Nucleon mass in covariant baryon chiral perturbation theory at leading two-loop order

Ze-Rui Liang (梁泽锐)

Hebei Normal University

Z.-R. Liang, H.-X. Chen, F.-K. Guo, Z.-H. Guo, and De-Liang Yao, *JHEP* 04 (2025) 192

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Outline

- Introduction
- Leading two-loop result of the nucleon mass
- Chiral extrapolation of lattice QCD data
- Summary and outlook

Origin of the nucleon mass

- Decomposition of the nucleon mass (classical view)

$$m_N = \langle N(p) | T^\mu{}_\mu | N(p) \rangle \sim \langle N(p) | \underbrace{\frac{\beta}{2g} G^a_{\mu\nu} G^{\mu\nu}_a}_{\text{field energy}} + \underbrace{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s}_{\text{Higgs}} | N(p) \rangle$$

  - How large is the trace anomaly term?
  - Sigma terms: $\sigma_{\pi N} \sim \hat{m}(\bar{u}u + \bar{d}d)$, $\sigma_s = m_s \bar{s}s$

- Feynman-Hellman Theorem
  - three-point to two-point problem
  - High-precision determination of the quark mass dependence of the nucleon mass is crucial.

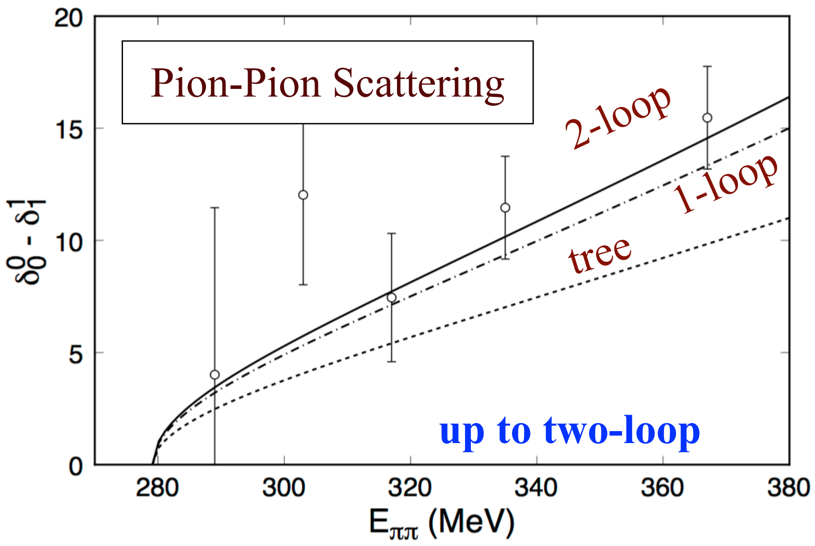
$$\sigma_{\pi N} = \hat{m} \frac{\partial m_N}{\partial \hat{m}} = M_\pi^2 \frac{\partial m_N}{\partial M_\pi^2} \ , \quad \hat{m} = (m_u + m_d)/2 \ .$$

Chiral perturbation theory beyond one-loop is necessary!

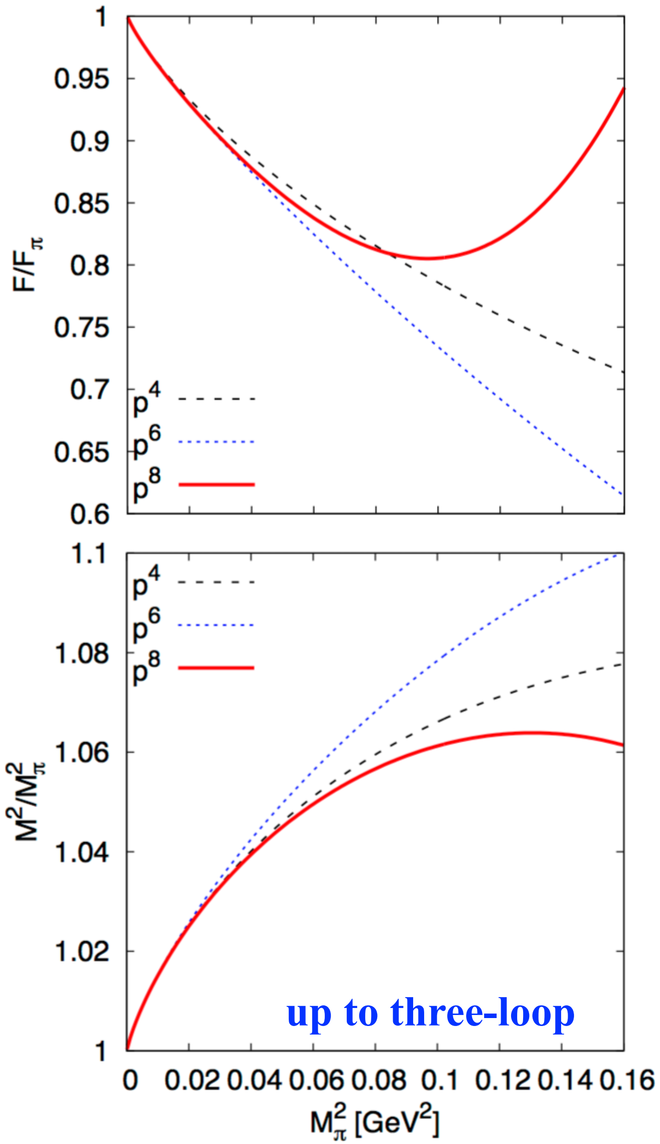
Status of ChPT beyond one-loop

- **Pure meson sector:** a great triumph has been achieved.
  - ➡ Two-loop calculations become standard and good convergence can be seen.
    - Pion mass
    - Pion decay constant
    - Electromagnetic form factor
    - $\pi\pi$ scattering
    - πK scattering
    - $K_{\ell 4}$
    - …

[Bijnens, et al., PLB (1996)]



[Bijnens and Truedsson, JHEP (2017)]



