第十届手征有效场论研讨会@南师大随园

Potential Method in Lattice QCD: An EFT Perspective

Lu Meng（孟璐）

Southeast University

Oct. 20, 2025

Based on papers in preparation

Together with Evgeny Epelbaum (RUB)

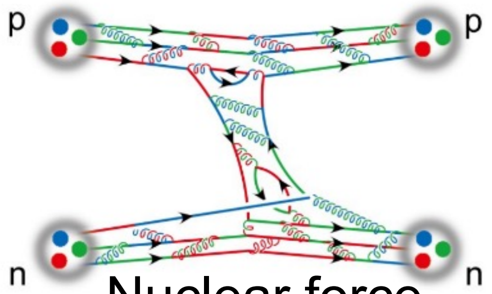
Background

From QCD to nuclei and neutron stars

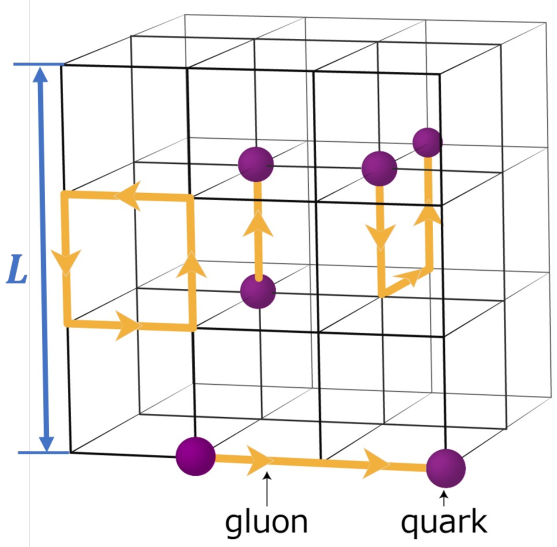
QCD: $\mathcal{L}_{QCD} = \sum_f \bar{q}_f(i\not{D} - \mathcal{M}q_f) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}$

Phenomenological model

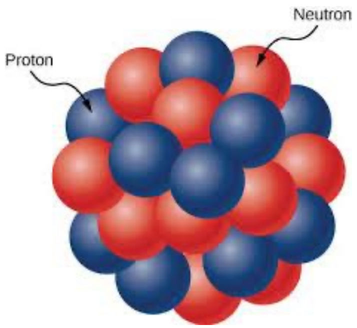
Chiral effective field theory



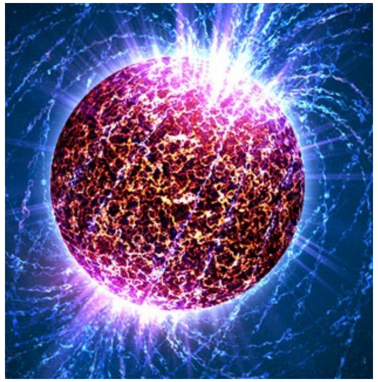
Nuclear force



Lattice QCD



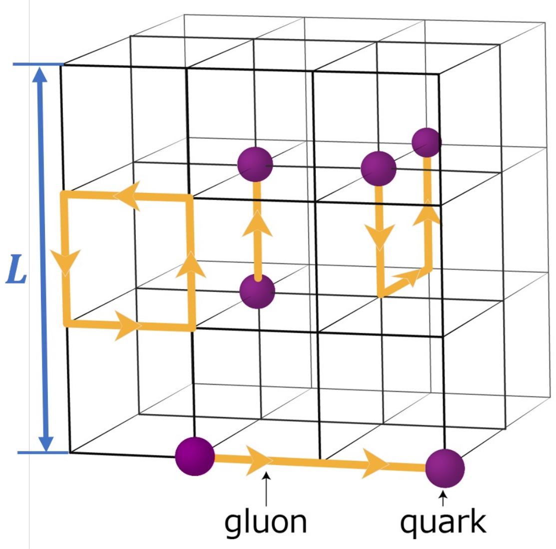
nucleus



neutron star

Hadronic molecule

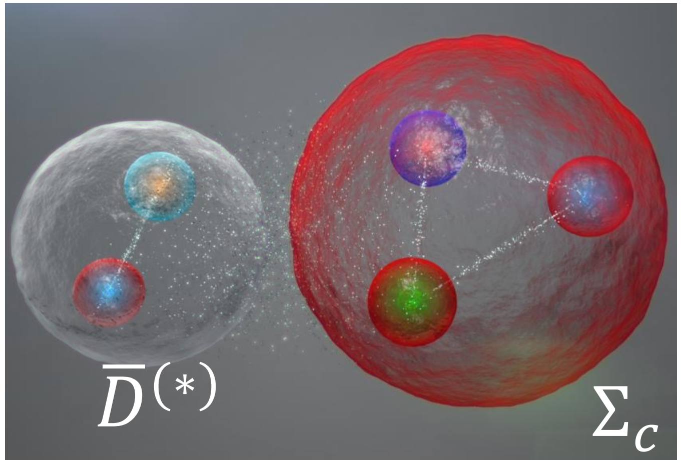
QCD: $\mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M} q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$



Lattice QCD

Phenomenological model

Chiral effective field theory

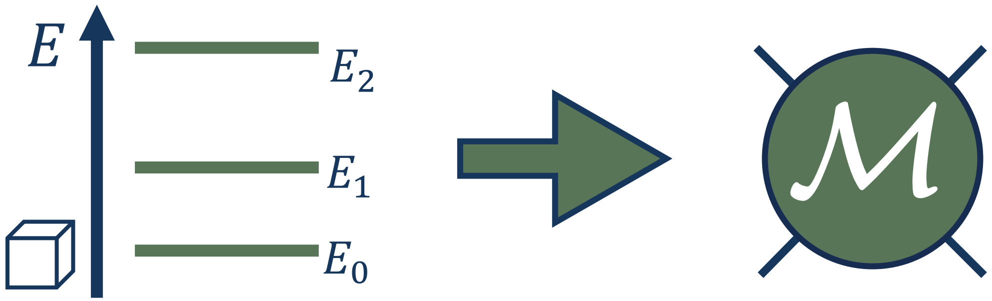


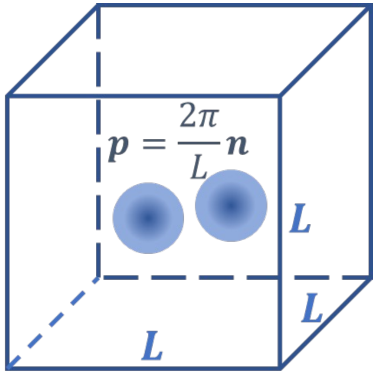
hadronic molecule

LHCb:2015yax

Energy level method

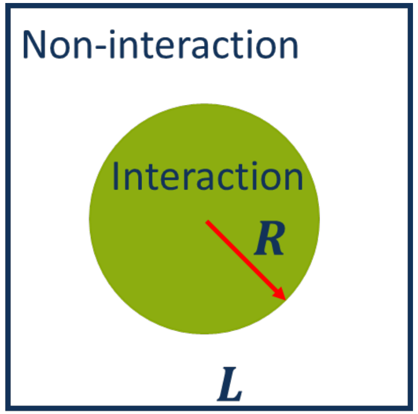
- Raw data: finite volume energy levels E^{FV} Luscher:1990ux
- Get observables: Lüscher's formula
  - ▶ $E^{FV} \sim \delta(E^{FV})$



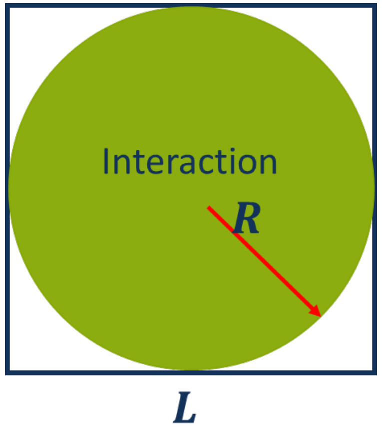


Periodic boundary condition

  - ▶ Valid condition



$$\frac{L}{2} \gg R$$



Long-range interaction and small box

https://indico.itp.ac.cn/event/199/
L. Meng et al, PRD109 (2024) 7, L071506;
Hansen et al. JHEP 06 (2024) 051;
Rishabh Bubna et al. JHEP 05 (2024) 168
Talks of Jia-Jun, Jing-Yi, Kang Yu

Potential method

东南大学 SOUTHEAST UNIVERSITY

- Raw data: Nambu-Bethe-Salpeter wave function (NBS WFs) Ishii:2006ec,Aoki:2009ji,Aoki:2012tk

AKA: HAL QCD method

Time-dependent

Time-independent

Nambu-Bethe-Salpeter wave function (raw data)

Derivative expansion

Potential → Observables

Used for NN,YN,DD^*, ...

See Yan Lyu's talk

- Time-independent HAL QCD (set $m=1$, 1D case as an example)

$$\int dr' V(r,r')\psi_{k_i}(r') = \left(\frac{d^2}{dr^2} + k_i^2\right)\psi_{k_i}(r)$$

- Time-dependent HAL QCD: correlation without ground state saturation Ishii:2012ssm

$$R(r,t) = \sum_n a_n \psi_{k_n}(r) e^{-(2\sqrt{m_N^2+k_n^2}-2m_N)t} \qquad \left(-\frac{\partial}{\partial t} + \frac{1}{4m_N}\frac{\partial^2}{\partial t^2}\right) R(r,t) = \left(\hat{H}_0 + \hat{V}\right) R(r,t)$$

- General problem: get potential $V(r,r')$ once $\{\Psi_{k_i}(r)\}$ or $\{R_i(r)\}$ are given

$$\boxed{\int dr'\, V(r,r') R^{(i)}(r') = K^{(i)}(r)}$$

$R^{(i)}(r)$ and $K^{(i)}(r)$ are known

To bind or not to bind

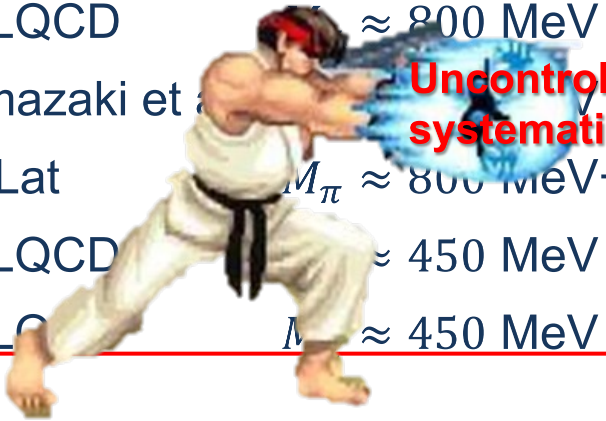
With (deeply) bound NN

1

2006	NPLQCD	First dynamical calculations
2011	NPLQCD	$M_\pi \approx 390$ MeV

2

2012	Yamazaki et al.	$M_\pi \approx 510$ MeV
2015	NPLQCD	$M_\pi \approx 800$ MeV
2015	Yamazaki et al.	
2015	CalLat	$M_\pi \approx 800$ MeV+P,D,F waves
2015	NPLQCD	$M_\pi \approx 450$ MeV
2020	NPLQCD	$M_\pi \approx 450$ MeV



Uncontrolled systematics

Without bound NN (or inconclusive)

2012	HAL QCD	$M_\pi \approx 710$ **MeV**
2012	HAL QCD	$M_\pi \approx 469 - 1171$ MeV

3

2019	"Mainz"	$M_\pi \approx 960$ MeV
2020	CoSMoN	$M_\pi \approx 714$ MeV
2021	NPLQCD	$M_\pi \approx 800$ MeV
2025	BaSc	$M_\pi \approx 714$ MeV

- However, we are observing a preponderance of evidence that the older methods with present statistics, are yielding qualitatively incorrect spectrum —
  I believe the old results are wrong (including those I was involved with)
  I believe the di-nucleon system unbinds at pion masses heavier than physical

Talk of A.Walker-Loud in lattice2023:https://indico.fnal.gov/event/57249/contributions/271301/

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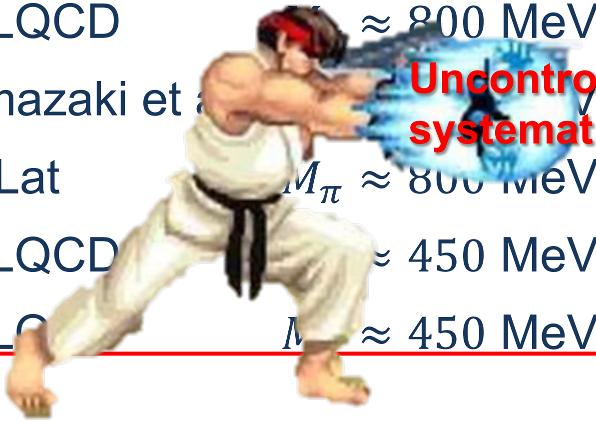

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CD $M_\pi \approx 800$ MeV

$M_\pi \approx 714$ MeV

3

Di-nucleons do not form bound states at heavy pion mass

BaSc Collaboration • John Bulava (Ruhr U., Bochum) Show All(19)

May 8, 2025

30 pages

e-Print: 2505.05547 [hep-lat]

Report number: LLNL-JRNL-2005660

View in: ADS Abstract Service

A benchmark of two methods

功夫两个字，一横一竖。
错了，躺下喽。
站着的才有资格讲话。

ound NN (or inconclusive)

CD	$M_\pi \approx 710$ **MeV**
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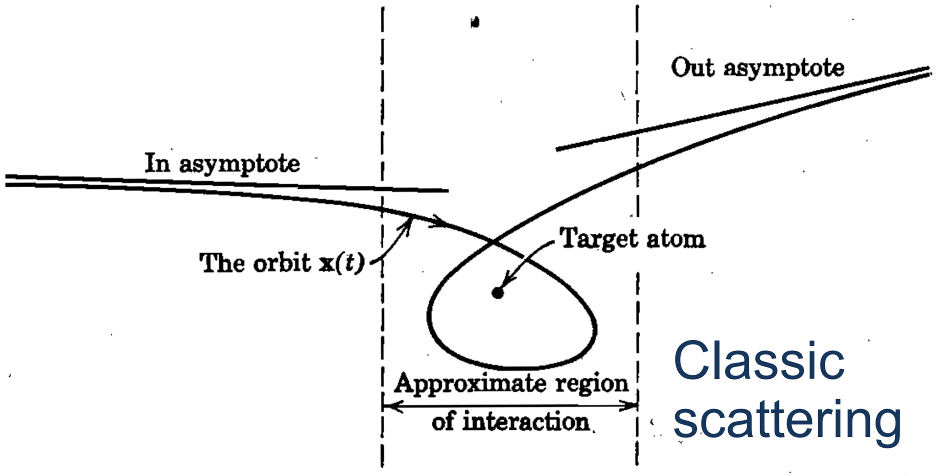
	$M_\pi \approx 960$ Me
N	$M_\pi \approx 714$ MeV
D	$M_\pi \approx 800$ MeV
	$M_\pi \approx 714$ MeV

Disclaimers:
- I am not the member of the HAL QCD group
- I will try my best to be fair

Basics of Potential Scattering

The only obseravble: cross section

- Scattering theory: interactions occur over a finite region
  - ► Inner region (interacting region): $V(r) \neq 0 \ (r < R)$
  - ► outer region (asymptotic region): $V(r) = 0 \ (r' > R)$
- Asymptotic states (trajectories): particles are free long before and after the collision
- The observable: cross section



Classic scattering

On-shell scattering amplitude Phase shift Asymptotic wave function	⟷	Cross section

- Non-observable
  - ► Non-asymptotic behavior of ψ
  - ► Off-shell T-matrix
  - ► Potential
  - ► Source term of femtoscopy

