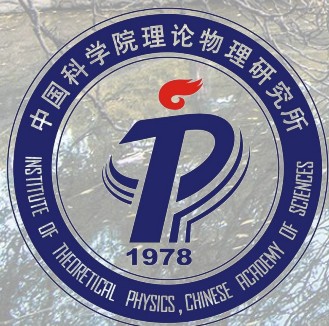


第十届手征有效场论研讨会, 南京

PRD 108 034009 (2023)
PRD 112 034042 (2025)
PRD 112 L031503 (2025)

Xiong-Hui Cao 曹雄辉

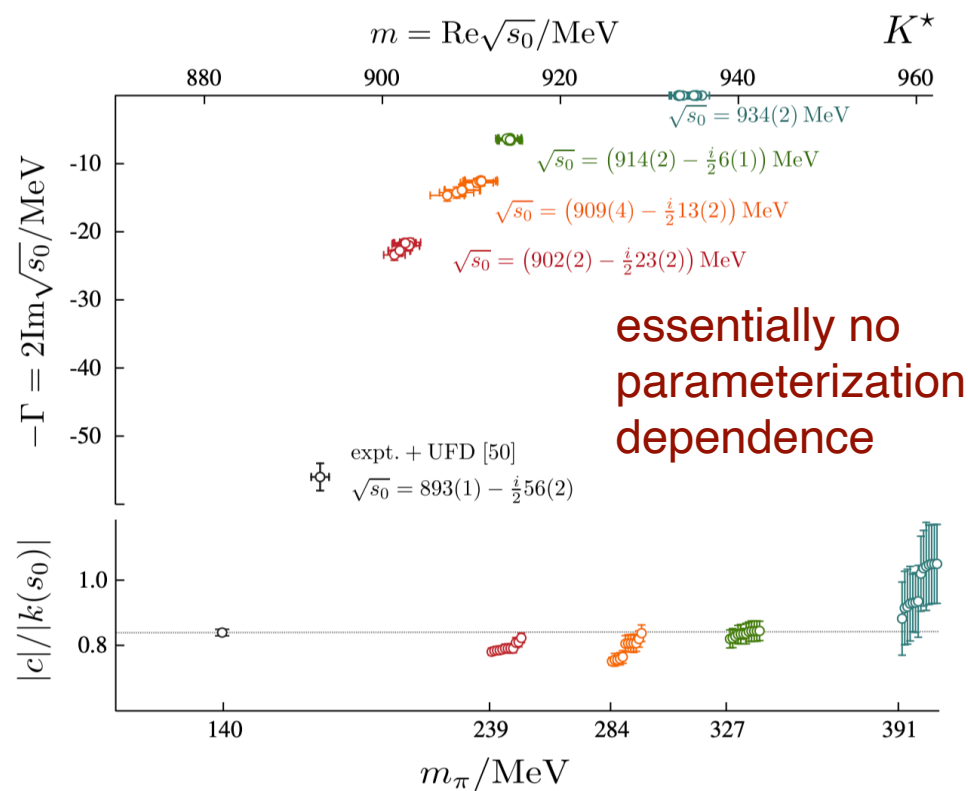
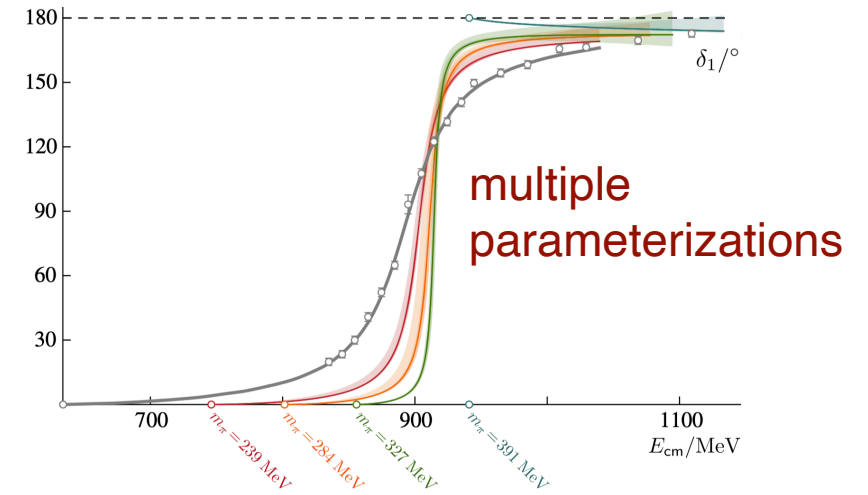
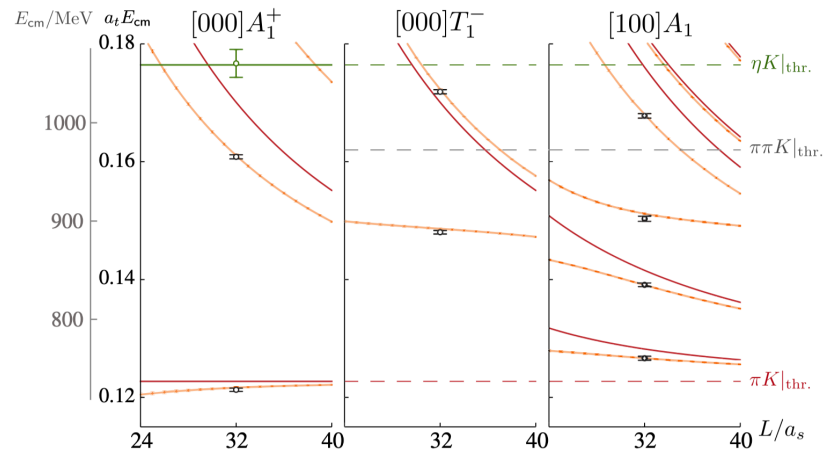
in collaboration with
Feng-Kun Guo, Zhi-Hui Guo,
Qu-Zhi Li and Han-Qing Zheng



2025年10月17日至21日

πK elastic scattering in P-wave

Hadron Spectrum Collaboration, PRL (2014); PRD (2015); PRL(2019)



