Nuclear Lattice Effective Field Theory without a Lattice

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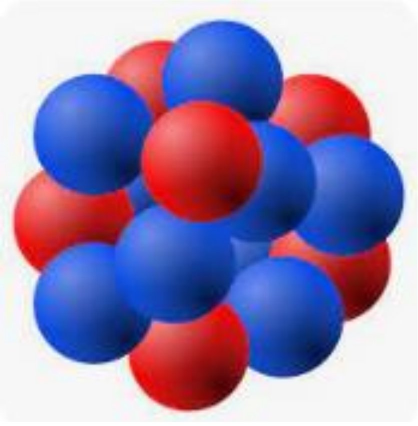


第十届手征有效场论研讨会

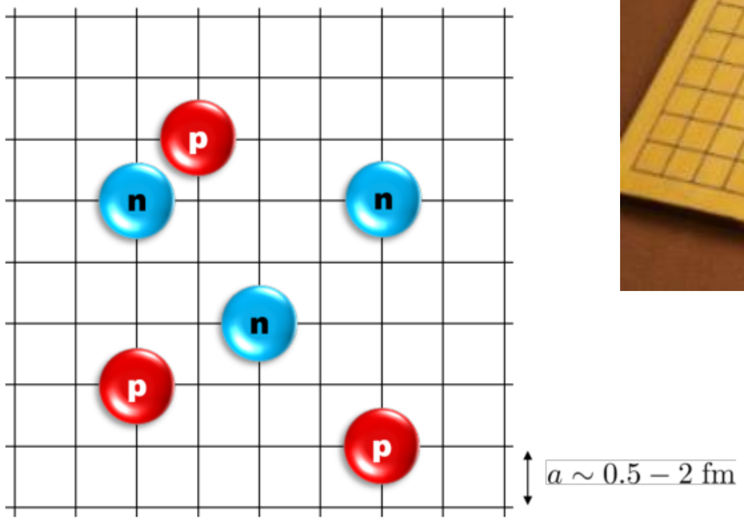
Oct 18, 2025

Lattice EFT: A many-body EFT solver

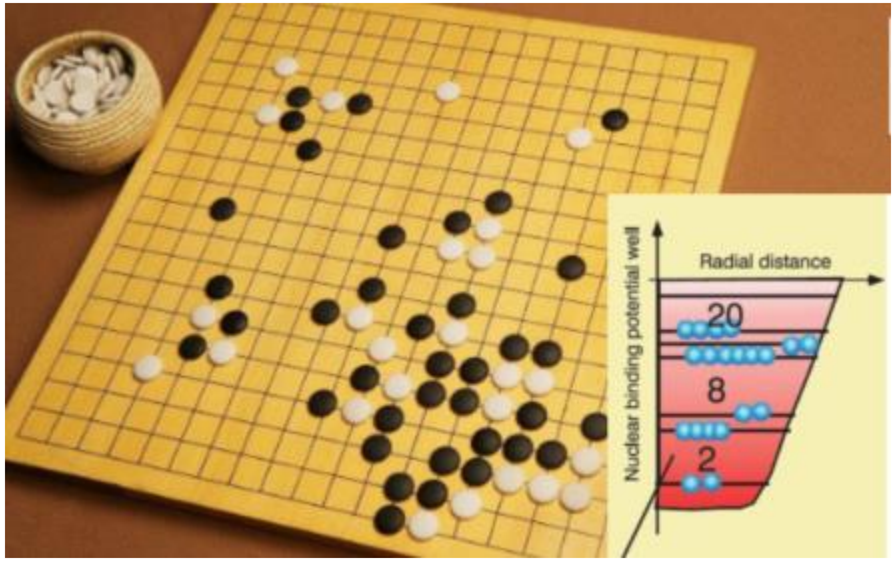
Lattice EFT $=$ **Chiral EFT** $+$ **Lattice** $+$ **Monte Carlo**

Review: Dean Lee, Prog. Part. Nucl. Phys. 63, 117 (2009),
Lähde, Meißner, “Nuclear Lattice Effective Field Theory”, Springer (2019)

- Discretized chiral nuclear force
- Lattice spacing $a \approx 1$ fm $=$ 620 MeV (∼chiral symmetry breaking scale)
- Protons & neutrons interacting via **short-range, δ-like** and **long-range, pion-exchange** interactions
- Exact method, polynomial scaling ($\sim A^2$)



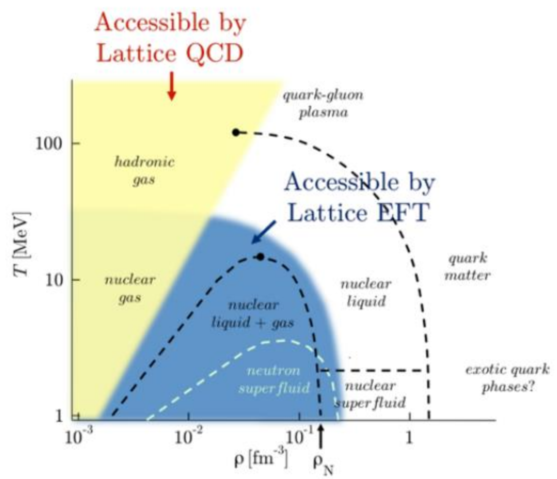
Lattice adapted for nucleus

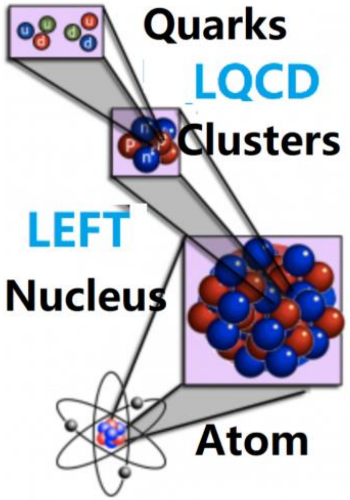


- Solve the non-perturbative nuclear many-body problem by sampling all configurations

