南京师范大学
NANJING NORMAL UNIVERSITY

第十届手征有效场论研讨会

Discontinuity calculus

analytic continuation and topological structure of two-body scattering amplitudes

Speaker: Hao-Jie Jing (SXU)

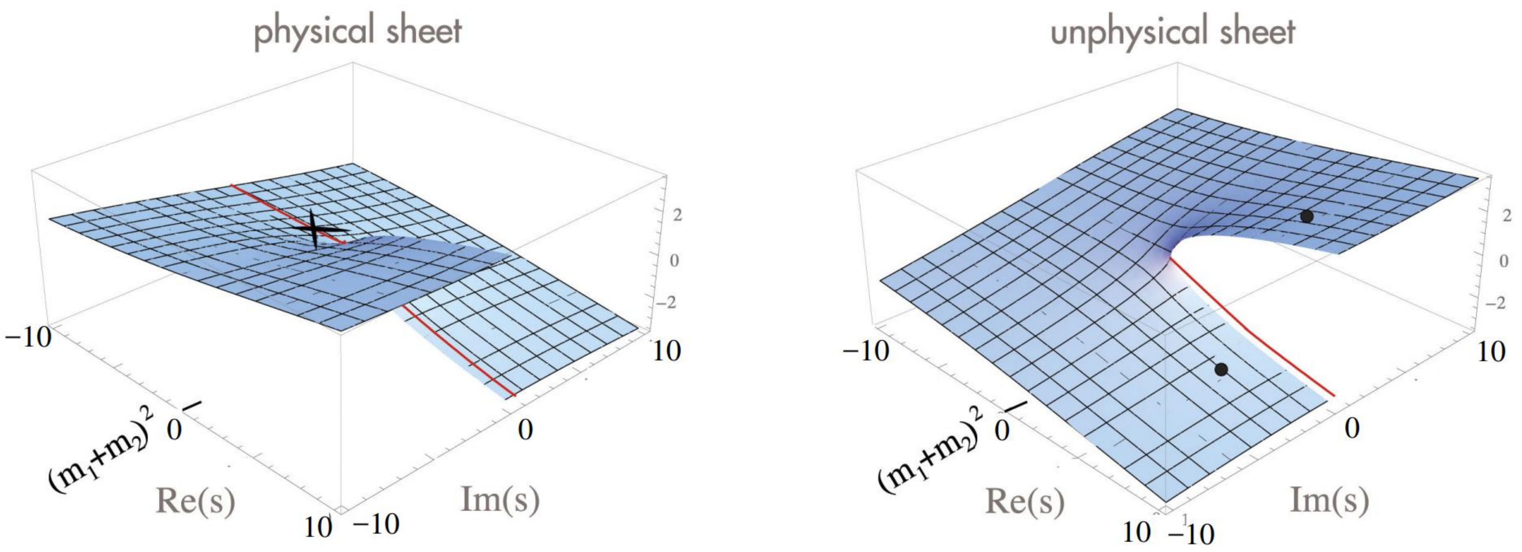
NJNU · Nanjing · 2025-10-20

Based on: HJJ, Xiong-Hui Cao and Feng-Kun Guo [arXiv:2507.06175]
[To be published in **Frontiers of Physics**]

Contents

Motivation

- S-matrix theory is a cornerstone tool for studying scattering processes
  - Bound states/Resonances correspond to poles in scattering amplitudes. The pole positions and residues encode crucial particle properties.
  - Causality constraint: Unstable resonance poles reside on unphysical Riemann sheets of the complex energy plane.
  - Discontinuity analysis of the S-matrix is indispensable: it extracts information about poles/zeros (discreteness) and branch cuts (discontinuity).

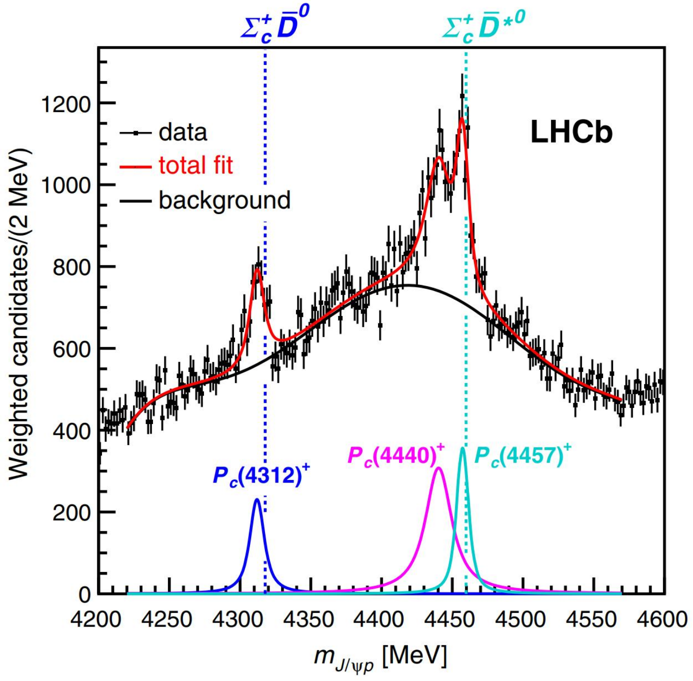


Feng-Kun Guo et al, Rev.Mod.Phys. 90 (2018).

Motivation

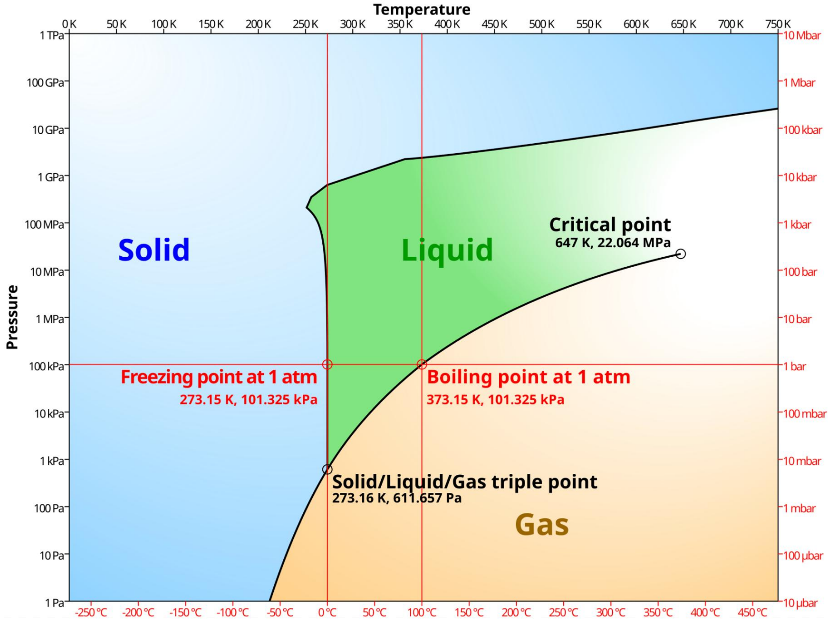
- Discreteness and discontinuity are ubiquitous in phenomena:

Hadron spectroscopy



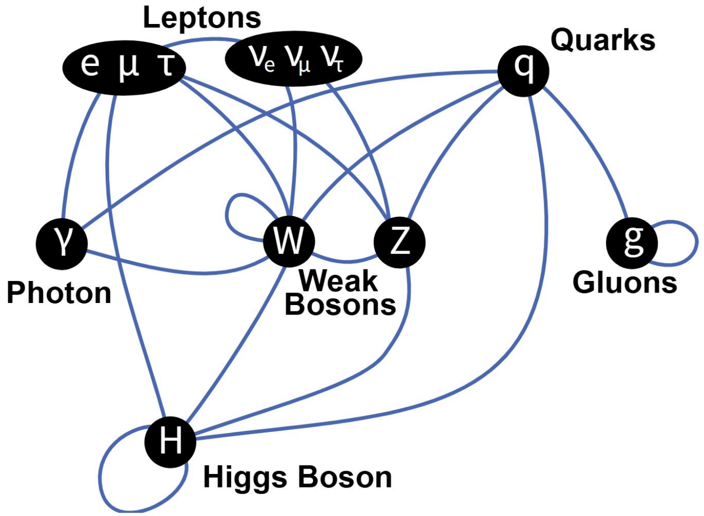
LHCb, Phys. Rev. Lett. 122(2019) 22, 222001

Statistical physics



Phase diagram of water

Standard model



Elementary particles mass gaps

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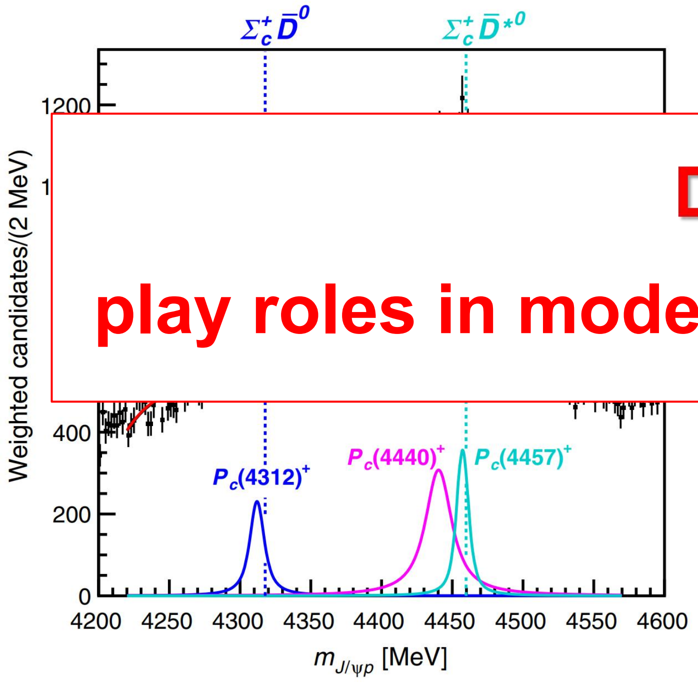

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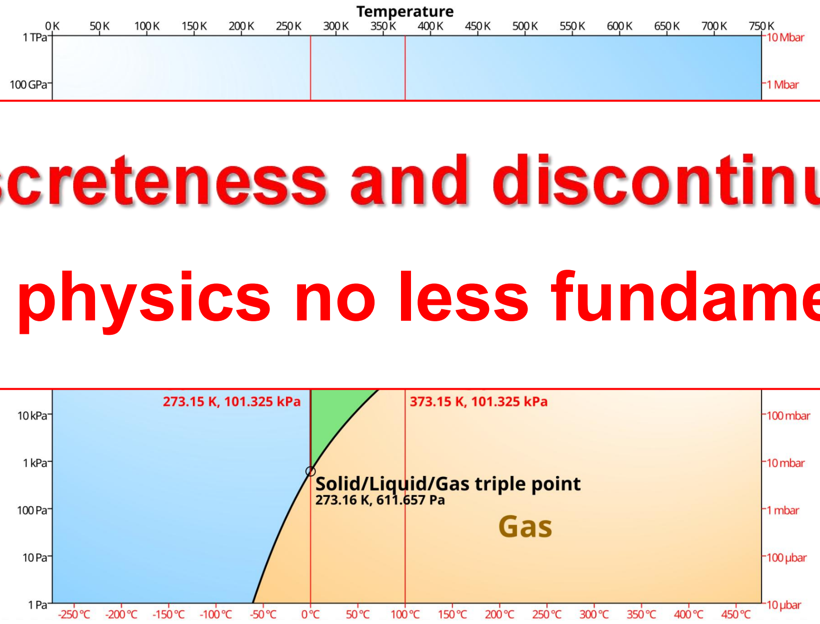

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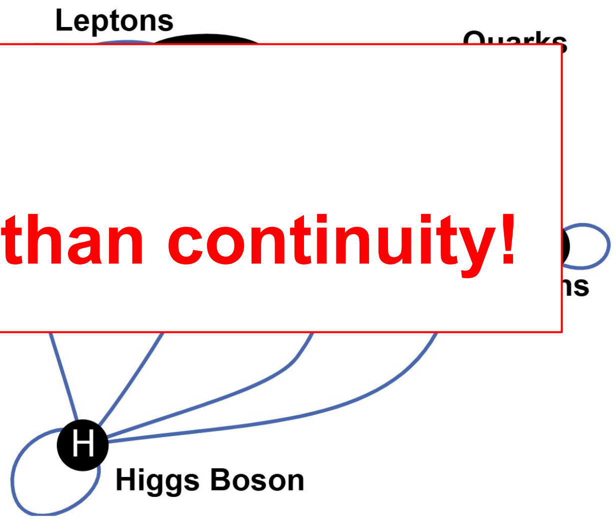

Elementary particles mass gaps

Discreteness and discontinuity

play roles in modern physics no less fundamental than continuity!