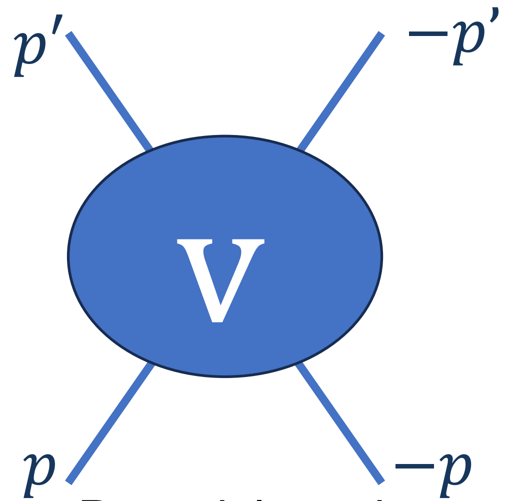
E. Epelbaum, et al, **Can the strong interactions between hadrons be determined using femtoscopy**?, arXiv:2504.08631

Local potential vs Nonlocal potential

- General potential in coordinate and momentum space: (1D as an example)

$$\mathcal{V}(p',p) = \langle p'|V|p\rangle,\ \langle p|\hat{V}|\psi\rangle = (2\pi)^{-1}\int dp'\mathcal{V}(p,p')\psi(p')$$

$$V(r',r) = \langle r'|\hat{V}|r\rangle,\ \langle r|\hat{V}|\psi\rangle = \int dr' V(r,r')\psi(r')$$



Potential matrix element in C.M.F

- Representation with derivative operator

$$\begin{aligned} V(r',r) &= (2\pi)^{-2}\int dp dp' \mathcal{V}(p',p) e^{i(p'r'-pr)} \\ &= (2\pi)^{-2}\int dk dq \bar{\mathcal{V}}(q,k) e^{i(ks+qR)} \\ &= (2\pi)^{-2}\int dk dq \bar{\mathcal{V}}(q,-i\partial_s) e^{i(ks+qR)} \\ &= \bar{V}(R,-i\partial_s)\delta(s) \end{aligned}$$

$$\begin{cases} R = \frac{r+r'}{2}, & q = p'-p, \\ s = r'-r, & k = \frac{p'+p}{2}. \end{cases}$$

exchanged

nonlocal

- No reason to reject nonlocal potentials: nonstatic, short-range, inelastic…

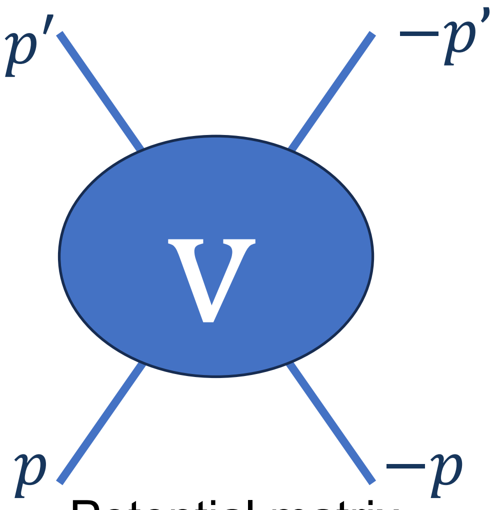
Tamagaki, R., & Watari, W. (1967). Potential model approach. *Prog. Theor. Phys. Suppl.*, *39*, 23–90

Local potential vs Nonlocal potential

- General potential in coordinate and momentum space: (1D as an example)

$$\mathcal{V}(p',p)=\langle p'|V|p\rangle,\ \langle p|\hat{V}|\psi\rangle=(2\pi)^{-1}\int dp'\mathcal{V}(p,p')\psi(p')$$

$$V(r',r)=\langle r'|\hat{V}|r\rangle,\ \langle r|\hat{V}|\psi\rangle=\int dr' V(r,r')\psi(r')$$



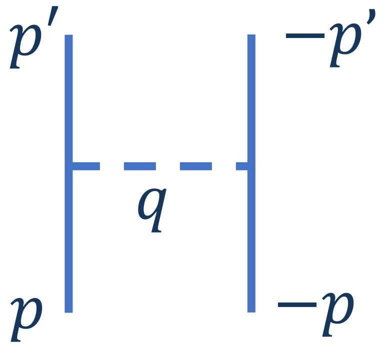
Potential matrix element in C.M.F

- Representation with derivative operator (local limit)

$$\begin{aligned} V(r',r) &= (2\pi)^{-2}\int dkdq\mathcal{V}(p',p)e^{i(ks+qR)} \\ &= (2\pi)^{-2}\int dkdq\bar{\mathcal{V}}(q)e^{i(ks+qR)} \\ &= \bar{V}(R)\delta(s) \\ &= \bar{V}(\frac{r+r'}{2})\delta(r'-r) \end{aligned}$$

$$\begin{cases} R=\frac{r+r'}{2}, & q=p'-p, \\ s=r'-r, & k=\frac{p'+p}{2}. \end{cases}$$

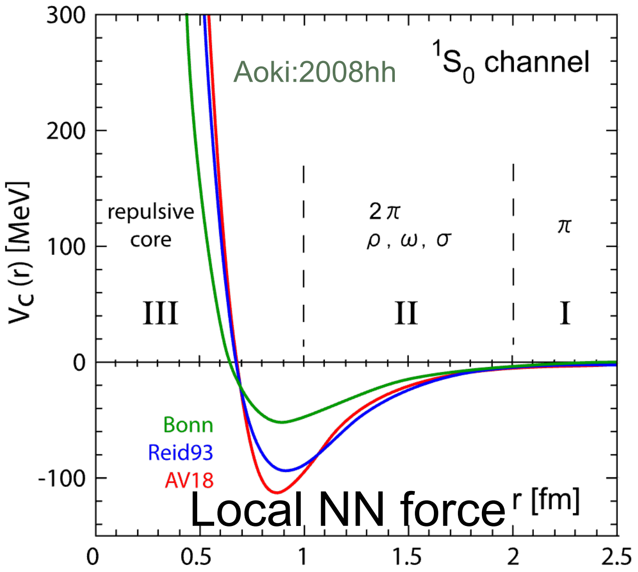
$$\langle r|\hat{V}|\psi\rangle=\int dr' V(r,r')\psi(r')=\bar{V}(r)\psi(r)$$



Static boson-exchange

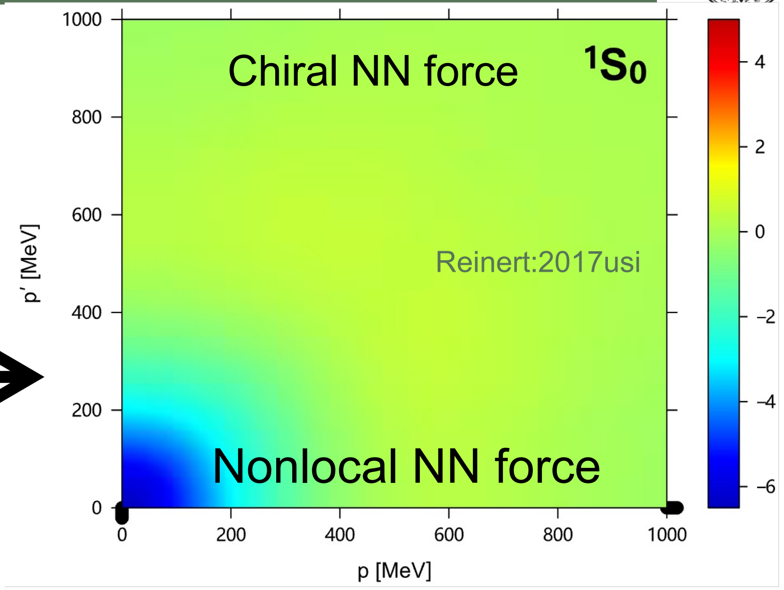
- No reason to reject nonlocal potentials: nonstatic, short-range, inelastic…

Tamagaki, R., & Watari, W. (1967). Potential model approach. *Prog. Theor. Phys. Suppl.*, *39*, 23–90



Phase shift equivalent

Unitary transformation



- Potential is not observable: cannot be determined uniquely by scattering experiments
  - Eg: high precision nuclear forces
- Observable-equivalent potentials are related by unitary trans. (UT) or field redefinition

$$H|\psi\rangle = E|\psi\rangle \Rightarrow UHU^{\dagger}U|\psi\rangle = EU|\psi\rangle \Rightarrow \tilde{H}|\tilde{\psi}\rangle = EU|\tilde{\psi}\rangle$$

  - Potential, Non-asymptotic ψ and off-shell T −matrix and are changed consistently
- UT can relate local potentials to nonlocal potentials

Ekstein:1960xkd

Query 1: potential is not observable

Query 1: Given the potential is not observable, is HAL QCD method still reasonable?

- In principle one may choose any composite operators with the same quantum numbers as the hadron to define the BS wave function
- Different operators give different BS wave functions and different hadron potentials
  - ► They are related by UT
  - ► We anticipate they lead to the same observables such as the δ and E_b

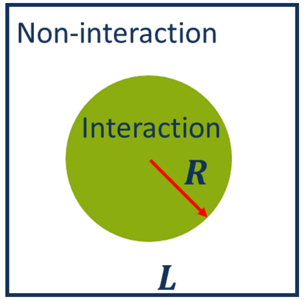
Aoki, S, et al, *PTEP*, *2012*, 01A105; S. Aoki's talk @Lattice 2019

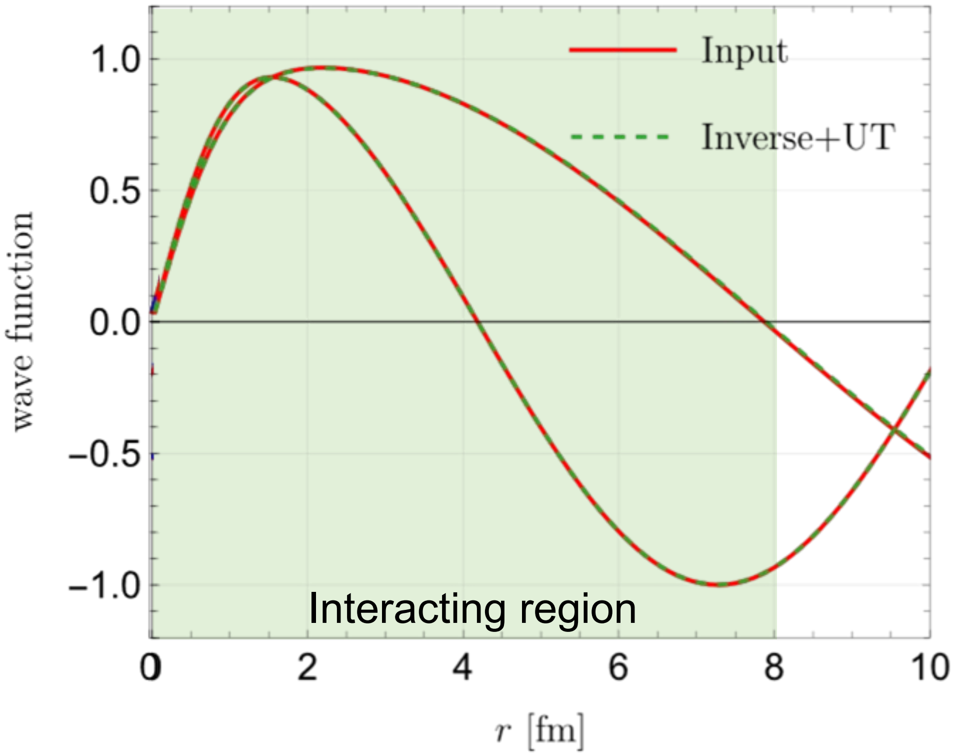
$$
\begin{tikzcd}
\boxed{\mathcal{O}_1} \arrow[r] & \boxed{\{\psi_1\}} \arrow[r] & \boxed{V_1} \arrow[dr] & \\
 & & & \boxed{\delta} \\
\boxed{\mathcal{O}_2} \arrow[r] & \boxed{\{\psi_2\}} \arrow[r] & \boxed{V_2} \arrow[ur] &
\end{tikzcd}
$$

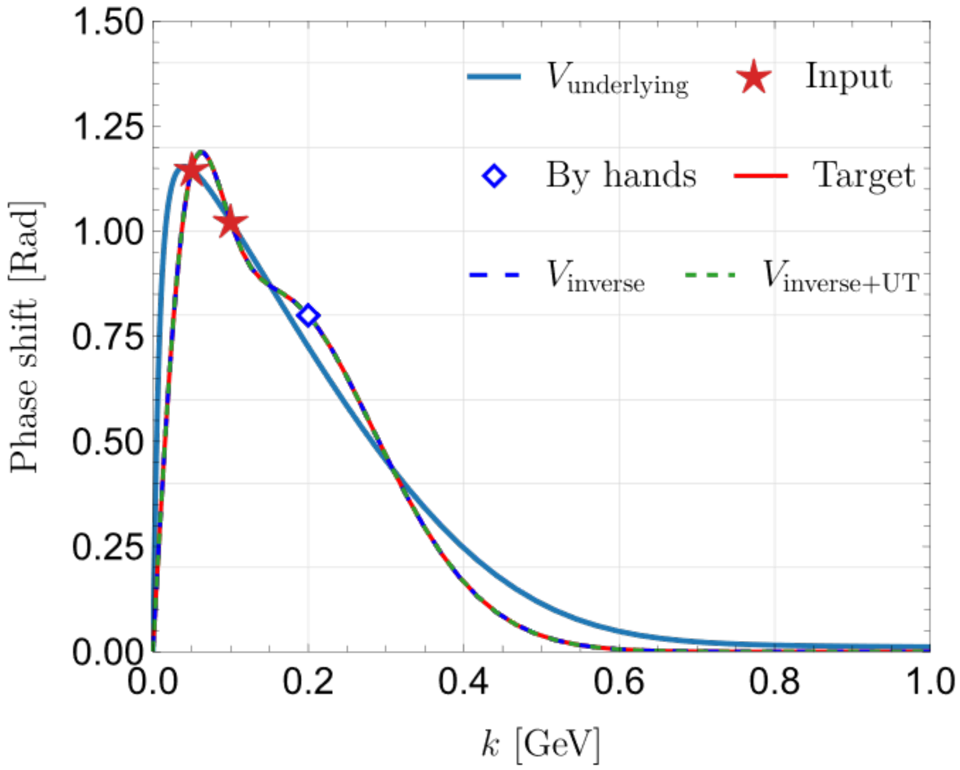
Query 2: Additional information from WFs of interacting region?