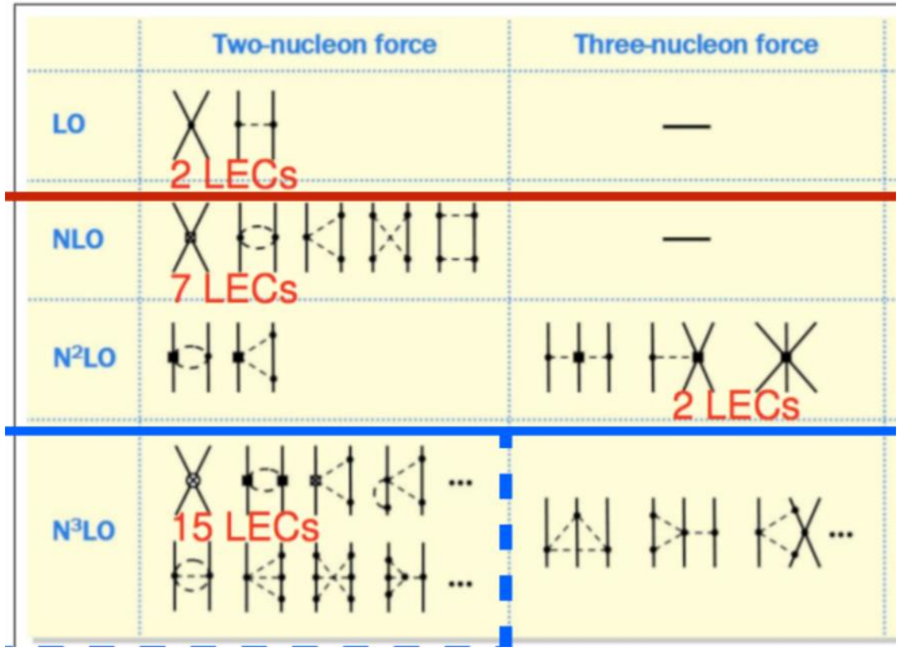
Compare Lattice EFT and Lattice QCD

	LQCD	LEFT
degree of freedom	quarks & gluons	nucleons and pions
lattice spacing	$\sim$0.1 fm	$\sim$1 fm
dispersion relation	relativistic	non-relativistic
renormalizability	renormalizable	effective field theory
continuum limit	yes	no
Coulomb	difficult	easy
accessibility	high T / low ρ	low T / $\rho_{\rm sat}$
sign problem	severe for $\mu > 0$	moderate





- Lattice EFT share a lot of common features with Lattice QCD. However,
  - Non-rel. → particle number conservation
  - Quadratic dispersion relation → no Fermion doubling problem
  - EFT contains non-renormalizable terms → no continuum limit



Naïve discretization of chiral Hamiltonian

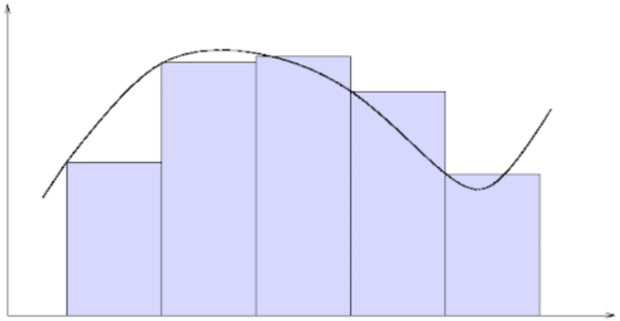
$$\langle \boldsymbol{p}_1', \boldsymbol{p}_2' | V_{N-N} | \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle = \Big\{ B_1 + B_2(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + C_1 q^2 + C_2 q^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + C_3 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + C_4 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

$$+ C_5 \frac{i}{2} (\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + C_6 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q}) + C_7 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

$$- \frac{g_A^2}{4F_\pi^2} \left[\frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})}{q^2 + M_\pi^2} + C_\pi \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right] (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + \cdots \Big\} \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2'),$$

$$f''(x) + O(a^{2N}) = c_0^{(N)} f(x) + a^{-2} \sum_{k=1}^{N} \frac{2(N!)^2(-1)^{k+1}}{k^2 (N+k)!(N-k)!} [f(x+ka) + f(x-ka)]$$

$$\int_a^b f(x)\,dx \approx \frac{1}{3} h \sum_{i=1}^{n/2} \left[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}) \right]$$



- Derivative → **Finite difference / FFT**
- Integration → **Summation**

Discretization error $O(a^N)$ → Systematically improvable

Dual role of the lattice

- Lattice is used to **discretize the action** and **regulate the delta-functions**

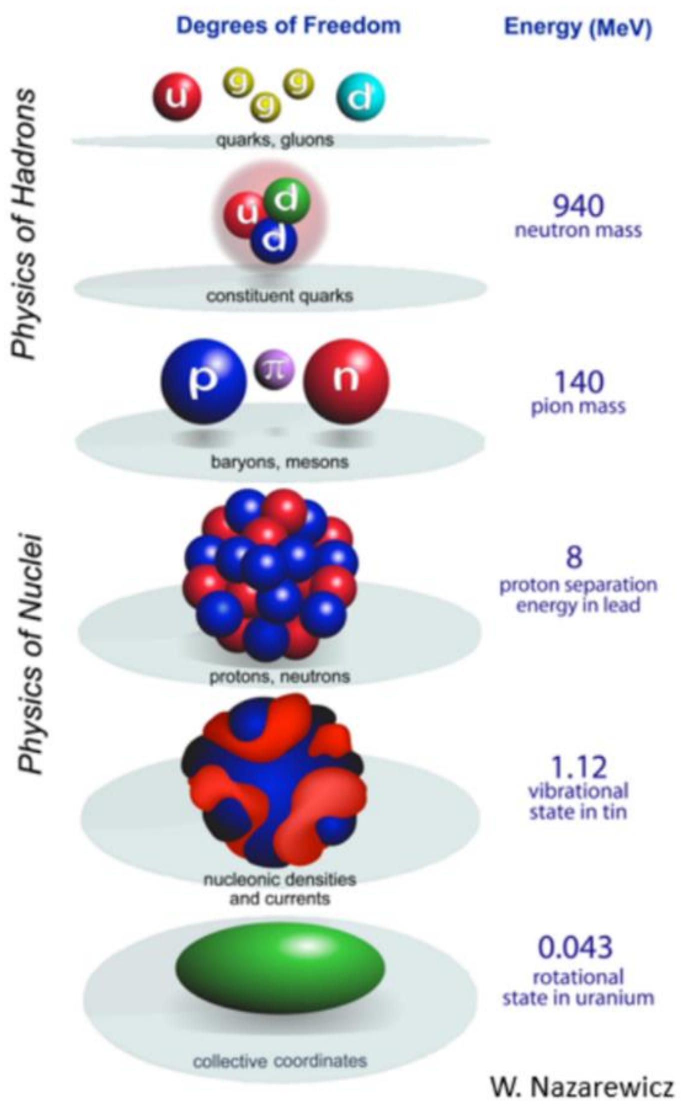
- QCD coupling is dimensionless
  → QCD is **renormalizable**
  → Safe to take the continuum limit $a \to 0$