Contents

Calculus vs Discontinuity Calculus

- Differential operator

1. Constant kernel: $\mathrm{d}C = 0$
2. Linearity:

$$\mathrm{d}\left[\alpha_1 f_1(z) + \alpha_2 f_2(z)\right] = \alpha_1 \mathrm{d} f_1(z) + \alpha_2 \mathrm{d} f_2(z)$$

3. Leibniz law:

$$\mathrm{d}\left[f_1(z) f_2(z)\right] = \mathrm{d} f_1(z) f_2(z) + f_1(z) \mathrm{d} f_2(z)$$

4. Chain rule:

$$\mathrm{d}F[f(z)] = \left.\frac{\mathrm{d}F(w)}{\mathrm{d}w}\right|_{w=f(z)} \mathrm{d}f(z)$$

5. Newton-Leibniz formula:

$$f(z) = C(z_0) + \int_{z_0}^{z} \mathrm{d}f(z')$$

Calculus vs Discontinuity Calculus

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- Discontinuity operator

1. Holomorphic kernel: $\mathbb{D}h(z) = 0$
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$$\mathbb{D}\left[\alpha_1 f_1(z) + \alpha_2 f_2(z)\right] = \alpha_1 \mathbb{D} f_1(z) + \alpha_2 \mathbb{D} f_2(z)$$

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$$\mathbb{D}F\left[f(z)\right] = F\left[f(z)\right] - \left[F(\omega) - \mathbb{D}F(\omega)\right]_{\omega = f(z) - \mathbb{D}f(z)}$$

5. Dispersion relation:

$$f(z) = h(z) + \frac{1}{2\pi i} \int \frac{\mathbb{D}f(z')}{z' - z}\, \mathrm{d}z'$$

Discontinuity Calculus

- Discontinuity of the square root function $f(z) = \sqrt{z}\ (z \in \mathbb{C})$

$$\begin{cases} \mathbb{D}(\sqrt{z})^2 = \mathbb{D}z = 0 \\ \mathbb{D}(\sqrt{z})^2 = 2\sqrt{z}\ \mathbb{D}\sqrt{z} - (\mathbb{D}\sqrt{z})^2 \end{cases}$$

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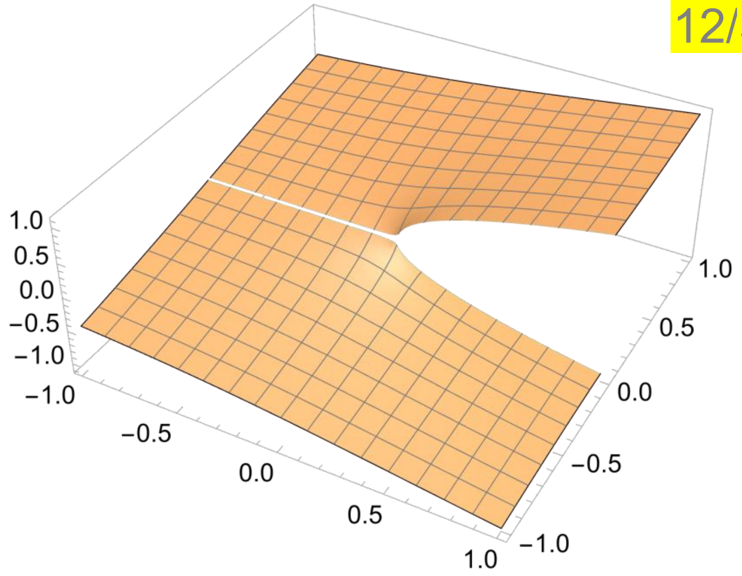

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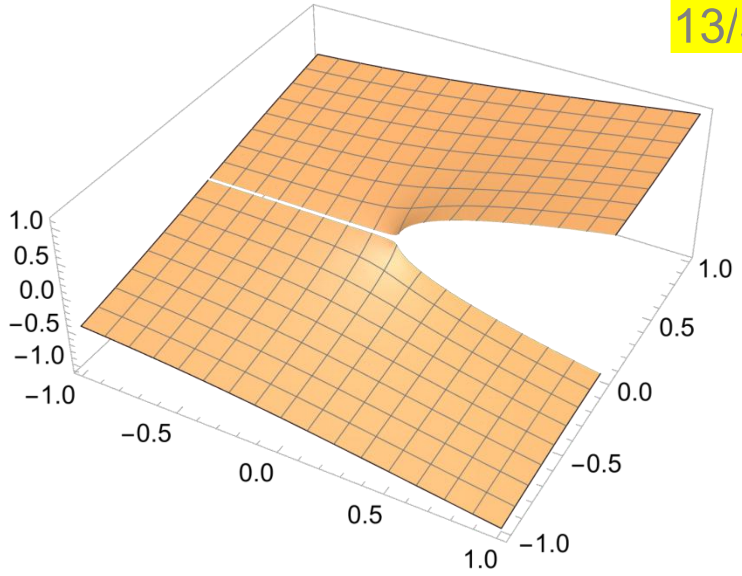

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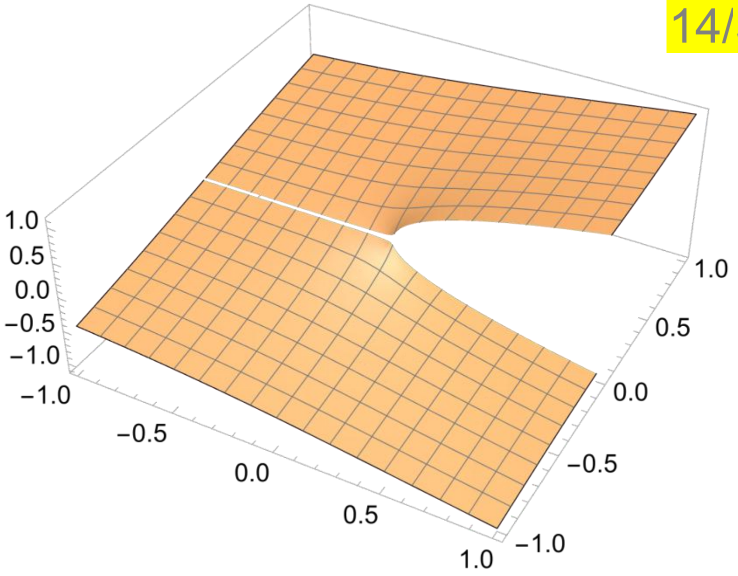

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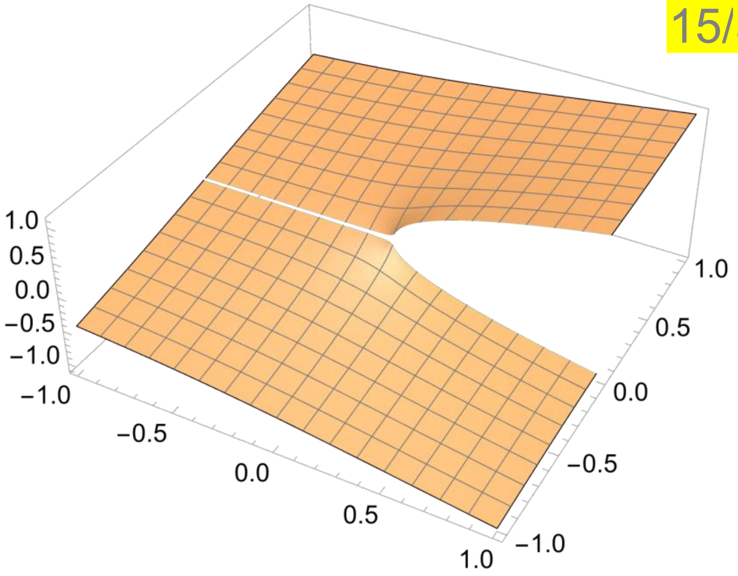

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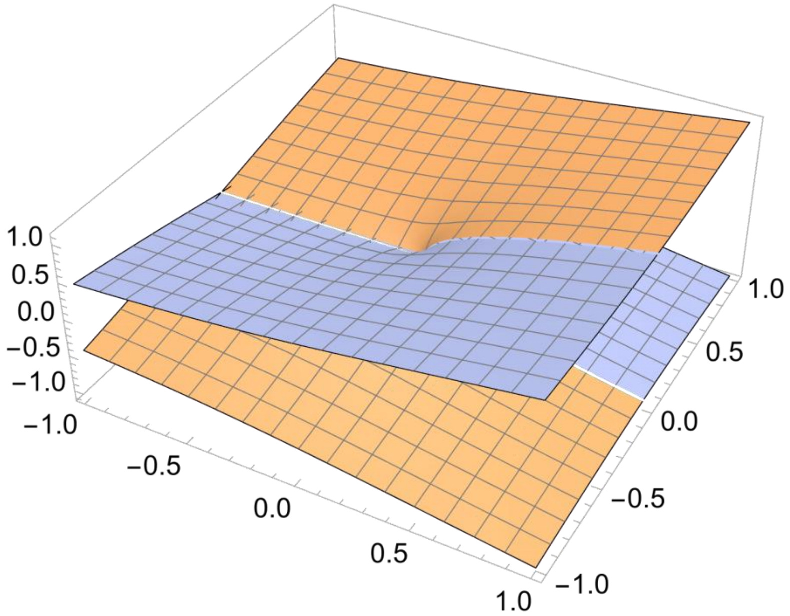

$$\boxed{\mathrm{RS}\{\sqrt{z}\} = \mathbb{C} \times \left\{\mathbb{T}_1 \mid \mathbb{T}_1^2 \sim 1\right\} \cong \mathbb{C} \times \mathbb{Z}_2}$$

