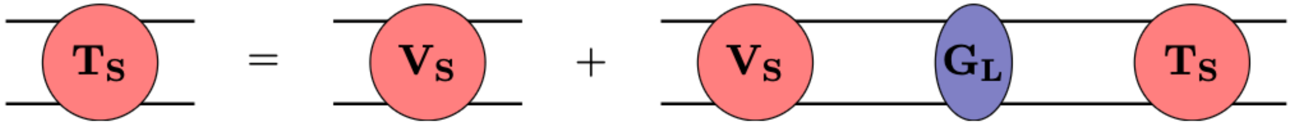
$$\langle q_0, \ell m | T_L | q_0, \ell m \rangle = \frac{1}{q_0 \cot \sigma_\ell - i q_0}.$$

- $|\psi^{(+)}(q_0^2), \ell m\rangle$: Partial-wave long-range basis
- δ_ℓ : phase shift of total interaction (need to determine)
- σ_ℓ : phase shift of long interaction (known)

02 Modified Lüscher formula (infinite volume part)

Effective short-range LS equation

➢ **Effective LS equation in long-range bases**



$$\boxed{T_S = V_S + V_S G_L T_S}$$

$$B(\mathbf{p},\mathbf{q};q_0^2+i\epsilon) = \tilde{V}_S(\mathbf{p},\mathbf{q}) + \int \frac{d^3k}{(2\pi)^3}\tilde{V}_S(\mathbf{p},\mathbf{k})\frac{1}{k^2-q_0^2-i\epsilon}B(\mathbf{k},\mathbf{q};q_0^2+i\epsilon)$$

- Effective amplitude $B(\mathbf{p},\mathbf{q};q_0^2+i\epsilon) = \langle\psi_{\mathbf{p}}^{(+)}|T_S(q_0^2+i\epsilon)|\psi_{\mathbf{q}}^{(+)}\rangle$
- Effective short-range potential $\tilde{V}_S(\mathbf{p},\mathbf{q}) = \langle\psi_{\mathbf{p}}^{(+)}|V_S|\psi_{\mathbf{q}}^{(+)}\rangle$
- Long-range propagator

Long-range propagator's spectral representation $G_L(q_0^2+i\epsilon) = \int \frac{d^3p}{(2\pi)^3}\frac{|\psi_{\mathbf{p}}^{(\pm)}\rangle\langle\psi_{\mathbf{p}}^{(\pm)}|}{p^2-q_0^2-i\epsilon}$ $\longrightarrow$ $\langle\psi_{\mathbf{p}}^{(\pm)}|G_L(q_0^2+i\epsilon)|\psi_{\mathbf{k}}^{(\pm)}\rangle = \frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{k})}{k^2-q_0^2-i\epsilon}$

02 Modified Lüscher formula (infinite volume part)

Separation of effective short-range potential

$$B(\mathbf{p},\mathbf{q};q_0^2+i\epsilon)=\tilde{V}_S(\mathbf{p},\mathbf{q})+\int\frac{d^3k}{(2\pi)^3}\tilde{V}_S(\mathbf{p},\mathbf{k})\frac{1}{k^2-q_0^2-i\epsilon}B(\mathbf{k},\mathbf{q};q_0^2+i\epsilon)$$

➢ **Effective short-range potential in long-range bases** $\tilde{V}_S(\mathbf{p},\mathbf{q})=\langle\psi_{\mathbf{p}}^{(+)}|V_S|\psi_{\mathbf{q}}^{(+)}\rangle$

Low-energy constants

- Polynomial representation of local potential

$$\langle\mathbf{p}|V_S|\mathbf{q}\rangle=4\pi\sum_{\ell m}\mathcal{Y}_{\ell m}(\mathbf{p})\Big[\sum_{a,b}C_\ell^{ab}(p^2)^a(q^2)^b\Big]\mathcal{Y}_{\ell m}^*(\mathbf{q})$$

- Potential in long-range bases

$$\tilde{V}_S^{(\ell)}(k,k')=\sum_{a,b}C_\ell^{ab}(A_\ell^a(k))^*\,A_\ell^b(k')$$

- Separation of long- and short-range interaction

UV integral of long-range wave-function $A_\ell^a(k)=\int\frac{d^3p}{(2\pi)^3}(p^2)^a p^\ell\psi_\ell^{(+)}(p;k)$

Separation of UV integral $A_\ell^a(k)=v_\ell^{(+)}(k)\mathfrak{p}_a(k^2)$ Local polynomials

$$v_\ell^{(+)}(k)=\frac{1}{4\pi}\int\frac{d^3p}{(2\pi)^3}p^\ell\psi_\ell^{(+)}(p;k)=\frac{k^\ell}{(2\ell+1)!f_\ell(k)}$$

Long-range Jost function

$$\tilde{V}_S^{(\ell)}(p,q)=\boxed{\frac{(pq)^\ell}{[(2\ell+1)!!]^2f_\ell^*(p)f_\ell(q)}}\bar{V}_S^{(\ell)}(p,q)$$

Long-range contribution in effective potential

Short-range kernel

$$\boxed{\bar{V}_S^{(\ell)}(p,q)=\sum_{a,b}C_\ell^{ab}\mathfrak{p}_a(p^2)\mathfrak{p}_b(q^2)}$$

02 Modified Lüscher formula (infinite volume part)

Solve effective LS equation in the infinite volume

➢ **Threshold expansion**

$$\boxed{B_\ell(p,q;q_0^2+i\epsilon)=\tilde{V}_S^{(\ell)}(p,q)+\int\frac{d^3k}{(2\pi)^3}\tilde{V}_S^{(\ell)}(p,k)\frac{1}{k^2-q_0^2-i\epsilon}B_\ell(k,q;q_0^2+i\epsilon).}$$

- Equation driven by short-range kernel $\bar{V}_S$

$$\tilde{V}_S^{(\ell)}(p,q)=\boxed{\frac{(pq)^\ell}{[(2\ell+1)!!]^2 f_\ell^*(p)f_\ell(q)}}\bar{V}_S^{(\ell)}(p,q)\qquad B_\ell(p,q;q_0^2+i\epsilon)=\boxed{\frac{(pq)^\ell}{[(2\ell+1)!!]^2 f_\ell^*(p)f_\ell(q)}}\bar{B}_\ell(p,q;q_0^2+i\epsilon)$$

$$\boxed{\bar{B}_\ell(p,q;q_0^2+i\epsilon)=\bar{V}_S^{(\ell)}(p,q)+\int\frac{d^3k}{(2\pi)^3}\frac{k^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\frac{\bar{V}_S^{(\ell)}(p,k)\bar{B}_\ell(k,q;q_0^2+i\epsilon)}{k^2-q_0^2-i\epsilon}}$$

Singularity-free

- Singularity-free R-equation

$$\boxed{R_\ell(p,q;q_0^2)=\bar{V}_S^{(\ell)}(p,q)+\int\frac{d^3k}{(2\pi)^3}\frac{k^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\frac{\bar{V}_S^{(\ell)}(p,k)R_\ell(k,q;q_0^2)-\bar{V}_S^{(\ell)}(p,q_0)R_\ell(k,q_0;q_0^2)}{k^2-q_0^2}}$$

- On-shell effective amplitude

$$B_\ell(q_0^2+i\epsilon)=\frac{q_0^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(q_0)|^2}\frac{1}{\boxed{R_\ell^{-1}(q_0,q_0;q_0^2)}-\underbrace{\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle}_{\text{Loop integral}}}$$

Loop integral $\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle=\int\frac{d^3k}{(2\pi)^3}\frac{1}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\frac{k^{2\ell}}{k^2-q_0^2-i\epsilon}$

02 Modified Lüscher formula (infinite volume part)

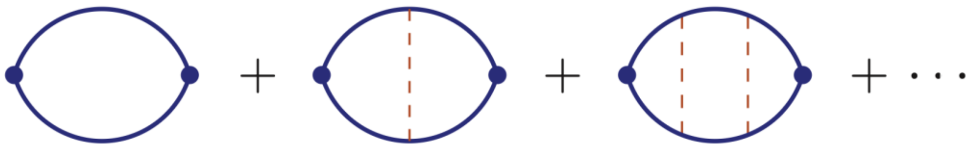
Renormalization in the infinite volume

➢ **Match equation in the infinite volume**

$$B_\ell(q_0^2+i\epsilon)=\frac{q_0^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(q_0)|^2}\frac{1}{R_\ell^{-1}(q_0,q_0;q_0^2)-\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle}\quad\xleftrightarrow{\text{Match}}\quad B_\ell(q_0^2+i\epsilon)=\frac{4\pi q_0^{2\ell}}{q_0^{2\ell+1}\cot(\delta_\ell-\sigma_\ell)-iq_0^{2\ell+1}}$$