- EFT couplings has negative mass dimensions
  →EFT is **non-renormalizable**
  →Break down at a small lattice spacing

Example: Wigner bound Wigner, PR98, 145 (1955)

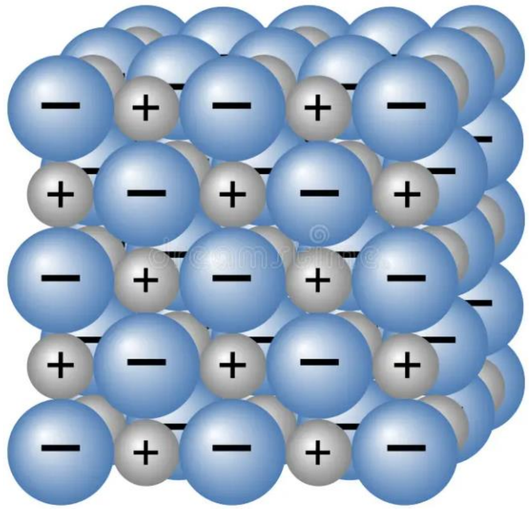
For $\Lambda \to \infty$, effective range $r_{\rm eff} \leq 0$

However, experimental n-p scattering $r_{\rm eff} \approx 1$ fm

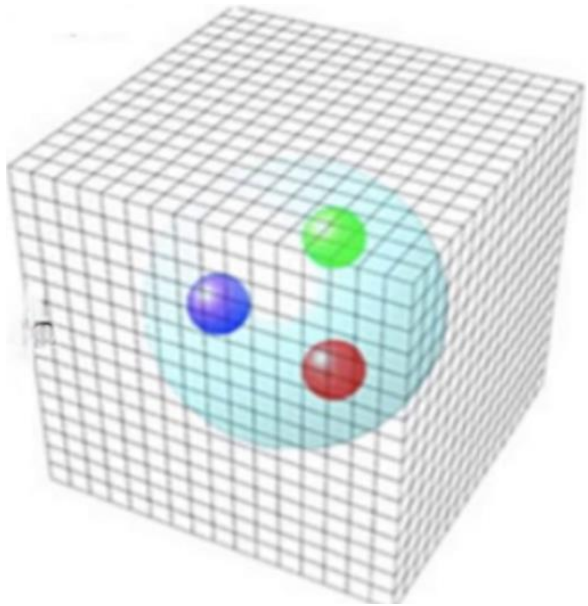


Decoupling of real physics from the lattice

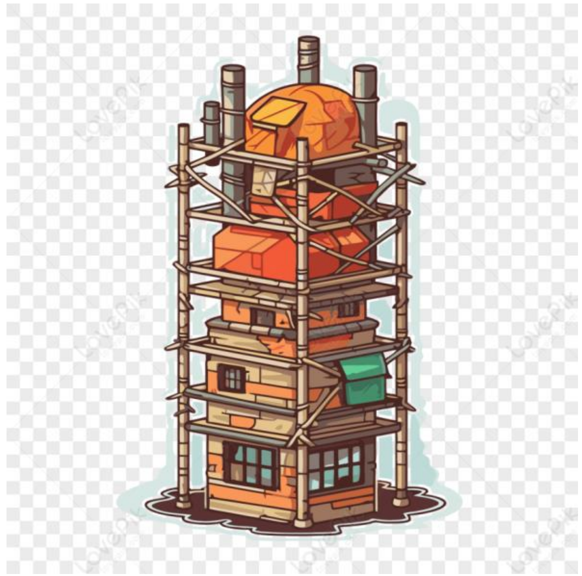
- In lattice field theories, the **lattices** are introduced **artificially**
  - Induce lattice artifacts → contaminations to observables
  - Predictions should be independent of the lattice spacing & geometry