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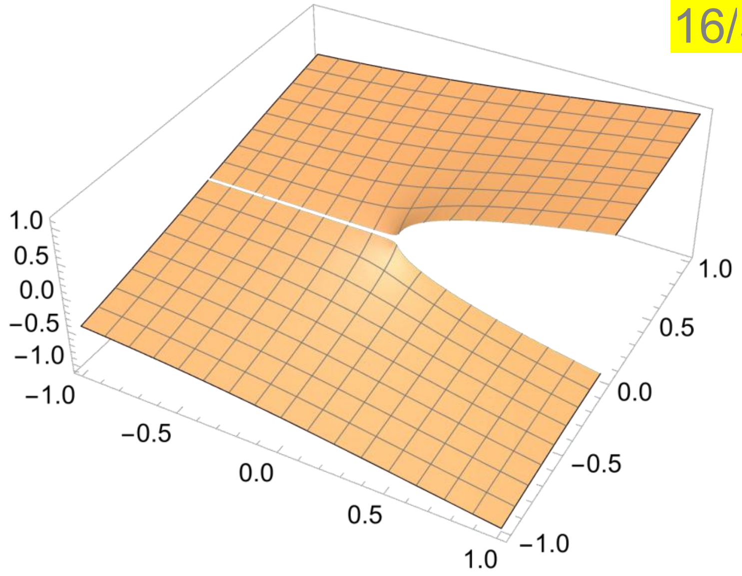

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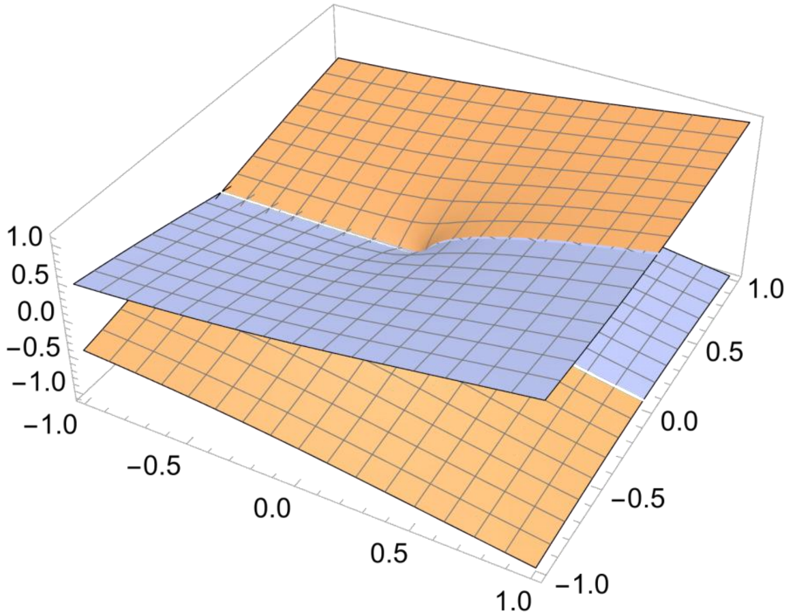


The Riemann surface for the n-th root function:

$$\mathrm{RS}\{\sqrt[n]{z}\} \cong \mathbb{C} \times \mathbb{Z}_n \text{ for } n \in \mathbb{N}^*$$

Discontinuity Calculus

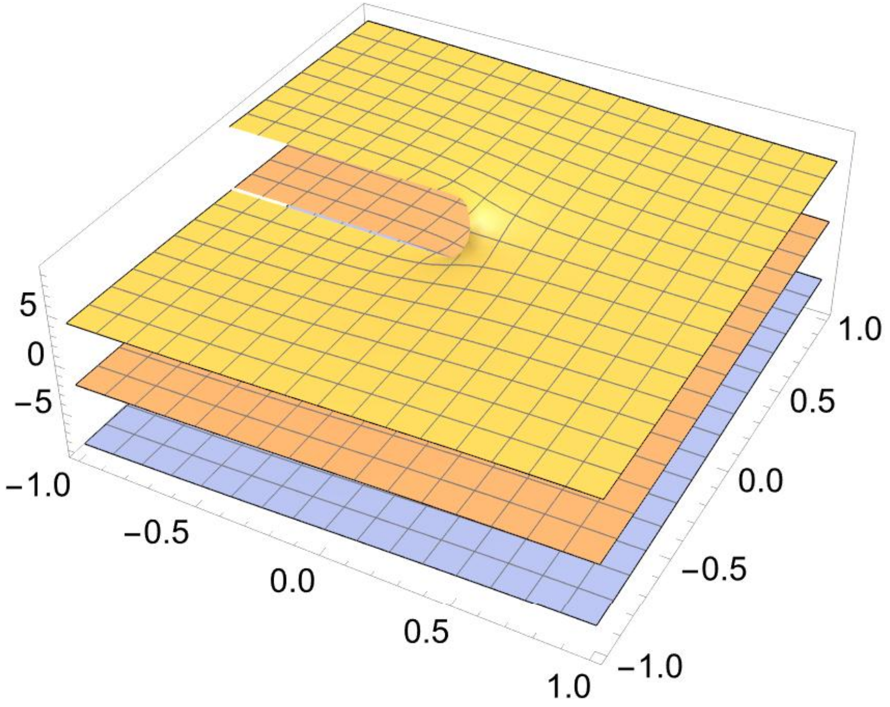
- Discontinuity of the logarithmic function $f(z) = \log z \;\; (z \in \mathbb{C}^* \equiv \mathbb{C}\backslash\{0\})$

$$\begin{cases} \mathbb{D}e^{\log z} = 0 \\ \mathbb{D}e^{\log z} = e^{\log z} - e^{\log z - \mathbb{D}\log z} \end{cases}$$

$$e^{\mathbb{D}\log z} = 1$$

$$\boxed{\mathbb{D}\log z = \pm 2\pi i\, \theta(-z)}$$

$$\mathrm{RS}\{\log z\} = \mathbb{C}^* \times \{\mathbb{T}_1, \mathbb{T}_{-1} \mid \mathbb{T}_1\mathbb{T}_{-1} \sim 1\} \cong \mathbb{C}^* \times \mathbb{Z}$$



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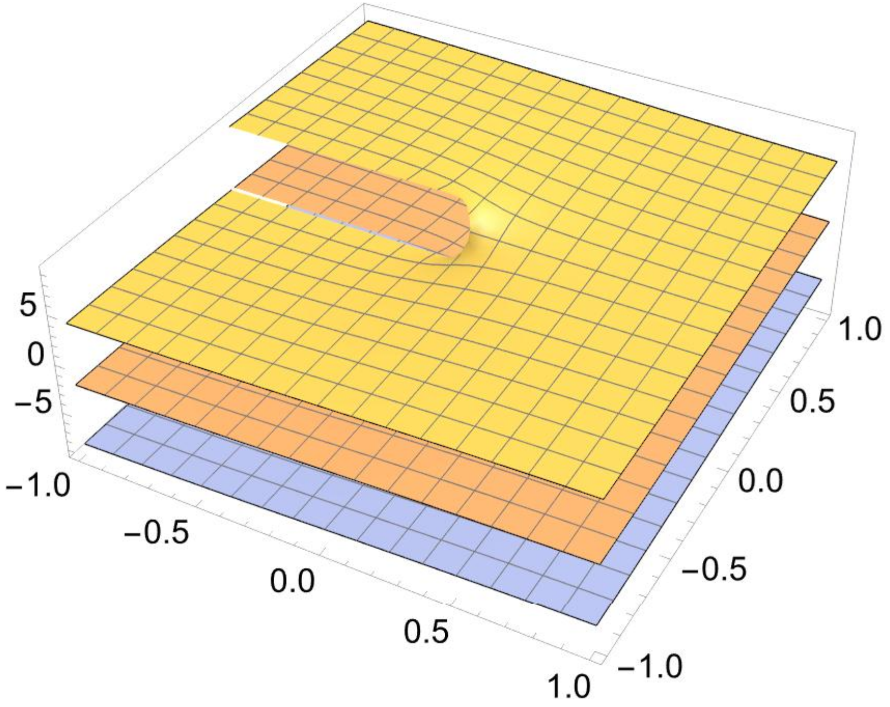

- Discontinuity of meromorphic functions

$$\mathbb{D}\left(\frac{1}{z}\right) = \mp 2\pi i\, \delta(z)$$

$$\mathbb{D}\left(\frac{1}{z^{n+1}}\right) = \mp 2\pi i \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\delta(z)$$

Discontinuity Calculus

- Discontinuity of complex-valued functions

$$\mathbb{D}f(z) = \sum_{i=1}^{n} \mathbb{K}_i f(z)\ \theta_i(z) + \sum_{j=1}^{n'} \alpha_j \delta_j(z)$$

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- Continuation of complex-valued functions

$$\boxed{\mathbb{K}_i f(z)\ :\ \text{the } \textbf{continuation kernel} \text{ of } f(z)}$$

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$\boxed{\mathbb{K}_i f(z) \,:\, \text{the } \textbf{continuation kernel} \text{ of } f(z)}$

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$$f(z) \;\rightarrow\; \mathbb{T}_i f(z) \;\rightarrow\; \mathbb{T}_j \mathbb{T}_i f(z) \;\rightarrow\; \mathbb{T}_k \mathbb{T}_j \mathbb{T}_i f(z) \;\rightarrow\; \cdots$$

Discontinuity Calculus

- Discontinuity of complex-valued functions

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- Continuation of complex-valued functions

Monodromy group

$\boxed{\mathbb{K}_i f(z) \,:\, \text{the } \textbf{continuation kernel} \text{ of } f(z)}$

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$$f(z) \;\to\; \mathbb{T}_i f(z) \;\to\; \mathbb{T}_j \mathbb{T}_i f(z) \;\to\; \mathbb{T}_k \mathbb{T}_j \mathbb{T}_i f(z) \;\to\; \cdots$$

- A discontinuity relation

$$\begin{cases} \mathbb{D}\left[\frac{1}{f(z)} \cdot f(z)\right] = 0 \\[2ex] \mathbb{D}\left[\frac{1}{f(z)} \cdot f(z)\right] = \mathbb{D}\left[\frac{1}{f(z)}\right] \cdot f(z) + \frac{1}{f(z)} \cdot \mathbb{D} f(z) - \mathbb{D}\left[\frac{1}{f(z)}\right] \cdot \mathbb{D} f(z) \end{cases} \Rightarrow \boxed{\mathbb{D} f(z) = -f(z) \cdot \mathbb{D}[1/f(z)] \cdot [f(z) - \mathbb{D} f(z)]}$$

Contents

Discontinuity Analysis of the Two-Point Green's Function

- The scalar two-point one-loop Green's function

$$G\left(p^2\right) \equiv \int_{\mathbb{R}^4} \frac{i\, \mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left[(p/2+q)^2 - m_1^2 + i\epsilon\right]\left[(p/2-q)^2 - m_2^2 + i\epsilon\right]}$$