- Renormalization of loop integral

$$\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle=\int\frac{d^3k}{(2\pi)^3}\frac{1}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\frac{k^{2\ell}}{k^2-q_0^2-i\epsilon}$$



$$\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle=\frac{M_\ell(q_0)+\mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

Conter-terms

$$M_\ell(q)=\frac{(-iq)^\ell}{2^\ell\ell!}\frac{1}{f_\ell(q)}\lim_{r\to0}\frac{d^{2\ell+1}}{dr^{2\ell+1}}\left[r^\ell f_\ell(q,r)\right]\qquad f_\ell(q)=\frac{(-iq)^\ell}{(2\ell-1)!!}\lim_{r\to0}\left[r^\ell f_\ell(q,r)\right]$$

- Match equation and modified K-matrix [Phys. Rev. A 26 (1982) 1218]

Renormalization condition

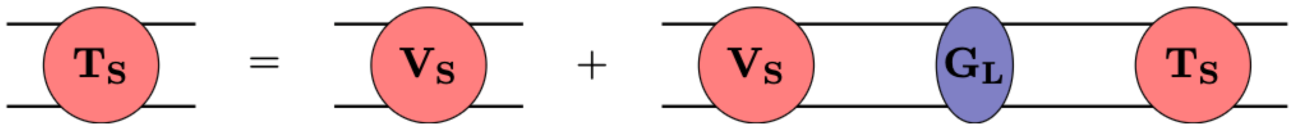
$$\boxed{4\pi[(2\ell+1)!!]^2R_\ell^{-1}(q_0,q_0;q_0^2)=K_\ell^M(q_0^2)+\mathcal{P}_\ell(q_0^2)}$$

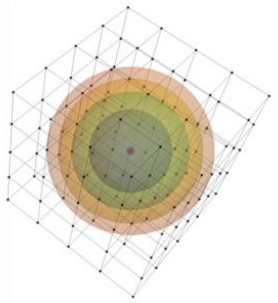
Modified K-matrix

$$\boxed{K_\ell^M(q_0^2)=M_\ell(q_0)+\frac{q_0^{2\ell+1}\cot(\delta_\ell-\sigma_\ell)-iq_0^{2\ell+1}}{|f_\ell(q_0)|^2}}$$

02 Modified Lüscher formula

Finite volume effective LS equation





Equation in the infinite volume

$$B(\mathbf{p},\mathbf{q};q_0^2+i\epsilon)=\tilde{V}_S(\mathbf{p},\mathbf{q})+\int\frac{d^3k}{(2\pi)^3}\tilde{V}_S(\mathbf{p},\mathbf{k})\frac{1}{k^2-q_0^2-i\epsilon}B(\mathbf{k},\mathbf{q};q_0^2+i\epsilon)$$

Equation in a finite volume ?

➢ **Finite volume basis**

$$B(\mathbf{p},\mathbf{q};q_0^2+i\epsilon)=\langle\psi_{\mathbf{p}}^{(+)}|T_S(q_0^2+i\epsilon)|\psi_{\mathbf{q}}^{(+)}\rangle$$

$$\tilde{V}_S(\mathbf{p},\mathbf{q})=\langle\psi_{\mathbf{p}}^{(+)}|V_S|\psi_{\mathbf{q}}^{(+)}\rangle$$

- Discrete long-range bases

$$|\psi_{\mathbf{n}}\rangle=\frac{1}{L^3}\sum_{\mathbf{p}}\psi_{\mathbf{n}}(\mathbf{p})|\mathbf{p}\rangle$$

$$\langle\psi_{\mathbf{n}}|\psi_{\mathbf{n}'}\rangle=\delta_{\mathbf{n}\mathbf{n}'},$$

$$\sum_{\mathbf{n}}|\psi_{\mathbf{n}}\rangle\langle\psi_{\mathbf{n}}|=1.$$

$$B(\mathbf{n},\mathbf{n}';q_0^2)=\langle\psi_{\mathbf{n}}|T_S(q_0^2)|\psi_{\mathbf{n}'}\rangle,$$

$$\tilde{V}_S(\mathbf{n},\mathbf{n}')=\langle\psi_{\mathbf{n}}|V_S|\psi_{\mathbf{n}'}\rangle,$$

- Finite volume effective LS equation in discrete long-range bases

Equation in a finite volume

$$\boxed{B(\mathbf{n},\mathbf{n}';q_0^2)=\tilde{V}_S(\mathbf{n},\mathbf{n}')+\sum_{\mathbf{k}}\tilde{V}_S(\mathbf{n},\mathbf{k})\frac{1}{q_{\mathbf{k}}^2-q_0^2}B(\mathbf{k},\mathbf{n}';q_0^2)}$$

02 Modified Lüscher formula

Separation of effective short-range potential in a finite volume

$$B(\mathbf{n},\mathbf{n}';q_0^2)=\tilde{V}_S(\mathbf{n},\mathbf{n}')+\sum_{\mathbf{k}}\tilde{V}_S(\mathbf{n},\mathbf{k})\frac{1}{q_{\mathbf{k}}^2-q_0^2}B(\mathbf{k},\mathbf{n}';q_0^2)$$

➢ **Finite volume effective short-range potential in long-range bases**

$$\tilde{V}_S(\mathbf{n},\mathbf{n}')=\langle\psi_{\mathbf{n}}|V_S|\psi_{\mathbf{n}'}\rangle$$

- Polynomial representation of local potential is the same in both inf. and fin. volume.

$$\langle\mathbf{p}|V_S|\mathbf{q}\rangle=4\pi\sum_{\ell m}\mathcal{Y}_{\ell m}(\mathbf{p})\Big[\sum_{a,b}C_\ell^{ab}(p^2)^a(q^2)^b\Big]\mathcal{Y}^*_{\ell m}(\mathbf{q})$$

- Potential in long-range bases

$$\tilde{V}_S(\mathbf{n},\mathbf{n}')=4\pi\sum_{\ell m}\sum_{a,b}C_\ell^{ab}(\mathcal{A}^a_{\ell m}(\mathbf{n}))^*\mathcal{A}^b_{\ell m}(\mathbf{n}')$$

UV summation of long-range wave-funct $\mathcal{A}^a_\ell(\mathbf{n})=\frac{1}{L^3}\sum_{\mathbf{p}}(p^2)^a\mathcal{Y}^*_{\ell m}(\mathbf{p})\psi_{\mathbf{n}}(\mathbf{p})$

- Separation of long- and short-range interaction

Separation of UV summation $\mathcal{A}^a_{\ell m}(\mathbf{n})=v_{\ell m}(\mathbf{n})\mathfrak{p}_a(q_{\mathbf{n}}^2)$

$$v_{\ell m}(\mathbf{n})=\frac{1}{L^3}\sum_{\mathbf{p}}\mathcal{Y}^*_{\ell m}(\mathbf{p})\psi_{\mathbf{n}}(\mathbf{p})$$

Local polynomials:
UV limits of V_L and its derivative
are NOT affected by finite vol.

$$\tilde{V}_S(\mathbf{n},\mathbf{n}')=4\pi\sum_{\ell m}v^*_{\ell m}(\mathbf{n})\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{n}'})v_{\ell m}(\mathbf{n}')$$

Long-range contribution
in effective potential

$$\boxed{\bar{V}_S^{(\ell)}(p,q)=\sum_{a,b}C_\ell^{ab}\mathfrak{p}_a(p^2)\mathfrak{p}_b(q^2)}$$

Short-range kernel NOT affected by finite vol.