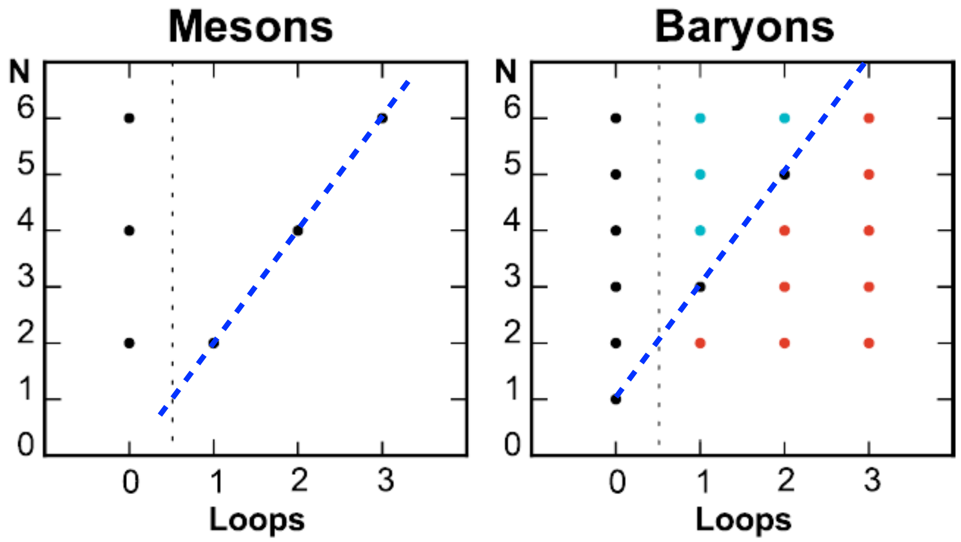
Status of ChPT beyond one-loop

- **Baryon sector:** multi-loop calculation is much more complex than pure meson sector
  - Dirac algebra & axial coupling between baryons and Goldstone bosons

    $\longrightarrow$ *More diagrams and master integrals*
  - Unequal mass

    $\longrightarrow$ *Hard to compute multi-loop integrals*
  - The introduction of non-zero baryon mass in the chiral limit

    $\longrightarrow$ *the notable Power Counting Breaking (PCB) problem!*

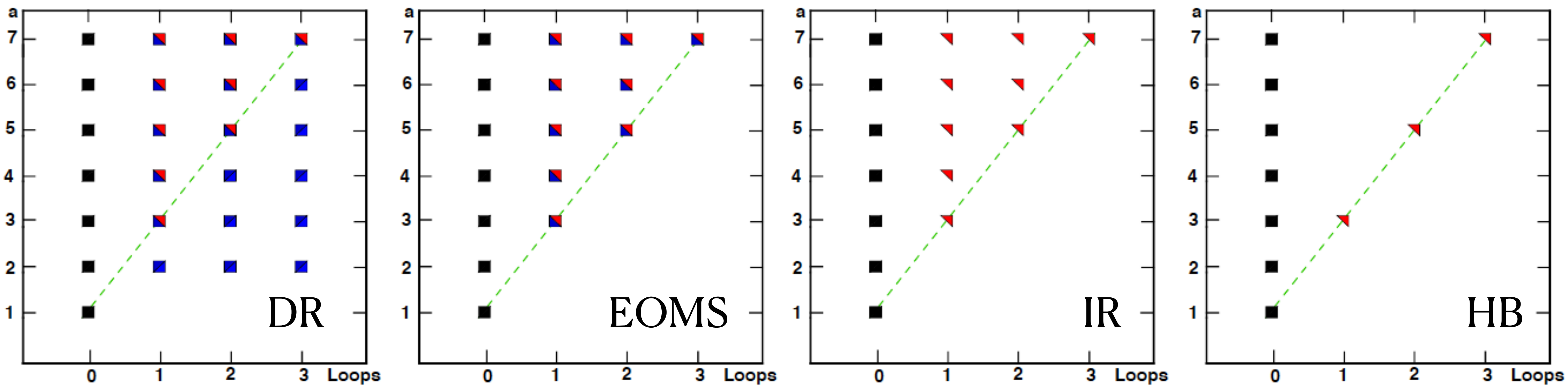


Naive power counting rule !!!

Red dots denote PCB terms!!!

Solutions to the PCB problem

- **Various approaches**
  - Heavy Baryon (HB) formalism [Jenkins and Manohar,PLB255'91]
  - Infrared Regularisation (IR) prescription [T.Becher and H.Leutwyler,Eur.Phys.J.C9'99]
  - Extended-On-Mass-Shell (EOMS) scheme [T.Fuchs,J.Gegelia,G.Japaridze and S.Scherer,PRD68'03]



philosophy: HB & IR: throwing away EOMS: reorganisation

Status of the nucleon mass at two loop

- **Previous works on the nucleon mass in ChPT**
  - 1999: HB formalism up to $\mathcal{O}(p^5)$ [McGovern & Birse, PLB446(1999)]
  - 2007/2008: IR prescription up to $\mathcal{O}(p^6)$ [truncated]

    [Schindeler, Djukanovic, Gegelia & Scherer, PLB649(2007), NPA803(2008)]
  - 2024: EOMS up to $\mathcal{O}(p^6)$ [truncated]

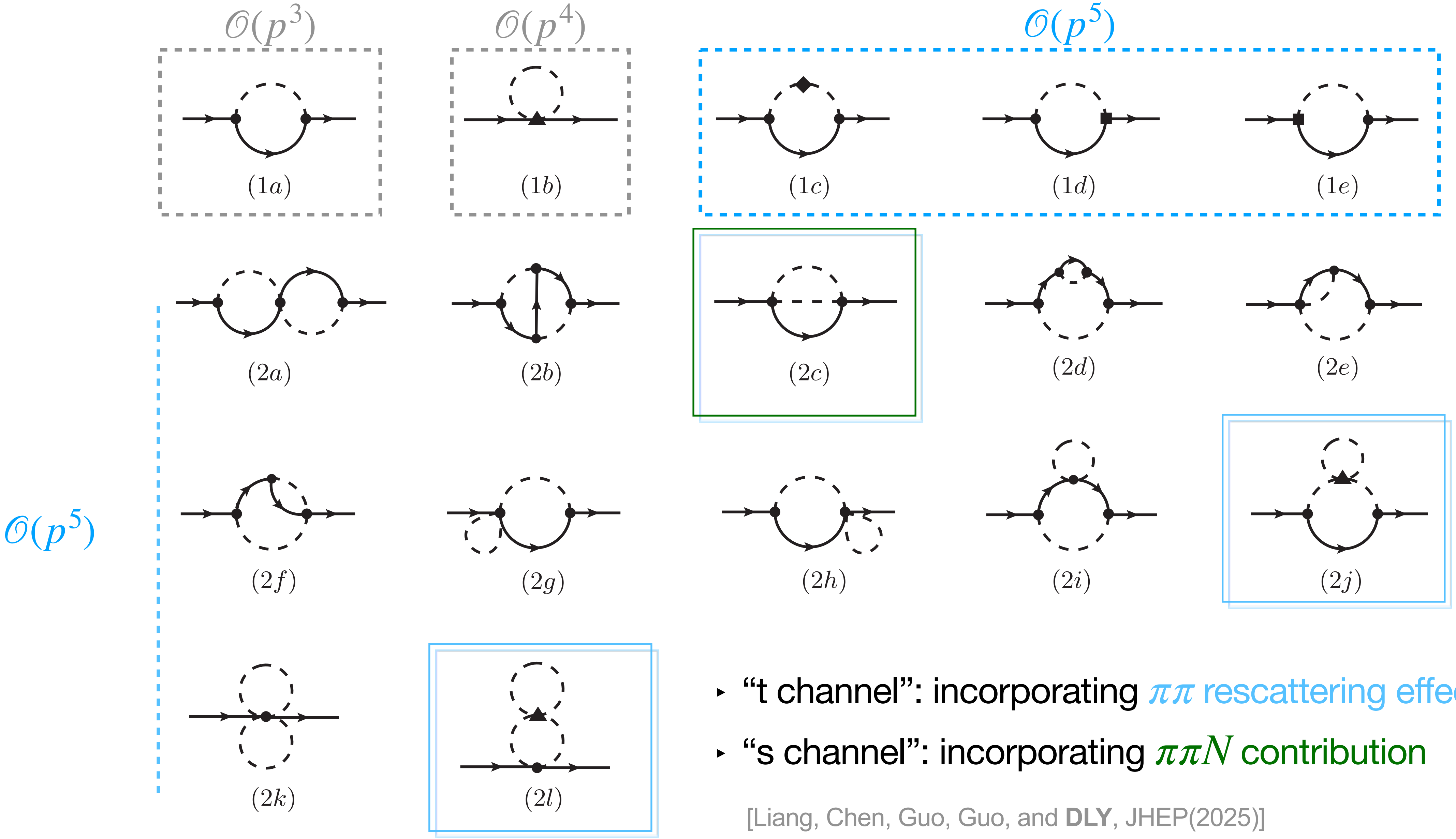
    [Conrad, Gasparyan & Epelbaum, PoS CD2021, 074 (2024), EPJ Web Conf. 303, 02001(2024)]