- Discontinuity analysis with explicit expression $(s = p^2)$

$$\boxed{G(s) = \frac{1}{(4\pi)^2}\left[a(\mu) + \log\left(\frac{m_1 m_2}{\mu^2}\right) + \frac{(m_1^2 - m_2^2)}{s}\log\frac{m_1}{m_2} + 8\pi\rho(s)\varphi(s)\right]}$$

direct calculation

$$\rho(s) = \frac{1}{16\pi s}\sqrt{(s-s_+)(s-s_-)} \qquad s_\pm = (m_1 \pm m_2)^2$$

$$\varphi(s) = \log\left[c - s - 16\pi s\rho(s)\right] - \log\left[c - s + 16\pi s\rho(s)\right]$$

$$\boxed{\mathbb{D}G(s) = -2i\rho(s)\theta\left(s - s_+\right)}$$

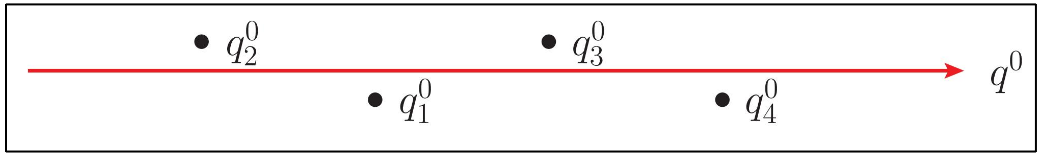
HJJ, Xiong-Hui Cao and Feng-Kun Guo [arXiv:2507.06175]

Discontinuity Analysis of the Two-Point Green's Function

- Discontinuity analysis from the definition

$$D_1 \equiv q^0 + E/2 - \omega_1 \quad D_3 \equiv q^0 - E/2 + \omega_2$$
$$D_2 \equiv q^0 + E/2 + \omega_1 \quad D_4 \equiv q^0 - E/2 - \omega_2$$
$$(E = \sqrt{s})$$

$$\boxed{G(s) = \int_{\mathbb{R}^4} \frac{id^4q}{(2\pi)^4} \frac{1}{(D_1 + i\epsilon)(D_2 - i\epsilon)(D_3 - i\epsilon)(D_4 + i\epsilon)}}$$



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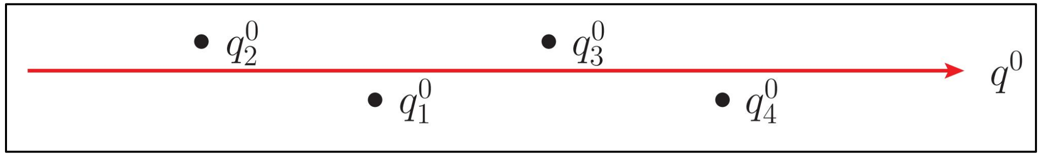
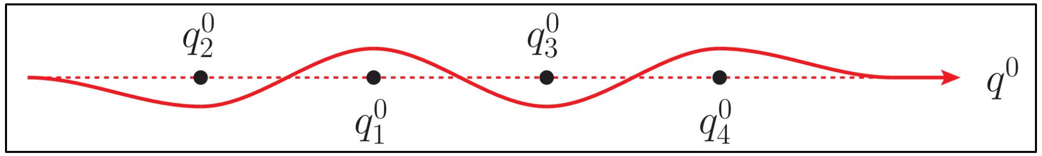
$$G(s) = \int_{\mathbb{R}^4} \frac{id^4 q}{(2\pi)^4} \frac{1}{(D_1 + i\epsilon)(D_2 - i\epsilon)(D_3 - i\epsilon)(D_4 + i\epsilon)}$$



$$\Downarrow$$

$$G(s) = \int_{\mathcal{C}\times\mathbb{R}^3} \frac{id^4 q}{(2\pi)^4} \frac{1}{D_1 D_2 D_3 D_4}$$



Discontinuity Analysis of the Two-Point Green's Function

- Discontinuity analysis from the definition

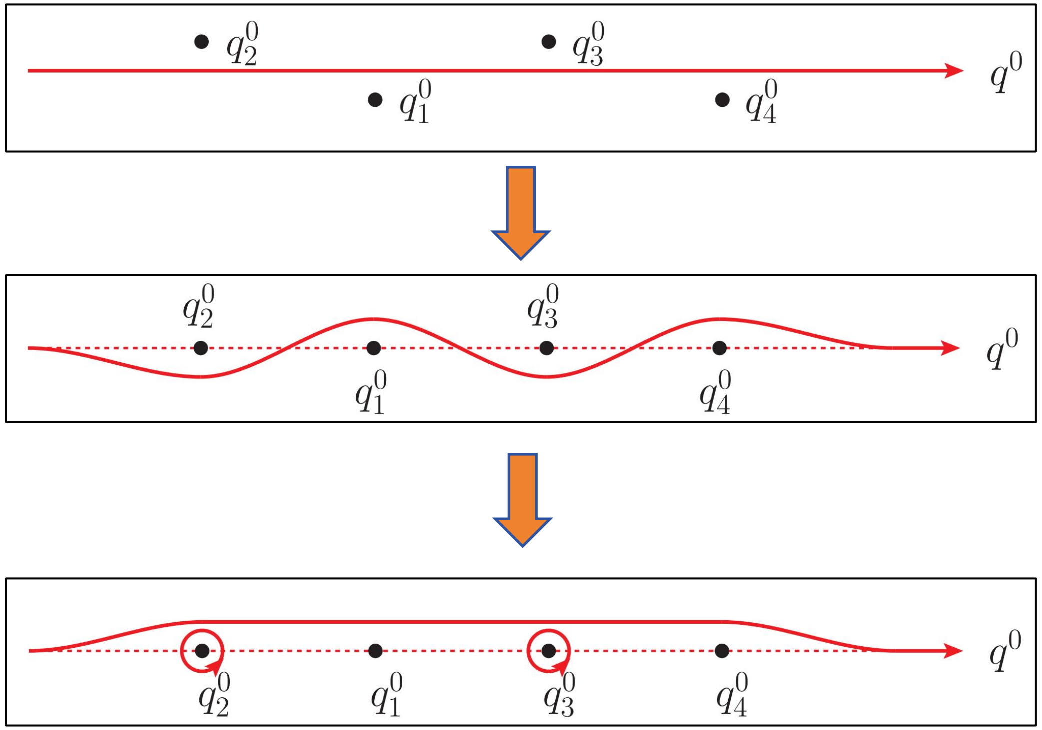
$$D_1 \equiv q^0 + E/2 - \omega_1 \quad D_3 \equiv q^0 - E/2 + \omega_2$$
$$D_2 \equiv q^0 + E/2 + \omega_1 \quad D_4 \equiv q^0 - E/2 - \omega_2 \qquad (E = \sqrt{s})$$

$$G(s) = \int_{\mathbb{R}^4} \frac{id^4q}{(2\pi)^4} \frac{1}{(D_1 + i\epsilon)(D_2 - i\epsilon)(D_3 - i\epsilon)(D_4 + i\epsilon)}$$

$$G(s) = \int_{\mathcal{C}\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4} \frac{1}{D_1 D_2 D_3 D_4}$$

$$G(s) = \int_{\mathcal{C}(\boldsymbol{n}_G)\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4} \frac{1}{D_1 D_2 D_3 D_4}$$

$$\boldsymbol{n}_G = (0, 1, 1, 0)$$



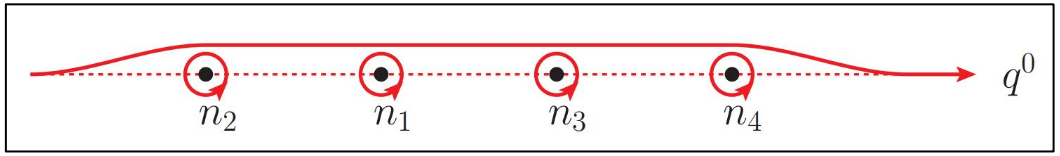
Discontinuity Analysis of the Two-Point Green's Function

- Discontinuity analysis from the definition

$$I(s,\boldsymbol{n}) = \int_{\mathcal{C}(\boldsymbol{n})\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4}\frac{1}{D_1D_2D_3D_4}$$

$$\mathbb{D}_E I(s,\boldsymbol{n}) = \int_{\mathcal{C}(\boldsymbol{n})\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4}\,\mathbb{D}_E\left(\frac{1}{D_1D_2D_3D_4}\right)$$

$$\mathbb{D}_E I(s,\boldsymbol{n}) = \int_{\mathcal{C}(\boldsymbol{n})\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4}\left[(2\pi i)^2\left(\frac{\delta_1\delta_3}{D_2D_4}+\frac{\delta_1\delta_4}{D_2D_3}+\frac{\delta_2\delta_3}{D_1D_4}+\frac{\delta_2\delta_4}{D_1D_3}\right)\right]$$



Homotopy class symbol

$$\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$$

$$\mathbb{D}_E\left(\frac{1}{D_1}\right) = -2\pi i\delta\left(q^0 + E/2 - \omega_1\right) \equiv -2\pi i\delta_1$$

$$\mathbb{D}_E\left(\frac{1}{D_2}\right) = -2\pi i\delta\left(q^0 + E/2 + \omega_1\right) \equiv -2\pi i\delta_2$$

$$\mathbb{D}_E\left(\frac{1}{D_3}\right) = +2\pi i\delta\left(q^0 - E/2 + \omega_2\right) \equiv +2\pi i\delta_3$$

$$\mathbb{D}_E\left(\frac{1}{D_4}\right) = +2\pi i\delta\left(q^0 - E/2 - \omega_2\right) \equiv +2\pi i\delta_4$$

Discontinuity Analysis of the Two-Point Green's Function

- Discontinuity analysis from the definition

$$\mathbb{D}_E I(s,\boldsymbol{n}) = \int_{\mathcal{C}(\boldsymbol{n})\times\mathbb{R}^3} \frac{id^4q}{(2\pi)^4}\left[(2\pi i)^2\left(\frac{\delta_1\delta_3}{D_2D_4}+\frac{\delta_1\delta_4}{D_2D_3}+\frac{\delta_2\delta_3}{D_1D_4}+\frac{\delta_2\delta_4}{D_1D_3}\right)\right]$$

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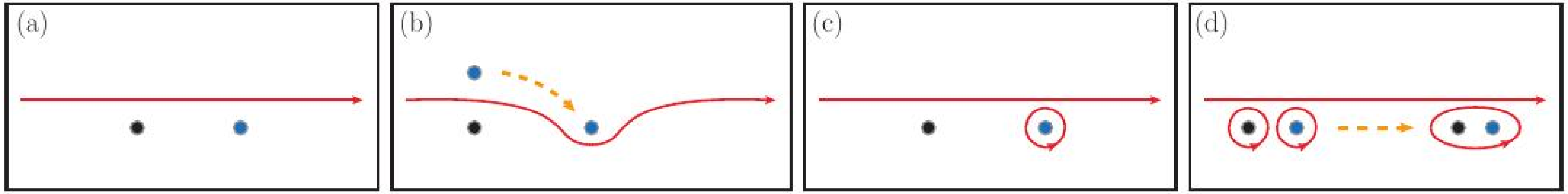
$$\boldsymbol{n} = (n_1,n_2,n_3,n_4)\in\mathbb{Z}^4$$



$$\mathbb{D}_E I(s,\boldsymbol{n}) = (2\pi i)^2\int_{\mathbb{R}^4}\frac{id^4q}{(2\pi)^4}\left[c_{13}(\boldsymbol{n})\frac{\delta_1\delta_3}{D_2D_4}+c_{14}(\boldsymbol{n})\frac{\delta_1\delta_4}{D_2D_3}+c_{23}(\boldsymbol{n})\frac{\delta_2\delta_3}{D_1D_4}+c_{42}(\boldsymbol{n})\frac{\delta_2\delta_4}{D_1D_3}\right]$$