02 Modified Lüscher formula

Solve effective LS equation in a finite volume

➢ **“Threshold expansion” in a finite volume**

- Finite volume effective LS equation in long-range bases

$$B(\mathbf{n},\mathbf{n}';q_0^2)=\tilde{V}_S(\mathbf{n},\mathbf{n}')+\sum_{\mathbf{k}}\tilde{V}_S(\mathbf{n},\mathbf{k})\frac{1}{q_{\mathbf{k}}^2-q_0^2}B(\mathbf{k},\mathbf{n}';q_0^2)$$

- Equation driven by short-range kernel $\bar{V}_S$

$$\tilde{V}_S(\mathbf{n},\mathbf{n}')=4\pi\sum_{\ell m}\underline{v_{\ell m}^*(\mathbf{n})}\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{n}'})\underline{v_{\ell m}(\mathbf{n}')}$$

$$B(\mathbf{n},\mathbf{n}';q_0^2)=4\pi\sum_{\ell m,\ell' m'}\underline{v_{\ell m}^*(\mathbf{n})}\bar{B}_{\ell m,\ell' m'}(q_{\mathbf{n}},q_{\mathbf{n}'};q_0^2)\underline{v_{\ell' m'}(\mathbf{n}')}$$

$$\begin{aligned}\bar{B}_{\ell m,\ell' m'}(q_{\mathbf{n}},q_{\mathbf{n}'};q_0^2)=&\,\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{n}'})\delta_{\ell m,\ell' m'}\\&+4\pi\sum_{j\nu}\sum_{\mathbf{k}}\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{k}})\frac{v_{\ell m}(\mathbf{k})v_{j\nu}^*(\mathbf{k})}{q_{\mathbf{k}}^2-q_0^2}\bar{B}_{j\nu,\ell' m'}(q_{\mathbf{k}},q_{\mathbf{n}'};q_0^2).\end{aligned}$$

- Singularity-free R-equation

$$\begin{aligned}R_{\ell m,\ell' m'}(q_{\mathbf{n}},q_{\mathbf{n}'};q_0^2)=&\,\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{n}'})\delta_{\ell m,\ell' m'}+4\pi\sum_{j\nu}\sum_{\mathbf{k}}v_{\ell m}(\mathbf{k})v_{j\nu}^*(\mathbf{k})\times\\&\times\boxed{\frac{\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_{\mathbf{k}})R_{j\nu,\ell' m'}(q_{\mathbf{k}},q_{\mathbf{n}'};q_0^2)-\bar{V}_S^{(\ell)}(q_{\mathbf{n}},q_0)R_{j\nu,\ell' m'}(q_0,q_{\mathbf{n}'};q_0^2)}{q_{\mathbf{k}}^2-q_0^2}}\end{aligned}$$

Singularity-free

- “On-shell” amplitude

$$\begin{aligned}\bar{B}_{\ell m,\ell' m'}(q_0,q_0;q_0^2)=&\,R_{\ell m,\ell' m'}(q_0,q_0;q_0^2)\\&+\sum_{j\nu,j'\nu'}R_{\ell m,j\nu}(q_0,q_0;q_0^2)H_{j\nu,j'\nu'}(q_0^2)\bar{B}_{j'\nu',\ell' m'}(q_0,q_0;q_0^2)\end{aligned}$$

Modified zeta-function $H_{\ell m,\ell' m'}(q_0^2)=4\pi\sum_{\mathbf{n}}\frac{v_{\ell m}(\mathbf{n})v_{\ell' m'}^*(\mathbf{n})}{q_{\mathbf{n}}^2-q_0^2}$

02 Modified Lüscher formula

Modified Lüscher formula

➢ **Determination of finite volume energy levels**

$$\bar{B}_{\ell m,\ell' m'}(q_0,q_0;q_0^2) = R_{\ell m,\ell' m'}(q_0,q_0;q_0^2) + \sum_{j\nu,j'\nu'} R_{\ell m,j\nu}(q_0,q_0;q_0^2) H_{j\nu,j'\nu'}(q_0^2)\bar{B}_{j'\nu',\ell' m'}(q_0,q_0;q_0^2)$$

$$\downarrow$$

$$\bar{B} = (R^{-1} - H)^{-1}$$

$$\det(R^{-1} - H) = 0$$

- Finite volume energy levels are determined by poles of T
- T's poles are equivalent to effective short-range part T_S
- T_S can be formulated by effective amplitude $\bar{B}$

➢ **Quantization condition**

$$\det(R^{-1} - H) = 0$$

Singularity-free ➡ Exponentially suppression

- Finite volume correction of R-function is exponentially suppressed $R_{\ell m,\ell' m'} = R_\ell \delta_{\ell m,\ell' m'} + O(e^{-E_{\text{th}} L})$
- Long-range interaction is encoded in modified zeta function H $H_{\ell m,\ell' m'}(q_0^2) = 4\pi \sum_{\mathbf{n}} \frac{v_{\ell m}(\mathbf{n}) v^*_{\ell' m'}(\mathbf{n})}{q_{\mathbf{n}}^2 - q_0^2} = \frac{4\pi}{L^6} \sum_{\mathbf{p},\mathbf{q}} \mathcal{Y}^*_{\ell m}(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q})$

02 Modified Lüscher formula

Renormalization in a finite volume

➢ **Regularization in quantization condition**

$$\boxed{\det \mathscr{A}(q_0^2) = 0}$$

quantization matrix $\quad \mathscr{A}_{\ell m,\ell' m'}(q_0^2) = R_\ell^{-1}(q_0,q_0;q_0^2)\delta_{\ell m,\ell' m'} - H_{\ell m,\ell' m'}(q_0^2)$

- Renormalization of R-function in the finite volume

Conter-terms

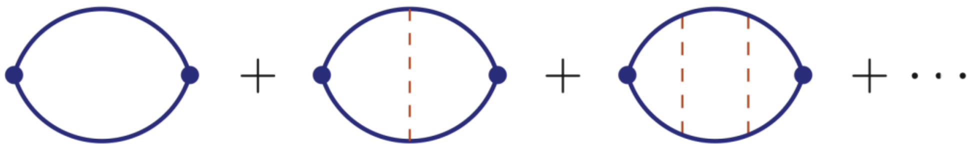
Renormalization condition $\quad \boxed{4\pi[(2\ell+1)!!]^2 R_\ell^{-1}(q_0,q_0;q_0^2) = K_\ell^M(q_0^2) + \mathcal{P}_\ell(q_0^2)}$

Modified K-matrix $\quad \boxed{K_\ell^M(q_0^2) = M_\ell(q_0) + \dfrac{q_0^{2\ell+1}\cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2}}$

- Regularization of modified zeta function H in a finite volume

$$H_{\ell m,\ell' m'}(q_0^2) = \frac{4\pi}{L^6}\sum_{\mathbf{p},\mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p})\langle \mathbf{p}|G_L(q_0^2)|\mathbf{q}\rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \xrightarrow{\text{Infinite vol. limit}}$$

$$\begin{aligned} H^\infty_{\ell m,\ell' m'}(q_0^2) &= 4\pi\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\mathcal{Y}_{\ell m}^*(\mathbf{p})\langle \mathbf{p}|G_L(q_0^2+i\epsilon)|\mathbf{q}\rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \\ &= \delta_{\ell m,\ell' m'}\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle, \end{aligned}$$



Regularization in the infinite volume

Conter-terms

$$\langle G_L^{(\ell)}(q_0^2+i\epsilon)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\frac{k^{2\ell}}{k^2-q_0^2-i\epsilon} = \frac{M_\ell(q_0)+\mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

sum-integral difference $\quad \Delta H_{\ell m,\ell' m'}(q_0^2) = H_{\ell m,\ell' m'}(q_0^2) - \Re H^\infty_{\ell m,\ell' m'}(q_0^2)$

Regularization $\quad H_{\ell m,\ell' m'}(q_0^2) = \Delta H_{\ell m,\ell' m'}(q_0^2) + \delta_{\ell m,\ell' m'}\dfrac{\Re M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$

02 Modified Lüscher formula

Modified Lüscher formula

➢ **Modified Lüscher formula**

$$\boxed{\det \mathscr{A}(q_0^2) = 0}$$

quantization matrix

$$\mathscr{A}_{\ell m,\ell' m'}(q_0^2) = R_\ell^{-1}(q_0, q_0; q_0^2)\delta_{\ell m,\ell' m'} - H_{\ell m,\ell' m'}(q_0^2)$$

- Regularized quantization matrix

$$R_\ell^{-1}(q_0, q_0; q_0^2) = \frac{K_\ell^M(q_0^2) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

$$H_{\ell m,\ell' m'}(q_0^2) = \Delta H_{\ell m,\ell' m'}(q_0^2) + \delta_{\ell m,\ell' m'}\frac{\Re M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

Conter-term cancellation

$$\mathscr{A}_{\ell m,\ell' m'}(q_0^2) = \frac{K_\ell^M(q_0^2) - \Re M_\ell(q_0)}{4\pi[(2\ell+1)!!]^2}\delta_{\ell m,\ell' m'} - \Delta H_{\ell m,\ell' m'}(q_0^2)$$