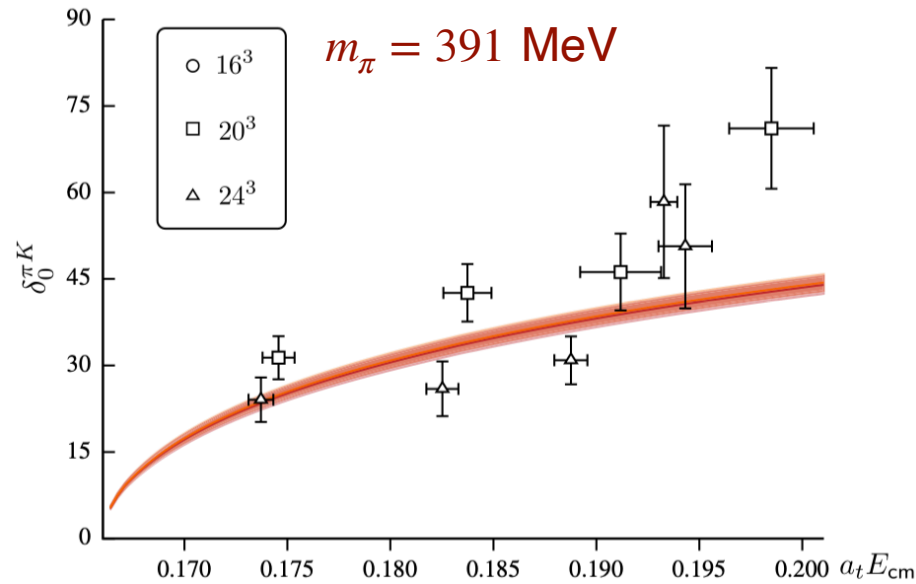
K^* four unphysical pion masses

essentially no
parameterization
dependence

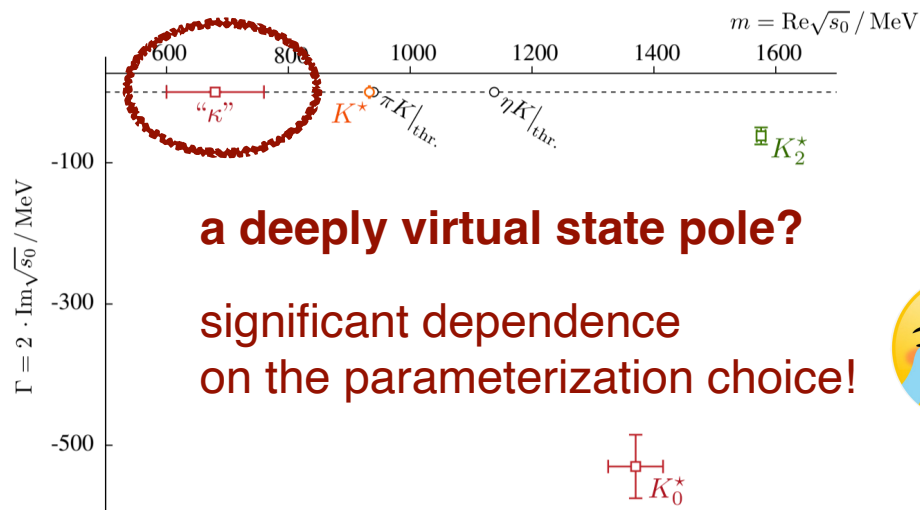
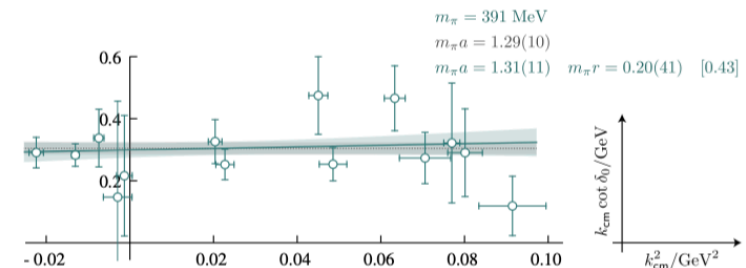
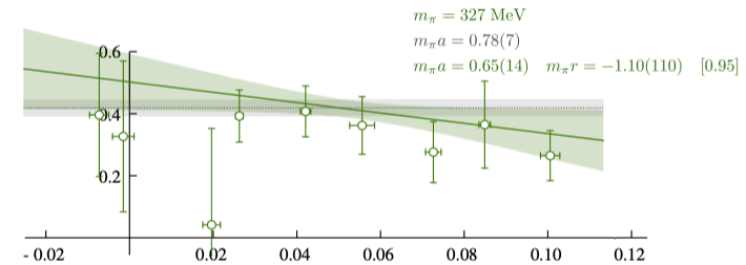
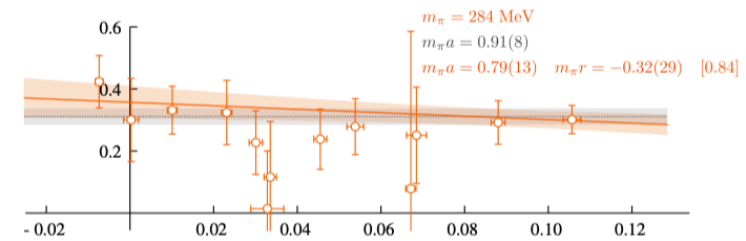
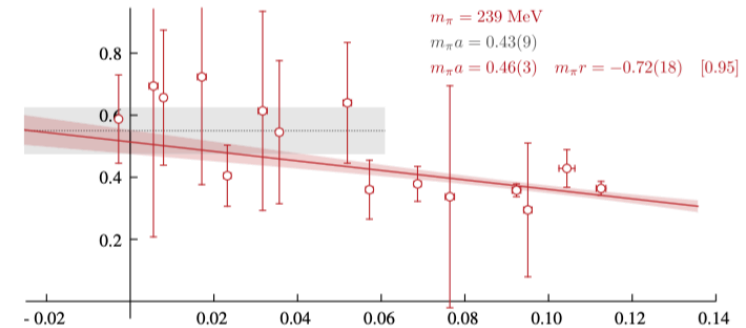


πK elastic scattering in S-wave

Hadron Spectrum Collaboration, PRL (2014); PRD (2015); PRL(2019)



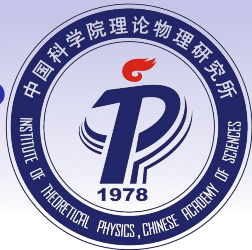
same story at several pion masses



a deeply virtual state pole?

significant dependence
on the parameterization choice!





what are these amplitude parameterizations you're using ?

in **elastic** case, easy to use wider variety ...

(variants) of effective range expansion: $k^{2\ell+1} \cot \delta_\ell = \left(\frac{1}{\hat{a}_\ell} + \frac{1}{2} \hat{r}_\ell k^2 + \dots \right)$

a conformal mapping in energy: $k^{2\ell+1} \cot \delta_\ell = \frac{\sqrt{s}}{2} \sum_n B_n \omega^n$

also Breit-Wigner...

in **coupled-channel** cases, generally K-matrix forms — ensure exact unitarity

or slightly improving the analytic properties,
a dispersively improved Chew-Mandelstam form

$$[t^{-1}]_{ij}(s) = [K^{-1}]_{ij}(s) - i\rho_i(s)\delta_{ij}$$

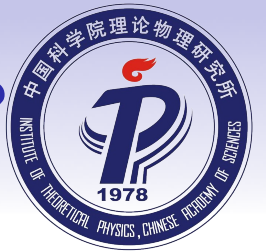
$$[t^{-1}]_{ij}(s) = [K^{-1}]_{ij}(s) + I_i(s)\delta_{ij}$$

$$\text{phase space } \rho_i(s) = \frac{2k_i(s)}{\sqrt{s}}$$

$$I_i(s) = I_i(s_0) + \frac{s - s_0}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{-\rho_i(s')}{(s' - s_0)(s' - s)}$$

K-matrix real and symmetric, commonly poles + polynomials

only trying to directly parameterize along the real energy axis above threshold
—let the data tell us what needs to be there



what are these amplitude parameterizations you're using ?