Real crystal

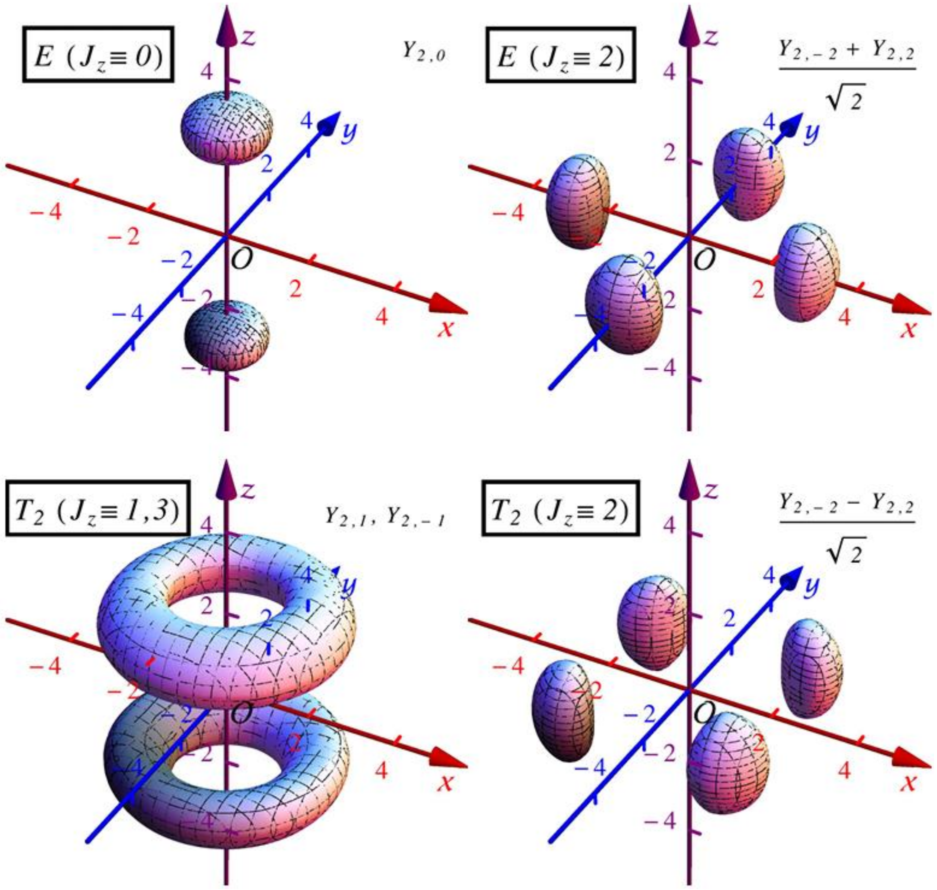


Lattice field theory



Scaffold

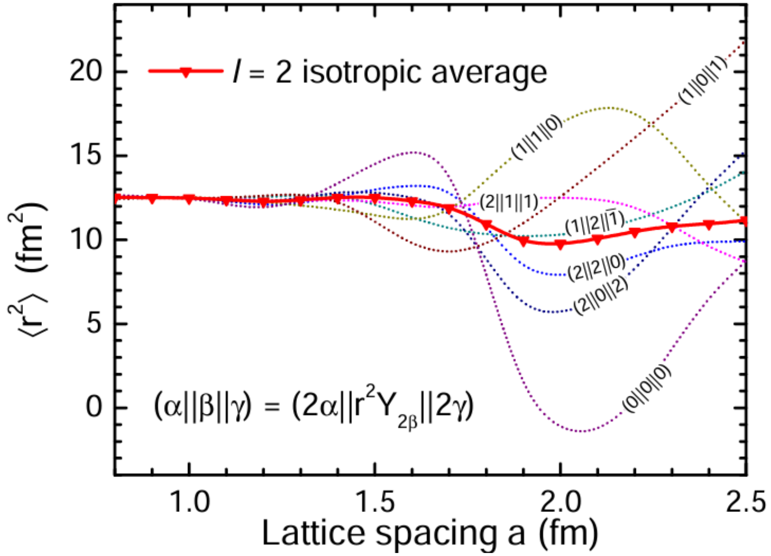
Symmetry breaking on the lattice



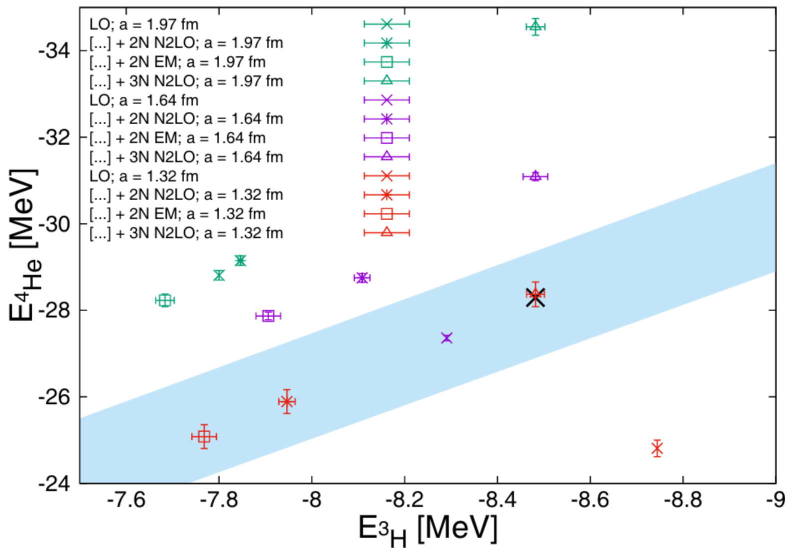
Lu et al., Phys. Rev. D 90, 034507 (2014); Phys. Rev. D 92, 014506 (2015)

- Lattice artifacts break **fundamental symmetries**
  - SO(3) rotational symmetry
  - Translational invariance
  - Galilean invariance
  - ? Chiral and Gauge
- Induce **systematic errors** difficult to quantify
- Symmetries can be restored
  - Average over orientations
  - Supplemental counter terms to Hamiltonian
    → **Symanzik improvement**
    Symanzik, Nucl. Phys. B 226, 187 (1983); Nucl. Phys. B 226, 205 (1983)
    Add non-renormalizable terms that vanish in continuum limit: $H \to H + aO^{(1)} + a^2O^{(2)} + a^3O^{(3)} + \cdots$
    Cancels lattice artifacts → Achieving continuum limit at finite lattice spacing

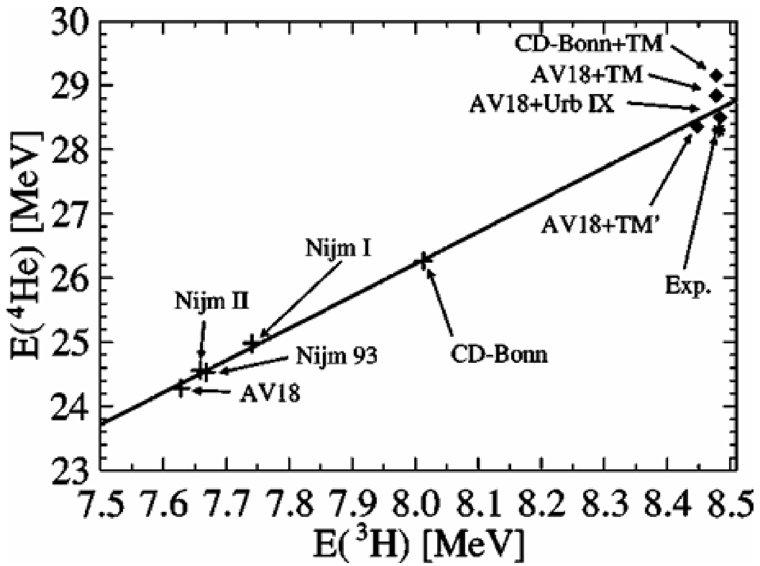
Consequences of lattice artifacts



Matrix elements of tensor operators
Lu et al., Phys. Rev. D 92, 014506 (2015)



Tjon line with varying lattice spacing
Klein et al., Eur.Phys. J. A 54, 121 (2018)



Tjon line is universal for modern nuclear forces
Tjon, Phys. Lett. B 56, 217 (1975)
Nogga et al., Phys. Rev. C 65, 054003 (2002)

- Symmetries and universalities are independent of details of interactions
  → Sensitive indicators for **non-physical artifacts**

Smearing as a regulator

- Smearing suppresses high-momentum modes

$$H_0 = K + \frac{1}{2} C_{\mathrm{SU4}} \sum_{\boldsymbol{n}} : \tilde{\rho}^2(\boldsymbol{n}) :$$

Elhatisari et al., Phys. Rev. Lett. 119, 222505 (2017)