Physical observable including both long- and short-range information

Infinite (finite) volume function determined by only long-range interaction

- Advantage of modified Lüscher formula

$$q_0^{2\ell+1}\cot\delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2}r_\ell q_0^2 + O(q_0^4)$$ (poor convergence for long-range force)

$$K_\ell^M(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1}\cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2} = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\tilde{r}_\ell q_0^2 + O(q_0^4)$$ (much larger convergence radius)

- Reproduce of standard Lüscher formula

Long-range interaction disappears

Quantization matrix in standard Lüscher formula

$$\boxed{\frac{q_0^{2\ell+1}\cot\delta_\ell}{4\pi}\delta_{\ell m,\ell' m'} - \frac{1}{L^3}\sum_{\mathbf{k}}\frac{4\pi\mathcal{Y}_{\ell m}^*(\mathbf{k})\mathcal{Y}_{\ell' m'}(\mathbf{k})}{k^2 - q_0^2}}$$

目录

CONTENTS

03 Zeta function

UV property of zeta function

➢ **Definition**

- ΔH is composed of free part and long-range part

$$\Delta H_{\ell m,\ell' m'}(q_0^2) = \Delta H^{(0)}_{\ell m,\ell' m'}(q_0^2) - \Delta H'_{\ell m,\ell' m'}(q_0^2) \quad \begin{cases} \Delta H^{(0)}_{\ell m,\ell' m'}(q_0^2) = \left(\frac{1}{L^3}\sum_{\mathbf{p}} - \mathrm{PV}\int \frac{d^3p}{(2\pi)^3}\right) \frac{4\pi \mathcal{Y}^*_{\ell m}(\mathbf{p})\mathcal{Y}_{\ell' m'}(\mathbf{p})}{p^2 - q_0^2} & \text{(Standard zeta function)} \\ \Delta H'_{\ell m,\ell' m'}(q_0^2) = \frac{4\pi}{L^6}\sum_{\mathbf{p},\mathbf{q}} \frac{\mathcal{Y}^*_{\ell m}(\mathbf{p})}{p^2 - q_0^2} T_L(\mathbf{p},\mathbf{q};q_0^2)\frac{\mathcal{Y}_{\ell' m'}(\mathbf{q})}{q^2 - q_0^2} - 4\pi\Re\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{\mathcal{Y}^*_{\ell m}(\mathbf{p})}{p^2 - q_0^2} T_L^\infty(\mathbf{p},\mathbf{q};q_0^2)\frac{\mathcal{Y}_{\ell' m'}(\mathbf{q})}{q^2 - q_0^2} \end{cases}$$

- Free part is regularized directly in all methods to calculate standard zeta function

➢ **UV property**

- Direct subtraction in long-range part $\Delta H'$ is numerically unstable

UV div. UV div.

$$\Delta H' = \boxed{\sum GT_L G} - \boxed{\int GT_L^\infty G}$$

UV div.-free
but numerically unstable

$$\begin{cases} T_L = V_L + \sum V_L G V_L + \cdots \longrightarrow H' = \boxed{\sum GV_L G} + \boxed{\sum GV_L G V_L G} + \cdots \\ T_L^\infty = V_L + \int V_L G V_L + \cdots \longrightarrow H'^\infty = \boxed{\int GV_L G} + \boxed{\int GV_L G V_L G} + \cdots \end{cases}$$

UV div. UV div.

- One-by-one Subtraction is not feasible in practice for un-separatable V_L

$$\Delta H' = \underbrace{(\sum - \int) GV_L G}_{\text{UV div.-free}} + \underbrace{(\sum - \int) GV_L G V_L G}_{\text{UV div.-free}} + \cdots$$

03 Zeta function

Fast convergent method for long-range part

➢ **Decomposition of propagator**

- G : singular and UV slowly convergent
- G_1 : regular but UV slowly convergent
- ΔG: singular but UV fast convergent
- Decomposition is μ-dept. but result is μ-indept.

$$G = G_1 + \Delta G \begin{cases} G(p) = \dfrac{1}{p^2 - q_0^2}; \\ G_1(p) = \dfrac{1}{p^2+\mu^2} + \dfrac{\mu^2+q_0^2}{(p^2+\mu^2)^2} + \dfrac{(\mu^2+q_0^2)^2}{(p^2+\mu^2)^3} + \cdots + \dfrac{(\mu^2+q_0^2)^{n-1}}{(p^2+\mu^2)^n}; \\ \Delta G(p) = \dfrac{1}{p^2-q_0^2}\dfrac{(\mu^2+q_0^2)^n}{(p^2+\mu^2)^n}. \end{cases}$$

➢ **Decomposition of long-range propagator**

- G_μ : regular
- G_f : UV fast convergent

regular $\boxed{T_1 = V_L + V_L G_1 T_1}$

$$\boxed{\tilde{G} = \Delta G + \Delta G T_L \Delta G}$$

$$T_L = V_L + V_L G T_L$$

$$\boxed{G_L = \underbrace{\boxed{G_1 + G_1 T_1 G_1}}_{G_\mu} + \underbrace{(\boxed{1 + G_1 T_1})\tilde{G}(\boxed{T_1 G_1 + 1})}_{G_f}}$$

03 Zeta function

Fast convergent method for long-range part

➢ **Decomposition of zeta function**

$$G_L = \underbrace{G_1 + G_1 T_1 G_1}_{G_\mu} + \underbrace{(1 + G_1 T_1)\tilde{G}(T_1 G_1 + 1)}_{G_f}$$

- G_μ is regular → $\Delta H^{(\mu)}$ is exponentially suppressed

$$\Delta H^{(\mu)} = H^{(\mu)} - H^{(\mu)\infty} = O(e^{-E_{\mathrm{th}} L})$$

- G_f is UV fast convergent → $H^{(f)}$ and $H^{(f)\infty}$ are both UV convergent

$$\Delta H_{\ell m,\ell' m'} = \boxed{\frac{4\pi}{L^6} \sum_{\mathbf{p},\mathbf{q}} \mathcal{Y}^*_{\ell m}(\mathbf{p}) G_f(\mathbf{p},\mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})} \quad \text{UV conv.}$$
$$- \boxed{4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}^*_{\ell m}(\mathbf{p}) G_f^\infty(\mathbf{p},\mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})} \quad \text{UV conv.}$$

- $H^{(f)}$ and $H^{(f)\infty}$ can be calculated separately

$$\begin{cases} H^{(f)}_{\ell m,\ell' m'} = \dfrac{4\pi}{L^6} \sum_{\mathbf{k},\mathbf{k}'} \mathcal{Y}^*_{\ell m}(\mathbf{k}) G_f(\mathbf{k},\mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') \\ H^{(f)\infty}_{\ell m,\ell' m'} = \dfrac{4\pi}{L^6} \int \dfrac{d^3 k}{(2\pi)^3} \dfrac{d^3 k'}{(2\pi)^3} \mathcal{Y}^*_{\ell m}(\mathbf{k}) G_f^\infty(\mathbf{k},\mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') \end{cases}$$

03 Zeta function

Fast convergent method for long-range part

➢ **Wave-function in modified zeta function**
$$G_f = (1 + G_1 T_1)\tilde{G}(T_1 G_1 + 1) = \Psi^* \tilde{G} \Psi$$

$$H^{(f)}_{\ell m,\ell' m'} = \frac{4\pi}{L^6} \sum_{\mathbf{k},\mathbf{k}'} \mathcal{Y}^*_{\ell m}(\mathbf{k}) G_f(\mathbf{k},\mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') = \frac{4\pi}{L^6} \sum_{\mathbf{p},\mathbf{p}'} \Psi^*_{\ell m}(\mathbf{p}) \tilde{G}(\mathbf{p},\mathbf{p}') \Psi_{\ell' m'}(\mathbf{p}')$$

$$H^{(f)\infty}_{\ell m,\ell' m'} = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \mathcal{Y}^*_{\ell m}(\mathbf{k}) G^\infty_f(\mathbf{k},\mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') = 4\pi \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \Psi^{\infty *}_{\ell m}(\mathbf{p}) \tilde{G}^\infty(\mathbf{p},\mathbf{p}') \Psi^\infty_{\ell' m'}(\mathbf{p}')$$

$$\tilde{G}(\mathbf{p},\mathbf{p}') = \Delta G(p) L^3 \delta_{\mathbf{p}\mathbf{p}'} + \Delta G(p) T_L(\mathbf{p},\mathbf{p}') \Delta G(p')$$

$$\tilde{G}^\infty(\mathbf{p},\mathbf{q}) = \Delta G(p)(2\pi)^3 \delta(\mathbf{p}-\mathbf{q}) + \Delta G(p) T^\infty_L(\mathbf{p},\mathbf{q}) \Delta G(q)$$