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phase
space $\rho_i(s) = \frac{2k_i(s)}{\sqrt{s}}$

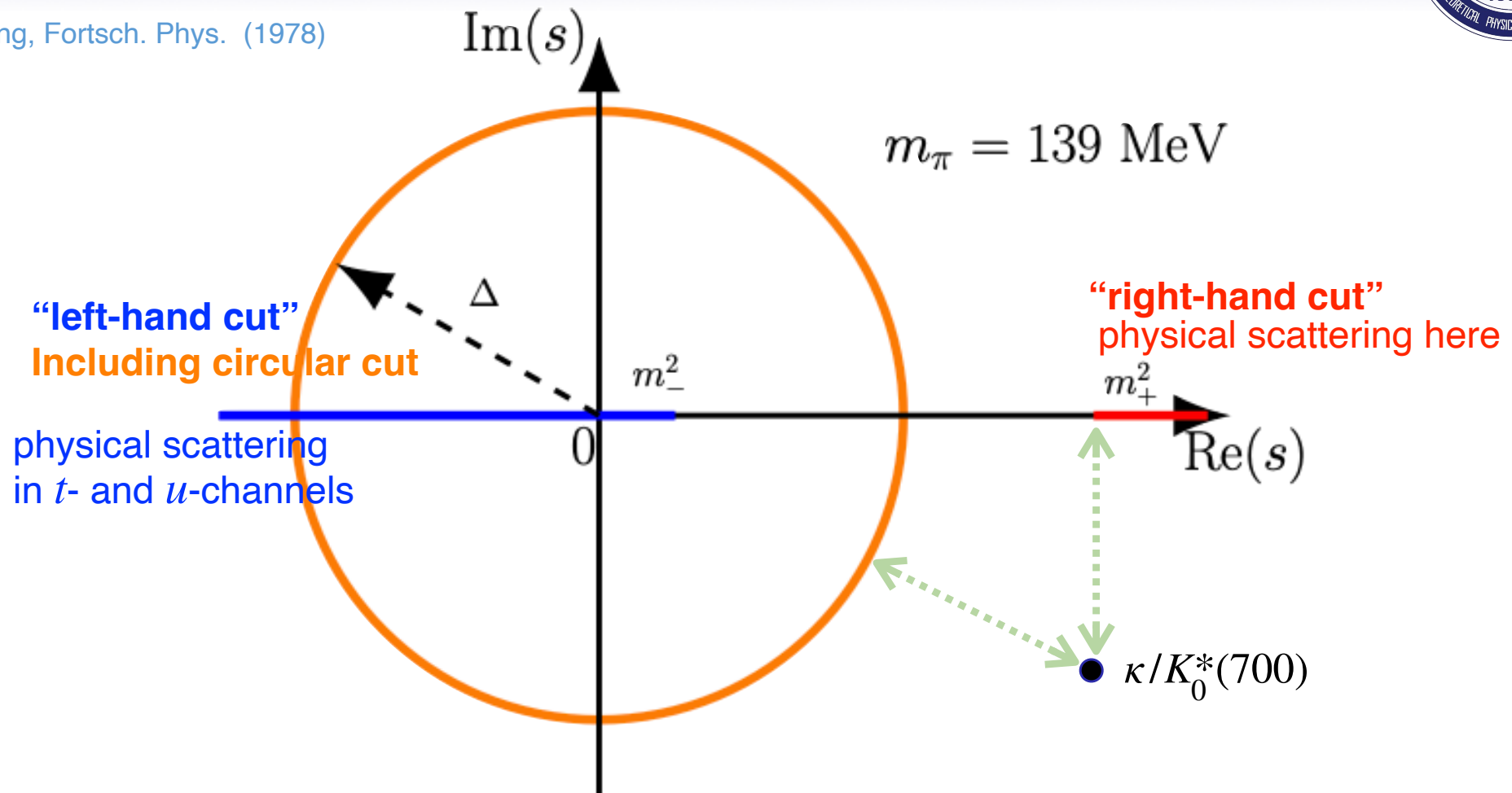
$$I_i(s) = I_i(s_0) + \frac{s - s_0}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{-\rho_i(s')}{(s' - s_0)(s' - s)}$$

K-matrix real and symmetric, commonly poles + polynomials

are we missing some important feature of the amplitudes?

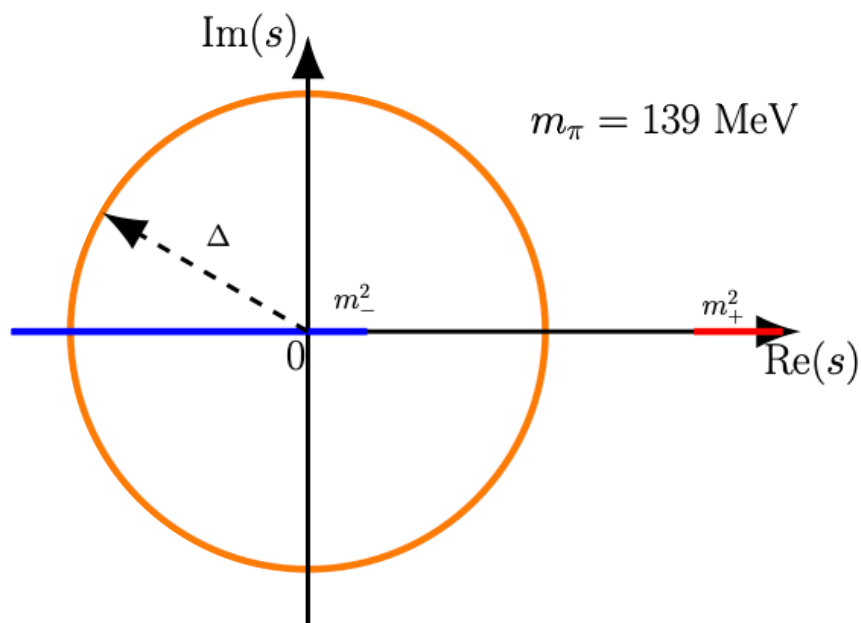
Cut structure of the πK partial-wave amplitudes

Lang, Fortsch. Phys. (1978)

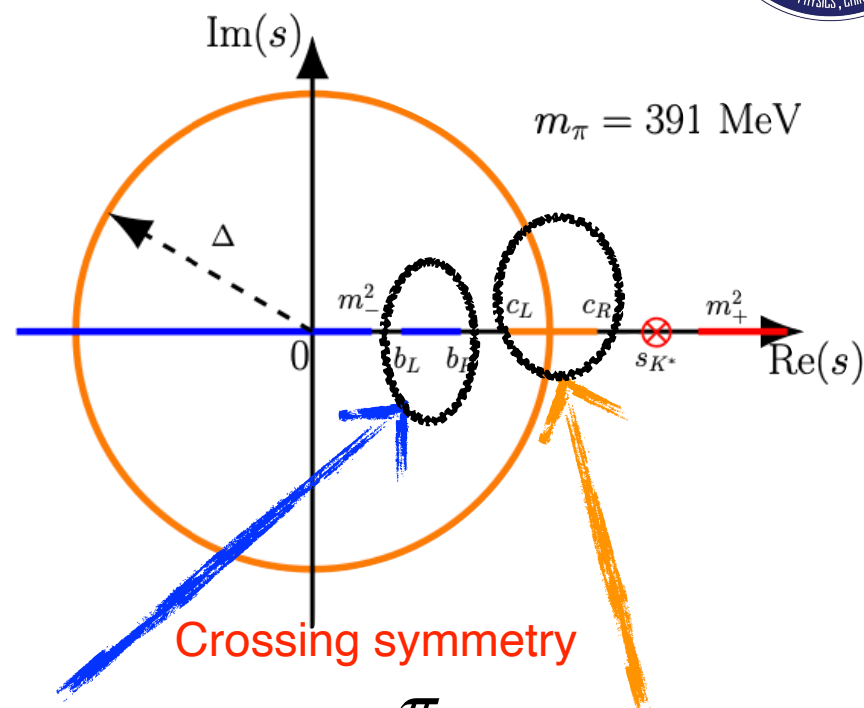


- for broad resonances, the left-hand cut may be just as close to physical cut
- simple K-matrix forms don't have a left-hand cut
- no real constraint on subthreshold amplitude or the left-hand cut