Winding number difference

$$c_{ij}(\boldsymbol{n}) = n_i - n_j$$

Physical case

$$\boldsymbol{n}_G = (0,1,1,0)$$

$$\mathbb{D}_E G(s) = \mathbb{D}_E I(s,\boldsymbol{n}_G) = (2\pi i)^2\int_{\mathbb{R}^4}\frac{id^4q}{(2\pi)^4}\left(-\frac{\delta_1\delta_3}{D_2^+D_4^+}-\frac{\delta_2\delta_4}{D_1^+D_3^+}\right) = -2i\rho(s)\,\theta\,(s-s_+)$$

The **Cutkosky rules** for computing discontinuities is a special case within discontinuity analysis.
R.E.Cutkosky, J.Math.Phys. 1 (1960) 429-433

Discontinuity Analysis of the Two-Point Green's Function

- Continuation of the two-point Green's function

$$\mathbb{D}G(s) = -2i\rho(s)\theta\left(s - s_+\right)$$

$$\mathbb{T}_1 G(s) = (1 - \mathbb{K}_1)G(s) = G(s) + 2i\rho(s)$$

$$\mathbb{D}\rho(s) = 2\rho(s)\theta\left[(s - s_+)(s - s_-)\right] - \frac{(m_1^2 - m_2^2)}{8i}\delta(s)$$

$$\theta[(s - s_+)(s - s_-)] = \theta(s - s_+) + \theta(s_- - s)$$

$$\mathbb{K}_1\rho(s) = \mathbb{K}_2\rho(s) = 2\rho(s)$$

$$\mathbb{K}_1 G(s) = -2i\rho(s) \qquad \mathbb{K}_2 G(s) = 0$$

$$\mathbb{T}_1 G(s) = G(s) + 2i\rho(s) \qquad \mathbb{T}_1\rho(s) = -\rho(s)$$

$$\mathbb{T}_2 G(s) = G(s) \qquad \mathbb{T}_2\rho(s) = -\rho(s)$$

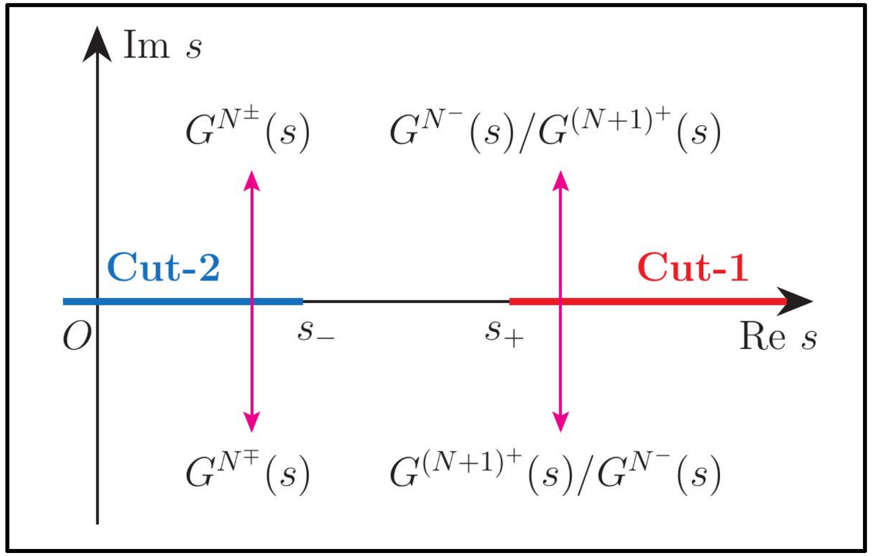
$$(\mathbb{T}_1\mathbb{T}_2)^m G(s) = G(s) + 2mi\rho(s) \qquad (m \in \mathbb{N})$$

$$(\mathbb{T}_2\mathbb{T}_1)^m G(s) = G(s) - 2mi\rho(s)$$

$$\boxed{G^{N^\pm}(s) \equiv G(s) \pm 2i\,(N-1)\,\rho(s)}$$

$$(N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \cdots)$$

$$G^{\mathrm{I}^\pm}(s) \equiv G(s)$$



Discontinuity Analysis of the Two-Point Green's Function

- Topological structure of the two-point Green's function

$$\boxed{\mathrm{RS}\{G(s)\} = \mathbb{C} \times \left\{\mathbb{T}_1, \mathbb{T}_2 \middle| \mathbb{T}_1^2 \sim 1, \mathbb{T}_2^2 \sim 1\right\} \cong \mathbb{C} \times \mathbb{Z}}$$

$$\mathbb{T}_1 G^{N^-}(s) = G^{(N+1)^+}(s), \quad \mathbb{T}_1 G^{(N+1)^+}(s) = G^{N^-}(s).$$
$$\mathbb{T}_2 G^{N^+}(s) = G^{N^-}(s), \quad \mathbb{T}_2 G^{N^-}(s) = G^{N^+}(s).$$

Red line: Cut-1 on physical sheet **Blue line: Cut-2 on unphysical sheets** **Orange line: Cut-1 on unphysical sheets**

Discontinuity Analysis of the Two-Point Green's Function

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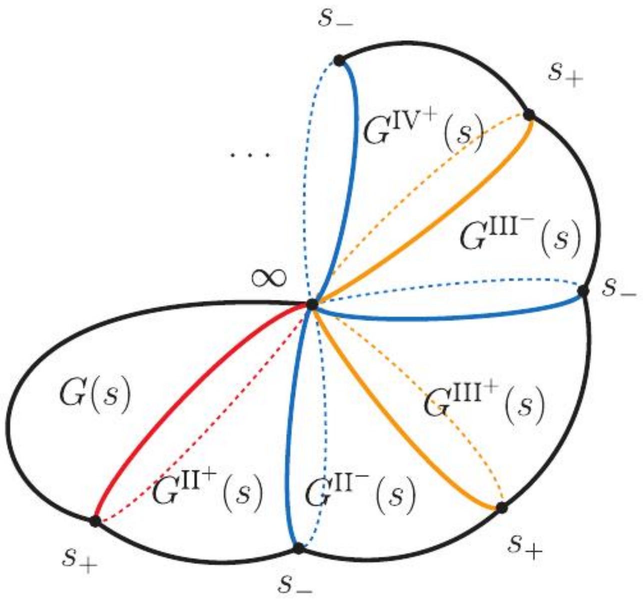

$\overline{\mathrm{RS}}\{G(s)\}$

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Discontinuity Analysis of the Two-Point Green's Function

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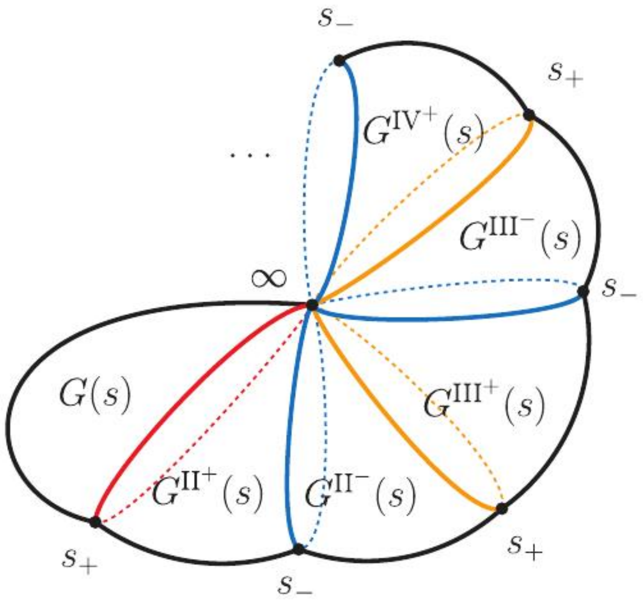
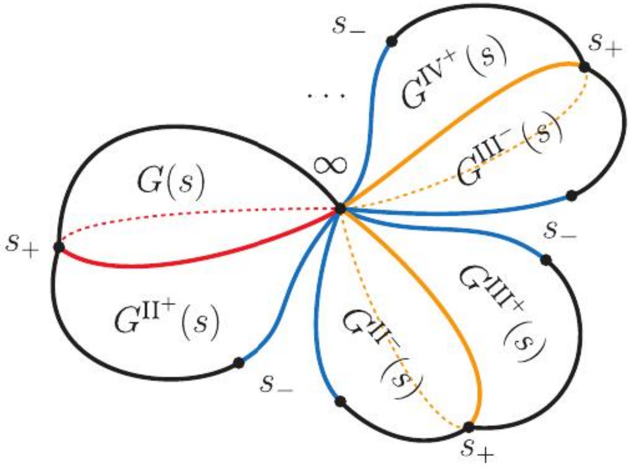


$\overline{\mathrm{RS}}\{G(s)\}$



$\mathrm{RS}\{G(s)\} \backslash \mathbb{T}_2 = \mathbb{C} \times \{\mathbb{T}_1 \mid \mathbb{T}_1^2 \sim 1\} \cong \mathbb{C} \times \mathbb{Z}_2$

Red line: Cut-1 on physical sheet **Blue line: Cut-2 on unphysical sheets** **Orange line: Cut-1 on unphysical sheets**

Discontinuity Analysis of the Two-Point Green's Function

- Topological structure of the two-point Green's function

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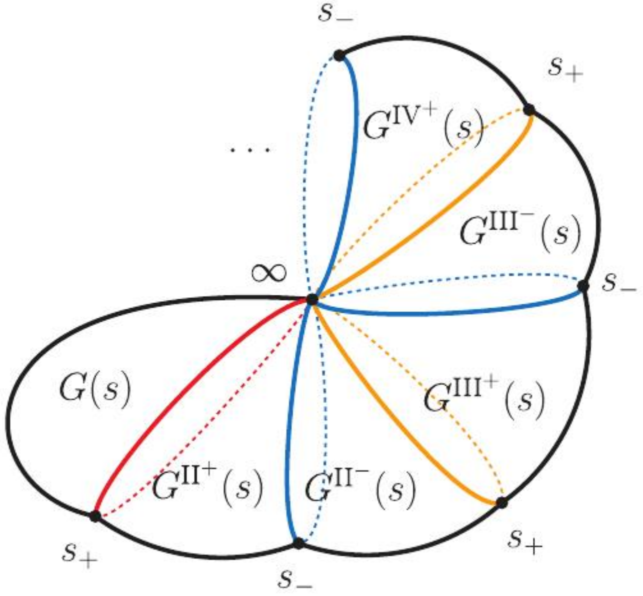
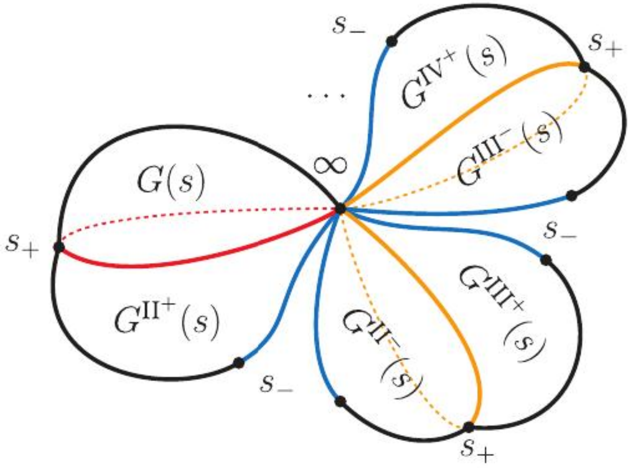