$$\left.\begin{aligned} \Psi^*_{\ell m}(\mathbf{p}) &= \frac{1}{L^3} \sum_{\mathbf{k}} \mathcal{Y}^*_{\ell m}(\mathbf{k}) \left(L^3 \delta_{\mathbf{k}\mathbf{p}} + G_1(k) T_1(\mathbf{k},\mathbf{p})\right); \\ \Psi_{\ell' m'}(\mathbf{p}') &= \frac{1}{L^3} \sum_{\mathbf{k}'} \left(T_1(\mathbf{p}',\mathbf{k}') G_1(k') + L^3 \delta_{\mathbf{p}'\mathbf{k}'}\right) \mathcal{Y}_{\ell' m'}(\mathbf{k}'). \end{aligned}\right\} \xrightarrow[\text{Finite vol. correction is exponentially suppressed}]{G_1 \text{ and } T_1 \text{ are regular}} \left\{\begin{aligned} \Psi^{\infty *}_{\ell m}(\mathbf{p}) &= \int \frac{d^3k}{(2\pi)^3} \mathcal{Y}^*_{\ell m}(\mathbf{k}) \left((2\pi)^3 \delta(\mathbf{k}-\mathbf{p}) + G_1(k) T^\infty_1(\mathbf{k},\mathbf{p})\right); \\ \Psi^\infty_{\ell' m'}(\mathbf{p}') &= \int \frac{d^3q'}{(2\pi)^3} \left(T^\infty_1(\mathbf{p}',\mathbf{k}') G_1(k') + (2\pi)^3 \delta(\mathbf{p}'-\mathbf{k}')\right) \mathcal{Y}_{\ell' m'}(\mathbf{k}'). \end{aligned}\right.$$

- Finite volume wave-function can be replaced by infinite volume one

03 Zeta function

Symmetric projection

➢ **Partial-wave projection**

- Rotational symmetry allows partial-wave projection in the infinite volume

$$T_1^{\infty}(\mathbf{p},\mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) T_{1,\ell}^{\infty}(p,q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\Psi_{\ell m}^{\infty *}(\mathbf{p}) = \Psi_{\ell}^{\infty *}(p)\mathcal{Y}_{\ell m}^*(\mathbf{p});$$
$$\Psi_{\ell m}^{\infty}(\mathbf{q}) = \mathcal{Y}_{\ell m}(\mathbf{q})\Psi_{\ell}^{\infty}(q).$$

$$\Psi_{\ell}^{\infty *}(p) = 1 + \int \frac{d^3k}{(2\pi)^3} k^{2\ell} G_1(k) T_{1,\ell}^{\infty}(k,p);$$

$$\Psi_{\ell}^{\infty}(q) = \int \frac{d^3k}{(2\pi)^3} T_{1,\ell}^{\infty}(q,k) G_1(k) k^{2\ell} + 1.$$

$$T_L^{\infty}(\mathbf{p},\mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) T_{L,\ell}^{\infty}(p,q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\tilde{G}^{\infty}(\mathbf{p},\mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) \tilde{G}_{\ell}^{\infty}(p,q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\tilde{G}_{\ell}^{\infty}(p,q) = \Delta G(p)\frac{(2\pi)^3\delta(p-q)}{4\pi p^2} + \Delta G(p) T_{L,\ell}^{\infty}(p,q)\Delta G(q)$$

- Infinite volume H $\quad H_{\ell m,\ell' m'}^{(f)\infty} = \delta_{\ell m,\ell' m'} H_{\ell}^{(f)\infty}$ with $H_{\ell}^{(f)\infty} = \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} \Psi_{\ell}^{\infty *}(p) p^{2\ell} \tilde{G}_{\ell}^{\infty}(p,q) q^{2\ell} \Psi_{\ell}^{\infty}(q)$

- Finite volume H $\quad H_{\ell m,\ell' m'}^{(f)} = \frac{4\pi}{L^6} \sum_{\mathbf{p},\mathbf{q}} \Psi_{\ell}^{\infty *}(p) \mathcal{Y}_{\ell m}^*(\mathbf{p}) \tilde{G}(\mathbf{p},\mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q}) \Psi_{\ell'}^{\infty}(q)$

NOT partial-wave expanded

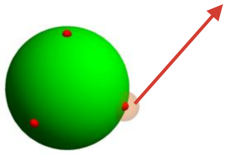
- Zeta function ΔH $\quad \Delta H_{\ell m,\ell' m'} = H_{\ell m,\ell' m'}^{(f)} - \delta_{\ell m,\ell' m'} H_{\ell}^{(f)\infty}$

03 Zeta function

Symmetric projection

➢ **Cubic projection**

- rotational symmetry → O_h group (rest frame) or little group (moving frame)
- 3-momentum state characterized by length + direction → shell + projection irreps.
- finite volume shell: $r\text{-shell} = \{\mathbf{p} | \mathbf{p} = g\mathbf{p}_0^{(r)}, g \in \mathcal{G}\}$



Cubic bases $v_{\beta,\alpha}^{(\Gamma)}(\mathbf{p}) = \sqrt{\frac{s_\Gamma}{\vartheta_r}} T_{\alpha\beta}^{(\Gamma)}(g)$, with $\mathbf{p} = g\mathbf{p}_0^{(r)}$

$$\tilde{G}(\mathbf{p},\mathbf{q}) = L^3 \sum_{\Gamma,\alpha} \sum_{\beta,\beta'} v_{\beta,\alpha}^{(\Gamma)}(\mathbf{p}) v_{\beta',\alpha}^{(\Gamma)*}(\mathbf{q}) \tilde{G}_{\beta\beta'}^{(\Gamma)}(r,r')$$

$$\tilde{G}_{\beta\beta'}^{(\Gamma)}(r,r') = \Delta G(p)\delta_{\beta\beta'}\delta_{rr'} + \Delta G(p) T_{L,\beta\beta'}^{(\Gamma)}(r,r')\Delta G(q)$$

$$T_L(\mathbf{p},\mathbf{q}) = L^3 \sum_{\Gamma,\alpha} \sum_{\beta,\beta'} v_{\beta,\alpha}^{(\Gamma)}(\mathbf{p}) v_{\beta',\alpha}^{(\Gamma)*}(\mathbf{q}) T_{L,\beta\beta'}^{(\Gamma)}(r,r')$$

$$T_{L,\beta\beta'}^{(\Gamma)}(r,r') = V_{L,\beta\beta'}^{(\Gamma)}(r,r') + \frac{1}{L^3}\sum_s \sum_\gamma V_{L,\beta\gamma}^{(\Gamma)}(r,s) G(p_s) T_{L,\gamma\beta'}^{(\Gamma)}(s,r')$$

Projected potential $V_{L,\beta\beta'}^{(\Gamma)}(r,r') = \frac{\sqrt{\vartheta_r \vartheta_{r'}}}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} T_{\beta\beta'}^{(\Gamma)}(g) \langle \mathbf{p}_0^{(r)} | V_L | g\mathbf{q}_0^{(r')} \rangle$

- Cubic harmonics $\mathcal{Y}_{\ell t,\alpha}^{(\Gamma)} = \sum_m c_{\Gamma \ell t,\alpha}^m \mathcal{Y}_{\ell m}$

$$\Delta H_{\ell t,\ell' t'}^{(\Gamma)} = \sum_{m,m'} c_{\Gamma \ell t,\alpha}^{m*} \Delta H_{\ell m,\ell' m'} c_{\Gamma \ell' t',\alpha}^{m'} = H_{\ell t,\ell' t'}^{(\Gamma,f)} - \delta_{\ell t,\ell' t'} H_\ell^{(f)\infty}$$

$$H_{\ell t,\ell' t'}^{(\Gamma,f)} = \frac{4\pi}{L^3} \sum_{r,r'} \sum_{\beta,\beta'} \tilde{\Psi}_{\ell t,\beta}^*(r) \tilde{G}_{\beta\beta'}^{(\Gamma)}(r,r') \tilde{\Psi}_{\ell' t',\beta'}(r') \quad \text{with} \quad \tilde{\Psi}_{\ell t,\beta}(r) = \sqrt{\frac{\vartheta_r}{s_\Gamma}} \mathcal{Y}_{\ell t,\beta}^{(\Gamma)}(\mathbf{p}_0^{(r)}) \Psi_\ell^\infty(p)$$