Cut structure of the πK partial-wave amplitudes



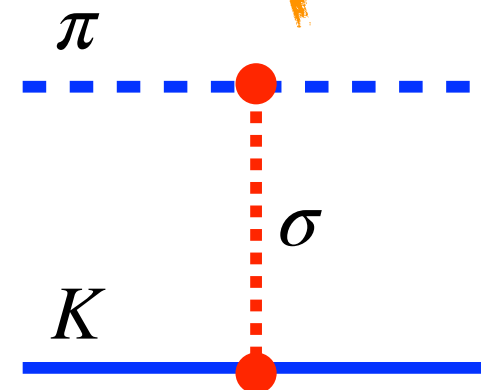
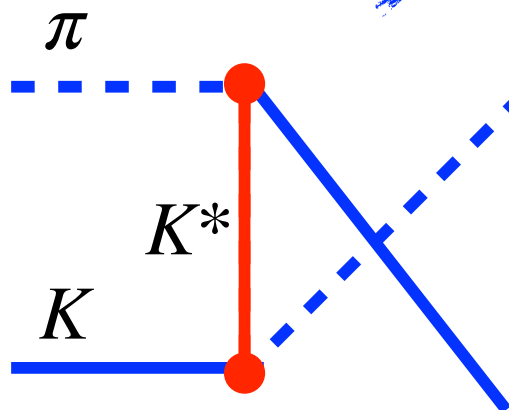
Lang, Fortsch. Phys. (1978)



Crossing symmetry

- $\sigma/f_0(500)$ and $K^*(892)$ as the S- and P-wave bound-states

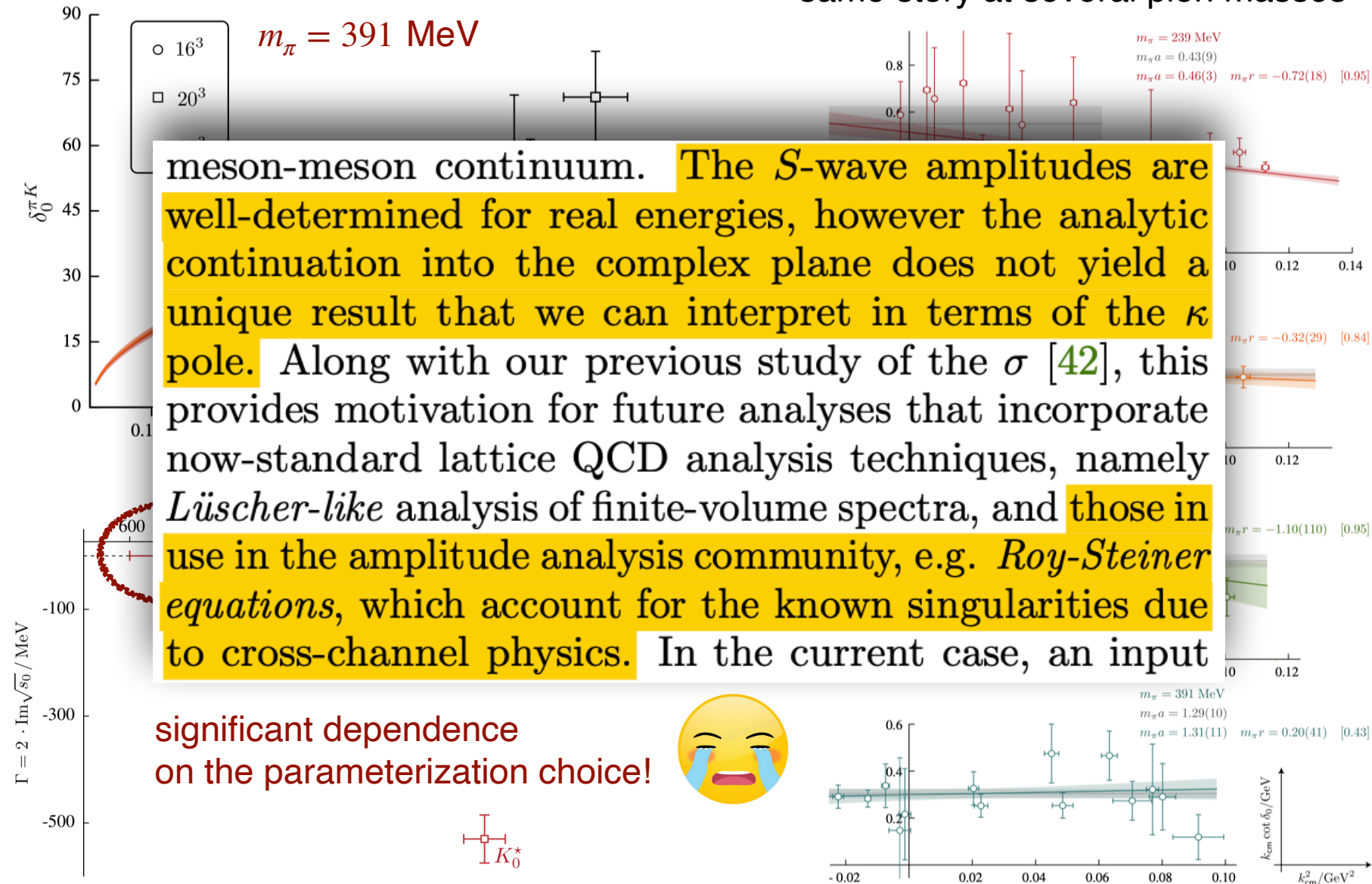
- New left-hand cuts



Why is the Roy-Steiner equation?

HSC, PRL (2014); PRD (2015); PRL(2019)

same story at several pion masses



Roy-Steiner-type equation



Roy-Steiner-type equations = Analyticity (Causality) + Unitarity + Crossing symmetry

*Crossing-symmetric
dispersive analyses*



S. Roy



F. Steiner

★ **Renaissance** caused by the development of ChPT

- $\pi\pi$ G. Colangelo, et al., NPB (2001); B. Ananthanarayan, et. al., Phys. Rept. (2001); I. Caprini, et al., PRL (2006); B. Moussallam, EPJC (2011); Garcia-Martin, et al., PRD (2011); PRL (2011); I. Caprini, et al., EPJC (2011); J. Pelaez, Phys.Rept. (2016); XHC et.al., PRD (2023); HSC, PRD (2024)...
- πK P. Buettiker, et al., EPJC (2004); S. Descotes-Genon, et al., EPJC (2006); J. Pelaez and A. Rodas, EPJC(2018); PRL (2020); Phys.Rept. (2022); J. Pelaez et.al., PRL (2023); XHC et.al., PRD (2025)...
- πN C. Ditsche, et al., JHEP (2012); M. Hoferichter et.al., JHEP (2012); M. Hoferichter, et al., PRL 115, 092301(2015); PRL 115, 192301 (2015); Phys. Rept. (2016); PLB (2016); EPJA (2016); J. Ruiz de Elvira et.al., JPG (2018); M. Hoferichter, et al., PRL (2018); XHC, et.al., JHEP (2022); M. Hoferichter, et al., PLB (2024)...
- $\gamma\pi \rightarrow \pi\pi$: T. Hannah, NPB (2001); M. Hoferichter et.al., PRD (2012); $\gamma\gamma \rightarrow \pi\pi$: M. Hoferichter et.al., EPJC (2011); $\gamma^*\gamma^* \rightarrow \pi\pi$: M. Hoferichter and P. Stoffer, JHEP (2019)...