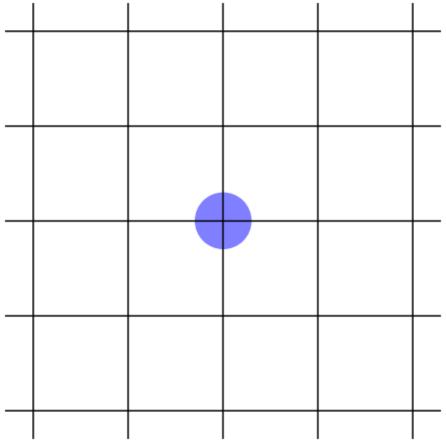
  - non-local smearing $\quad \tilde{a}_i(\boldsymbol{n}) = a_i(\boldsymbol{n}) + s_{NL} \sum a_i(\boldsymbol{n}')$
  - local smearing $\quad \tilde{\rho}(\boldsymbol{n}) = \sum_i \tilde{a}_i^\dagger(\boldsymbol{n}) \tilde{a}_i(\boldsymbol{n}) + s_L \sum_{|\boldsymbol{n}'-\boldsymbol{n}|=1} \sum_i \tilde{a}_i^\dagger(\boldsymbol{n}') \tilde{a}_i(\boldsymbol{n}')$
  - Gaussian smearing $\quad \tilde{a}(\boldsymbol{n}) = \sum_{\boldsymbol{n}'} \exp\left(-|\boldsymbol{n} - \boldsymbol{n}'|^2 / 2\Lambda^2\right) a(\boldsymbol{n}')$

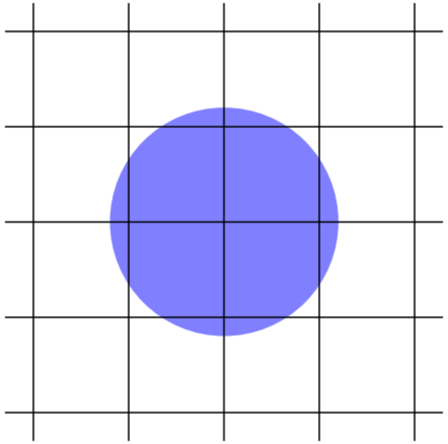
- Essential for building high-quality interactions

  Lu et al., Phys. Lett. B 797, 134863 (2019); Niu and Lu, arXiv:2506.12874 (2025)

- Smeared interactions are insensitive to lattice spacing

  Klein et al., Phys. Lett. B 747, 511 (2015)

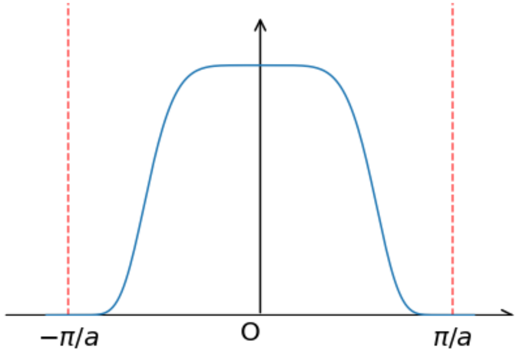




(Pseudo) Continuum limit of lattice EFT

- Smearing introduces an additional **soft regulator**

$$\tilde{a}(\boldsymbol{n}) = \sum_{\boldsymbol{n}'} f\left(|\boldsymbol{n} - \boldsymbol{n}'|\right) a(\boldsymbol{n}') \qquad f(r) = \int \frac{d^3p}{(2\pi)^3} \exp\left(-\frac{p^{2N}}{2\Lambda^{2N}}\right) \exp(i\boldsymbol{p}\cdot\boldsymbol{r})$$



- Smeared contact terms have well-defined **continuum limit**

$$\lim_{a\to 0} \sum_{\boldsymbol{n}} \tilde{a}(\boldsymbol{n})^\dagger \tilde{a}(\boldsymbol{n})^\dagger \tilde{a}(\boldsymbol{n}) \tilde{a}(\boldsymbol{n}) = \int d^3\boldsymbol{R} \int \prod d^3\boldsymbol{r}\, f(\boldsymbol{r}_1'-\boldsymbol{R}) f(\boldsymbol{r}_2'-\boldsymbol{R}) f(\boldsymbol{r}_1-\boldsymbol{R}) f(\boldsymbol{r}_2-\boldsymbol{R}) a(\boldsymbol{r}_1')^\dagger a(\boldsymbol{r}_2')^\dagger a(\boldsymbol{r}_2) a(\boldsymbol{r}_1)$$

In momentum space $\longrightarrow g(p_1')g(p_2')g(p_1)g(p_2), \qquad g(p) = \exp(-p^{2N}/2\Lambda^{2N})$

- Smearing regulators screen and eliminate most lattice artifacts
  - **Translational invariance** & **rotational symmetry** restored
  - **Galilean invariance** remains breaking

Application to real nuclei

- Smearing regulator works well combining with many-body algorithms

	E_0	δE_1	E_1	δE_2	E_2	E_{exp}
^{3}H	$-7.41(3)$	$+2.08$	$-5.33(3)$	-2.99	$-8.32(3)$	-8.48
^{4}He	$-23.1(0)$	-0.2	$-23.3(0)$	-5.8	$-29.1(1)$	-28.3
^{8}Be	$-44.9(4)$	-1.7	$-46.6(4)$	-11.1	$-57.7(4)$	-56.5
^{12}C	$-68.3(4)$	-1.8	$-70.1(4)$	-18.8	$-88.9(3)$	-92.2
^{16}O	$-94.1(2)$	-5.6	$-99.7(2)$	-29.7	$-129.4(2)$	-127.6
^{16}O†	$-127.6(4)$	$+24.2$	$-103.4(4)$	-24.3	$-127.7(2)$	-127.6
^{16}O‡	$-161.5(1)$	$+56.8$	$-104.7(2)$	-22.3	$-127.0(2)$	-127.6

Second order perturbative Quantum Monte Carlo calculation
Lu et al., Phys. Rev. Lett. 128, 242501 (2022)

- $a = 1.0$ fm → $\Lambda_a = \frac{\pi}{a} \approx 600$ MeV
- Soft cutoff $\Lambda = 350$ MeV

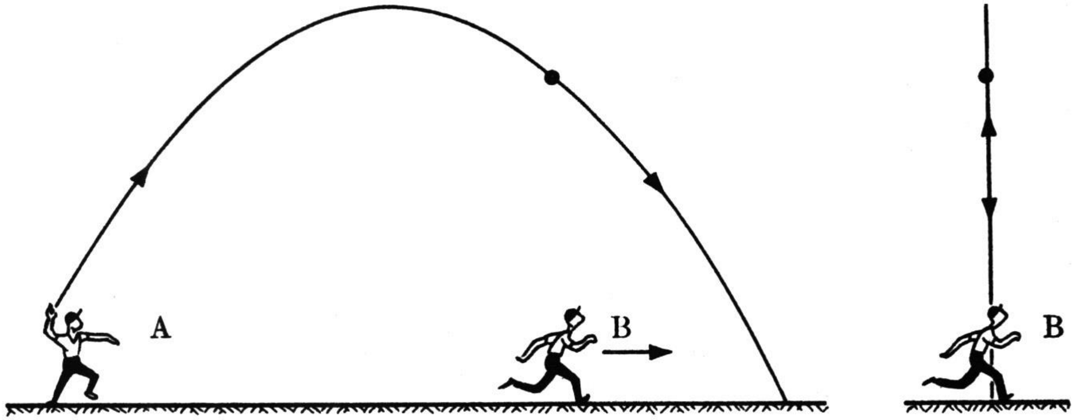
$\Lambda \ll \Lambda_a$ → near **continuum limit**