$\overline{\mathrm{RS}}\{G(s)\}$



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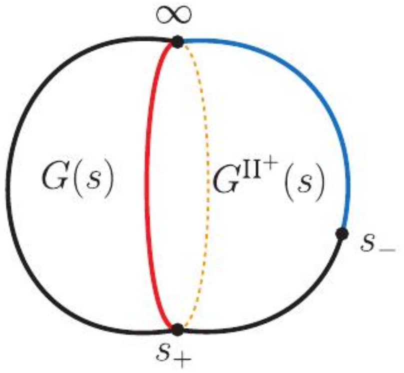

$\overline{\mathrm{RS}}\{G(s)\}\backslash\mathbb{T}_2 \cong S^2$

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Contents

Discontinuity Analysis of the Partial-Wave Scattering Matrix

- Unitarity condition

$$S\,S^\dagger = 1 \xrightarrow{S = 1 + iT} T - T^\dagger = iT\,T^\dagger$$

- Analytical condition

$$\mathcal{A}_{f,i} - \mathcal{A}^*_{i,f} = i \sum_\alpha \int \mathrm{d}\Phi_{n_\alpha}\, \mathcal{A}_{f,\alpha}\, \mathcal{A}^*_{i,\alpha}$$

$$\mathcal{A}_{f,i} = \mathcal{A}_{\mathscr{T}i,\mathscr{T}f}$$

$$\mathcal{A}_{f,i} - \mathcal{A}^*_{\mathscr{T}f,\mathscr{T}i} = i \sum_\alpha \int \mathrm{d}\Phi_{n_\alpha}\, \mathcal{A}_{f,\alpha}\, \mathcal{A}^*_{\mathscr{T}\alpha,\mathscr{T}i}$$

- Unitarity condition for the partial-wave scattering matrix

$$\int \mathrm{d}\Phi_{n_i} F^*_{ia} F_{ib} = 2\rho_a \delta_{ab}$$

$$\varrho(s) = \sum_a \varrho_a(s)\,\theta(s - s_a)$$

$$[\varrho_a(s)]_{bc} = 2\rho_a(s)\delta_{ab}\delta_{ac}$$

$$\mathcal{A}_{fi} = \sum_{a,b} F^*_{fa} F_{ib}\, \boldsymbol{T}_{ab}$$

$$\boxed{\boldsymbol{T}(s) - \boldsymbol{T}^*(s) = i\, \boldsymbol{T}(s) \cdot \varrho(s) \cdot \boldsymbol{T}^*(s)}$$

Discontinuity Analysis of the Partial-Wave Scattering Matrix

- Discontinuity

$$\boldsymbol{T}^*(s+i0^+) = \boldsymbol{T}(s-i0^+) = \boldsymbol{T}(s) - \mathbb{D}\boldsymbol{T}(s)$$

$$\mathbb{D}\boldsymbol{T}(s) = i\,\boldsymbol{T}(s)\cdot\varrho(s)\cdot[\boldsymbol{T}(s) - \mathbb{D}\boldsymbol{T}(s)]$$

$$\mathbb{D}f(z) = -f(z)\,\mathbb{D}[1/f(z)]\,[f(z) - \mathbb{D}f(z)]$$

$$\mathbb{D}\boldsymbol{T}^{-1}(s) = -i\varrho(s)$$

$$\boldsymbol{G}(s) \equiv \mathrm{diag}\{G_1(s), G_2(s), \cdots, G_{n_c}(s)\}$$

$$\mathbb{D}\boldsymbol{T}^{-1}(s) = \mathbb{D}\boldsymbol{G}(s)$$

$$\boldsymbol{T}(s) = [\boldsymbol{h}(s) + \boldsymbol{G}(s)]^{-1}$$

Discontinuity Analysis of the Partial-Wave Scattering Matrix

- Continuation

$$\mathbb{T}_i \boldsymbol{T}(s) = [\boldsymbol{h}(s) + \mathbb{T}_i \boldsymbol{G}(s)]^{-1} \quad\Rightarrow\quad \mathrm{RS}\{\boldsymbol{T}(s)\} \cong \mathrm{RS}\{\boldsymbol{G}(s)\}$$

$$\mathbb{D}\boldsymbol{G}(s) = -i\sum_{a=1}^{n_c}\theta_a(s)\sum_{b=1}^{a}\varrho_b(s) \qquad \theta(s-s_a) = \sum_{b=a}^{n_c}\theta_{[s_b,s_{b+1}]}(s) \equiv \sum_{b=a}^{n_c}\theta_b(s)$$

$$\mathbb{K}_{1a}\boldsymbol{G}(s) = -i\sum_{b=1}^{a}\varrho_b(s) \qquad (a=1,\cdots,n_c) \qquad \mathbb{T}_{1a}\boldsymbol{G}(s) = \boldsymbol{G}(s) + i\sum_{b=1}^{a}\varrho_b(s)$$

$$\mathbb{D}\left[\mathbb{T}_{1a}\boldsymbol{G}(s)\right] = i\sum_{b=1}^{n_c}\theta_b(s)\left[\sum_{c=1}^{\min\{a,b\}}\varrho_c(s) - \sum_{c>\min\{a,b\}}^{b}\varrho_c(s)\right] + \boxed{\text{“Cut-2 terms”} + \text{“Pole-terms”}}$$

HJJ, Xiong-Hui Cao and Feng-Kun Guo [arXiv:2507.06175]

$$\mathbb{K}_{1b}\left[\mathbb{T}_{1a}\boldsymbol{G}(s)\right] = i\left[\sum_{c=1}^{\min\{a,b\}}\varrho_c(s) - \sum_{c>\min\{a,b\}}^{b}\varrho_c(s)\right] \qquad \mathbb{T}_{1b}\left[\mathbb{T}_{1a}\boldsymbol{G}(s)\right] = \boldsymbol{G}(s) + i\sum_{c>\min\{a,b\}}^{\max\{a,b\}}\varrho_c(s)$$

$$\Rightarrow\quad \mathbb{T}_{1b}\left[\mathbb{T}_{1a}\boldsymbol{G}(s)\right] = \mathbb{T}_{1a}\left[\mathbb{T}_{1b}\boldsymbol{G}(s)\right] \qquad \mathbb{T}_{1a}\left[\mathbb{T}_{1a}\boldsymbol{G}(s)\right] = \boldsymbol{G}(s)$$

$$\mathrm{RS}\{\boldsymbol{G}(s)\} = \mathbb{C}\times\left\{\mathbb{T}_{11},\cdots,\mathbb{T}_{1n_c}\middle|\mathbb{T}_{1a}^2\sim 1, \mathbb{T}_{1a}\mathbb{T}_{1b}\sim\mathbb{T}_{1b}\mathbb{T}_{1a}\,(\text{for any } a,b)\right\} \cong \mathbb{C}\times\mathbb{Z}_2^{n_c}$$

Discontinuity Analysis of the Partial-Wave Scattering Matrix

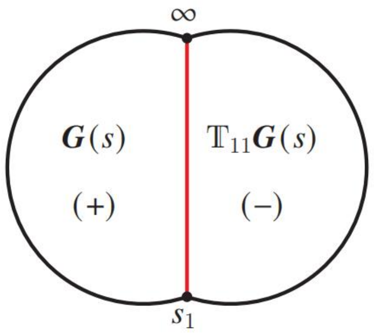
- The topological structure of the Riemann surface

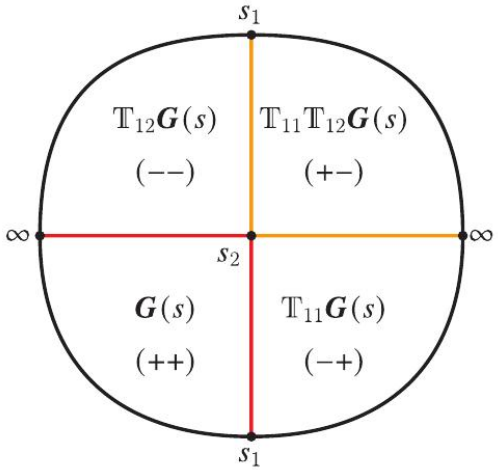
$$n_c = 1$$

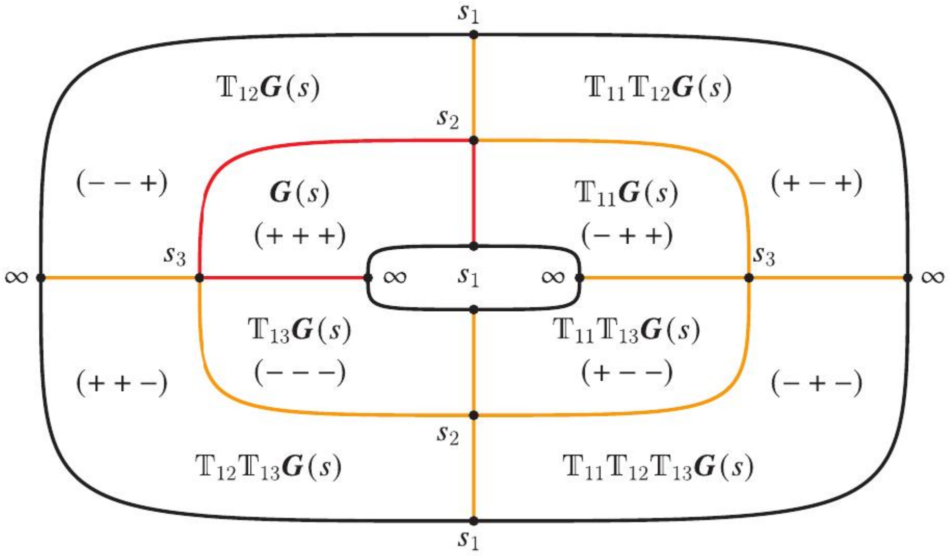
$$\mathrm{RS}\{\boldsymbol{G}(s)\} = \ \mathbb{C} \times \left\{\mathbb{T}_{11} \middle| \mathbb{T}_{11}^2 \sim 1\right\} \cong \mathbb{C} \times \mathbb{Z}_2$$

$$n_c = 3$$

$$\mathrm{RS}\{\boldsymbol{G}(s)\} = \ \mathbb{C} \times \left\{\mathbb{T}_{11}, \mathbb{T}_{12}, \mathbb{T}_{13} \middle| \mathbb{T}_{1a}^2 \sim 1, \mathbb{T}_{1a}\mathbb{T}_{1b} \sim \mathbb{T}_{1b}\mathbb{T}_{1a}\,(a,b=1,2,3)\right\} \cong \mathbb{C} \times \mathbb{Z}_2^3$$