- Asymptotic region WF: several $\delta(k_i)$, similar to Luscher
- Interacting region WFs ⇒ Potential ⇒ $\delta(k)$
  - ▶ scheme-dependent

Yamazaki, T., & Kuramashi, Y. PRD*96*(2017), 114511.
And subsequent reply and comment
PRD98(2018), 038501; *PRD98*(2018), 0385012







- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

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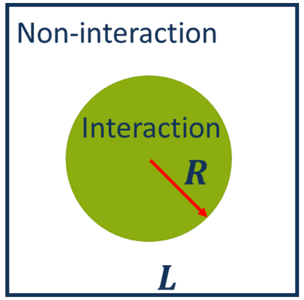

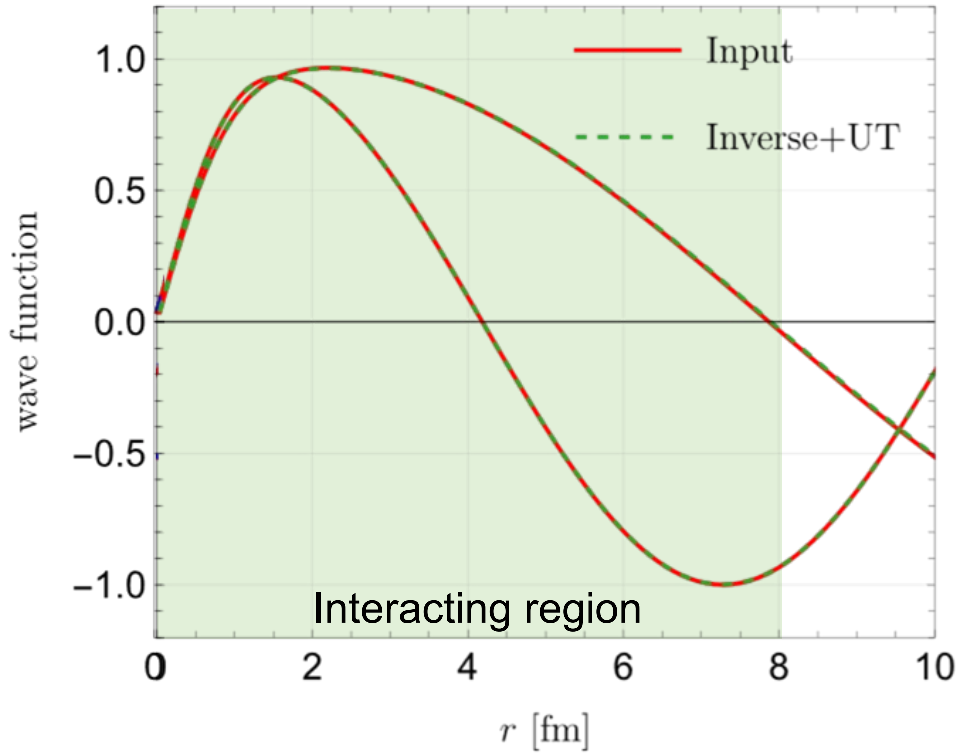

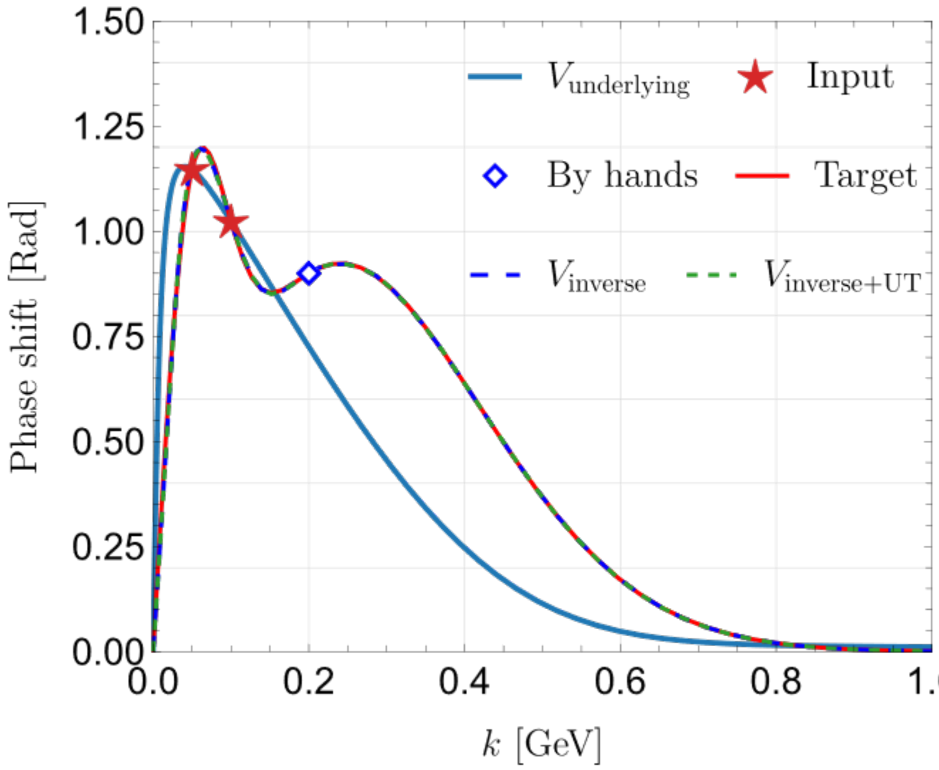

- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

Query 2: Additional information from WFs of interacting region?

- Asymptotic region WF: several $\delta(k_i)$, similar to Luscher
- Interacting region WFs $\Rightarrow$ Potential $\Rightarrow$ $\delta(k)$
  - ▶ scheme-dependent

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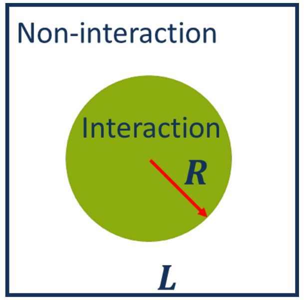
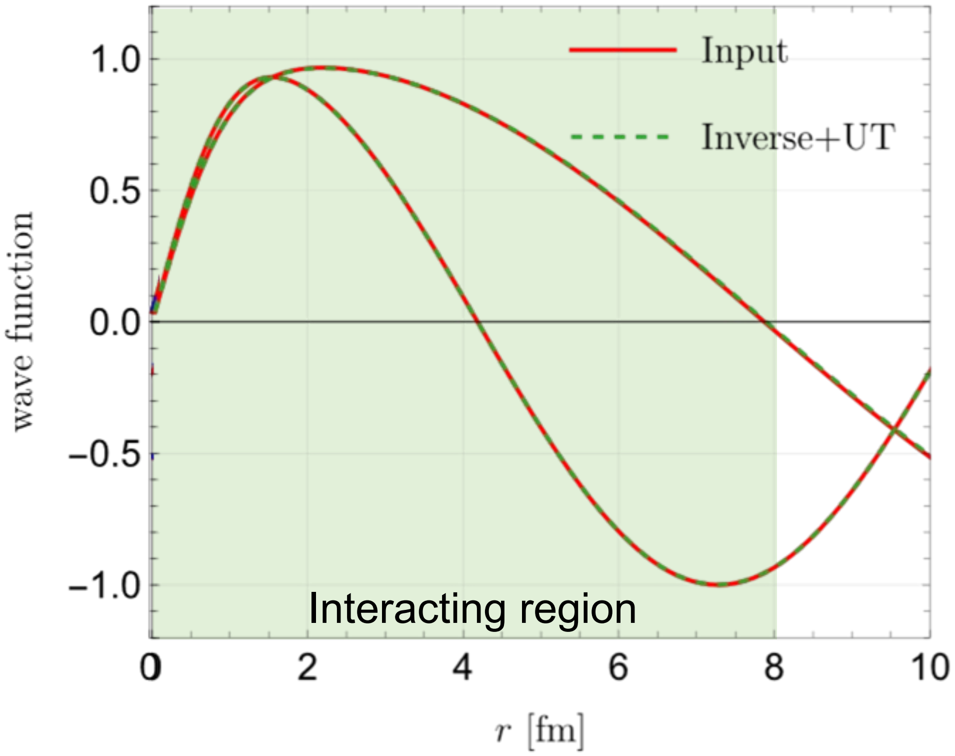
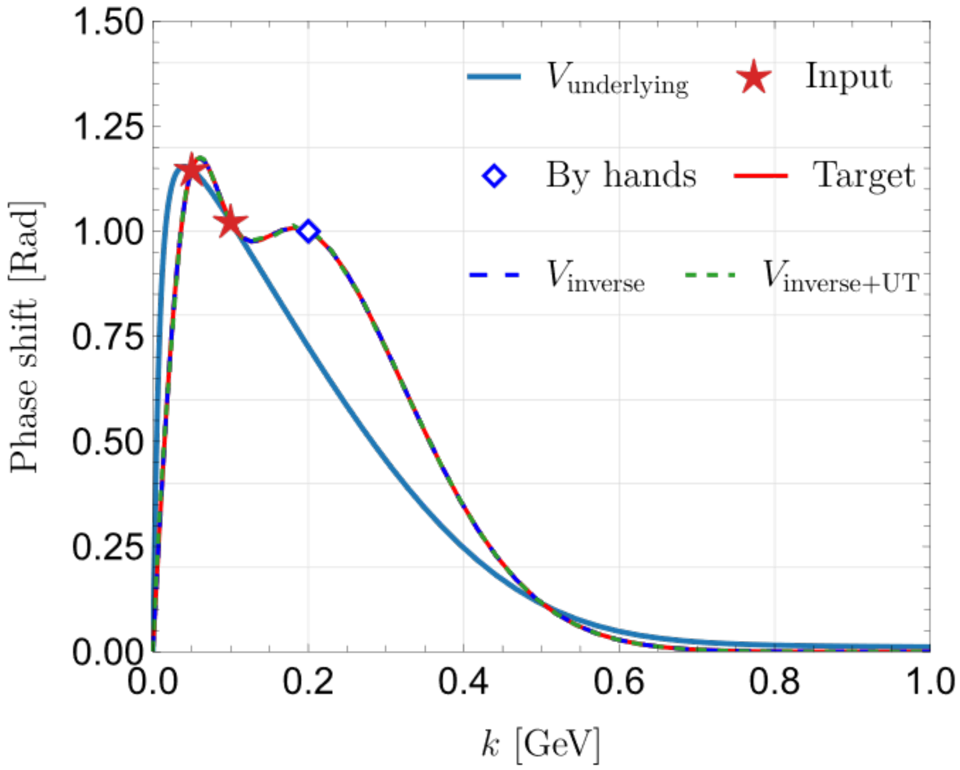






- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

Derivative expansion

Derivative expansion

- Derivative expansion

$$V(r,r') = V_0(r)\delta(r-r') + V_1(r)\delta(r-r')\frac{d^2}{dr'^2} + V_2(r)\delta(r-r')\frac{d^4}{dr'^4} + \dots$$

- LO

$$V_0(r)R^{(1)}(\vec{r}) = K^{(1)}(\vec{r}) \Rightarrow V_0(r) = \frac{K^{(1)}(\vec{r})}{R^{(1)}(\vec{r})}$$

- NLO

$$\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2}R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2}R^{(2)}(r) \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(2)}(r) \end{pmatrix}$$

Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

- Advantages:
  - ► Efficient for long-range potential: pionic dynamics, chiral symmetry

    E.g two-pion tail of DD^* interaction Y. Lyu *et al.*, Phys. Rev. Lett. **131**, 161901 (2023).
- Several possible problems beyond the LO

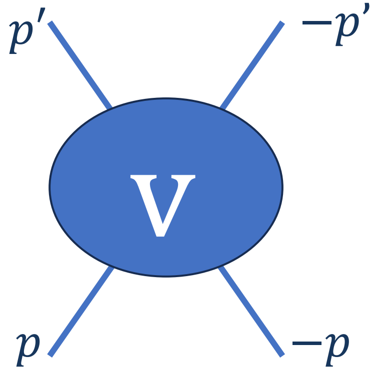
Hermitian

- Derivative expansion of HAL QCD group

$$V(r,r') = V_0(r)\delta(r-r') + V_1(r)\delta(r-r')\frac{d^2}{dr'^2} + V_2(r)\delta(r-r')\frac{d^4}{dr'^4} + \ldots$$

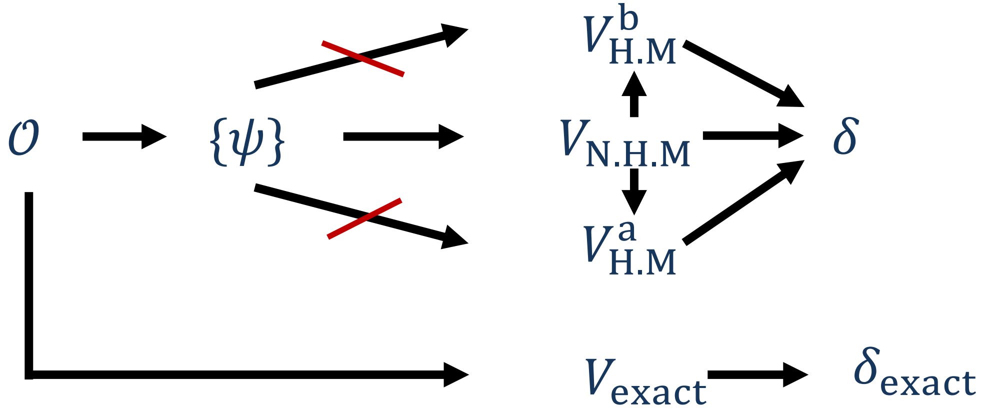
  - ▶ Non-Hermitian: all the differential operators acting to the right
  - ▶ $\mathrm{d}^2 R(r)/dr^2$ can be calculated directly ⇒ algebra eq.
  - ▶ Equivalent expansion in momentum space

$$V(p',p) = \tilde{V}(q,p) = \tilde{V}_0(q) + \tilde{V}_1(q)p + \frac{1}{2}\tilde{V}_2(q)p^2 + \ldots$$



$$q = p' - p,$$
$$k = \frac{p'+p}{2}.$$

- Hermitization: scheme-dependent, inconsistent with the WFs

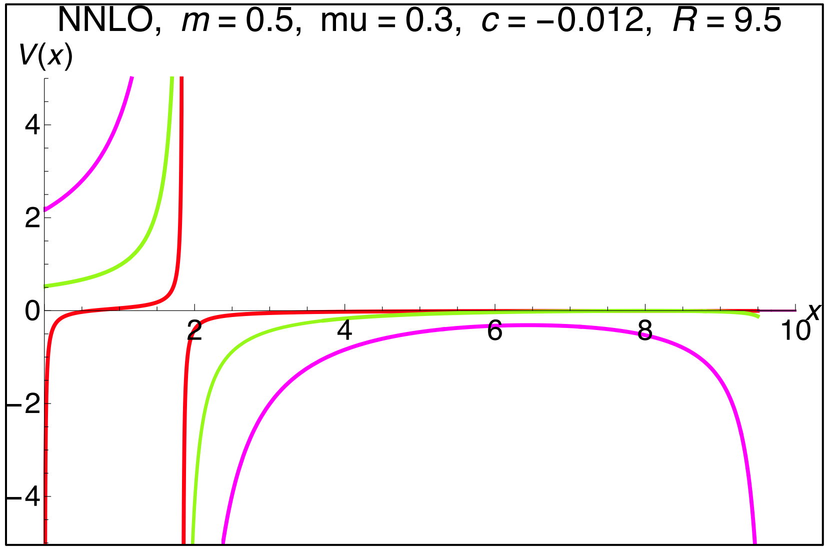


Singular potential

- Singular potential
- NLO derivative expansion

$$\boxed{\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2}R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2}R^{(2)}(r) \end{pmatrix}} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(2)}(r) \end{pmatrix}$$

- singular at the zero of det of the coefficient matrix
- An example from toy model
  - ▶ In simulation, it is challenging to handle the singularity
  - ▶ Wave functions are obtained at discrete point.