    [Chen, Hu, Mo & Jia, arXiv:2406.040124 [hep-ph]]

$$\boxed{m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6}$$

- **Our work: full EOMS chiral result up to $\mathcal{O}(p^5)$**
  - All the necessary analytical two-loop structure
  - Avoid too many unknown LECs

The nucleon mass at two-loop order



- "t channel": incorporating $\pi\pi$ rescattering effect
- "s channel": incorporating $\pi\pi N$ contribution

[Liang, Chen, Guo, Guo, and **DLY**, JHEP(2025)]

Two-loop nucleon mass in BChPT

- **Chiral expression up to $\mathcal{O}(p^5)$**
  - Tree + one loop + two loop $\quad m_N = -4c_1M^2 - 2e_mM^4 + \Delta m_N^{(1)} + \Delta m_N^{(2)}$
  - One-loop: $\quad \Delta m_N^{(1)} = \boxed{m_N^{(1a)}} + m_N^{(1b)} + m_N^{(1c)} + m_N^{(1d)} + m_N^{(1e)}\ ,$
  - Two-loop: $\quad \Delta m_N^{(2)} = \boxed{m_N^{(2a)}} + m_N^{(2b)} + \cdots + m_N^{(2l)} + m_N^{(2')} + m_N^{\text{sub.}}\ .$

$$m_N^{(1a)} = \frac{3g^2m\,\kappa}{2iF^2}\left[J_{01} + M^2J_{11}\right]$$

$$m_N^{(2a)} = \frac{3g^2\,\kappa^2}{16mF^4}\Big[8m^2M^4I_{11011} + 8m^2M^2(I_{01011} + I_{10011}) + 2m^2\big[4I_{00011} + I_{11(-1)01} + I_{11(-1)10} - I_{110(-1)1} - I_{1101(-1)}\big] + 2(M^2 - 2m^2)I_{11000} - I_{110(-1)0} - I_{1100(-1)}\Big],$$

✓ $J_{\nu_1\nu_2}$ and $I_{\nu_1\nu_2\nu_3\nu_4\nu_5}$ are one- and two-loop Feynman integrals.

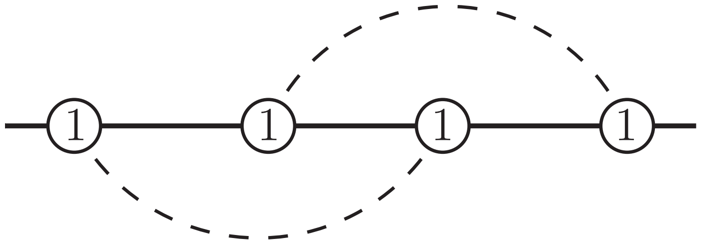
Two-loop Feynman integrals

- **Definition of two-loop integrals**

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int\int \frac{\mathrm{d}^d\ell_1}{(i\pi^{d/2})}\frac{\mathrm{d}^d\ell_2}{(i\pi^{d/2})}\frac{1}{\mathscr{D}_1^{\nu_1}\mathscr{D}_2^{\nu_2}\mathscr{D}_3^{\nu_3}\mathscr{D}_4^{\nu_4}\mathscr{D}_5^{\nu_5}}$$

  - ν_i integers; scalar integrals: all $\nu_i \geq 0$; tensor integrals: $\exists \nu_i \leq 0$
  - Number of irreducible scalar product: $N = L \times E + L(L+1)/2$

$$\mathscr{D}_1 = \ell_1^2 - M^2$$
$$\mathscr{D}_2 = \ell_2^2 - M^2$$
$$\mathscr{D}_3 = (p + \ell_1 + \ell_2)^2 - m^2$$
$$\mathscr{D}_4 = (p + \ell_1)^2 - m^2$$
$$\mathscr{D}_5 = (p + \ell_2)^2 - m^2$$



IBP reduction and master integrals

- IBP reduction: e.g. diagram (2c)

$$m_N^{(2c)} = \frac{-3\,\kappa^2}{32 m F^4}\left[2M^2\left(2M^2 I_{11100} + 2I_{01100} + 2I_{10100} - I_{11000}\right) + I_{110(-1)0} + I_{1100(-1)} - 4I_{111(-1)(-1)}\right]$$

Scalar integrals in d dim.

Tensor integrals in d dim.

Integration by parts (IBP) **Laporta algorithm**

Master integrals

Dimensional Shift Relations (DSR)

Scalar integrals in d+2 dim.