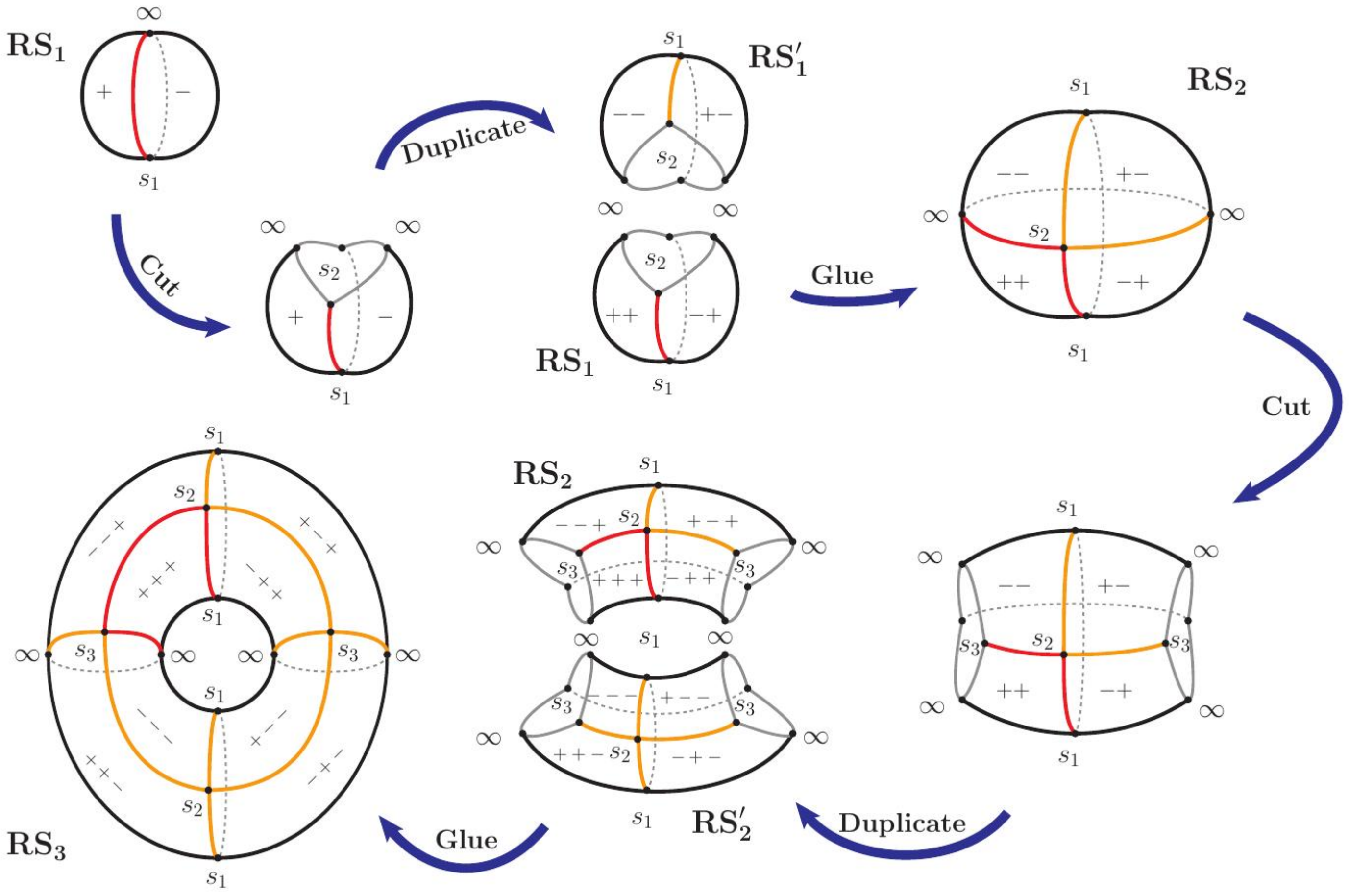


$$n_c = 2$$

$$\mathrm{RS}\{\boldsymbol{G}(s)\} = \ \mathbb{C} \times \left\{\mathbb{T}_{11}, \mathbb{T}_{12} \middle| \mathbb{T}_{1a}^2 \sim 1\ (a=1,2), \mathbb{T}_{11}\mathbb{T}_{12} \sim \mathbb{T}_{12}\mathbb{T}_{11}\right\} \cong \mathbb{C} \times \mathbb{Z}_2^2$$

Discontinuity Analysis of the Partial-Wave Scattering Matrix

- The topological structure of the Riemann surface



$$g_{n_c+1} = 2g_{n_c} + 2^{n_c-1} - 1$$

$$g_1 = 0$$

$$g_{n_c} = (n_c - 3)2^{n_c-2} + 1$$

n_c	1	2	3	4	5	$\cdots$
g_{n_c}	0	0	1	5	17	$\cdots$

Contents

Uniformization of the Partial-Wave Scattering Matrix

- Scattering amplitudes are **multivalued functions of invariant masses**.
  - In the complex energy plane, **right-hand cuts**, **left-hand cuts**, **circular cuts**, and other complex analytic structures pose challenges for amplitude analysis.
  - Origin of multivaluedness: The Riemann surface for scattering amplitudes forms **a multiple covering of the complex plane**.

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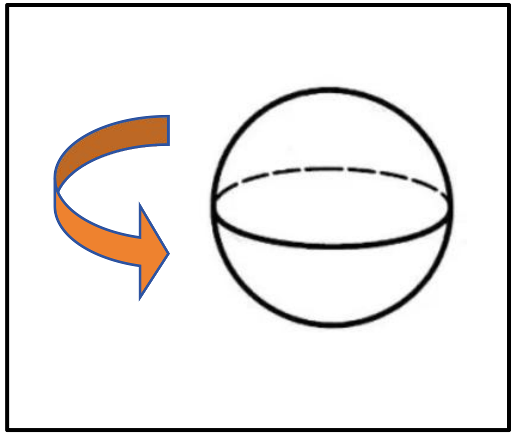
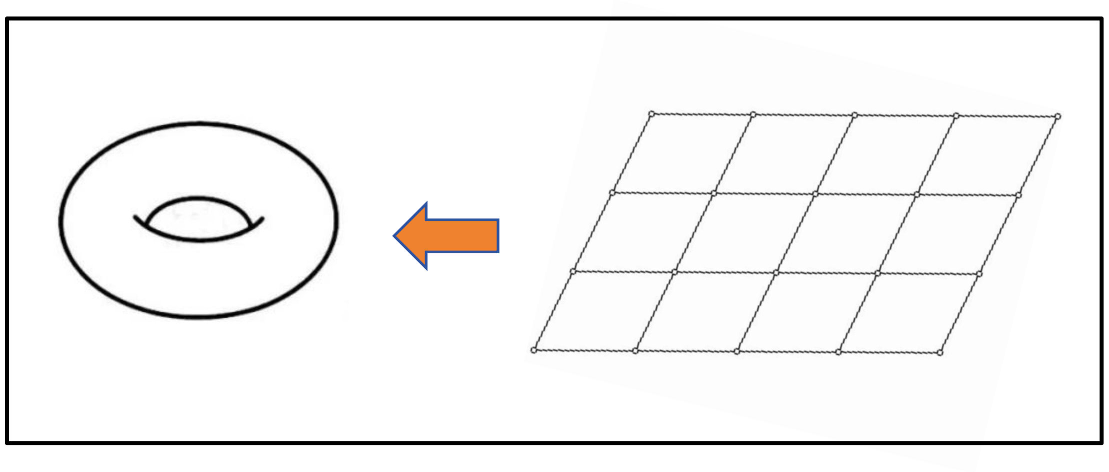
- This is mathematically equivalent to finding the **simply connected covering space** of the Riemann surface for the scattering amplitude — the **Universal Covering**.

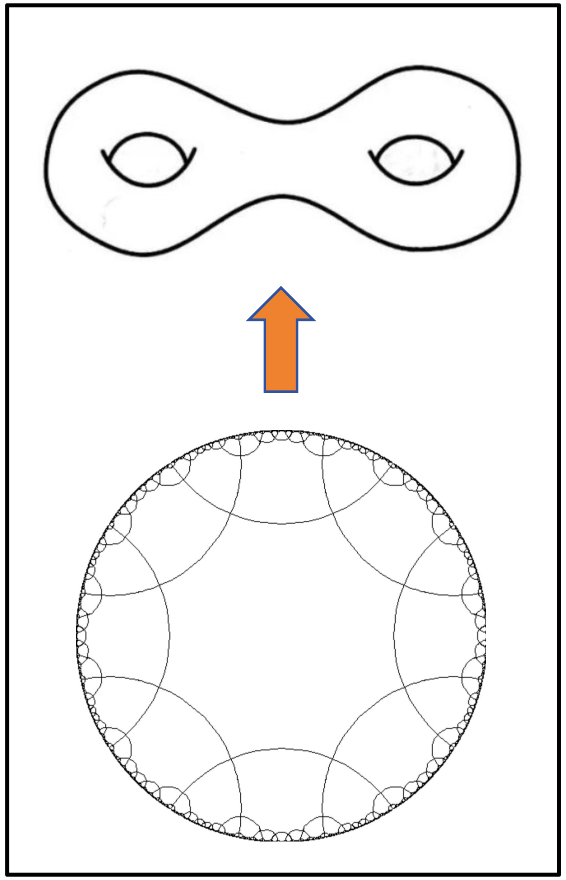
Uniformization of the Partial-Wave Scattering Matrix

- **Uniformization theorem** for closed orientable Riemann surfaces

Genus g	Universal covering	Covering mapping
0	Riemann sphere $\hat{\mathbb{C}}$	Rational function
1	Complex plane $\mathbb{C}$	Elliptic function
≥ 2	Unit disk $\mathbb{D}$	Automorphic function





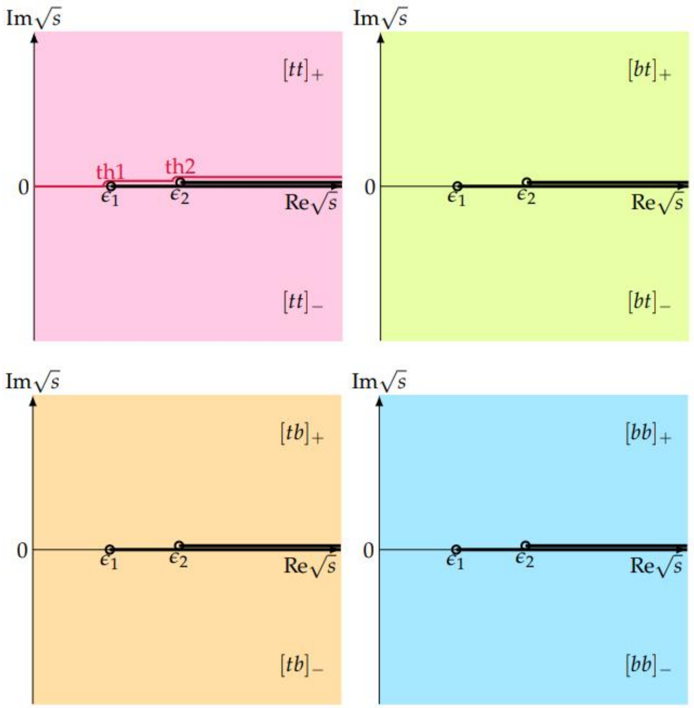


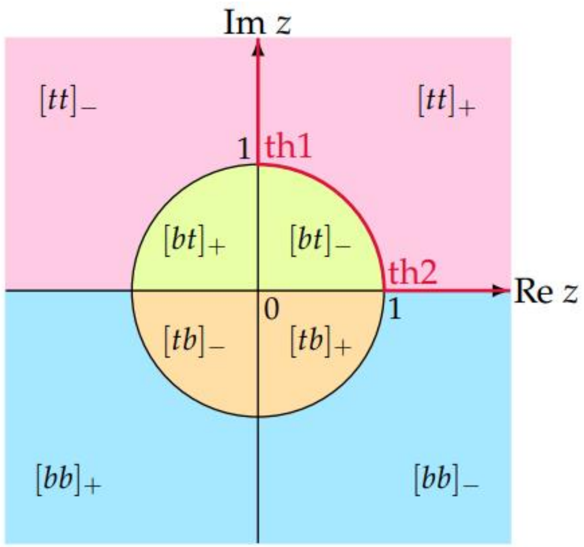
Uniformization of the Partial-Wave Scattering Matrix