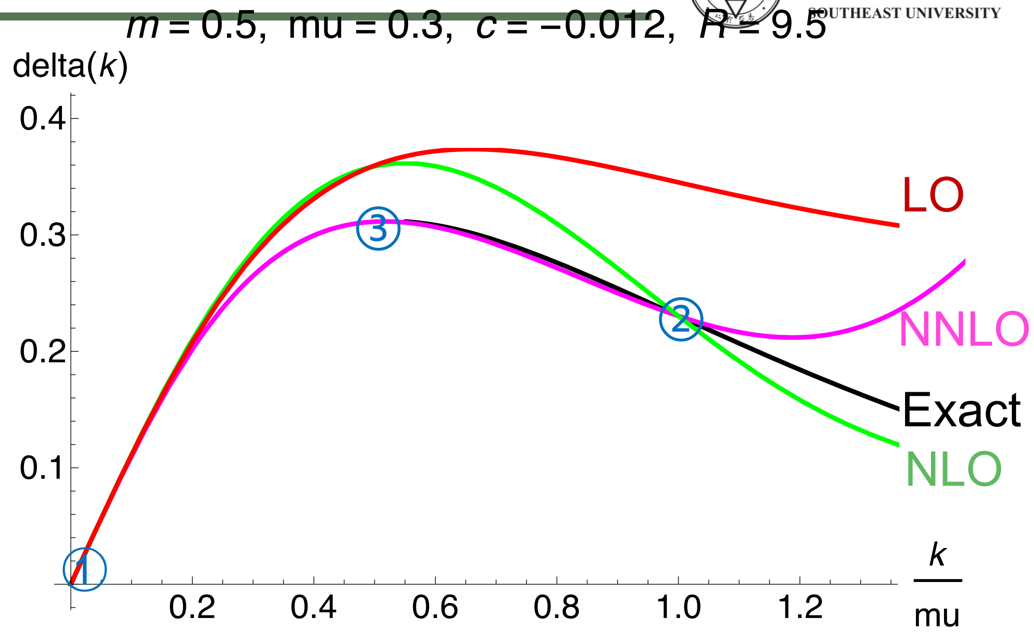
Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

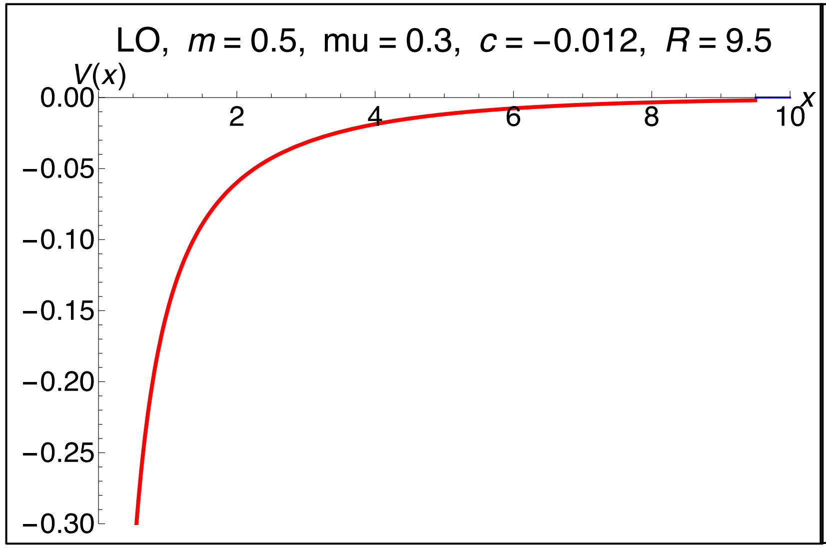
Convergence

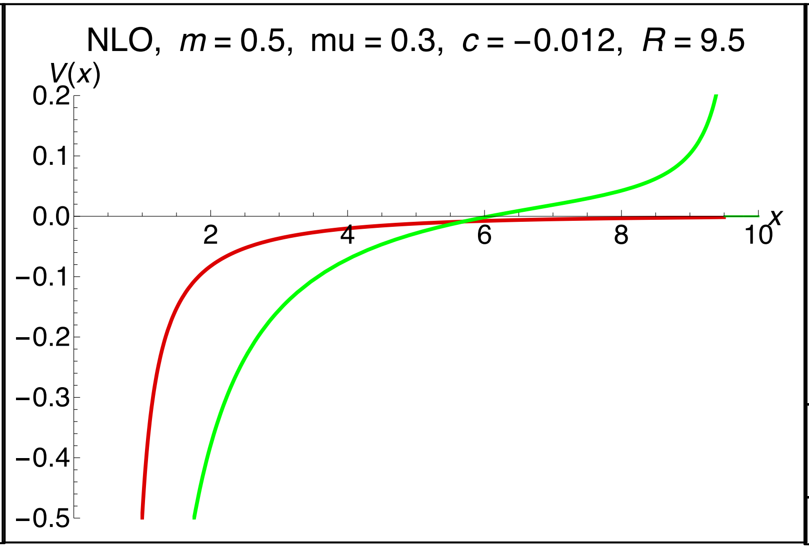
- Derivative exansion of the separable interaciton
- Three wave functions ①②③ as input in order
- The 2-body phase shift is improved with order

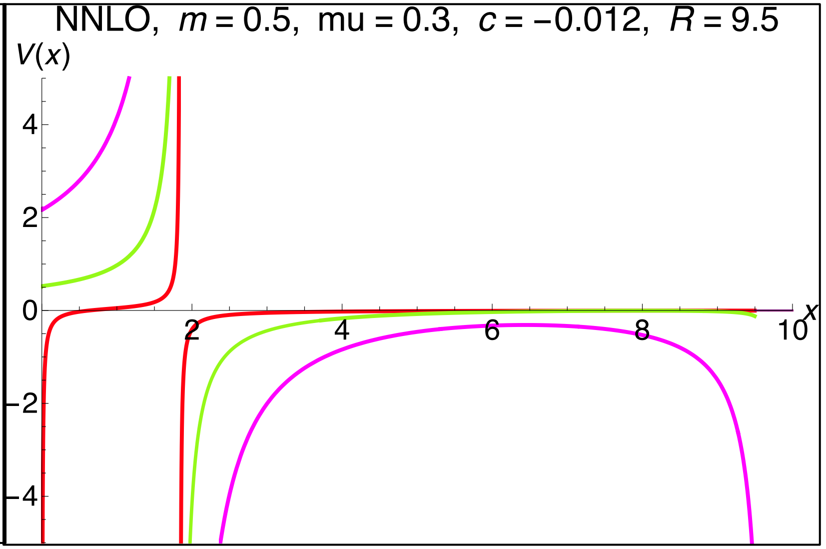
However,

- The potential shape does not convergent
- The 3-body observables?









Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

EST expansion

Separable representation

- The problem: $V|R^{(i)}\rangle = |K^{(i)}\rangle$

$$K^{(i)} = (E - \hat{H}_0)R^{(i)}$$

Ernst, D. J., Shakin, C. M., & Thaler, R. M. (1973). *Phys. Rev. C*, *8*, 46–52.

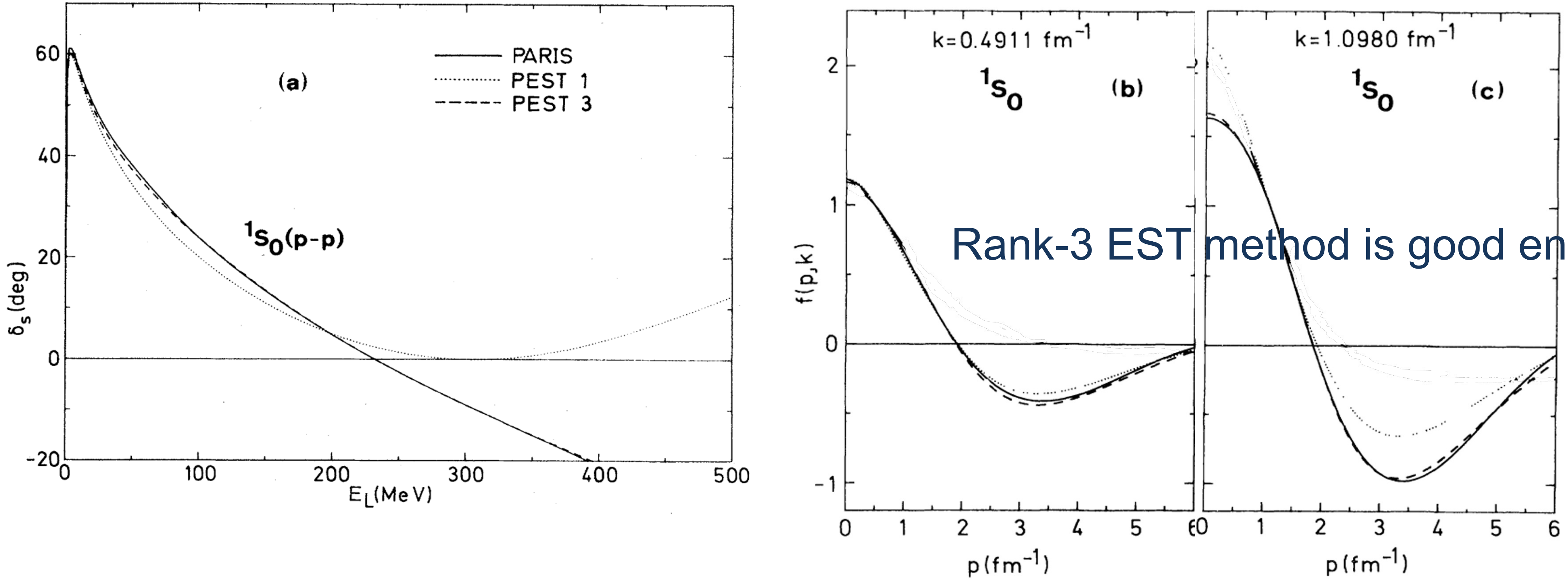
- Ernst-Shakin-Thaler (EST) method:

$$\boxed{\hat{V} = \sum_{mn} |K^{(m)}\rangle \Lambda_{mn} \langle K^{(n)}|, \quad \sum_{n} \Lambda_{mn} \langle K^{(n)}|R^{(i)}\rangle = \delta_{mi}}$$

- ▶ Hermitian potential
- ▶ No singularity
- ▶ On-shell and off-shell equivalent: three-body observable convergent

Separable representation

- Application: on-shell and off-shell equivalent separable potentials of Paris potentials
  - ► Purpose: to solve the 3-body Fadeev function
  - ► Convergent 3-body observables
  - ► High efficiency



Rank-3 EST method is good enough

Haidenbauer, J., & Plessas, W. (1984). *Phys. Rev. C*, *30*, 1822–1839.

Two underlying potentials

- Separable potential Aoki:2021ahj

$$V(\boldsymbol{r},\boldsymbol{r}') = \omega \frac{e^{-\mu r}}{r}\frac{e^{-\mu r'}}{r'}$$

- LO chiral nuclear force Reinert:2017usi

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}}, \quad V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

  - Separable contact interaction + local one-pion exchange interaction
- For simplicity: S-wave and $^1S_0$ NN interaction

- Solve the time-(in)dependent Schrodinger equation to get wave functions
- Time-independent method
  - Choose $\{\psi_{k_i}\}$ as inputs
- Time-dependent method
  - Initial wave functions

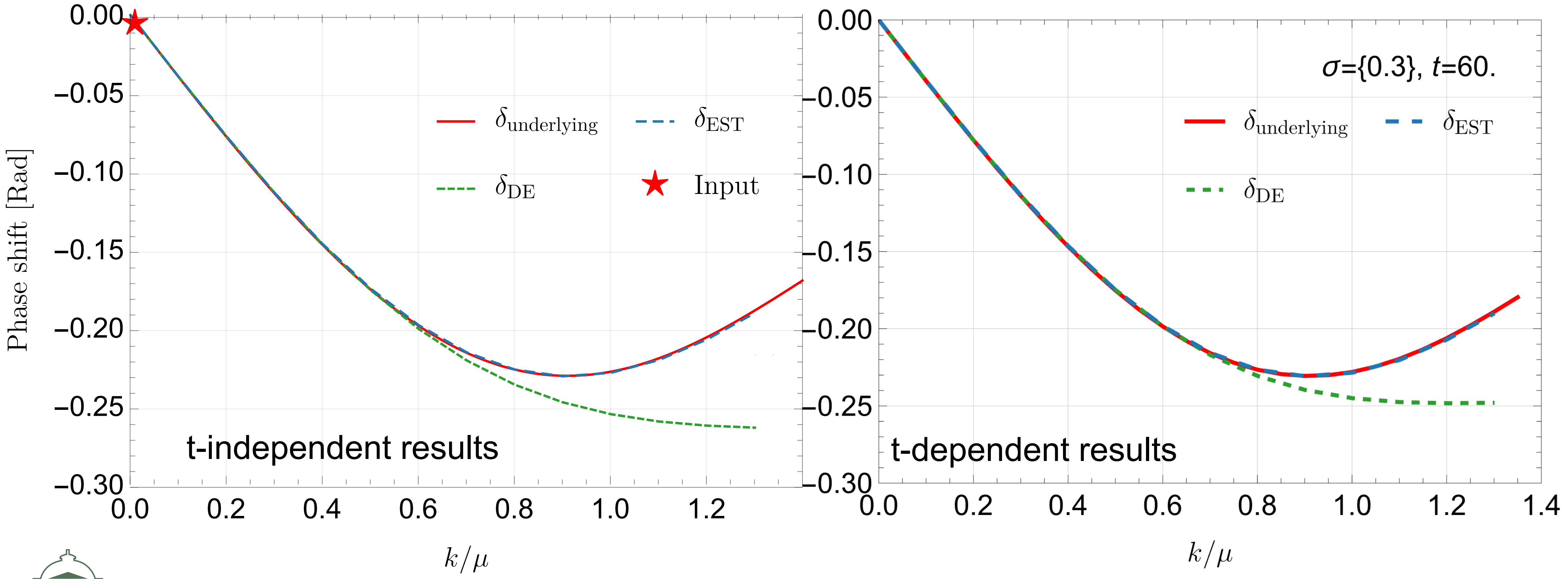
$$\tilde{R}(t=0,x) = \frac{\sigma^2 e^{-\sigma x}}{4\pi}$$

  - Evaluate t=60
  - Two $\sigma = \{0.3, 0.6\}$ as two inputs

Separatable interaction

- The EST methods give the accurate potential in LO
- The DE method is convergent

$$V(\boldsymbol{r},\boldsymbol{r}') = \omega \frac{e^{-\mu r}}{r}\frac{e^{-\mu r'}}{r'}$$