目录

CONTENTS

04 Numerical result

OPE potential and energy levels

➢ **Decomposition of potential**

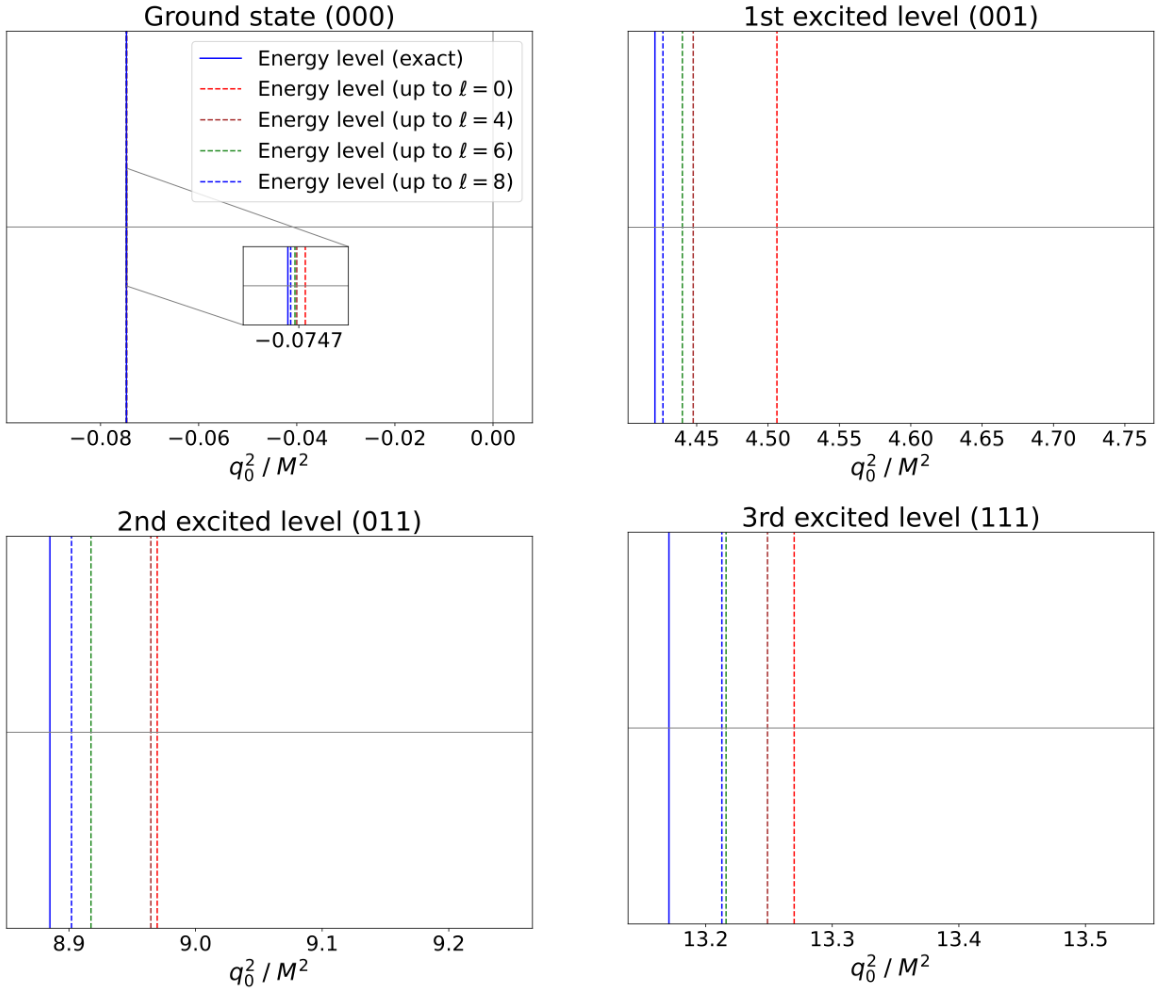
$$V(\mathbf{p},\mathbf{q}) = \underbrace{\frac{4\pi g}{M^2+(\mathbf{p}-\mathbf{q})^2}}_{V_L} + \underbrace{\frac{4\pi g_S}{M_S^2+(\mathbf{p}-\mathbf{q})^2}}_{V_S}$$

- One-pion-exchange potential
- Everything scaled by pion mass M
- Long-range coupling (attractive) $g = 0.489M$
- Short-range coupling (repulsive) $g_S = -1.34M$
- Two sets of short-range interaction $M_S/M = 2, 10$

➢ **Energy levels in a finite volume**

- Lattice size $ML = 3$
- Ground state is below threshold
- Partial-wave mixing in the excited states

$M_S/M = 10 \quad ML = 3$



04 Numerical result

Modified zeta functions

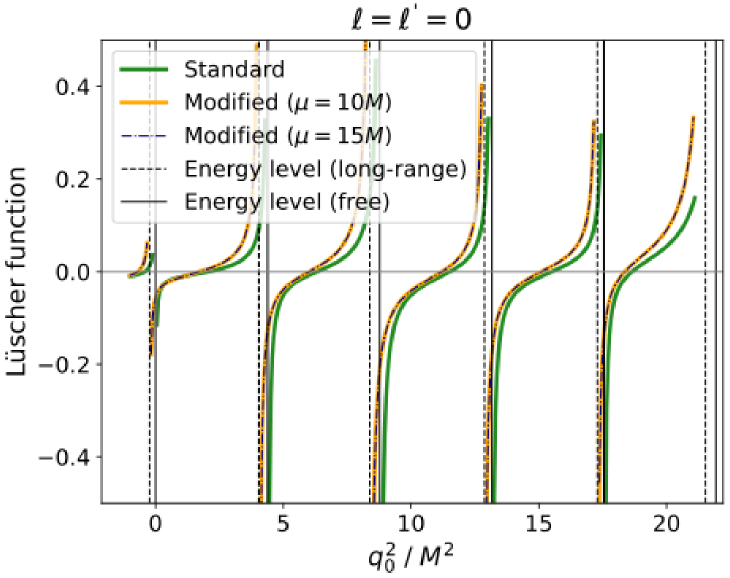
➢ **Definition of zeta functions**

$$\Delta H^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = \frac{4\pi}{L^6}\sum_{\mathbf{p},\mathbf{q}} \mathcal{Y}^{(\Gamma)*}_{\ell t,\alpha}(\mathbf{p}) G_L(\mathbf{p},\mathbf{q};q_0^2)\mathcal{Y}^{(\Gamma)}_{\ell' t',\alpha}(\mathbf{q}) - 4\pi\Re\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\mathcal{Y}^{(\Gamma)*}_{\ell t,\alpha}(\mathbf{p}) G_L^{\infty}(\mathbf{p},\mathbf{q};q_0^2)\mathcal{Y}^{(\Gamma)}_{\ell' t',\alpha}(\mathbf{q})$$

- ΔH: modified zeta function

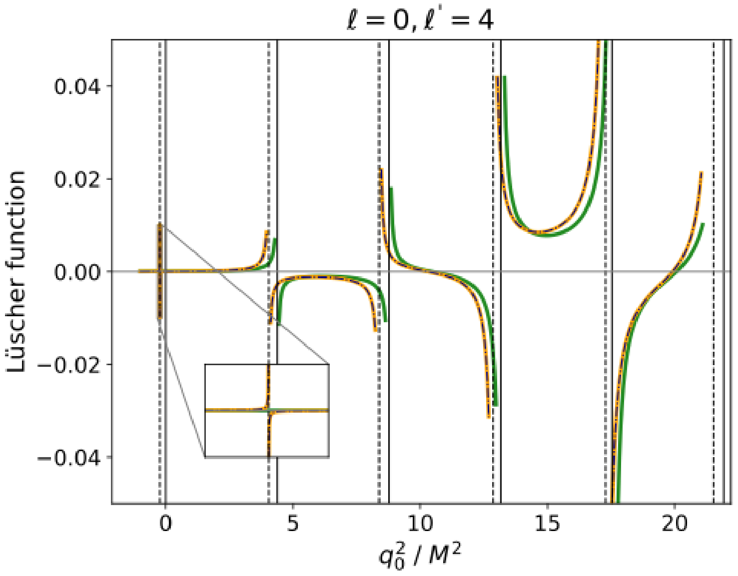
$$Z^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = 4\pi\left(\frac{1}{L^3}\sum_{\mathbf{p}} - \mathrm{PV}\int\frac{d^3p}{(2\pi)^3}\right)\frac{\mathcal{Y}^{(\Gamma)*}_{\ell t,\alpha}(\mathbf{p})\mathcal{Y}^{(\Gamma)}_{\ell' t',\alpha}(\mathbf{p})}{p^2 - q_0^2}$$