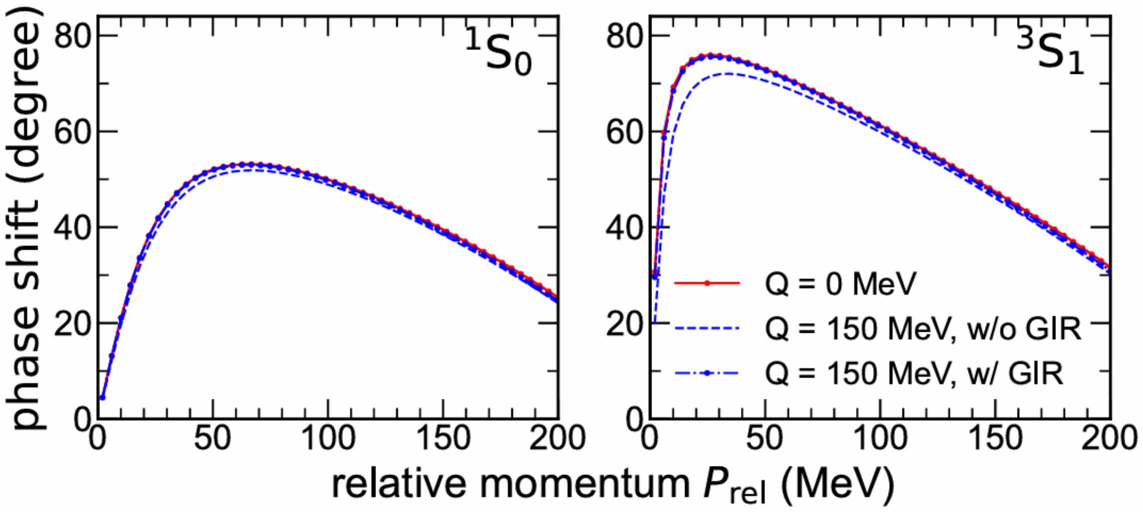
- N^{2}LO, 2NF + 3NF + OPEP
- 2NF fitted to **n-p phase shift**
- 3NF fitted to **^{3}H mass**

Is the nice reproduction a coincidence?

- **Galilean invariance broken**
- **Dependence on Λ** → RG problem
- Lack charge-symmetry-breaking

Galilean invariance restoration





- Galilean invariance
  → Observables only depend on **relative momenta**

- Galilean invariance restoration (GIR) terms
  $V_{\mathrm{GIR}} = \left[g_1 Q^2 + g_2 Q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right] f_{2\mathrm{N}}(\{p_i, p_i'\})$
  Center of mass momentum $\boldsymbol{Q} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2'$

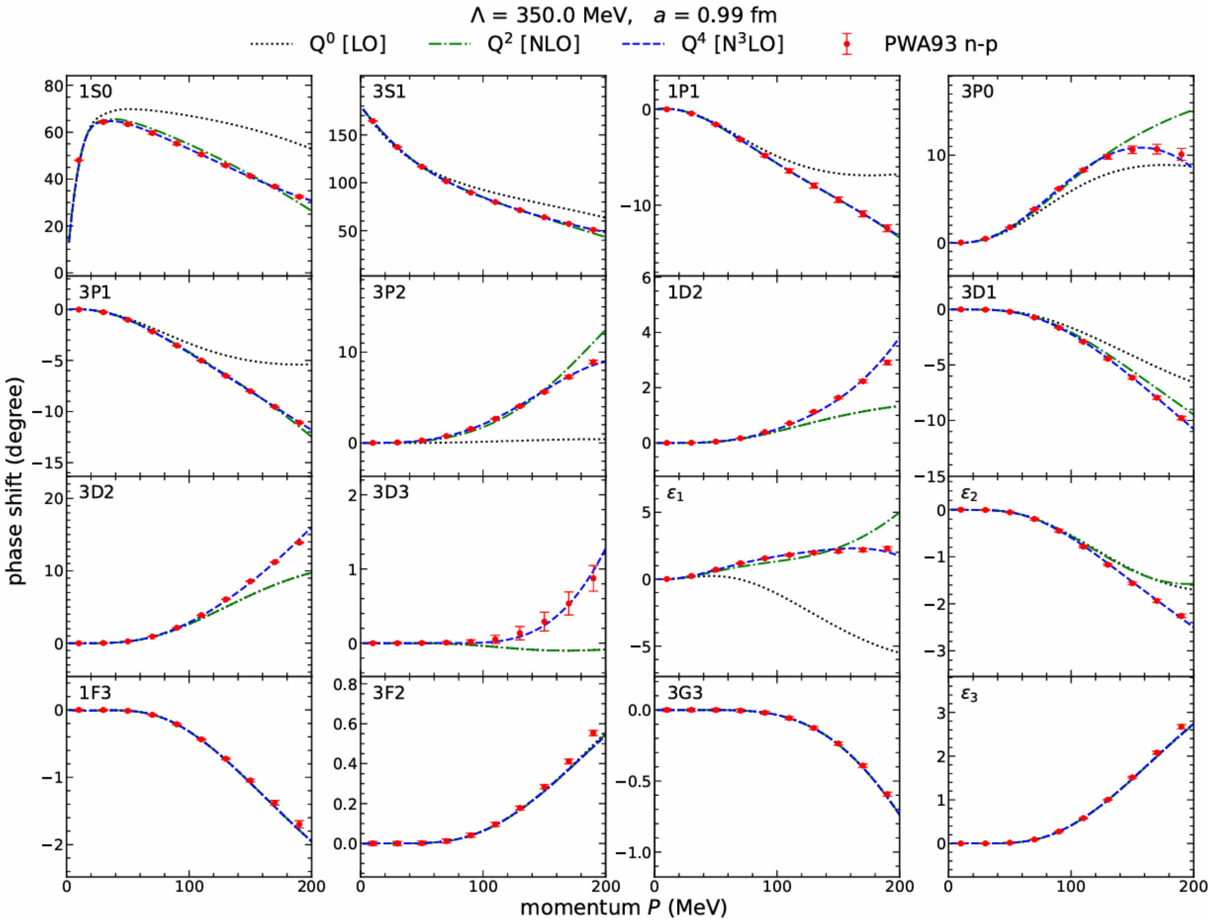
- Systematic determination of GIR terms
  Li et al., Phys. Rev. C 99,064001 (2019)