ISP-basis

Dot-basis ($\nu_i \geq 0$) → **suited for implementing EOMS scheme!**

- Master integrals: 3 one-loop + 10 two-loop (dot-basis)

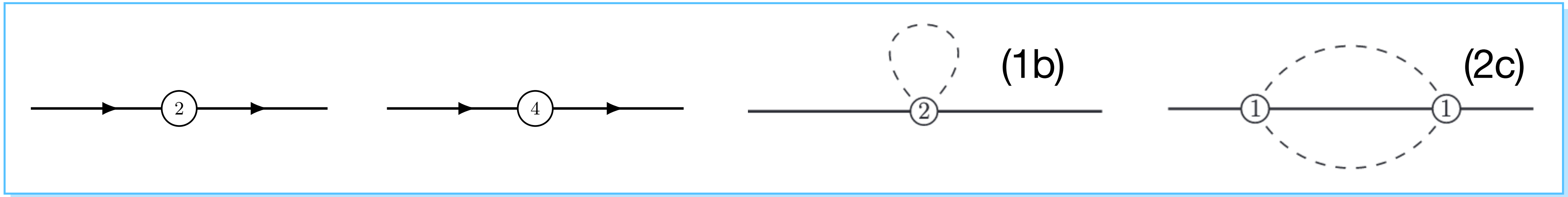
$$J^{\text{1-loop}}_{\text{master}} = \{J_{10},\, J_{01},\, J_{11}\}\ .$$

$$\begin{aligned} I^{\text{2-loop}}_{\text{master}} = \{&I_{11111},\, I_{11110} \leftrightarrow I_{11101},\, I_{10111} \leftrightarrow I_{01111},\, I_{21100} \leftrightarrow I_{12100},\, I_{11100},\, I_{00111},\\ &I_{10101} \leftrightarrow I_{01110},\, I_{10110} \leftrightarrow I_{01101},\, I_{10100} \leftrightarrow I_{01100},\, I_{00110} \leftrightarrow I_{00101}\}\ . \end{aligned}$$

- Differential equation method [Lotikov, PLB254 (1991), PLB267(1991)] [Henn, PRL110(2013)]

Renormalization of non-renormalizable EFT

- **Non-local UV divergences**
  - Pedagogic example with $g_A = 0$



- One-loop: $$m_N^{(1b)} = -\frac{3(d(c_3-2c_1)+c_2)M^4}{F^2 d}\left[-\frac{1}{\Lambda\epsilon}+\frac{1}{\Lambda}\log\frac{M^2}{\mu^2}+\mathcal{O}(\epsilon)\right]$$
- Two-loop: $$m_N^{(2c)} = \frac{3\left(6M^4m+8M^2m^3-3m^5\right)}{32F^4\Lambda^2\epsilon^2}+\frac{132M^4m-16M^2m^3-9m^5}{128F^4\Lambda^2\epsilon}$$ $$+\frac{3m^3\left(3m^2-8M^2\right)}{16F^4\Lambda^2\epsilon}\log\frac{m^2}{\mu^2}-\frac{9M^4m}{8F^4\Lambda^2\epsilon}\log\frac{M^2}{\mu^2}+\text{finite}+\mathcal{O}(\epsilon)$$

✓ Local UV div. tree-loop counter term

✓ How to deal with non-local UV div.?

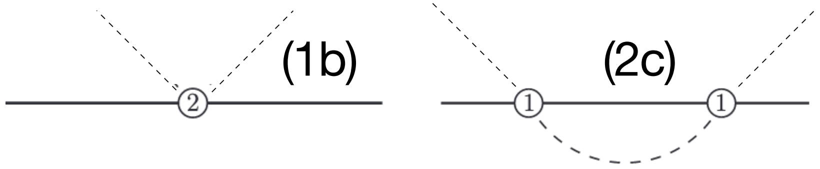
Renormalization of non-renormalizable EFT

- **UV renormalization at two-loop level**
  - Non-local UV: split the bare LECs in one-loop diagram

$$m_N^{(1b)} = -\frac{3(d(c_3-2c_1)+c_2)M^4}{F^2 d}\left[-\frac{1}{\Lambda\epsilon} + \boxed{\frac{1}{\Lambda}\log\frac{M^2}{\mu^2}} + \mathcal{O}(\epsilon)\right]$$

$$X = \boxed{-\frac{\beta_X^{(1)}}{\epsilon\Lambda}} + X^r + \mathcal{O}(\epsilon)\,, \quad X \in \{c_1, c_2, c_3\}$$

Pion-nucleon scattering up to $\mathcal{O}(\epsilon)$



$$\beta_{c_1}^{(1)} = 0\,, \quad \beta_{c_2}^{(1)} = \frac{m}{2F^2}\,, \quad \beta_{c_3}^{(1)} = \frac{m}{4F^2}$$

  - Local UV: split the bare LECs in the tree diagrams

$$m_N = \underbrace{m + 4c_1M^2 - e_mM^4}_{\text{tree}} + m_N^{(1b)} + m_N^{(2c)}$$

$$X = \frac{\beta_X^{(22)}}{\epsilon^2\Lambda^2} - \frac{\beta_X^{(21)}}{\epsilon\Lambda^2} - \frac{\beta_X^{(11)}}{\epsilon\Lambda} + X^r\,, \quad X \in \{m, c_1, e_m\}$$

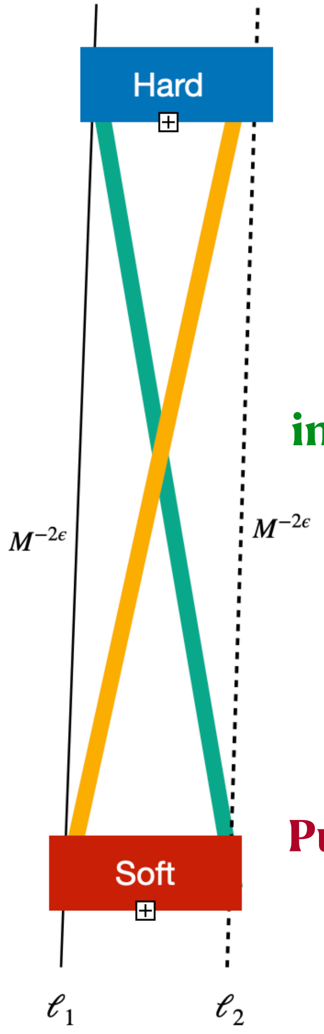
Two-loop UV renormalised parameters!

Chiral expansion of two-loop integrals

- **Dimensional counting analysis**
    - i.e. Equivalent to strategy of regions

[Gegelia, Japaridze and Turashvili, Theor. Math. Phys. 101(1994)]

[Beneke and Smirnov, NPB522(1998)]



Pure infrared regular part

[Local PCB term]

infrared regular-singular part

[Non-local PCB term]

Pure infrared singular part

[PCR-conserving term (IR)]

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int\int \frac{\mathrm{d}^d\ell_1}{(i\pi^{d/2})} \frac{\mathrm{d}^d\ell_2}{(i\pi^{d/2})} \frac{1}{\mathscr{D}_1^{\nu_1}\mathscr{D}_2^{\nu_2}\mathscr{D}_3^{\nu_3}\mathscr{D}_4^{\nu_4}\mathscr{D}_5^{\nu_5}}$$

$$\ell_i \longrightarrow M^{\alpha_i}\tilde{\ell}_i$$

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \sum_{\alpha_1,\alpha_2} M^{\varphi(\alpha_1,\alpha_2)} \times I^{(\alpha_1,\alpha_2)}_{\nu_1\nu_2\nu_3\nu_4\nu_5}$$

$$(\alpha_1, \alpha_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$$

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = I^{(0,0)}_{\dots} + M^{-2\epsilon} I^{(0,1)}_{\dots} + M^{-2\epsilon} I^{(1,0)}_{\dots} + M^{-4\epsilon} I^{(1,1)}_{\dots}$$