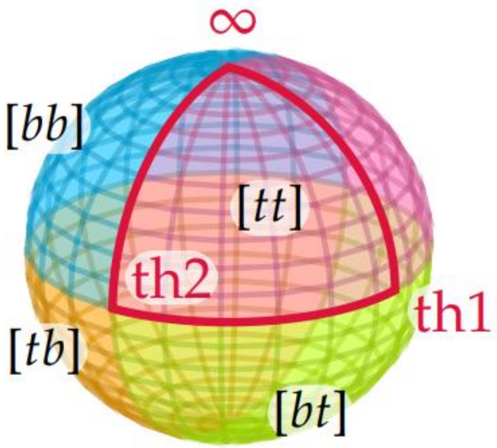
- 2-channel case g=0

M.Kato, Annals Phys. 31 (1965) 1, 130-147

$$q_1 = \frac{\Delta_{12}}{2}\left(z + \frac{1}{z}\right) \qquad q_2 = \frac{\Delta_{12}}{2}\left(z - \frac{1}{z}\right) \qquad \Delta_{ij} = \sqrt{|s_i - s_j|} \qquad q_i = \sqrt{s - s_i}$$







W.Yamada, O.Morimatsu, and T.Sato, Phys. Rev. Lett. 129, 192001 (2022)

Uniformization of the Partial-Wave Scattering Matrix

- 3-channel case g=1 **W.Yamada, O.Morimatsu, and T.Sato, Phys. Rev. Lett. 129, 192001 (2022)**

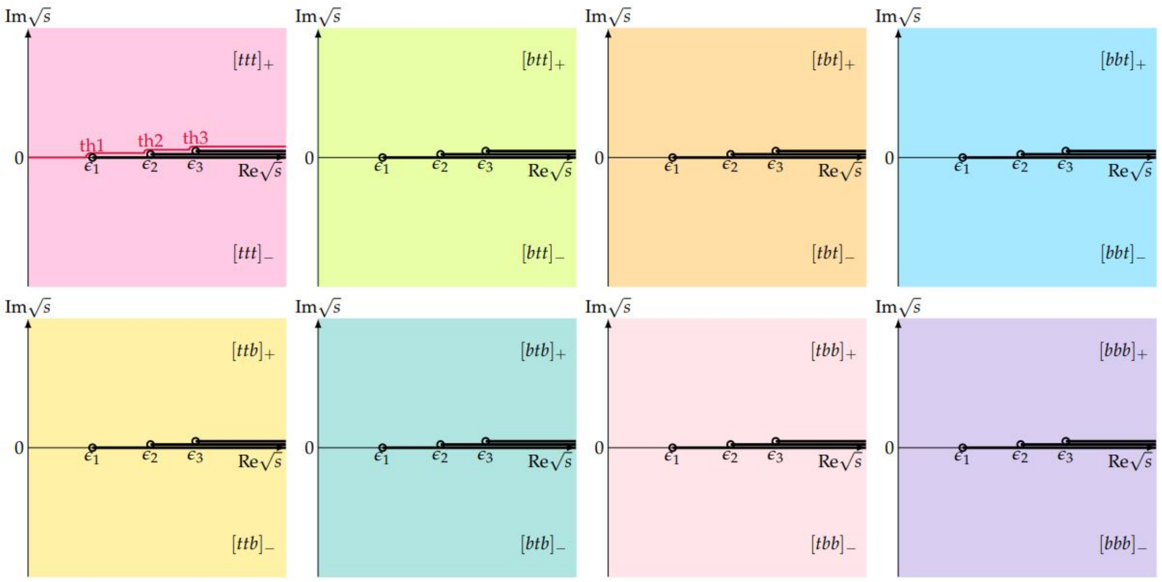
$$q_1 = \frac{\Delta_{12}}{2}\left(\frac{\mathrm{sn}\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}{\gamma} + \frac{\gamma}{\mathrm{sn}\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}\right)$$

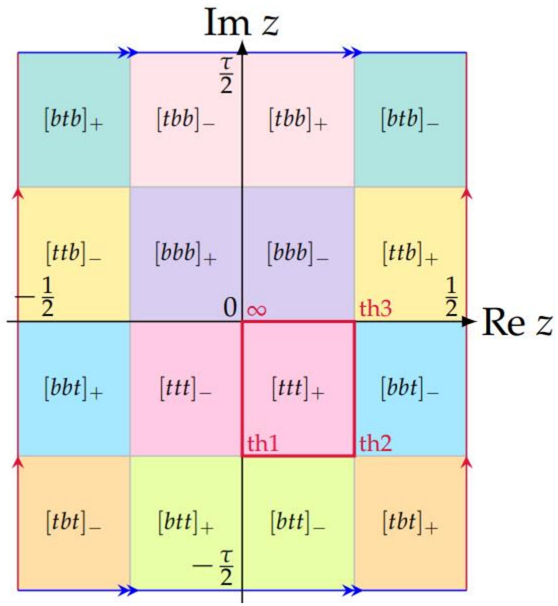
$$q_2 = \frac{\Delta_{12}}{2}\left(\frac{\mathrm{sn}\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}{\gamma} - \frac{\gamma}{\mathrm{sn}\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}\right)$$

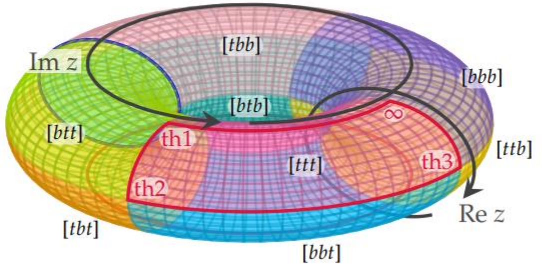
$$q_3 = \frac{\Delta_{12}}{2}\frac{\gamma\,\mathrm{sn}'\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}{\mathrm{sn}\left(4K\left(1/\gamma^2\right)z, 1/\gamma^2\right)}$$

$$\mathrm{sn}^{-1}(v,k) = \int_0^v \frac{dv'}{\sqrt{\left(1-v'^2\right)\left(1-k^2v'^2\right)}}$$

$$\gamma = \frac{\Delta_{13}+\Delta_{23}}{\Delta_{12}} \qquad K(k) = \int_0^1 \frac{dv}{\sqrt{\left(1-v^2\right)\left(1-k^2v^2\right)}}$$

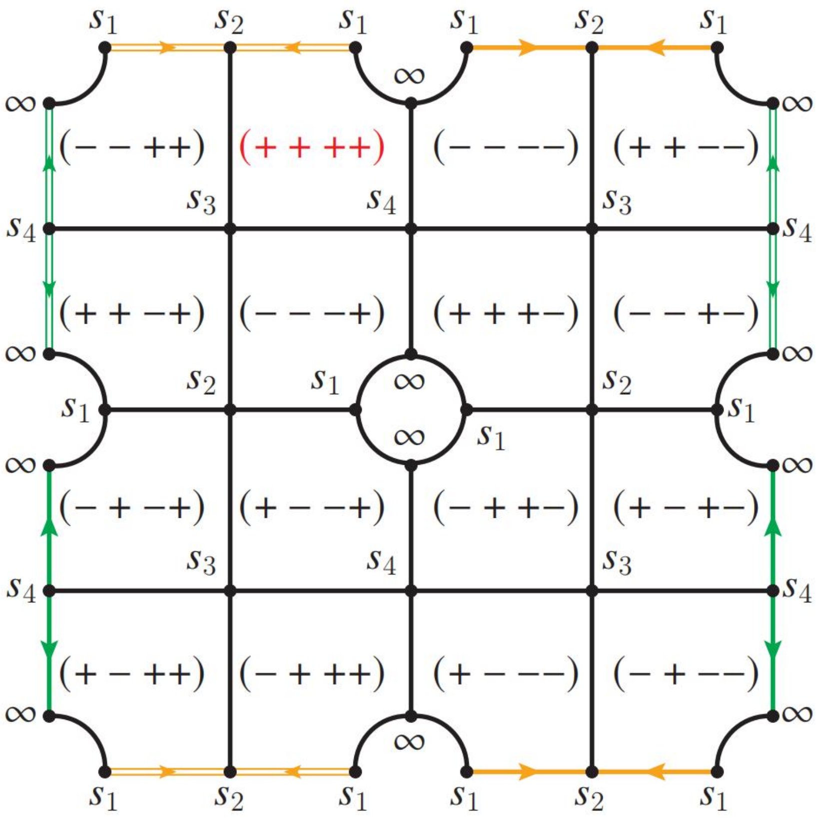






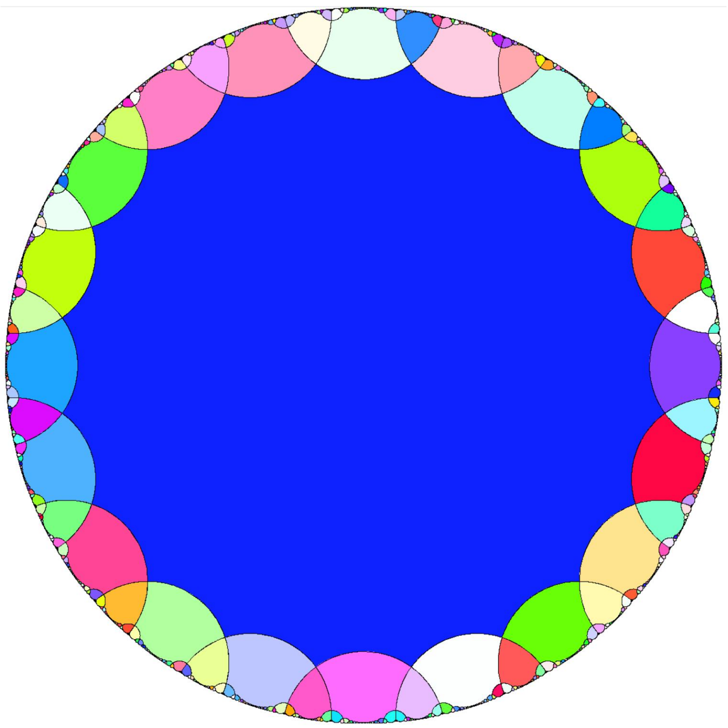
Uniformization of the Partial-Wave Scattering Matrix

- 4-channel case g=5
  - ➢ The Riemann surface comprises **16 sheets** and can be **conformally mapped** via **automorphic functions** onto a regular **hyperbolic 20-gon** within the unit disk



automorphic

functions



Hyperbolic Tessellations (clarku.edu)

Contents

Summary and Outlook

- **Discontinuity Calculus**: A systematic framework for handling analytic continuation of complex functions and the topological structure of Riemann surfaces.
- **Discontinuity analysis** applied to scattering amplitudes in two-body scattering problems.

- **Crossing symmetry**: Investigate discontinuities in scattering amplitudes within unphysical regions.
- **Three- and four-body processes**: Extend discontinuity analysis to multivariate complex functions.
- **Multi-loop amplitudes**: Analyze analytic continuation and topological structures of Riemann surfaces.
- … and beyond

Thank you for your attention!