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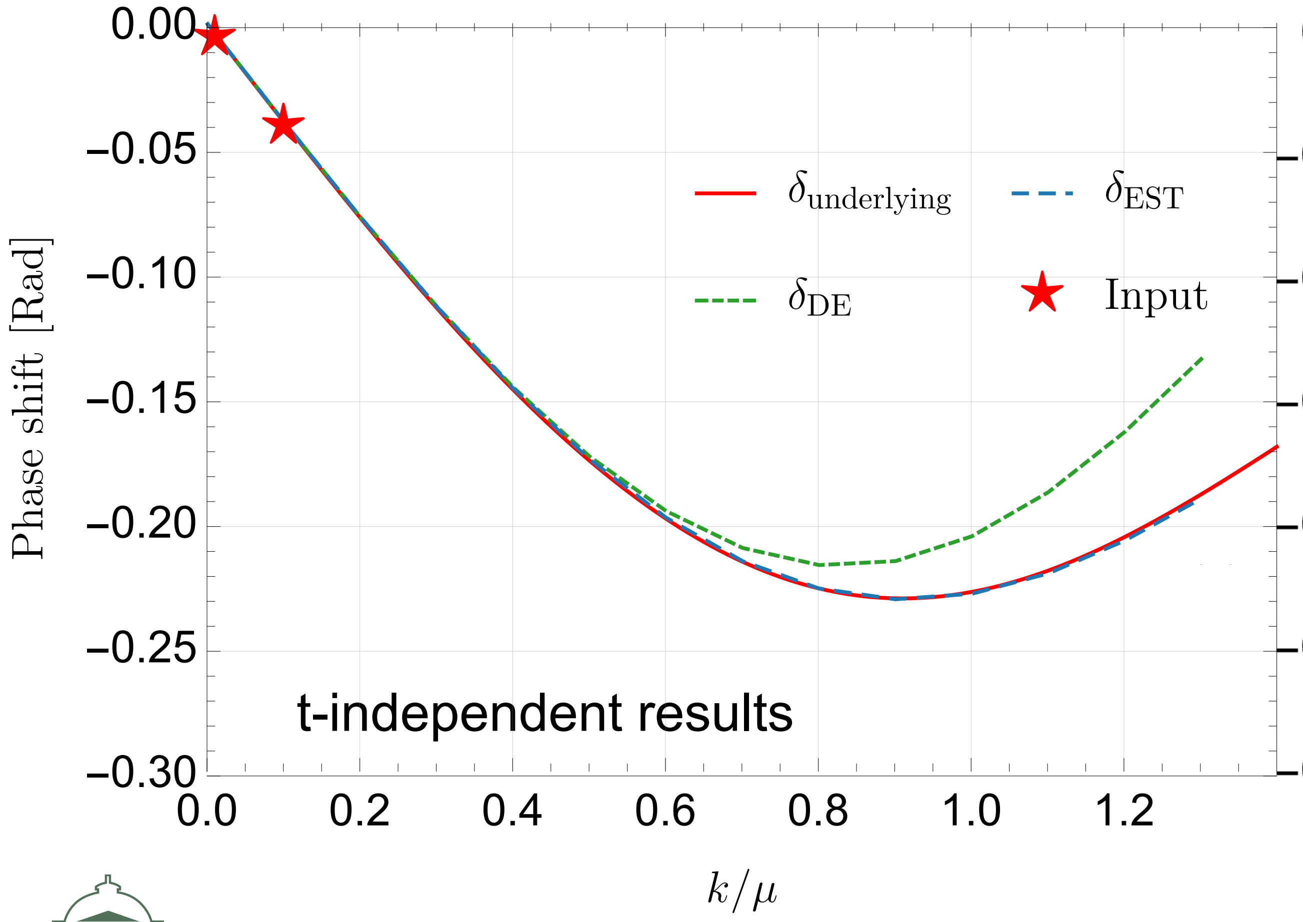
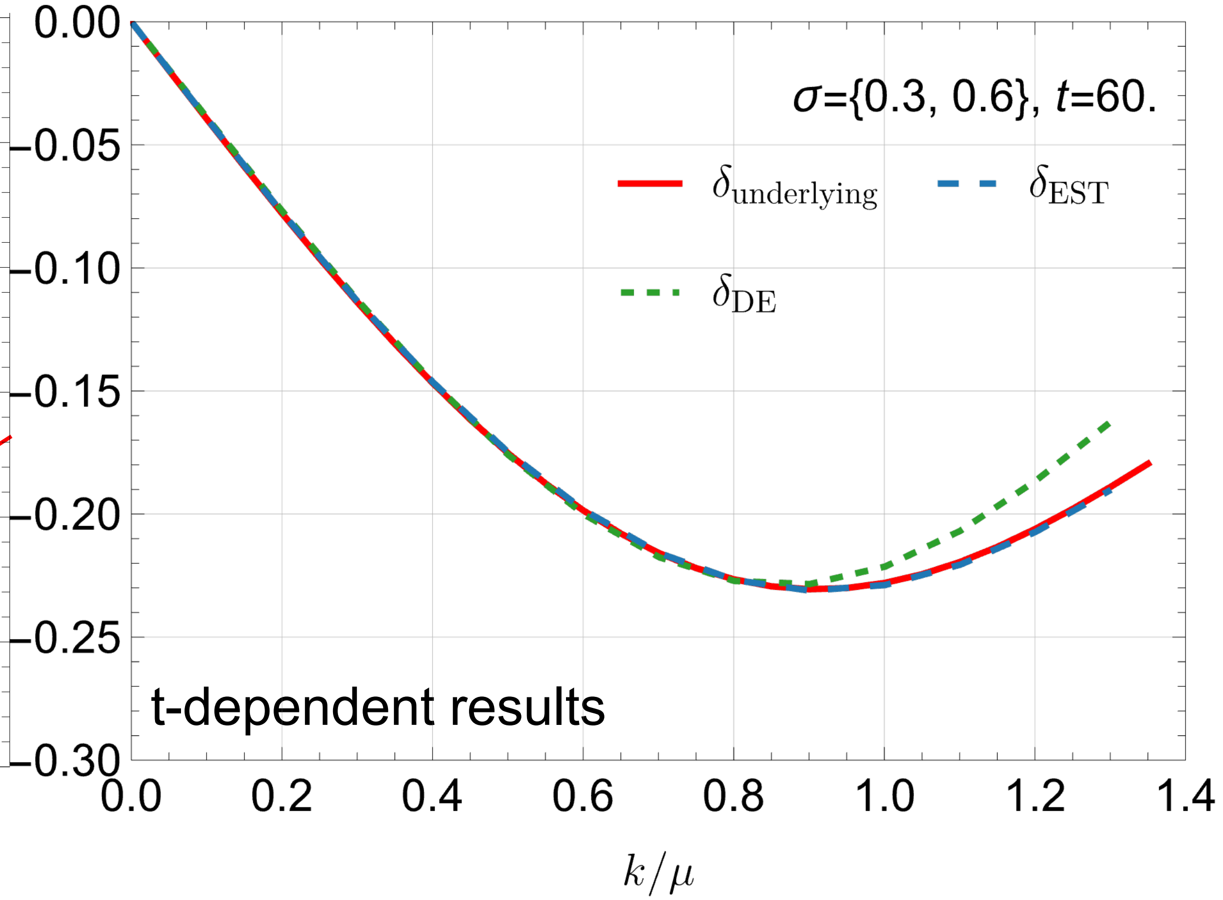


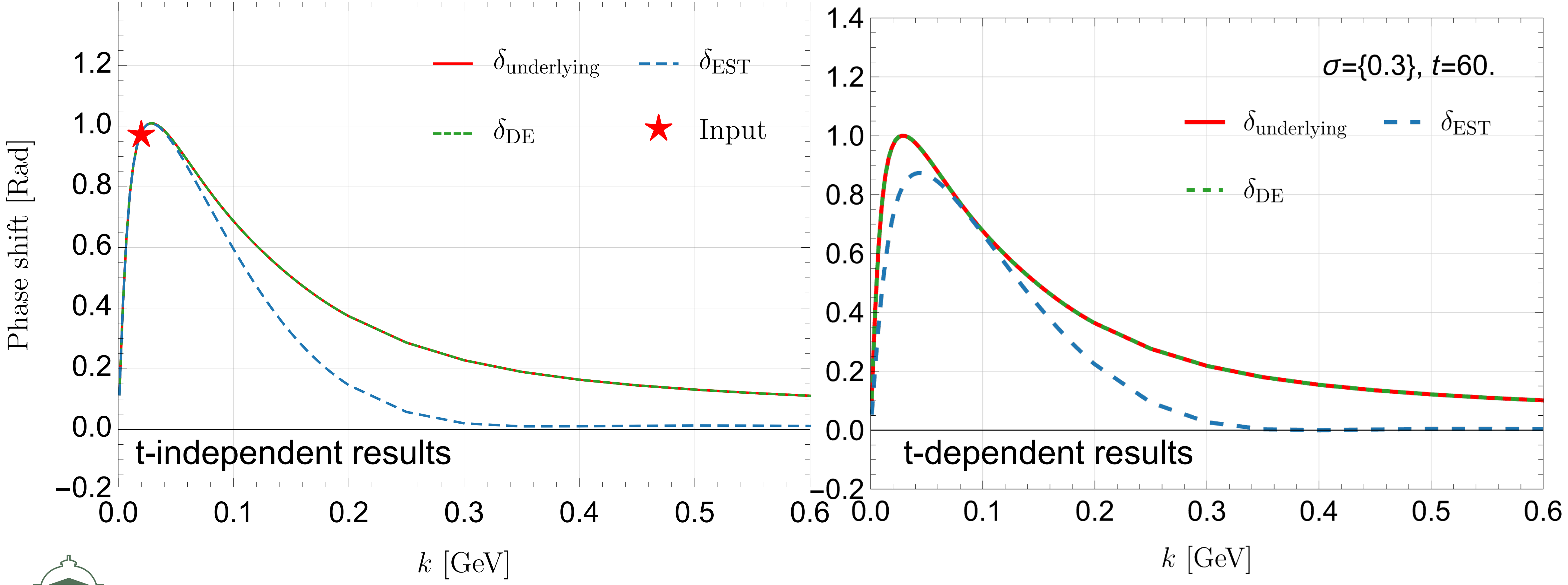


Local interaction

- The DE method gives the accurate results at LO
- Convergent EST results, not bad performance

$$\cancel{V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},}$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



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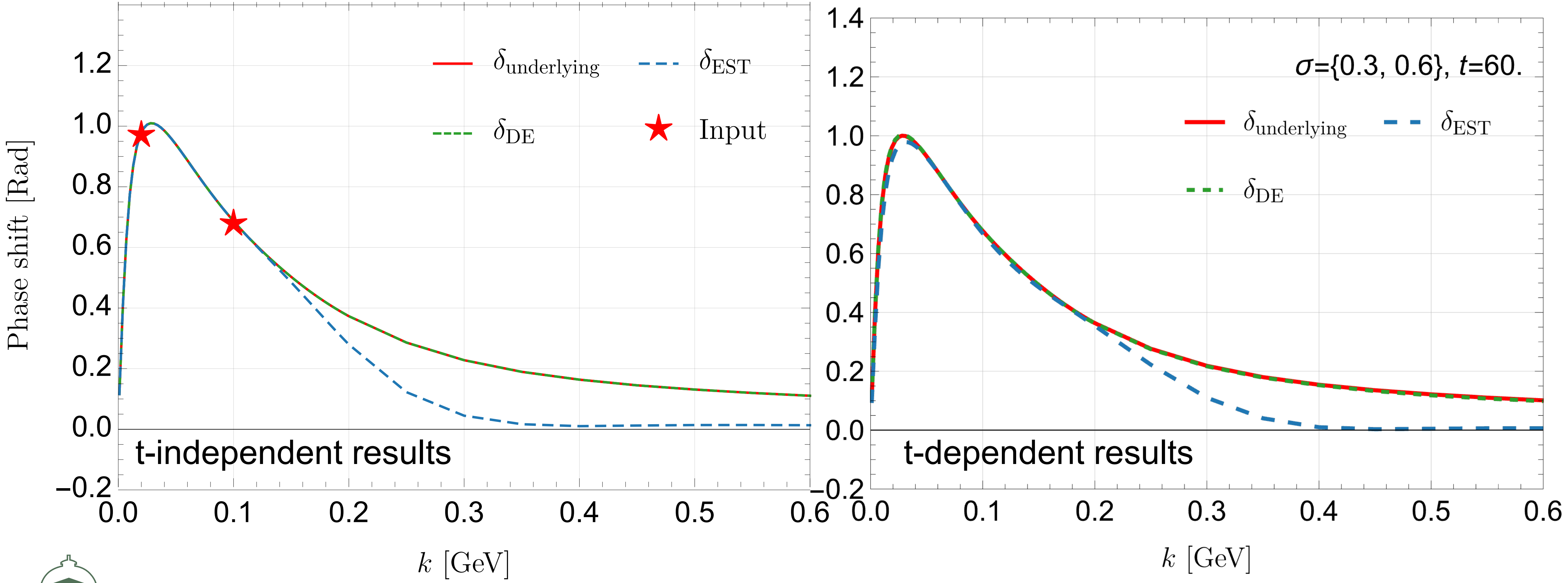
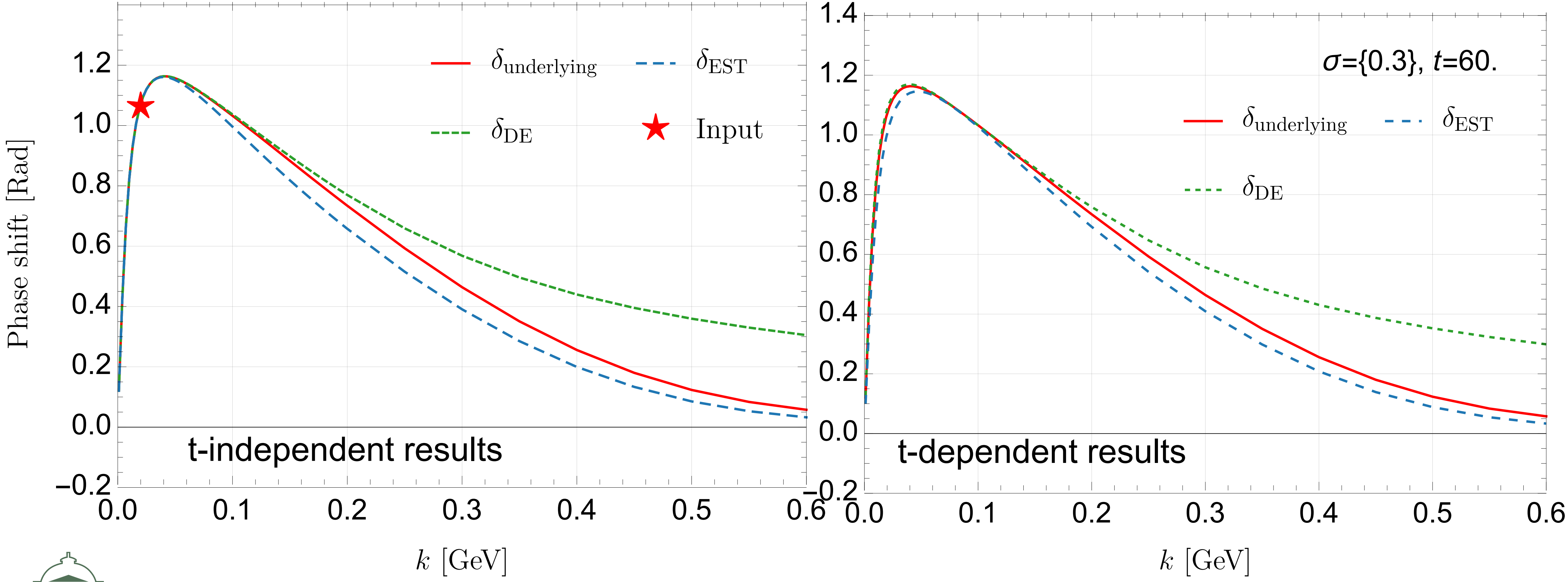


Physical interaction

- Including both separatable part and local part
- The performance of EST method is better
- In t-dependent methods, singular potential