Roy-Steiner-type equation

S.R. Roy, (1971)

F. Steiner et al., (1970, 1971, 1973)

$$\text{Re } t_J^I(s) = k_J^I(s) + \sum_{I'} \sum_{J'} \mathcal{P} \int_{4m_\pi^2}^{\infty} ds' \underbrace{K_{JJ'}^{II'}(s', s)}_{\frac{1}{\pi} \frac{\delta_{JJ'} \delta_{II'}}{s' - s} + \bar{K}_{JJ'}^{II'}(s, s')} \text{Im } t_{J'}^{I'}(s')$$

Roy equation for $\pi\pi$

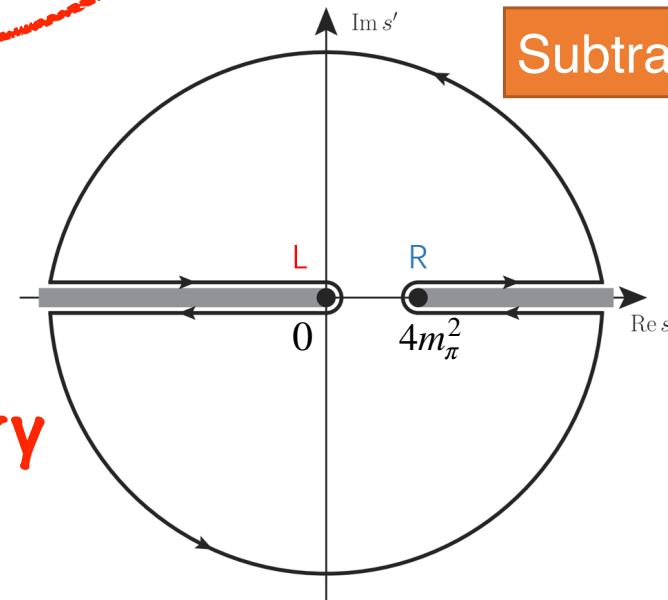
$$K_{00}^{00}(s, s') = \frac{1}{\pi(s' - s)} + \frac{2 \ln \left(\frac{s + s' - 4M_\pi^2}{s'} \right)}{3\pi(s - 4M_\pi^2)} - \frac{5s' + 2s - 16M_\pi^2}{3\pi s'(s' - 4M_\pi^2)}$$

Kernel

$$k_0^0(s) = a_0^0 + \frac{s - 4m_\pi^2}{12m_\pi^2} (2a_0^0 - 5a_0^2)$$

Subtraction term

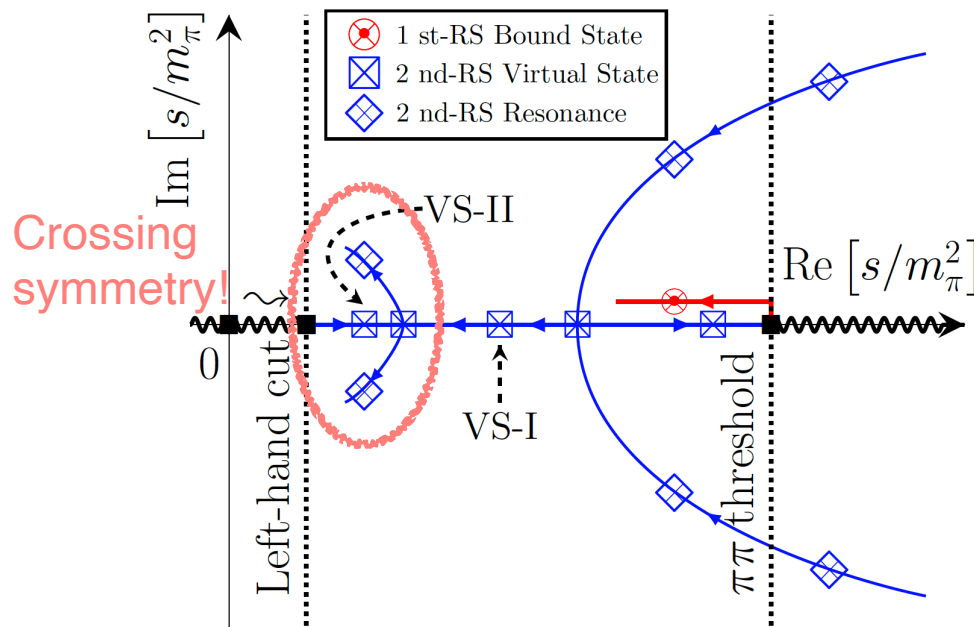
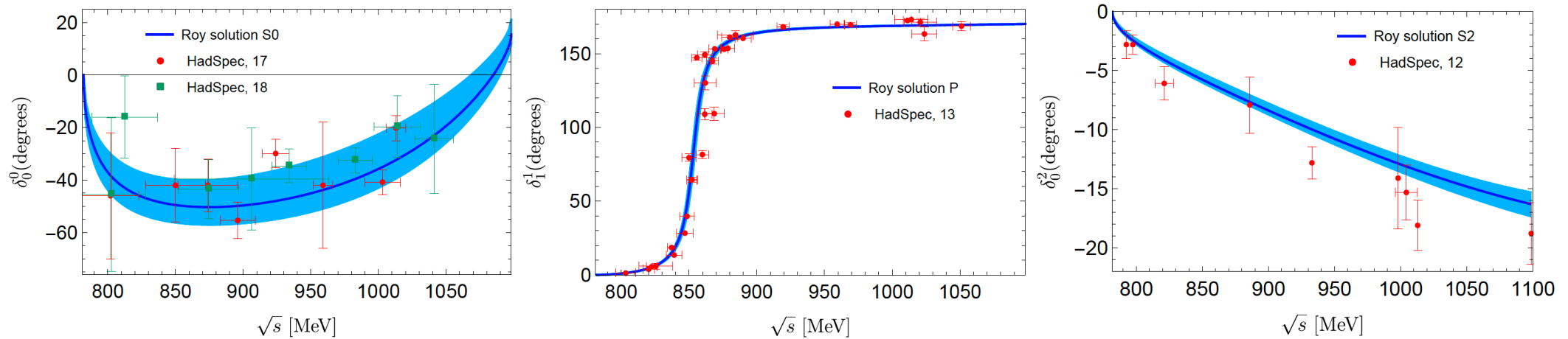
Left-hand cuts
from crossing symmetry



$\pi\pi$ scattering at $m_\pi = 391$ MeV

XHC, Q.-Z. Li, Z.-H. Guo and H.-Q. Zheng, PRD (2023)

$\sigma/f_0(500)$ $m_\pi = 391$ MeV $\Rightarrow$ bound state σ pole!