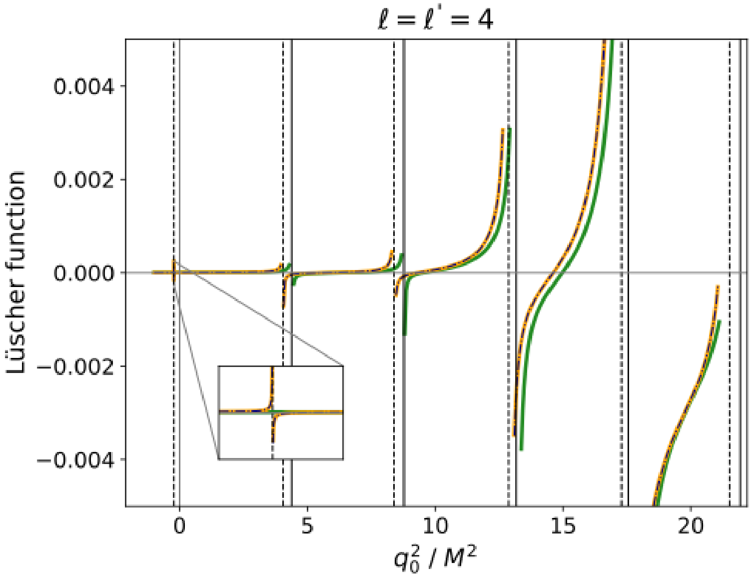
- Z: standard zeta function
- CMS A_1^+-irreps (implicity index t, t' and bases index are trivial)



➢ **Property of zeta functions**

- Modified zeta function is singular at long-range energy level
- Standard zeta function is singular at free energy level
- Modified zeta function is μ-indept.





04 Numerical result

Modified Lüscher formula

➢ **Quantization condition** $\det \mathscr{A}^{(\Gamma)}(q_0^2) = 0$

- Modified quantization matrix

$$\boxed{\mathscr{A}^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = \frac{K_\ell^M(q_0^2) - \Re M_\ell(q_0)}{4\pi[(2\ell+1)!!]^2}\delta_{\ell t,\ell' t'} - \Delta H^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2)}$$

- Standard quantization matrix

$$\boxed{\mathscr{A}^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = \frac{q_0^{2\ell+1}\cot\delta_\ell}{4\pi}\delta_{\ell t,\ell' t'} - Z^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2)}$$

- A_1^+-irreps truncated at S-wave

Modified $\boxed{D_S^{(A_1^+)} = \mathscr{K}_S - \Delta H_S^{(A_1^+)}}$ $\left\{ \begin{array}{l} \mathscr{K}_S(q_0^2) = \dfrac{q_0\cot(\delta_0 - \sigma_0)}{4\pi|f_0(q_0)|^2} \\ \Delta H_S^{(A_1^+)}(q_0^2) = \Delta H_{00}^{(A_1^+)}(q_0^2) \end{array} \right.$

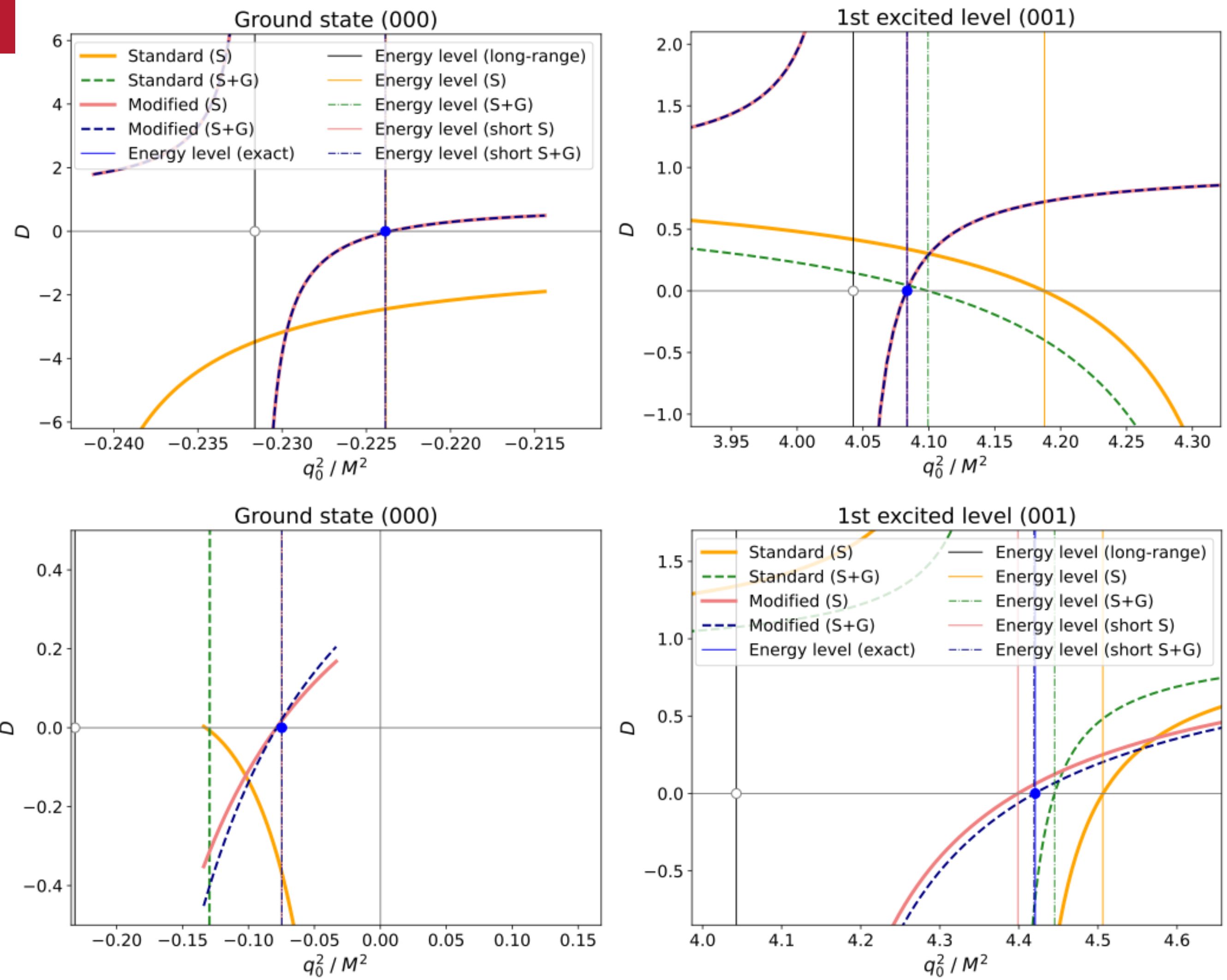
Standard $\boxed{D_S^{(A_1^+)} = \dfrac{q_0\cot\delta_0}{4\pi} - Z_S^{(A_1^+)}}$ with $Z_S^{(A_1^+)}(q_0^2) = Z_{00}^{(A_1^+)}(q_0^2)$

- A_1^+-irreps truncated at G-wave

Modified $\boxed{D_{S+G}^{(A_1^+)} = \mathscr{K}_S - \Delta H_{S+G}^{(A_1^+)}}$ with $\mathscr{K}_G(q_0^2) = \dfrac{q_0^9\cot(\delta_4 - \sigma_4)}{4\pi[9!!]^2|f_4(q_0)|^2}$

$$\Delta H_{S+G}^{(A_1^+)} = \Delta H_{00}^{(A_1^+)} + \Delta H_{44}^{(A_1^+)}\frac{\mathscr{K}_S}{\mathscr{K}_G} - \frac{\Delta H_{00}^{(A_1^+)}\Delta H_{44}^{(A_1^+)} - \left(\Delta H_{04}^{(A_1^+)}\right)^2}{\mathscr{K}_G}$$

Standard $\boxed{D_{S+G}^{(A_1^+)} = \dfrac{q_0\cot\delta_0}{4\pi} - Z_{S+G}^{(A_1^+)}}$ with $Z_{S+G}^{(A_1^+)}(q_0^2) = Z_{00}^{(A_1^+)}(q_0^2) + Z_{44}^{(A_1^+)}(q_0^2)\dfrac{q_0\cot\delta_0}{q_0^9\cot\delta_4} - \dfrac{4\pi}{q_0^9\cot\delta_4}\left[Z_{00}^{(A_1^+)}Z_{44}^{(A_1^+)} - \left(Z_{04}^{(A_1^+)}\right)^2\right]$