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



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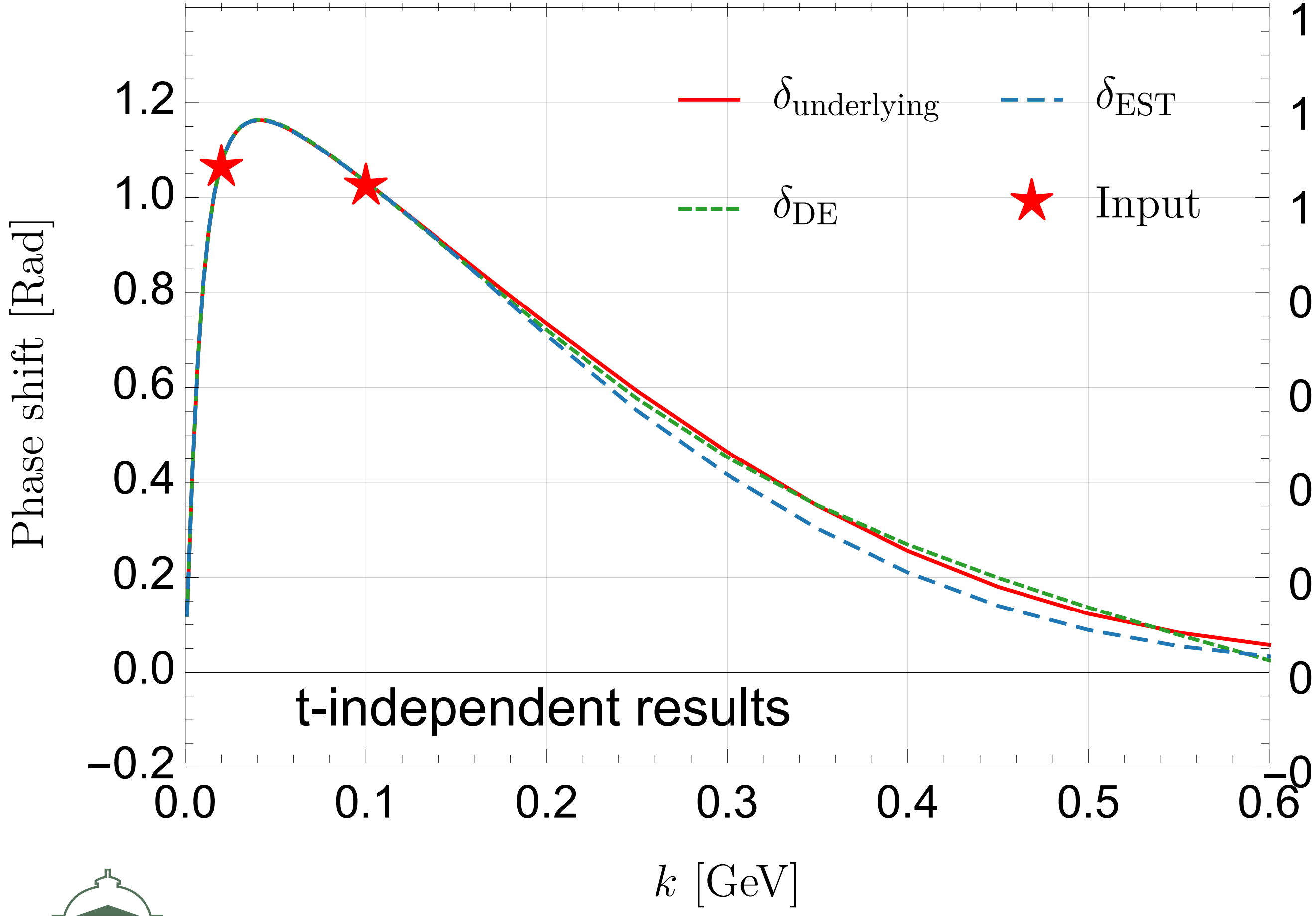
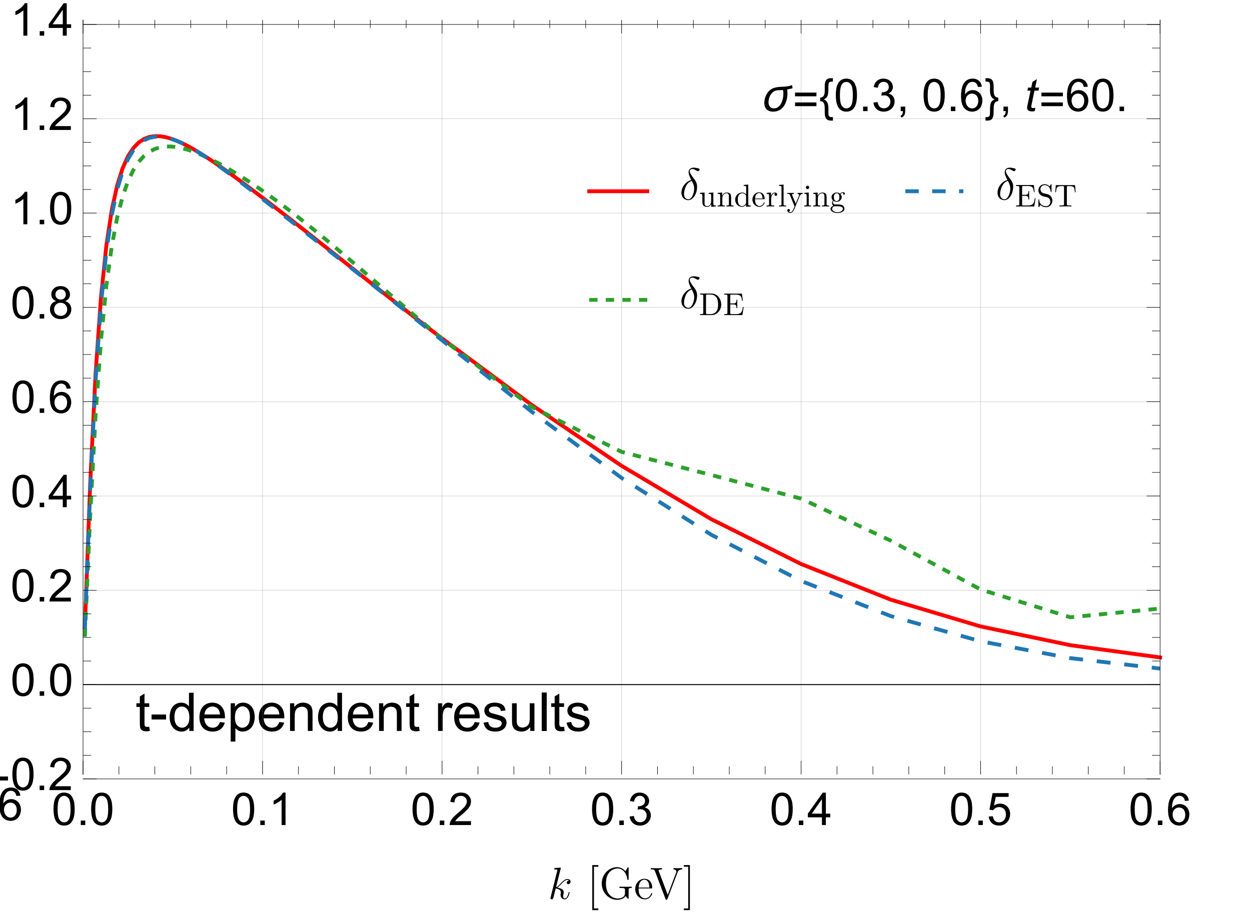




Combine local and seperable parameterizaiton

- Chiral nuclear force
  - Long-range: pionic exchange interaction + Short-range: separable interaction

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

- Mixed strategy: $\Psi = \Psi_{\rm long} + \Psi_{\rm resi}$
  - Long-range part: local potential
  - Short-range part: EST method

Summary and outlook

NBS WFs → Potential → Observables

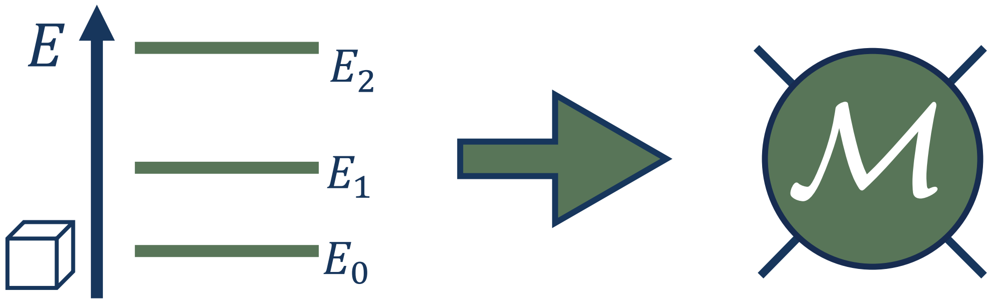
- Re-emphasize some concepts
  - ► Potential and non-asymptotic wave function are not observable
  - ► Interpolating operators ⇒ potential
  - ► Cannot rule out the nonlocal potential either in principle or phenomenologically
  - ► Different V parameterizations to get uncertainties
- Derivative expansion beyond LO:
  - ► Highly efficient for long-range interaction
  - ► Beyond LO: singular, non-Hermitian, convergent of potential and 3-body observable
- EST expansion:
  - ► Hermitian, no-singular, convergent potential and 3-body observable
  - ► Efficient and robust; EST perform better for chiral nuclear force,
  - ► Systemic uncertainty
- Outlook
  - ► Combing EST and DE: short-range: EST, long-range: DE
  - ► Check: 3-body observable, peripheral phase shift

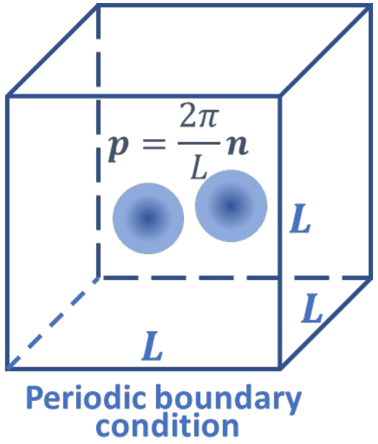
Thanks for your attention!

BACKUP

Energy level method

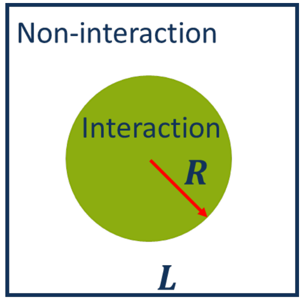
- Raw data: finite volume energy levels E^{FV} Luscher:1990ux
- Get observables: Lüscher's formula
  - ▶ $E^{FV} \sim \delta(E^{FV})$



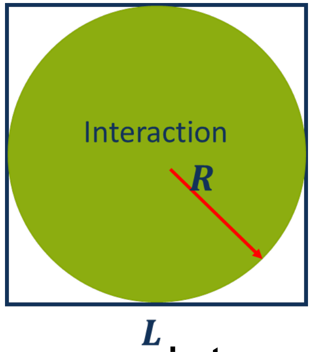


- Quantization condition
  - ▶ Boundary condition: $\psi(r) = \psi(r + L)$
  - ▶ Asymptotic behavior: $\psi_k(r) \sim \frac{e^{i\delta(k)}\sin[kr+\delta(k)]}{kr}$

$$(\nabla^2+k^2)\psi(r) = 0 \text{ for } r > R$$



$$\frac{L}{2} \gg R$$



Long-range interaction and small box

L. Meng et al, PRD109 (2024) 7, L071506;
Hansen et al. JHEP 06 (2024) 051;
Rishabh Bubna et al. JHEP 05 (2024) 168
Kang Yu's talk

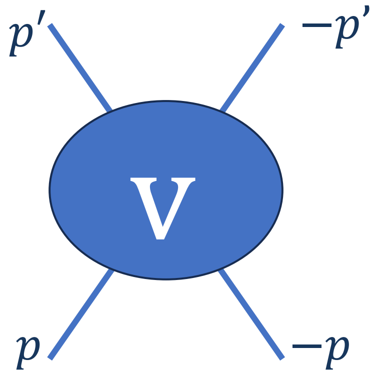
Hermitian

- Derivative expansion of HAL QCD group

$$V(r,r') = V_0(r)\delta(r-r') + V_1(r)\delta(r-r')\frac{d^2}{dr'^2} + V_2(r)\delta(r-r')\frac{d^4}{dr'^4} + \dots$$

  - ► Non-Hermitian: all the differential operators acting to the right
  - ► $\mathrm{d}^2R(r)/dr^2$ can be calculated directly ⇒ algebra eq.
  - ► Equivalent expansion in momentum space

$$V(p',p) = \tilde{V}(q,p) = \tilde{V}_0(q) + \tilde{V}_1(q)p + \frac{1}{2}\tilde{V}_2(q)p^2 + \dots$$



$$q = p' - p,$$
$$k = \frac{p'+p}{2}.$$

- Optional hermitian Derivative expansion:

$$V(p',p) = \bar{V}(q,k) = \bar{V}_0(q) + \bar{V}_1(q)k + \frac{1}{2}\bar{V}_2(q)k^2 + \dots$$

  - ► Have to solve differenttial eq.

$$V_1(r)\frac{\overrightarrow{d}^2}{dr} + \frac{\overleftarrow{d}^2}{dr}V_1(r) = V_1''(r) + 2V_1'(r)\frac{\overrightarrow{d}}{dr} + 2V_1(r)\frac{\overrightarrow{d}^2}{dr^2}$$

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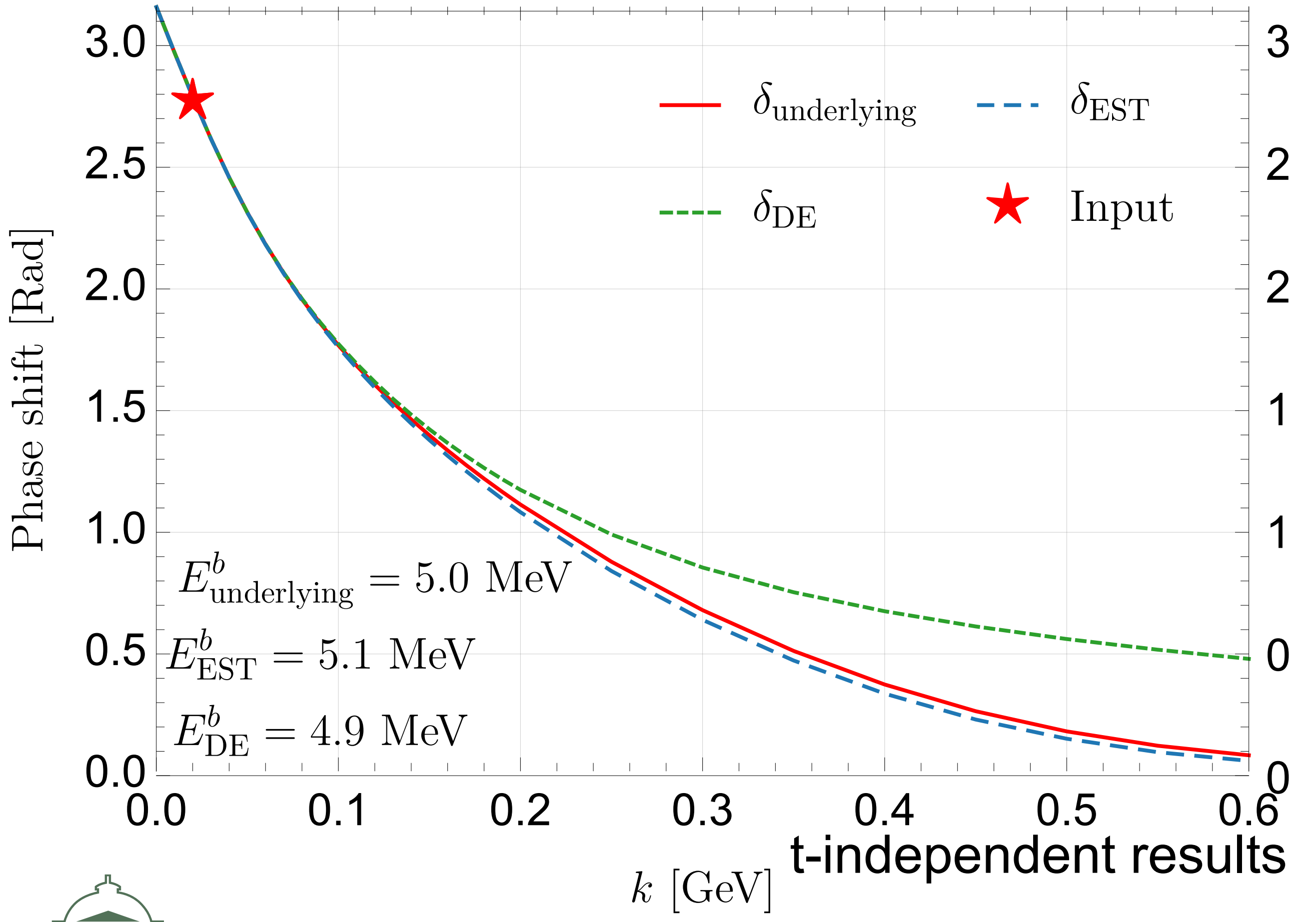
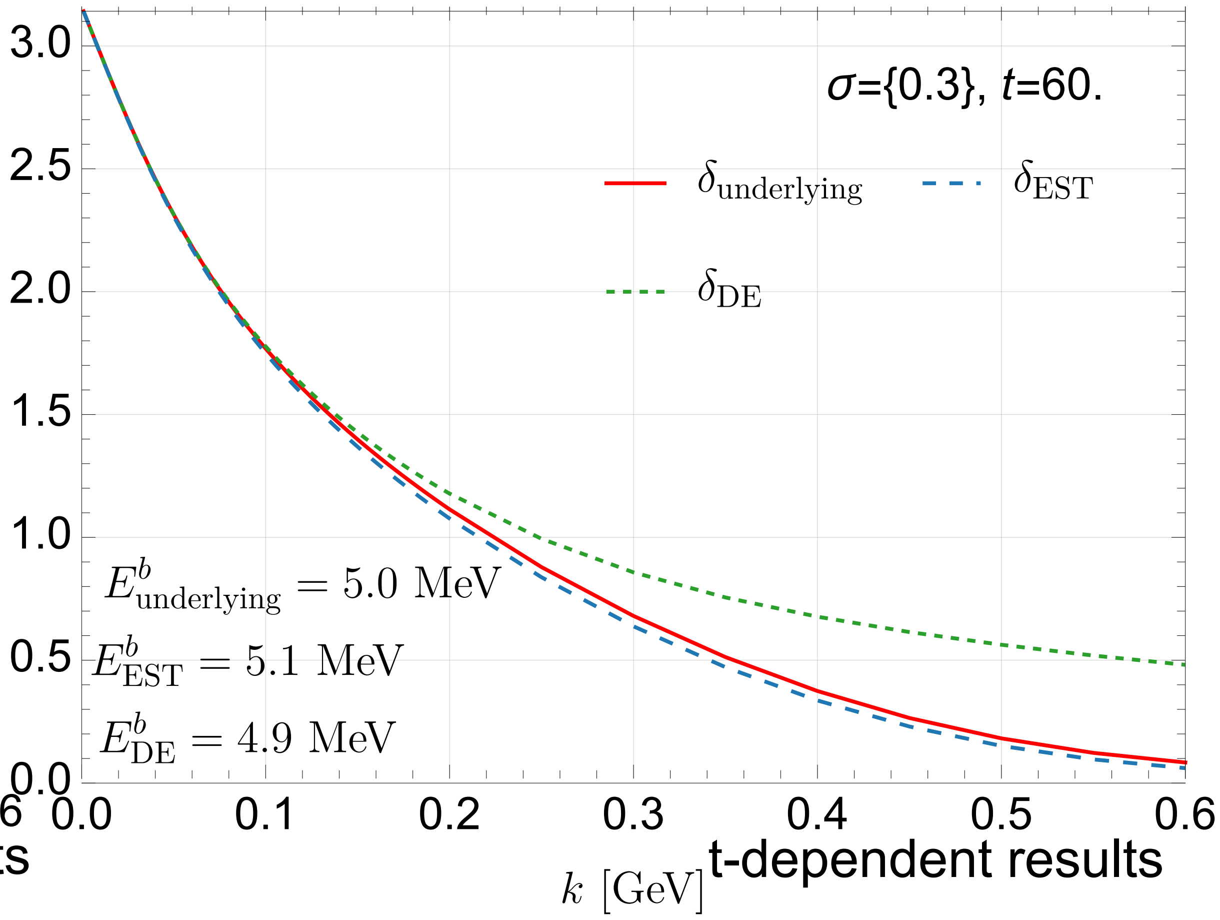
- At LO, both EST and DE method give reasonable binding energy
- The EST method perform better in phase shift
- Singular potential in DE at NLO