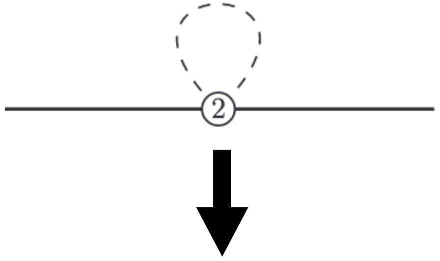
PCB renormalisation

- **EOMS scheme at two-loop:** absorption of PCB terms by LECs

Two-loop

non-local PCB term

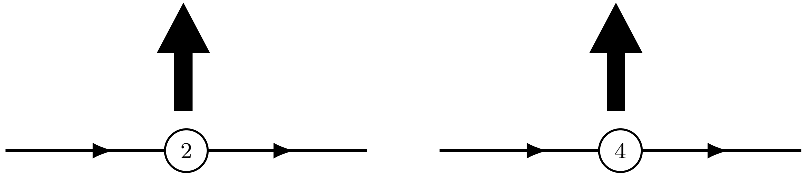
One-loop



$$X^r = \widetilde{X} + \delta_X^{(1)}\ , \quad X \in \{c_1, c_2, c_3\}$$

2-loop local PCB term

$$X^r = \widetilde{X} + \delta_X^{(21)} + \delta_X^{(22)}\ , \quad X \in \{m, c_1, e_m\}$$



Tree

- $\delta_X^{(21)}$: cancel 1-loop local PCB term
- $\delta_X^{(22)}$: cancel 2-loop local PCB term

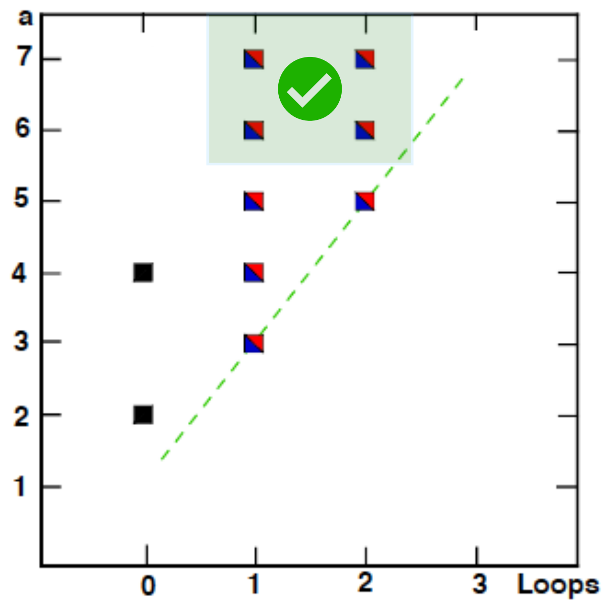
Final two-loop representation

- **The renormalised nucleon mass up to $\mathcal{O}(p^5)$**

$$m_N = \widetilde{m} \underbrace{- 4\widetilde{c}_1 M^2}_{\mathcal{O}(p^2)} + \underbrace{\bar{m}_N^{(1a)}}_{\mathcal{O}(p^3)} \underbrace{- 2\widetilde{e}_m M^4 + \bar{m}_N^{(1b)}}_{\mathcal{O}(p^4)} + \underbrace{\bar{m}_N^{(1c)} + 2\bar{m}_N^{(1d)} + \bar{m}_N^{2-\mathrm{loop}} + \bar{m}_N^{\mathrm{sub-diag.}}}_{\mathcal{O}(p^5)}$$

$$m_N^{\text{sub-diag.}} \sim \sum_X \left\{ \left(- \frac{\beta_X^{(1)}}{\epsilon \Lambda} \right) \times \left[\text{linear term in } \epsilon \text{ from one loops}\right] \right\}$$

- **Merits**
  - Faithful structure: respect original analytic property
  - Controllable accuracy: possess correct power counting rule
  - Renormalisation scale independent



Alternative two-loop representation

- **EOMS truncated at** $\mathcal{O}(p^5)$

$$\boxed{m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5}$$

$$k_1^{\mathrm{EOMS}} = -4c_1 \,,$$

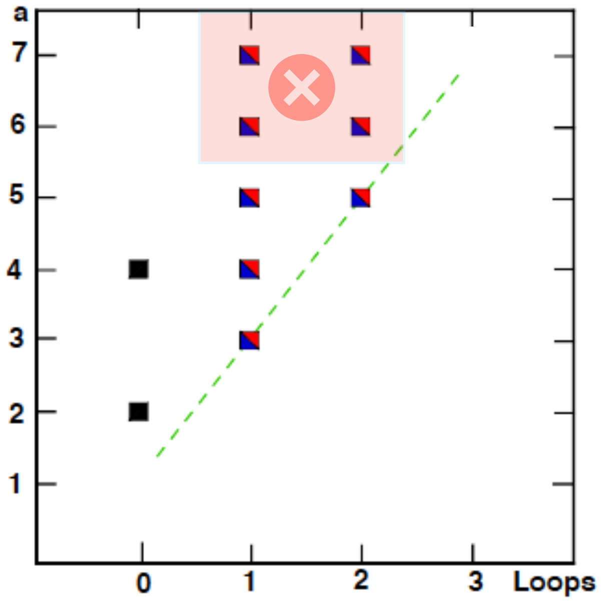
$$k_2^{\mathrm{EOMS}} = -\frac{3g^2\pi}{2F^2\Lambda} \,,$$

$$k_3^{\mathrm{EOMS}} = -\frac{3g^2}{2F^2 m\Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda} \,,$$

$$k_4^{\mathrm{EOMS}} = -2e_m + \frac{3g^2}{2F^2 m\Lambda} + \frac{3c_2}{8F^2\Lambda} \,,$$

$$k_5^{\mathrm{EOMS}} = \frac{3g^2(16g^2-3)\pi}{4F^4\Lambda^2} \,,$$

$$k_6^{\mathrm{EOMS}} = \frac{19\pi g^4}{4F^4\Lambda^2} + \frac{6\pi g(d_{18}-2d_{16})}{F^2\Lambda} + g^2\left[\frac{6\pi}{F^4\Lambda^2} + \frac{3\pi\left(F^2 + 8m^2(2\ell_4 - 3\ell_3)\right)}{16F^4\Lambda m^2}\right] \,.$$