04 Numerical result

Modified Lüscher formula

➢ **Quantization condition**

- A_1^+-irreps truncated at S-wave

Modified $\boxed{D_S^{(A_1^+)} = \mathscr{K}_S - \Delta H_S^{(A_1^+)}}$ Standard $\boxed{D_S^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_S^{(A_1^+)}}$

- A_1^+-irreps truncated at G-wave

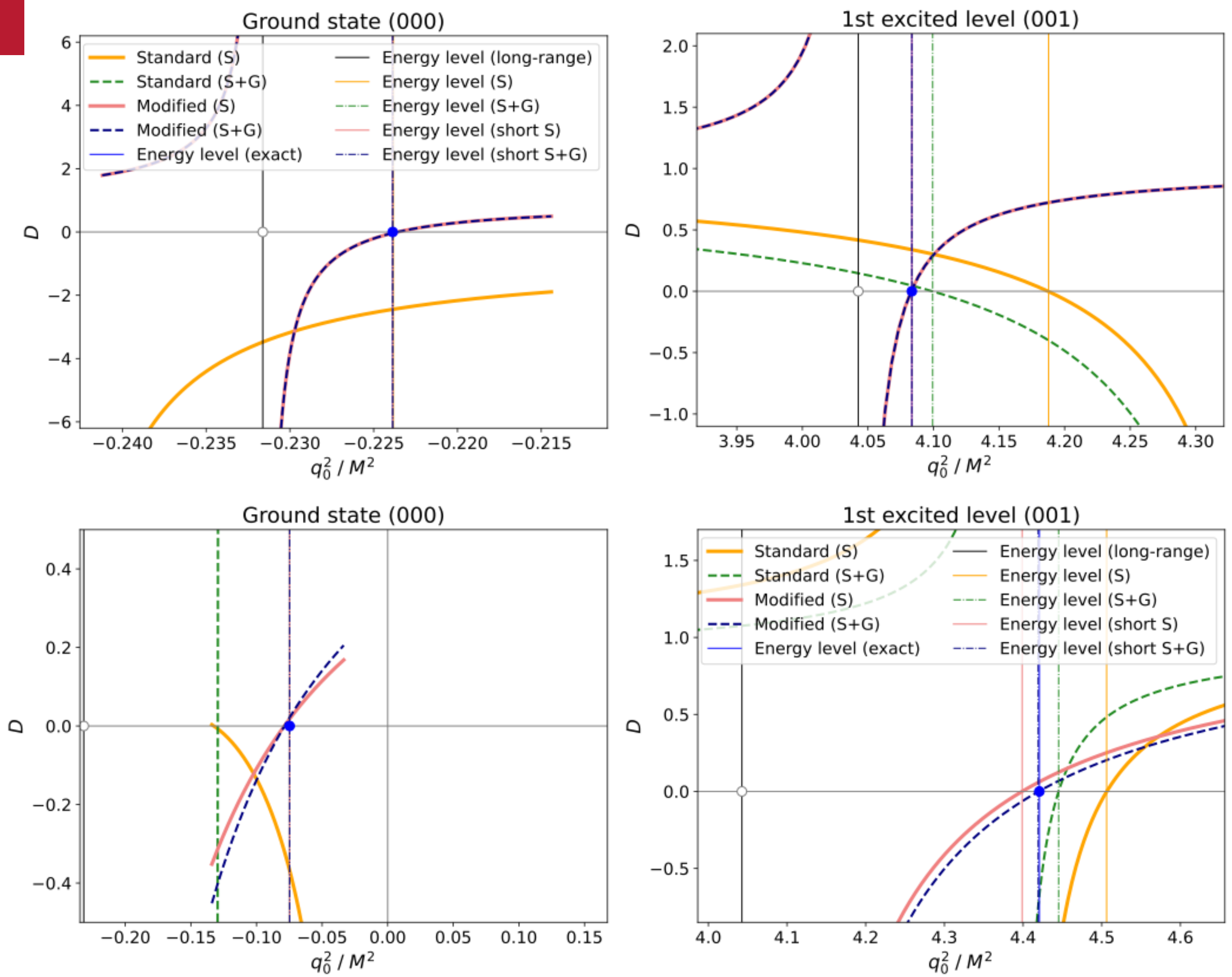
Modified $\boxed{D_{S+G}^{(A_1^+)} = \mathscr{K}_S - \Delta H_{S+G}^{(A_1^+)}}$ Standard $\boxed{D_{S+G}^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_{S+G}^{(A_1^+)}}$

➢ **Ground state**

- Left-handed branch cut began with $q_0^2/M^2 = -0.25$
- Standard Lüscher formula does not work near left-handed cut
- Modified Lüscher formula predicts the ground state exactly and shows good convergence of partial-wave expansion

➢ **1st excited state**

- For $M_S\ /\ M = 10$, modified Lüscher formula predicts the 1st excited state and shows good convergency of partial-wave expansion however standard one shows serious partial-wave mixing
- For $M_S\ /\ M = 2$, there is limited partial-wave mixing in modified Lüscher formula but it still works well up to G-wave



04 Numerical result

Modified phase shift term

➢ **Quantization condition** $\det \mathscr{A}^{(\Gamma)}(q_0^2) = 0$

- Modified quantization matrix

$$\boxed{\mathscr{A}^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = \frac{K^M_\ell(q_0^2) - \Re M_\ell(q_0)}{4\pi[(2\ell+1)!!]^2}\delta_{\ell t,\ell' t'} - \Delta H^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2)}$$

- Standard quantization matrix

$$\boxed{\mathscr{A}^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2) = \frac{q_0^{2\ell+1}\cot\delta_\ell}{4\pi}\delta_{\ell t,\ell' t'} - Z^{(\Gamma)}_{\ell t,\ell' t'}(q_0^2)}$$

➢ **Advantage of modified phase shift term**

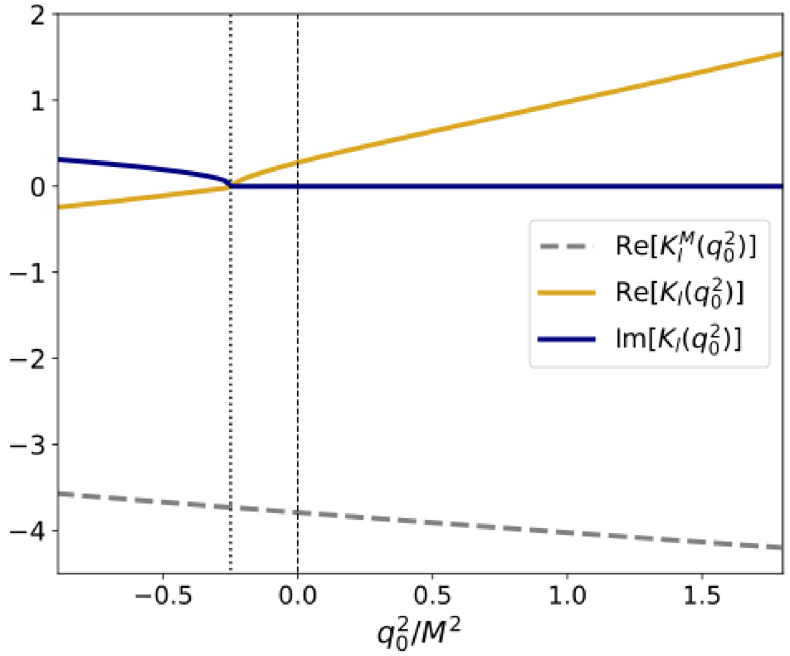
- Modified phase shift term

$$K^M_\ell(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1}\cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2} = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\tilde{r}_\ell q_0^2 + O(q_0^4)$$

(much larger convergence radius, analytic continue below left-handed cut)

- Standard quantization matrix

$$q_0^{2\ell+1}\cot\delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2}r_\ell q_0^2 + O(q_0^4)$$ (poor convergence for long-range force and not work below left-handed cut)



目录

CONTENTS

05 Conclusion and outlook

➢ **Build up modified Lüscher formula**

- Extract correct physical information near left-hand cut
- Fast convergency in partial-wave expansion
- Modified effective range expansion has much larger convergence radius

➢ **Give fast convergent method to calculate modified zeta function**

➢ **Apply to OPE potential**

➢ **Outlook: apply modified Lüscher formula to NN and $T_{cc}^{+}(3875)$**

谢谢各位老师和同学！