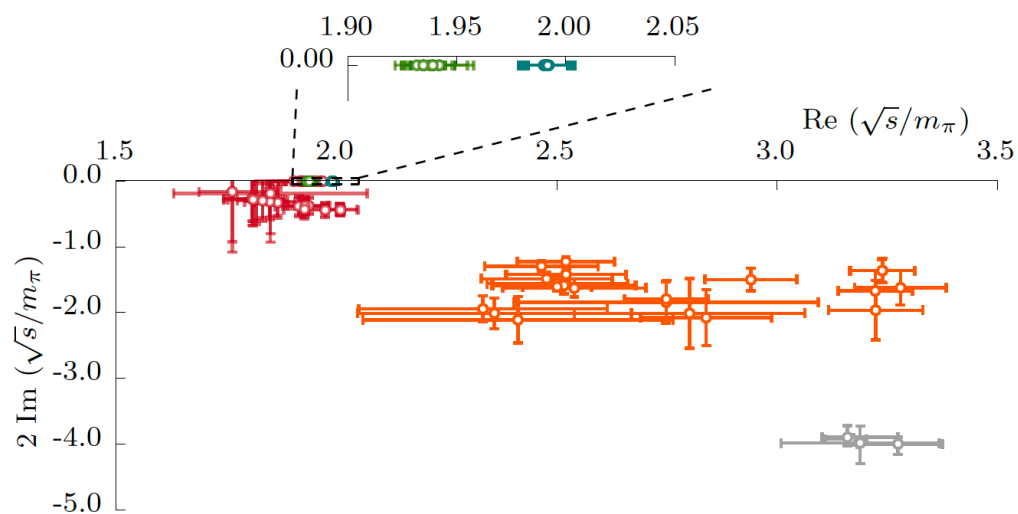
★ Bound state and virtual state pole trajectories of σ as a function of m_π

K-matrix v.s. dispersive analysis

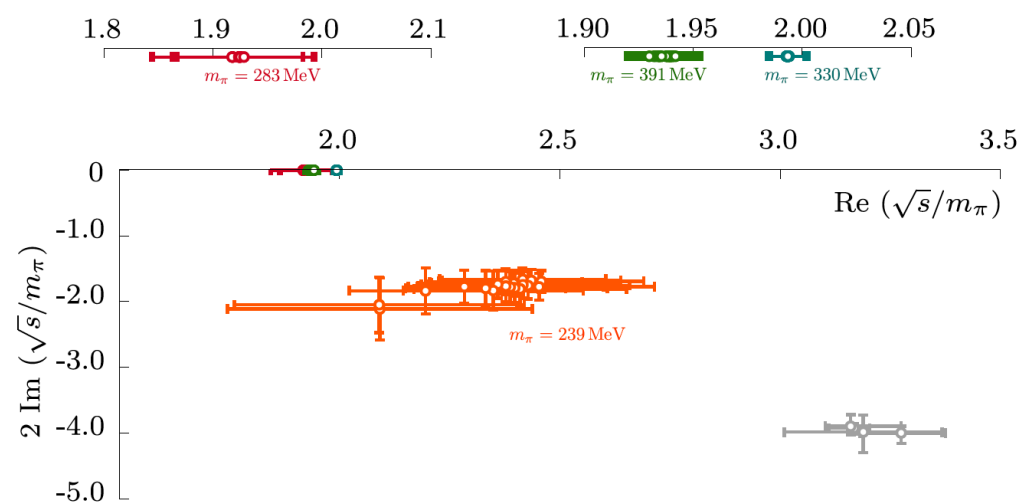


$$\sigma/f_0(500)$$

New σ pole positions via preliminary Roy equation analyses [XHC et.al., PRD \(2023\)](#); [HSC, PRD \(2024\)](#)

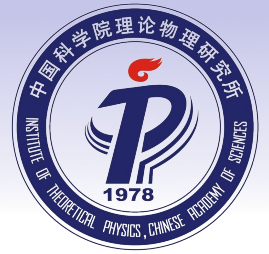


K-Matrix analysis



Roy equation analysis

Roy-Steiner-type equation for πK scattering



► t -channel Roy-Steiner-type equations

$$\pi\pi \rightarrow K\bar{K}$$

$$g_J^I(t) = \Delta_J^I(t) + \frac{t}{\pi} \int_{t_\pi}^{t_m} \frac{dt'}{t'} \frac{\text{Im } g_J^I(t')}{t' - t},$$

$$(I, J) = (0, 0), (1, 1)$$

► s -channel Roy-Steiner-type equations

$$\pi K \rightarrow \pi K$$

Sub. Pole s - and t -channel driving terms

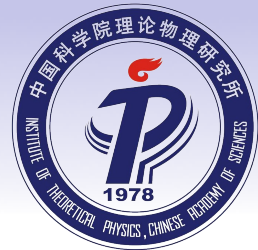
$$f_J^I(s) = \text{ST}_J^I(s) + \text{PT}_J^I(s) + s\text{DT}_J^I(s) + t\text{KT}_J^I(s)$$

$$+ \frac{1}{\pi} \int_{m_\pi^2}^{s_m} ds' \left\{ K_{J,0}^{I,\frac{1}{2}}(s, s') \text{Im } f_0^{\frac{1}{2}}(s') \right.$$

$$\left. + K_{J,1}^{I,\frac{1}{2}}(s, s') \text{Im } f_1^{\frac{1}{2}}(s') + K_{J,0}^{I,\frac{3}{2}}(s, s') \text{Im } f_0^{\frac{3}{2}}(s') \right\},$$

$$(I, J) = \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{3}{2}, 0\right)$$

t -channel $\pi\pi \rightarrow K\bar{K}$ partial wave amplitudes



- t -channel RS-type equations as a Muskhelishvili-Omnès problem

$$\text{Im } g_J^I(t) = [t_J^I(s)]^* \rho_{\pi\pi}(t) g_J^I(t)$$

- Single-channel: Omnès solution, $\Omega_J^I(t) = \exp \left\{ \frac{t}{\pi} \int_{t_\pi}^{t_m} dt' \frac{\phi_J^I(t')}{t'(t' - t)} \right\}$

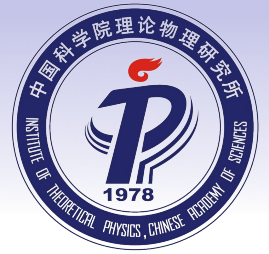
$$g_J^I(t) = \underbrace{\Delta_J^I(t)}_{\text{Crossing}} + t \underbrace{\Omega_J^I(t)}_{\text{Unitarity}} \left[\frac{1}{\pi} \int_{t_\pi}^{t_m} dt' \frac{\Delta_J^I(t') \sin \phi_J^I(t')}{|\Omega_J^I(t')| t' (t' - t)} + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{|g_J^I(t')| \sin \phi_J^I(t')}{|\Omega_J^I(t')| t' (t' - t)} \right]$$

s -channel πK scattering partial waves and t -channel $\pi\pi \rightarrow K\bar{K}$ partial waves where $I'J' \neq IJ$