Alternative two-loop representation

- **IR truncated at $\mathcal{O}(p^5)$**

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 \ .$$

$$k_1^{\rm IR} = -4c_1 \ ,$$

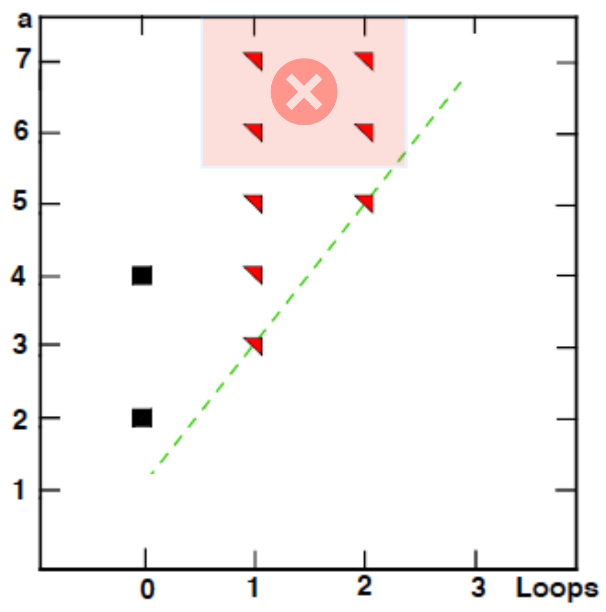
$$k_2^{\rm IR} = -\frac{3g^2\pi}{2F^2\Lambda} \ ,$$

$$k_3^{\rm IR} = -\frac{3g^2}{2F^2 m\Lambda} + \frac{3(8c_1 - c_2 - 4c_3)}{2F^2\Lambda} \ ,$$

$$k_4^{\rm IR} = -2e_m - \frac{3g^2}{4F^2 m\Lambda} + \frac{3c_2}{8F^2\Lambda} \ ,$$

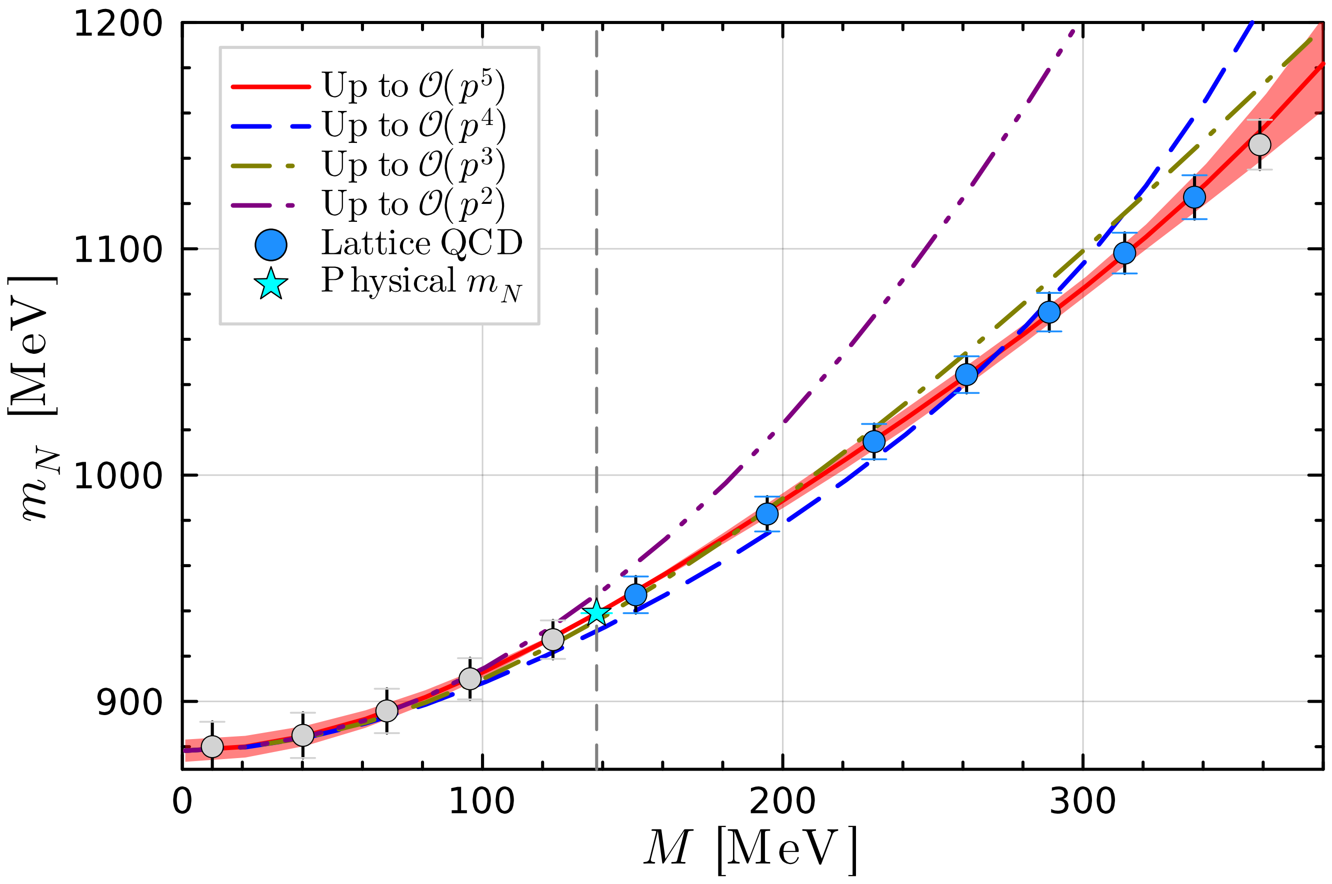
$$k_5^{\rm IR} = \frac{3g^2(16g^2-3)\pi}{4F^4\Lambda^2} \ ,$$

$$k_6^{\rm IR} = \frac{3\pi g^4}{F^4\Lambda^2} + \frac{6\pi g(d_{18}-2d_{16})}{F^2\Lambda} + \frac{3\pi g^2\left(F^2 + 8m^2(2\ell_4 - 3\ell_3)\right)}{16F^4\Lambda m^2} \ .$$