- Role of GIR terms in renormalizing 1-d bosons
  Klein et al., Eur. Phys. J. A 54, 233 (2018)

- Wavefunction matching with GIR terms
  Elhatisari et al., Nature 630, 59 (2024)

Lattice chiral forces

- Lattice chiral forces with **varying lattice spacing**
  - N^{2}LO chiral force
    Alarcon et al., Eur. Phys. J. A 53, 83 (2017)
  - N^{3}LO chiral force
    Li et al., Phys. Rev. C 98, 044002 (2018)
  - Charge-symmetry-breaking N^{3}LO chiral force
    Li et al., Phys. Rev. C 112, 014009 (2025)
- Typically 1 fm $\leq a \leq$ 2 fm
  → 300 MeV $\leq \Lambda_a \leq$ 600 MeV
- Excellent agreement with empirical phase shifts



Lattice chiral forces with soft regulators

$$H = K + V_{Q^0} + V_{Q^2} + V_{1\pi} + V_{\text{cou}} + V_{\text{CSB}}^{\text{pp}} + V_{\text{CSB}}^{\text{nn}} + V_{\text{GIR}} + V_{3\text{N}}$$

$$V_{Q^0} = [B_1 + B_2(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)] f_{2\text{N}}(\{p_i, p_i'\}),$$

$$V_{Q2} = [C_1 q^2 + C_2 q^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + C_3 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + C_4 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + C_5 \frac{i}{2}(\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + C_6 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q}) + C_7 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)] f_{2\text{N}}(\{p_i, p_i'\}),$$

$$V_{1\pi} = -\frac{g_A^2 f_\pi(q^2)}{4F_\pi^2}\left[\frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})}{q^2 + M_\pi^2} + C_\pi' \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\right](\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

$$V_{\text{cou}} = \frac{\alpha}{q^2}\left(\frac{1+\tau_{1z}}{2}\right)\left(\frac{1+\tau_{2z}}{2}\right) f_{\text{cou}}(q^2)$$

$$V_{\text{CSB}}^{\text{pp}} = c_{\text{pp}}\left(\frac{1+\tau_{1z}}{2}\right)\left(\frac{1+\tau_{2z}}{2}\right) f_{2\text{N}}(\{p_i, p_i'\})$$

$$V_{\text{CSB}}^{\text{nn}} = c_{\text{nn}}\left(\frac{1-\tau_{1z}}{2}\right)\left(\frac{1-\tau_{2z}}{2}\right) f_{2\text{N}}(\{p_i, p_i'\})$$

$$V_{\text{GIR}} = \left[g_1 Q^2 + g_2 Q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right] f_{2\text{N}}(\{p_i, p_i'\})$$

$$V_{3\text{N}} = \frac{c_E}{2F_\pi^4 \Lambda_\chi} f_{3\text{N}}(\{p_i, p_i'\})$$

$$f_{2\text{N}}(\{p_i, p_i'\}) = \exp\left[-\sum_{i=1}^{2}\left(p_i^6 + p_i'^6\right)/(2\Lambda^6)\right]$$

$$f_{3\text{N}}(\{p_i, p_i'\}) = \exp\left[-\sum_{i=1}^{3}\left(p_i^6 + p_i'^6\right)/(2\Lambda^6)\right]$$

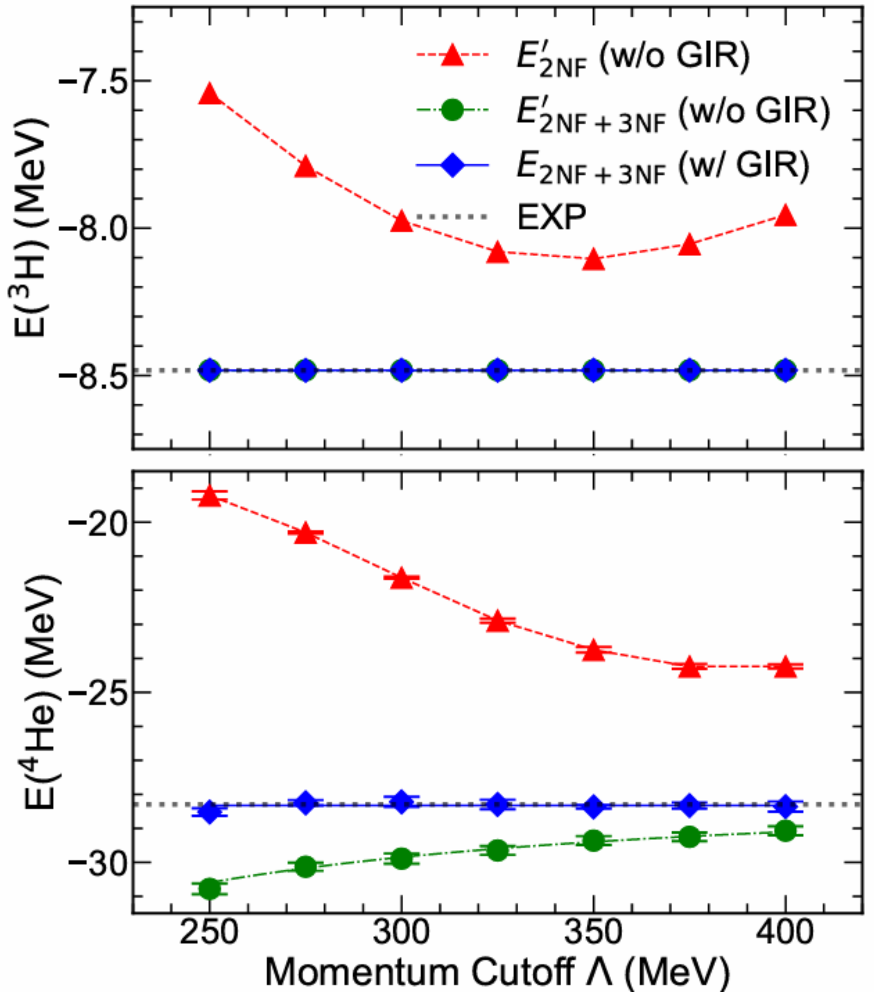
- Include all contact terms up to $O(Q^2)$ consistent with symmetry
- Take cutoff $\Lambda = 250, 275, 300, \dots 400$ MeV
- For each Λ, determine LECs from
  - Neutron-proton phase shifts
  - Proton-proton & neutron-neutron scattering lengths
  - Galilean invariance of S-wave scattering lengths
  - ^{3}H binding energy