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



t-independent results



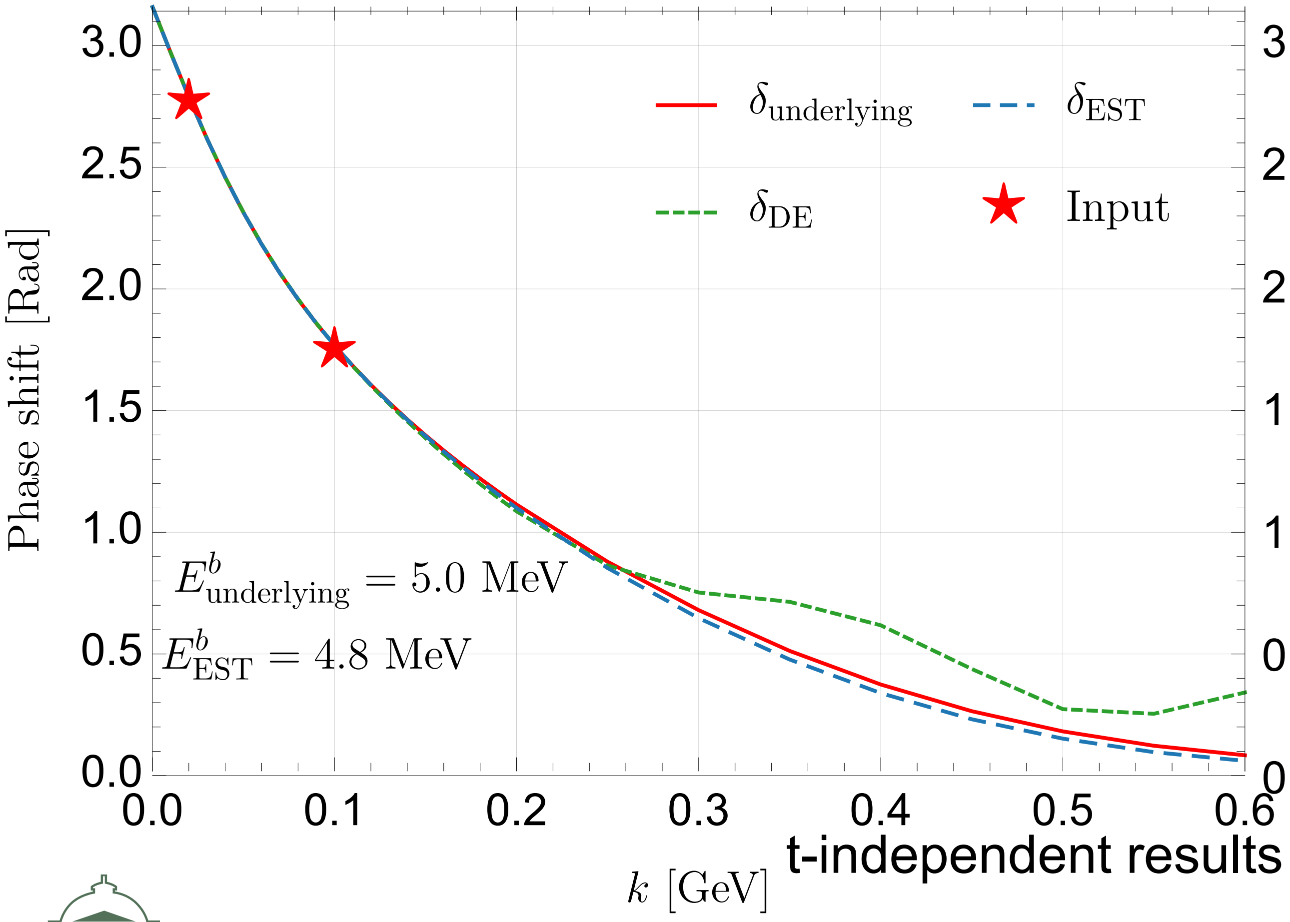
t-dependent results

Bound state

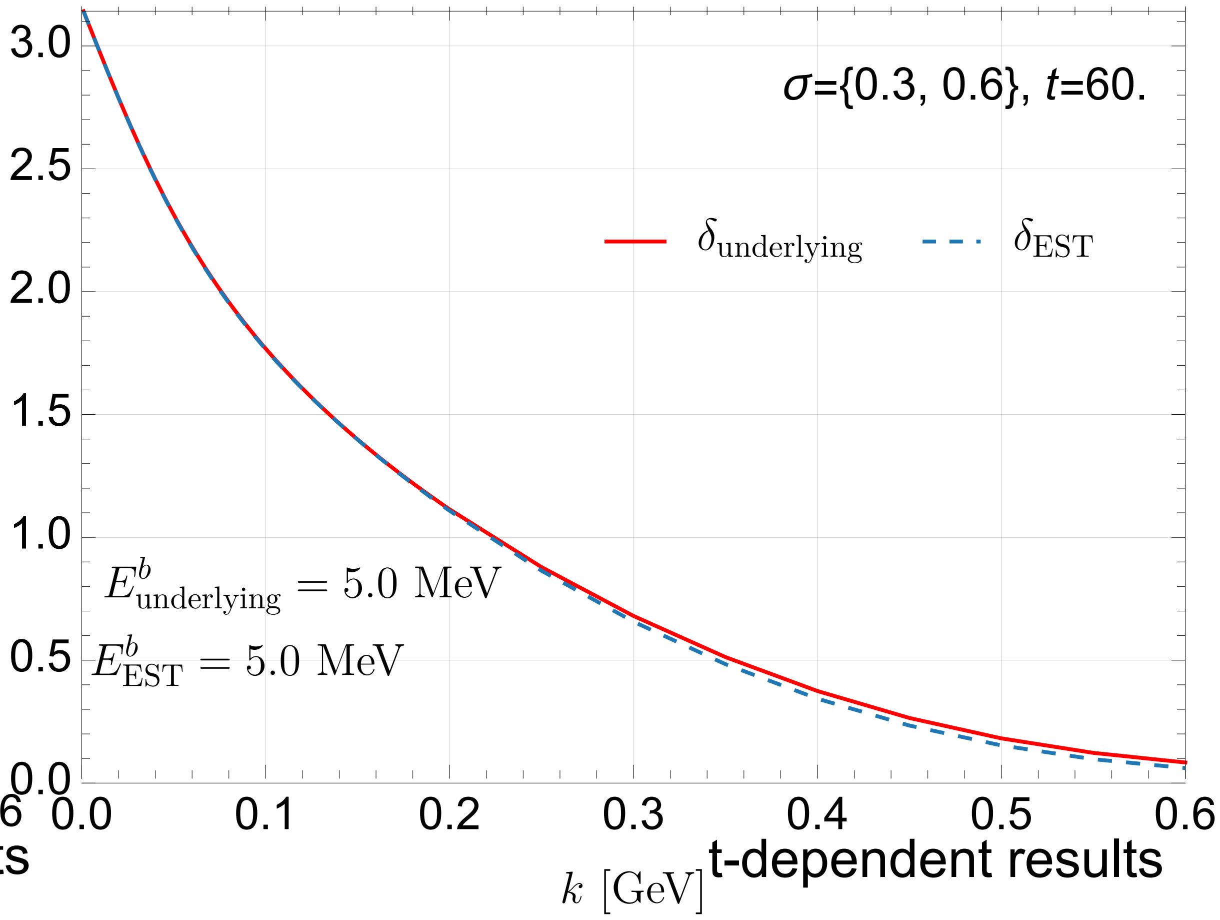
- At LO, both EST and DE method give reasonable binding energy
- The EST method perform better in phase shift
- Singular potential in DE at NLO

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = C e^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right) e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

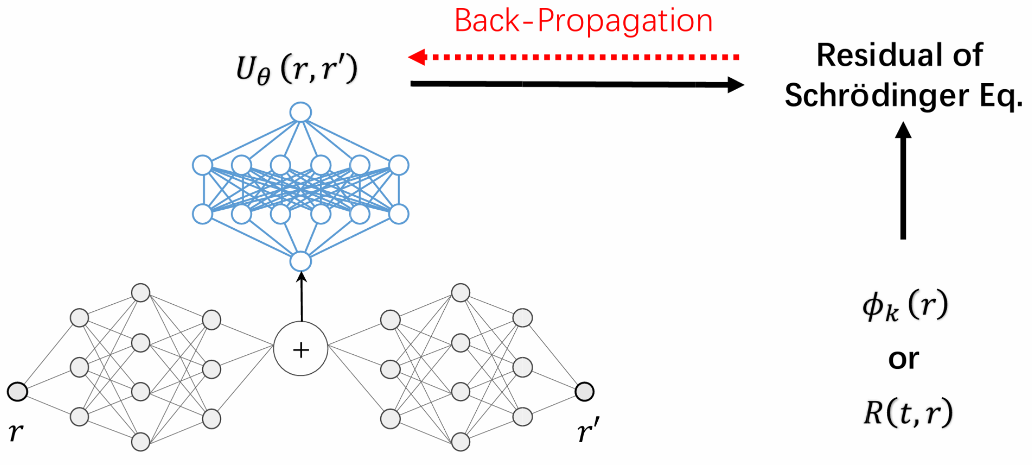


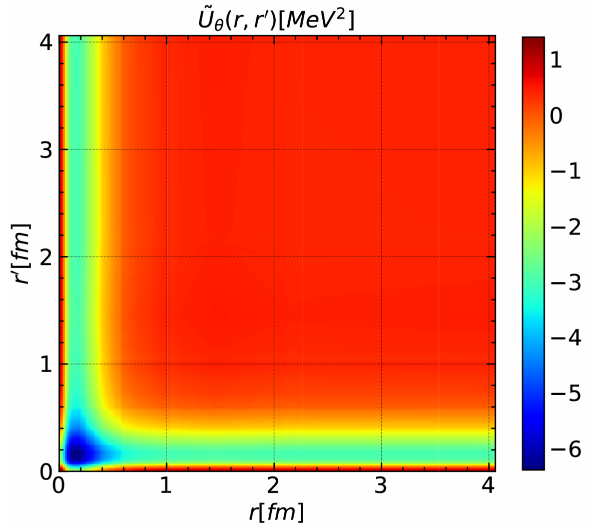
t-independent results



t-dependent results

Machine learning offer a solution?





Wang, L., Doi, T., Hatsuda, T., & Lyu, Y. (2025). *PoS*, *LATTICE2024*, 076

- Deep neutral network as the representation of potential
  - ▶ General potential function: nonlocal potential
- Cannot alter the fact that the solution is not unique

$$\int dr' V(r,r') R^{(i)}(r) = K^{(i)}(r) \Rightarrow \mathbb{V}_{N\times N} R^{(i)}_{N\times 1} = K^{(i)}_{N\times 1}$$

N: # of quadrature points
N wave functions to fix potential matrix $\mathbb{V}_{N\times N}$

- Promising approach the systemic uncertainties of the different potential parameterization

discrepancy

ceive significant contributions from excited states. For single hadrons, the excited state gap is roughly set by $2m_\pi$. Thus, the ground state is resolvable over a Euclidean time of roughly $1-2$ fm. In contrast, for NN calculations, the excited state gap is roughly set by the quantized momentum modes of the nucleon. With periodic spatial boundary conditions, these are $\Delta E_n = 2\sqrt{m_N^2+p_n^2} - 2m_N \approx p_n^2/m_N$ with $p_n = \sqrt{n}(2\pi/L)$ for integer n. For typical sizes of the lattice L, these energy gaps are $\mathrm{O}(20-50)$ MeV and thus, without a suppression of the excited states, the ground state is not resolvable until $4-10$ fm in time while the S/N for NN calculations has degraded around $t\sim 2$ fm. Thus, for correlation func-

This discrepancy was believed to be due to uncontrolled systematic uncertainties with the HAL QCD Potential method [19–24]. At the same time, subsequent work by HAL QCD raised serious concerns about the calculations that observed bound states [16–18, 25]. This was followed up by two-baryon calculations that utilized a set of operators that enabled the use of momentum space creation operators, in contrast to the local hexa-quark (HX) creation operators used in Ref. [9–15] (these operators will be defined subsequently). These new methods found significantly reduced binding energies in the case of the h-dibaryon [26, 27], or a lack of bound states in the case of di-nucleons [28, 29]. This discrepancy has long plagued progress toward the physical point and led to a lack of confidence in results reported for further two- and higher- nucleon quantities [30, 31].

Di-nucleons do not form bound states at heavy pion mass

BaSc Collaboration • John Bulava (Ruhr U., Bochum) Show All(19)

May 8, 2025

30 pages

e-Print: 2505.05547 [hep-lat]

Report number: LLNL-JRNL-2005660

View in: ADS Abstract Service

Time-independent HAL QCD

- The equal-time BS amplitude (BS wave function, BSWF)

CP-PACS:2005gzm

$$\psi(\vec{x};\vec{k}) = \langle 0|\pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\pi_1(\vec{k}),\pi_2(-\vec{k});in\rangle$$

- Asymptotic behavior of BS wave function

$$\psi(\vec{x};\vec{k}) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3}\frac{T(p;k)}{p^2-k^2-i\epsilon}e^{i\vec{p}\cdot\vec{x}}$$

  - ► $T(p;k)$ is the half-on-shell T-matrix
  - ► $\psi(\vec{x};\vec{k})$ satisfy the Lippmann-Schwinger eq. as the non-relativistic scattering wave function
  - ► Using Nishijima, Zimmermann and Haag (NZH) reduction formula for composite local operators
  - ► No non-relativistic approximation, just a formal resemblance

- The BSWF at different energies $\{k_i\}$ in the lattice are the raw data of t-independent HAL QCD

- The general problem: $\psi_{k_i}(\vec{x}) \Rightarrow V$ from Schrodinger-like equations

Observable or not?