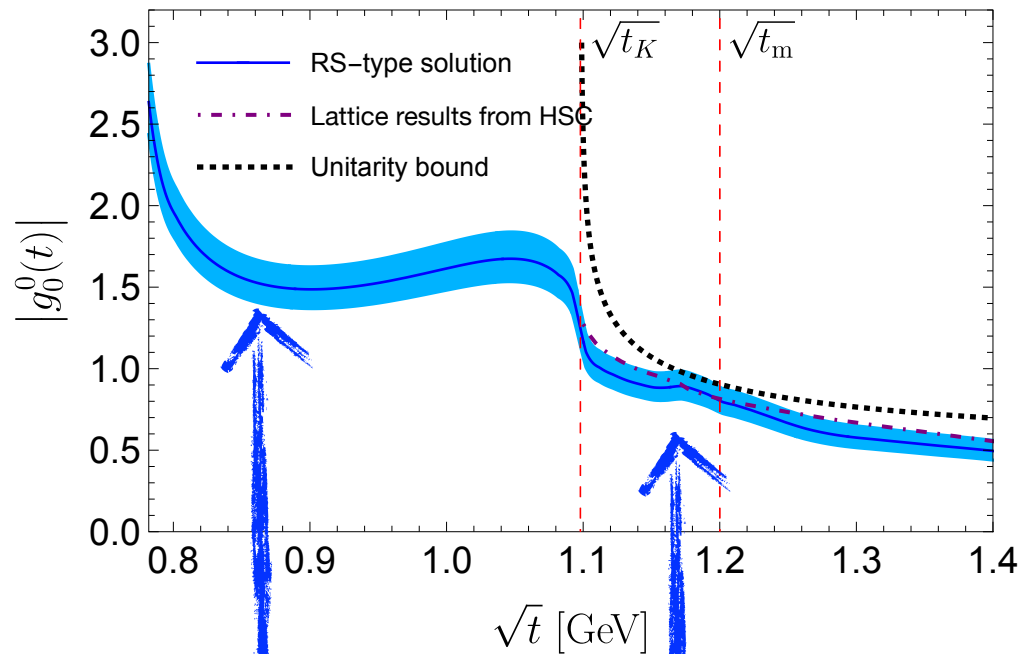
Phase-shifts from $\pi\pi$ scattering:
LQCD data + Roy eq. analyses

XHC et.al., PRD (2023)

t -channel $\pi\pi \rightarrow K\bar{K}$ partial wave amplitudes

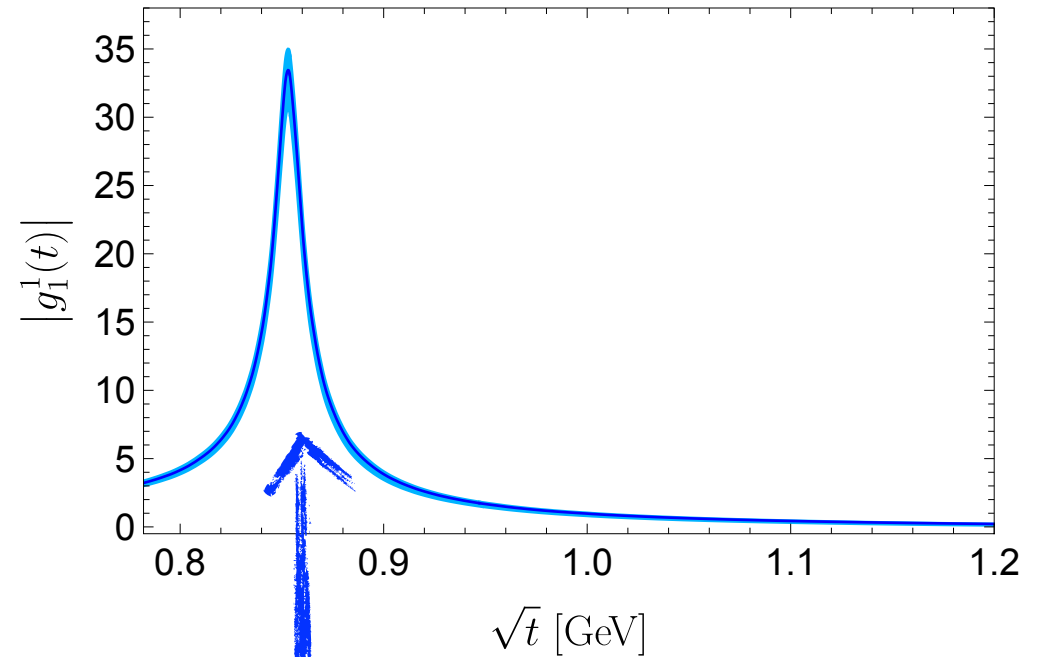


✓ t -channel solution from Roy-Steiner-type equations



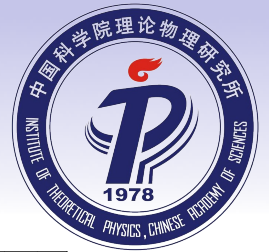
σ bound state

$f_0(980)$



$\rho(770)$ dominance

s -channel $\pi K \rightarrow \pi K$ partial wave amplitudes



✓ s -channel solution from Roy-Steiner-type equations

● $I = \frac{1}{2}$ S-wave:

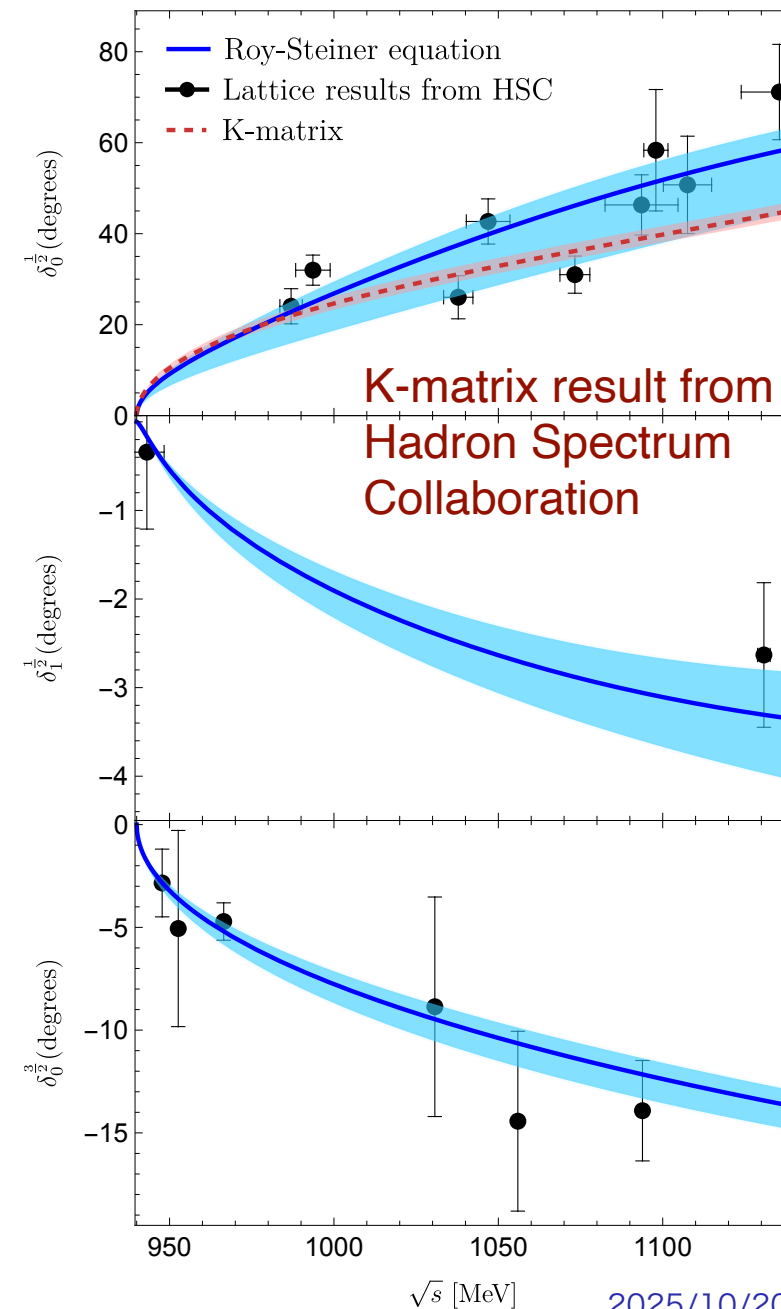
no sharp features that signal the presence of a nearby pole