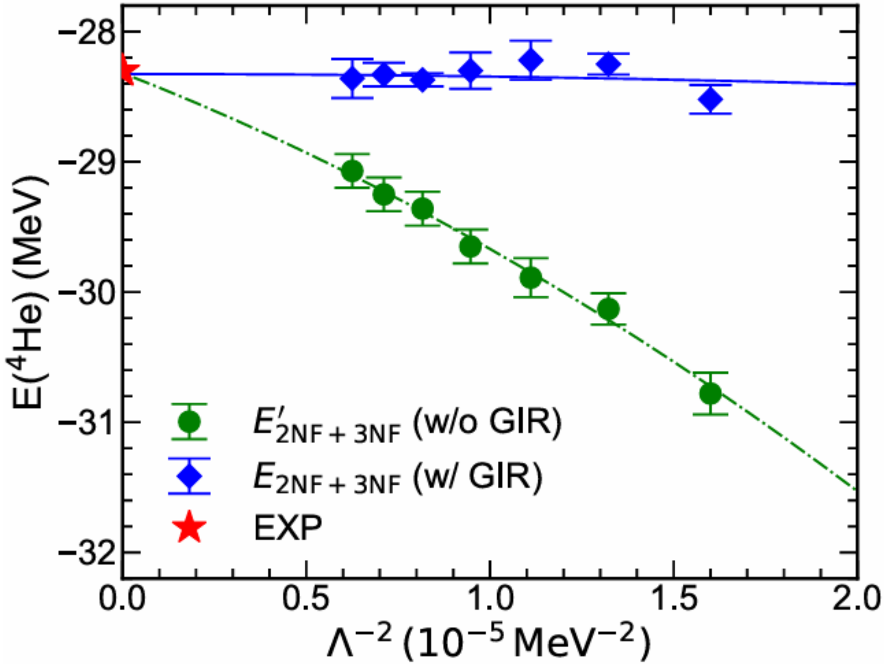
Shi et al., arXiv:2509.02953

LECs with running cutoff

Λ (MeV)	250	275	300	325	350	375	400
B_1	-4.931	-4.762	-4.592	-4.435	-4.285	-4.140	-3.997
B_2	-0.369	-0.326	-0.285	-0.246	-0.206	-0.164	-0.119
C_1	0.363	0.432	0.454	0.456	0.449	0.440	0.432
C_2	0.063	0.011	-0.017	-0.032	-0.041	-0.048	-0.056
C_3	-0.002	-0.024	-0.033	-0.039	-0.042	-0.047	-0.051
C_4	0.008	-0.029	-0.049	-0.060	-0.067	-0.073	-0.078
C_5	0.997	0.939	0.901	0.875	0.856	0.841	0.829
C_6	0.024	0.015	0.007	0.002	-0.002	-0.004	-0.005
C_7	-0.288	-0.267	-0.255	-0.248	-0.245	-0.245	-0.247
c_{nn}	0.074	0.068	0.063	0.060	0.058	0.057	0.057
c_{pp}	0.174	0.155	0.142	0.133	0.127	0.124	0.124
g_1	-1.299	-0.802	-0.501	-0.312	-0.193	-0.119	-0.078
g_2	-0.224	-0.137	-0.082	-0.048	-0.027	-0.014	-0.005
3NF w/ GIR c_E	1.863	0.941	0.504	0.313	0.245	0.247	0.289
3NF w/o GIR c'_E	5.389	2.928	1.661	0.995	0.653	0.496	0.459

^{4}He binding energy against Λ





Extrapolation to $\Lambda = \infty$ with

$$E(\Lambda) = E(\infty) + \frac{c_2}{\Lambda^2} + \frac{c_4}{\Lambda^4}$$

Shi et al., arXiv:2509.02953

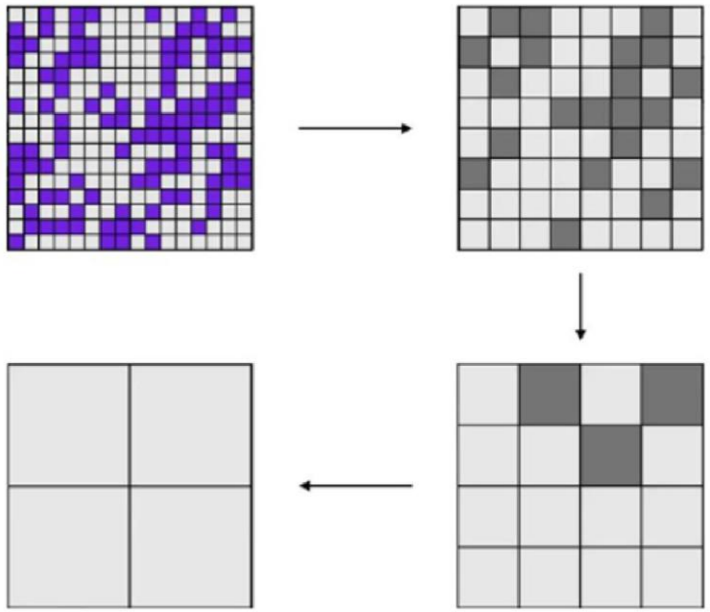
- w/o GIR:
  $E(^4\text{He})$ converge at $\Lambda = \infty$
  $E(\infty) = -28.32(33)$ MeV
- w/ GIR:
  $E(^4\text{He})$ agrees for all Λ
  $E(\infty) = -28.33(6)$ MeV
- Experiment:
  $E(^4\text{He}) = -28.3$ MeV

All LECs fixed with $A \leq 3$

Predict $E(^4\text{He})$ within 0.1 MeV
No adjustable parameter

Summary and reflections

- **Nuclear lattice EFT** can be built **without a lattice**
  (Though still solved with a lattice)
- Lattice discretization introduces more complications, e.g., symmetry breaking
  - Restore **translational** & **rotational** invariances using a smearing regulator
  - Restore **Galilean** invariance using counter terms (Similar to Symanzik program)
- Wilson's viewpoint:
  - All contact terms compatible with symmetries should be included to achieve RG invariance
    → **A stringent test for many-body methods**
- EFT principles are beliefs
  however, requiring verifications (done for $A \leq 4$)



Thanks for your attention!