Chiral extrapolation

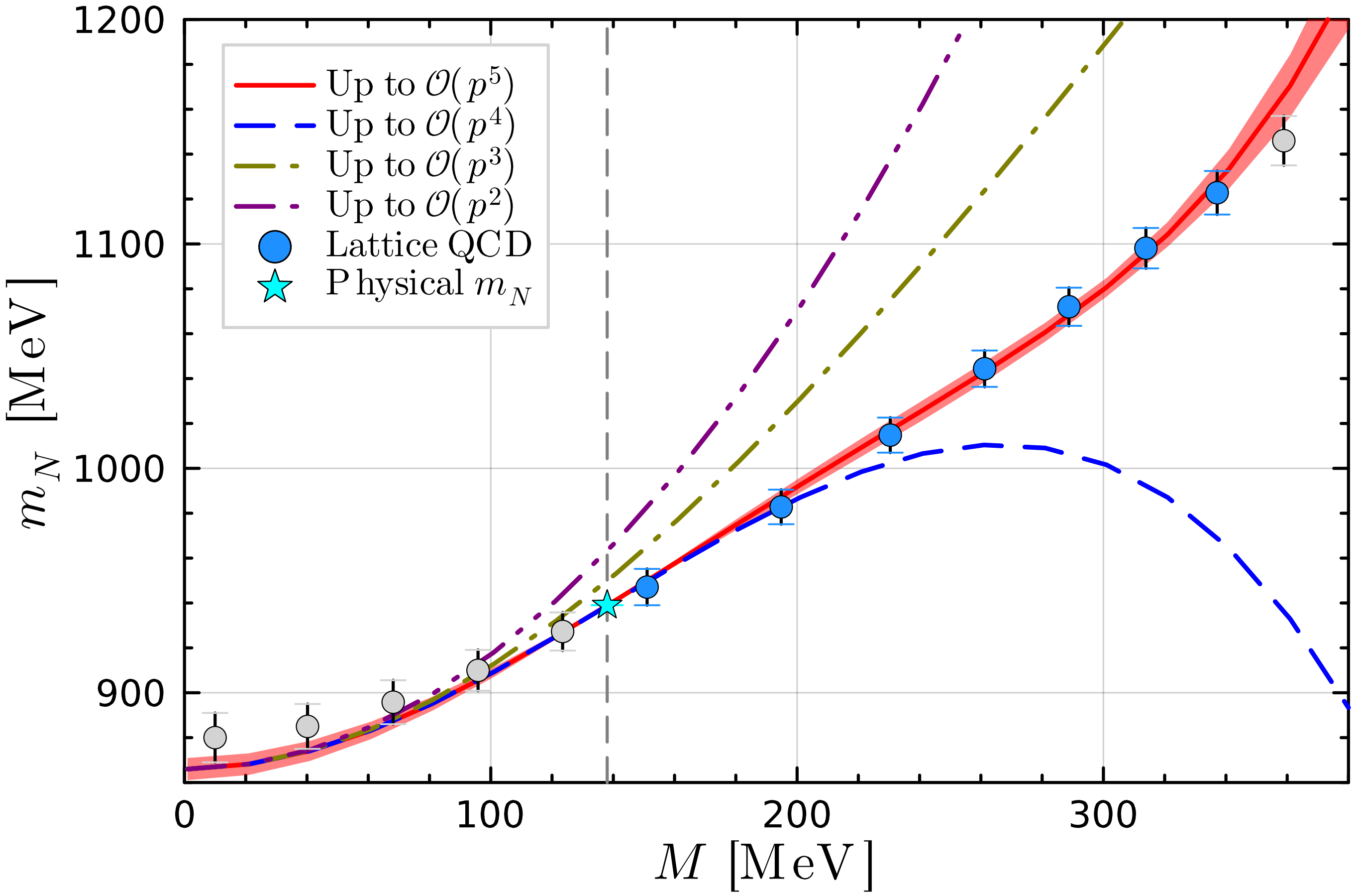
- Full EOMS



  - Original analyticity guarantees the successfulness of chiral extrapolation
  - Relativistic renormalisation scheme possesses good convergency