● $I = \frac{1}{2}$ P-wave:

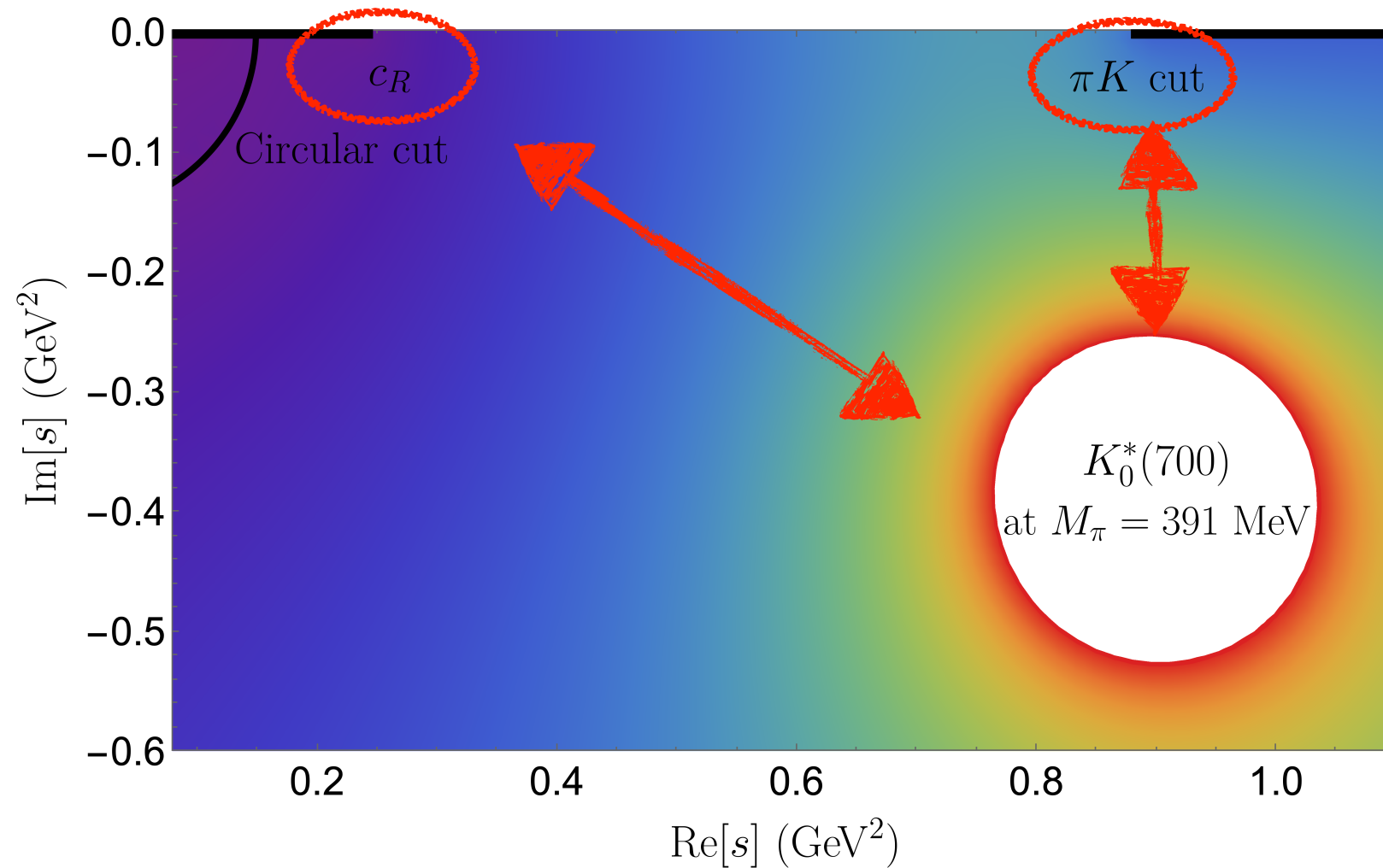
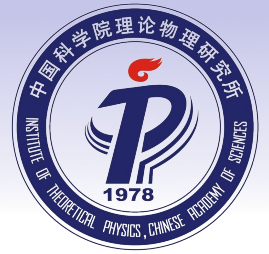
shallow vector bound state $K^*(892)$

● $I = \frac{3}{2}$ S-wave:

repulsive channel



Dispersive determination of $\kappa/K_0^*(700)$ from LQCD data



**A broad resonance instead of
a virtual state pole**

Summary and outlook

- ☒ The unity of dispersive techniques and lattice QCD data is powerful to investigate low energy hadron physics
- ☒ Widely-used unitarization methods such as K-matrix, etc., are not good in light meson & baryon studies
- ☒ Dispersive approaches, Muskhelishvili-Omnès formalism, Roy-Steiner-type equations, Khuri-Treiman equations, etc. are necessary
- ☐ πD scattering at physical & unphysical m_π : $D_0^*(2300)$, two pole structure
- ☐ Dispersive determination of three-body resonances.....

Thank you for your attention!

Roy-Steiner-type equation

$$\text{Re } t_J^I(s) = k_J^I(s) + \sum_{I'} \sum_{J'} \mathcal{P} \int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s', s) \text{Im } t_{J'}^{I'}(s')$$

Phase shifts: $t_J^I(s) = \frac{\eta_J^I(s) e^{2i\delta_J^I(s)} - 1}{2i\rho_{\pi\pi}(s)}$

Inelasticity (input)

Unitarity

Roy equation for $\pi\pi$

$$\frac{\eta_J^I(s) \sin 2\delta_J^I(s)}{2\rho(s)} = k_J^I(s) + \sum_{I'} \sum_{J'} \mathcal{P} \int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s', s) \frac{1 - \eta_{J'}^{I'}(s') \cos 2\delta_{J'}^{I'}(s')}{2\rho(s')}$$

- **Nonlinear, inhomogeneous and (Cauchy) singular integral equations**
- **Nonlinear Fredholm integral equations of the second kind**

T.P. Pool, Nuovo Cim. (1978)
 C. Pomponiu and G. Wanders, NPB (1976)
 D. Atkinson and R.L. Warnock, PRD (1977)
 L. Epele and G. Wanders, NPB (1978), PLB (1978)

 J. Gasser and G. Wanders, EPJC (1999)
 G. Wanders, EPJC (2000)

Numerical strategy



- Numerical strategy to solve Roy-Steiner-type equations