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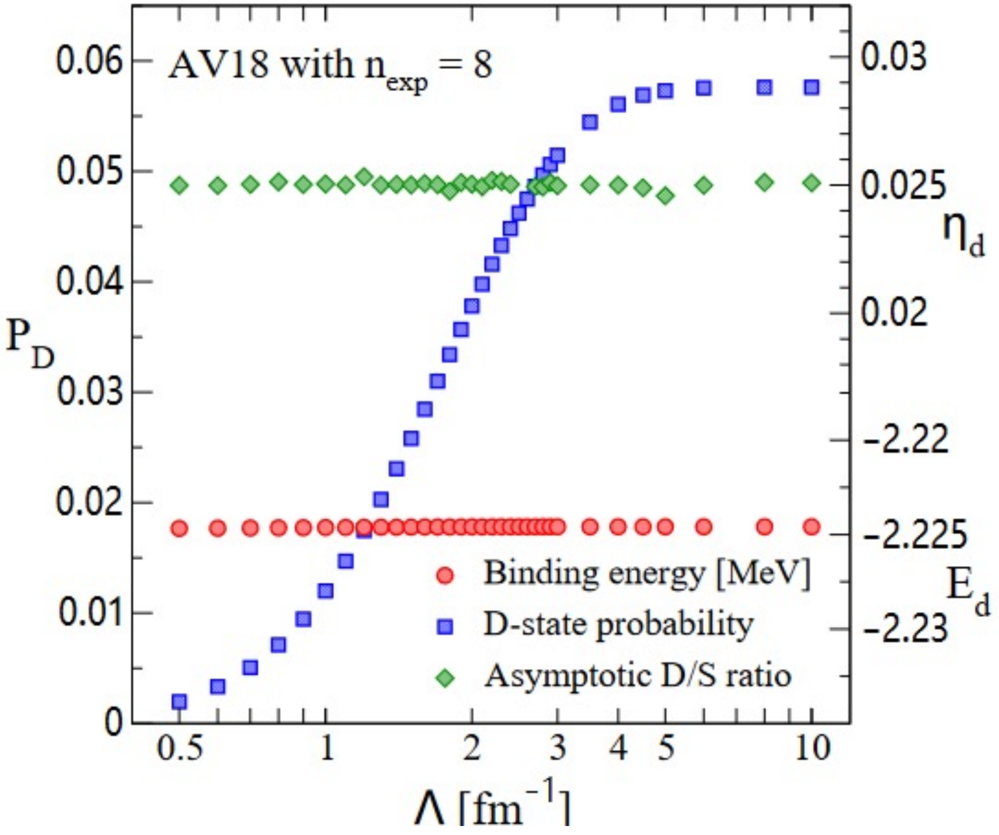
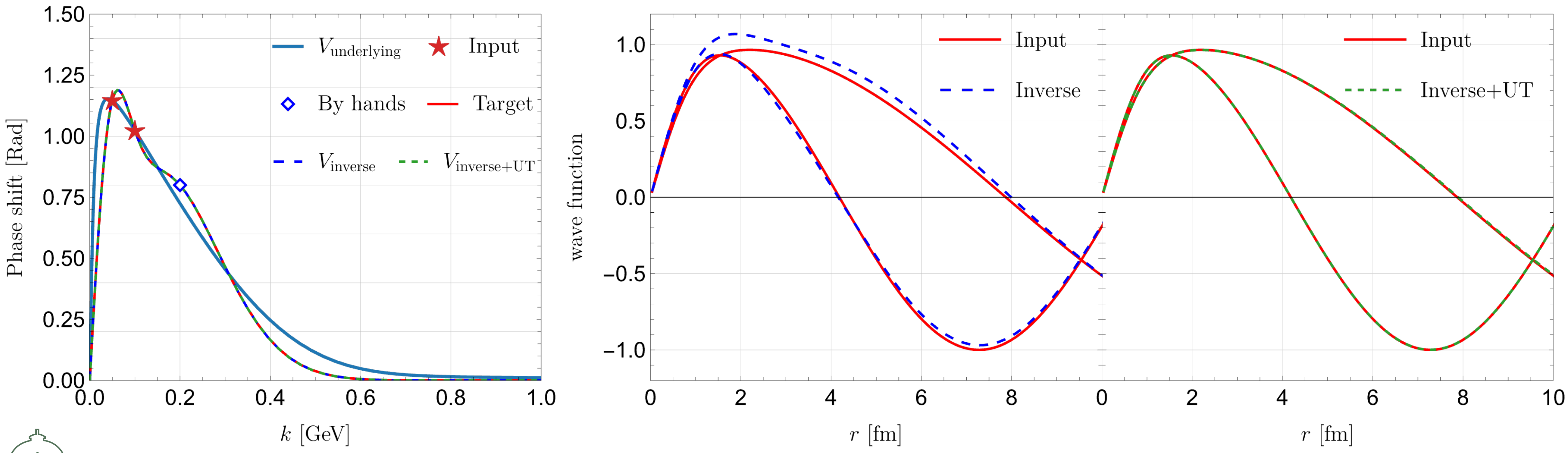
- Non-observables
  - ▶ Non-asymptotic behavior of ψ, e.g. the deuteron D-state probability
  - ▶ Off-shell T-matrix
  - ▶ Potential
  - ▶ Source functions.
- Observables
  - ▶ Asymptotic behavior of ψ
  - ▶ Phase shift
  - ▶ On-shell T-matrix
  - ▶ Correlation functions

Amghar:1995av



$$
\begin{tikzcd}
V_{\text{underlying}} \arrow[r] \arrow[d] & \{\delta_{k_1},\ \delta_{k_2}\} \arrow[r] & \{\delta_{k_1},\ \delta_{k_2},\ \delta_{k_3}^{\text{by hand}}\} \arrow[r] & \delta(k) \arrow[d] \\
\{\psi_{k_1}(r),\ \psi_{k_2}(r)\} & \neq & \{\psi_{k_1}^{\text{inverse}}(r),\ \psi_{k_2}^{\text{inverse}}(r)\} & V_{\text{inverse}} \arrow[l] \arrow[d] \\
 & & & V_{\text{inverse}+\text{UT}} \arrow[ulll]
\end{tikzcd}
$$



$$
\begin{tikzcd}
V_{\text{underlying}} \arrow[r] \arrow[d] & \{\delta_{k_1},\ \delta_{k_2}\} \arrow[r] & \{\delta_{k_1},\ \delta_{k_2},\ \delta_{k_3}^{\text{by hand}}\} \arrow[r] & \delta(k) \arrow[d] \\
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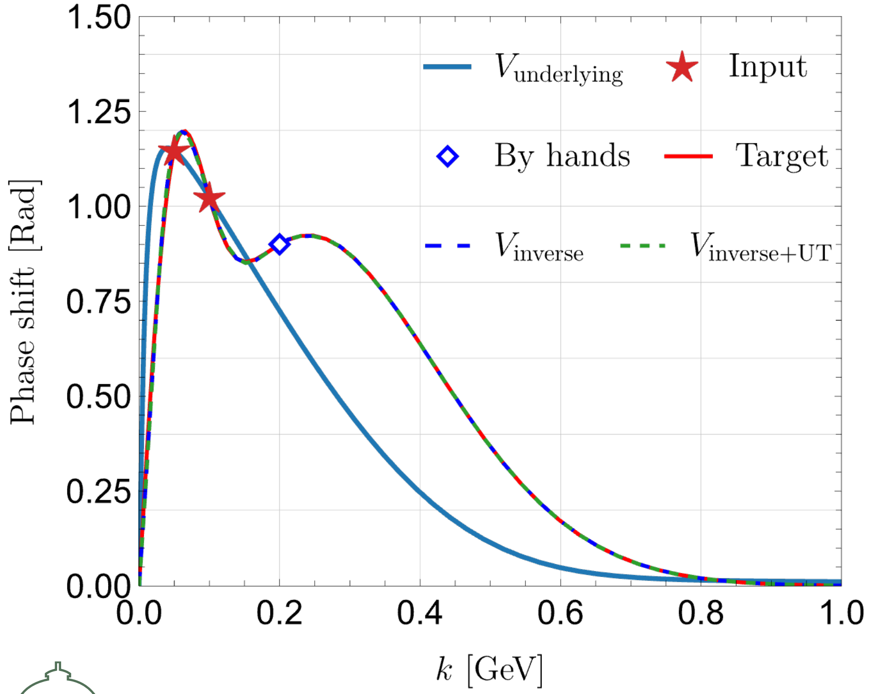
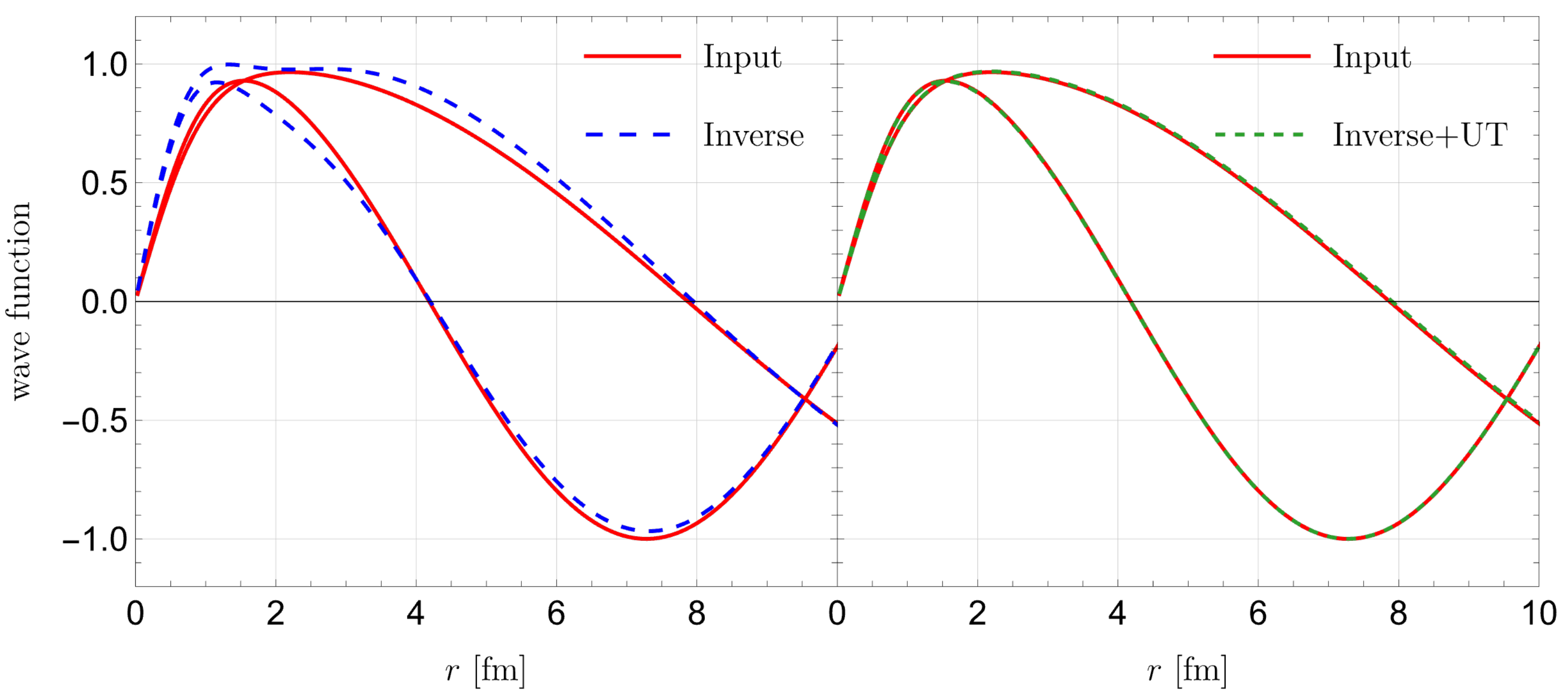
 & & & V_{\text{inverse}+\text{UT}} \arrow[lllu]
\end{tikzcd}
$$





$$
\begin{tikzcd}
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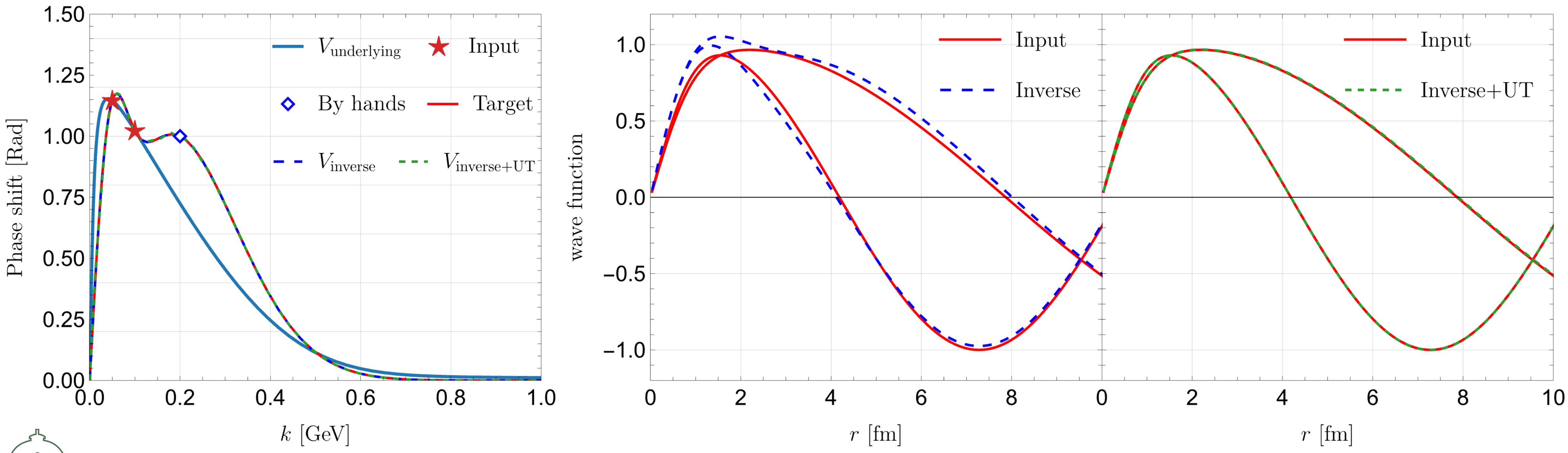
\{\psi_{k_1}(r),\ \psi_{k_2}(r)\} \arrow[r, phantom, "\neq"] & \{\psi^{\text{inverse}}_{k_1}(r),\ \psi^{\text{inverse}}_{k_2}(r)\} & & V_{\text{inverse}} \arrow[ll] \arrow[d] \\
& & & V_{\text{inverse}+\text{UT}} \arrow[lllu]
\end{tikzcd}
$$



Some tips

Two different expansion

- using the potential to perform formal expansion
  - ► A regular potential could be singular after expansion
- Using the potential generate the wave function and then use the wave function to generate potential

- Why only even term

odd derivative terms are absent in the hermitian potential with rotational and time-reversal symmetries, we do not need such terms to describe scattering phase shift.

Proof of hermitian

$$\hat{V}|R^{(i)}\rangle = K^{(i)}$$

$$K^{(i)} = (E - \hat{H}_0)R^{(i)}$$

$$\hat{V} = \sum_{mn} |K^{(m)}\rangle \Lambda_{mn} \langle K^{(n)}|, \quad \sum_{n} \Lambda_{mn} \langle K^{(n)}|R^{(i)}\rangle = \delta_{mi}$$

Hermitian

$$\hat{V}^{\dagger} = \sum_{mn} |K^{(m)}\rangle \Lambda^{*}_{nm} \langle K^{(n)}|$$

the matrix $\{\langle K^{(n)}|R^{(i)}\rangle\}$ is Hermitian and so it is $\{\Lambda_{mn}\}$

$$\begin{aligned} \langle K^{(m)}|R^{(n)}\rangle &= \langle (E - \hat{H}_0)R^{(m)}|R^{(n)}\rangle \\ &= \langle R^{(m)}|E - \hat{H}_0|R^{(n)}\rangle \end{aligned}$$