Chiral extrapolation

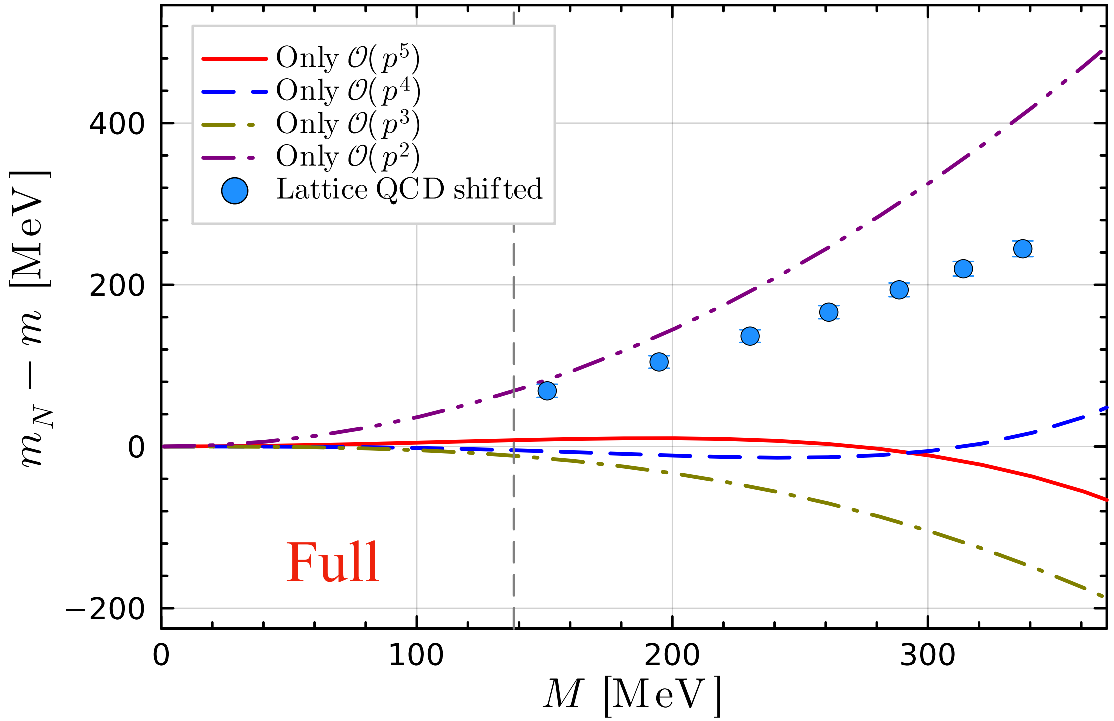
- EOMS truncated at $\mathcal{O}(p^5)$

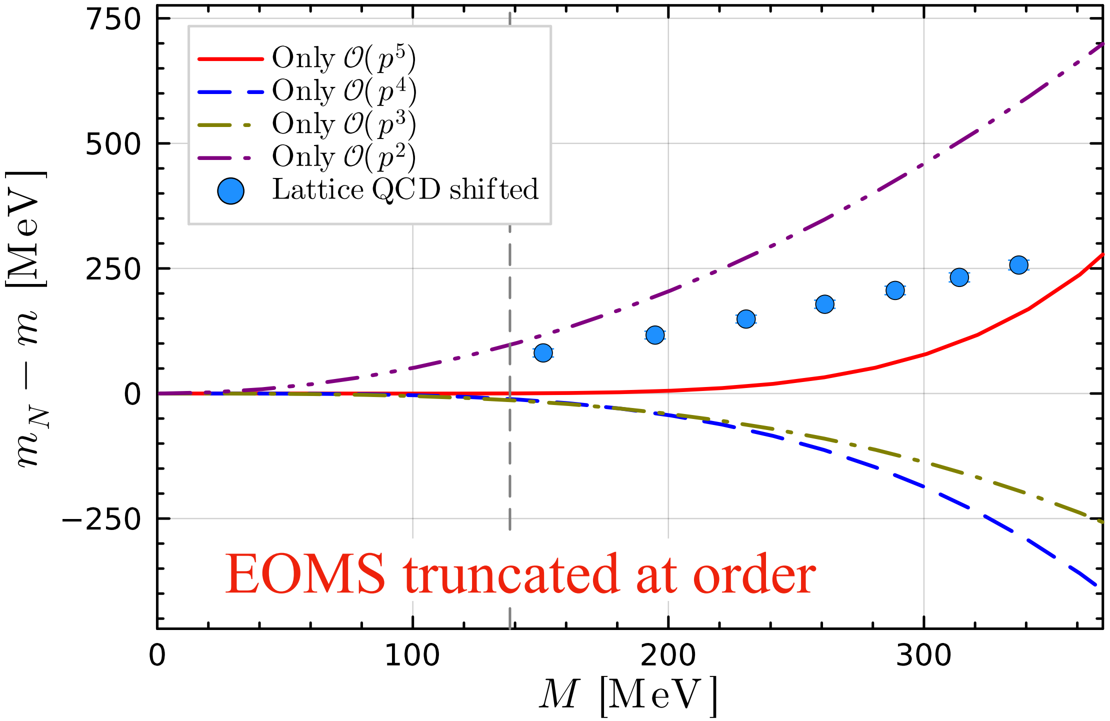


  - Agree well in the fitting range, but deviate beyond
  - Weird behaviour of the pion mass dependence.

Convergency property

- Assess the convergency of the chiral expansion





    - Two-loop contribution is small, approximately 10 MeV, for pion mass < 300 MeV;

Summary and outlook

- **Chiral representation of the nucleon mass in full-EOMS BChPT at two-loop order for the first time**
  - **Validity of EOMS at two-loop level:** the notable PCB issue addressed via dimensional counting analysis.
  - **Convergency:** the $\mathcal{O}(p^5)$ contribution from the full EOMS is small around 10 MeV, implying that the chiral series in the full EOMS scheme converges very well.

$$m_N = \Big\{878.2 + \underbrace{68.8}_{\mathcal{O}(p^2)} + \underbrace{[-11.4]}_{\mathcal{O}(p^3)} + \underbrace{[-4.6]}_{\mathcal{O}(p^4)} + \underbrace{7.9}_{\mathcal{O}(p^5)}\Big\}\ \text{MeV} = 938.9\ \text{MeV}$$

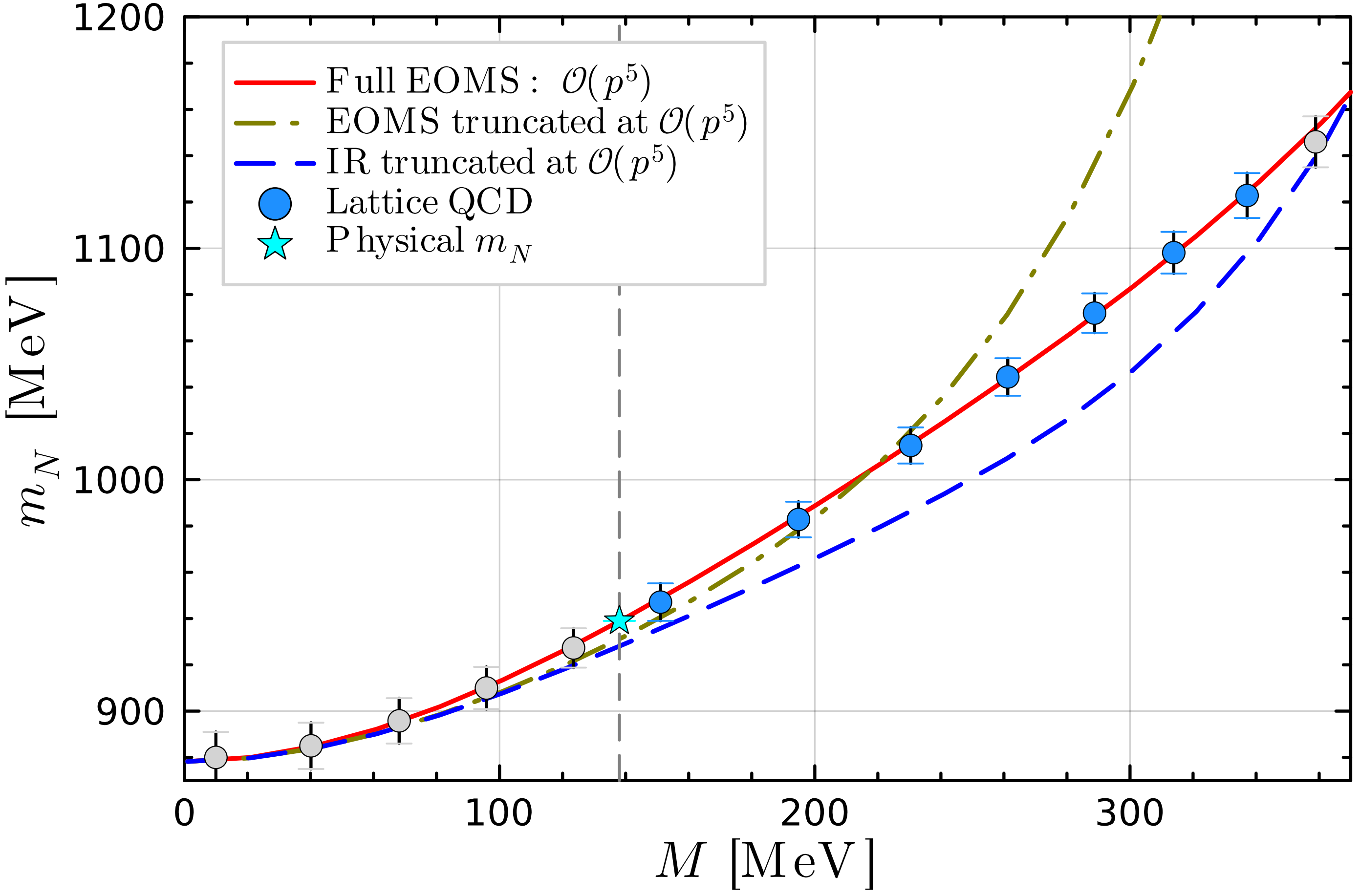
  - **Comparison:** the EOMS-truncated and IR-truncated results change the analytic structure.

- **Outlook**
  - Provide reliable and high-order chiral expression for chiral extrapolation of lattice QCD data.
  - Step into a promising era of two-loop BChPT, where the nucleon property can be explored with high precision!

Thanks!

Comparison of various schemes

- Full EOMS, EOMS truncated, IR truncated



  - IR-truncated: the discarded terms due to the $\mathcal{O}(p^5)$ truncation contribute sizeably.
  - EOMS-truncated: starts to deviate at pion mass ~220MeV.