

**QPT 2025**

2025.10.26, Guilin

# **Analytical Solution and Lie algebra of Relativistic Boltzmann Equation**

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Ref: JHEP 2025 (2025) 07, 066, **Jin Hu**

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

Lin-yuan Wei, Yi Wang, Bao-yi Chen, **Jin Hu**, to appear

# Outline

- Introduction
- Analytical Solution to Relativistic Boltzmann Equation
- Lie algebra of Boltzmann Equation
- Summary and outlook

# Introduction

Relativistic Boltzmann equation describes the non-equilibrium evolution of a weakly coupled system, e.g. heavy-ion collisions and cosmology etc.

$$p^\alpha \partial_\alpha f(x, p) = C[f] \quad \text{Nonlinear integro-differential equation}$$

$$C[f] \equiv \int dK dP_f dK_f \left( f(x, p_f^\mu) f(x, k_f^\mu) - f(x, p^\mu) f(x, k^\mu) \right) W_{p, k \rightarrow p_f, k_f}$$

Generally, it is very hard to solve, let alone find an analytical solution.

# Introduction

In rare cases, we could seek an analytical solution:

1. Equilibrium state, important and well-known.

local equilibrium+ killing conditions = global equilibrium

2. BKW solution, homogeneous, isotropic, self-similar

Non-relativistic: R. S. Krupp, M.Sc. thesis, MIT (1967) 1. M. H. Ernst, J. Stat. Phys. 34, 1001 (1984)

A. V. Bobylev, Sov. Phys. Dokl. 20:820, 822 (1976), 1. M. Krook and T. T. Wu, Phys. Rev. Lett. 36:1107 (1976)

Relativistic: D. Bazow etc, Phys. Rev. Lett. 116, 022301 (2016)

# Introduction

By the way,

A.V.**B**obylev : discovering Bobylev instability within  
Burnett equation.

M.**K**rook : BGK model (RTA)

T.T.**W**u(吴大峻): finding that the mathematic structure within  
the gauge field theory is fiber bundle in collaboration with  
Prof.C.N.Yang. May Prof. Yang rest in peace.

# Introduction

## 0. Boltzmann equation (1872)

## 1. Maxwell-Boltzmann Distribution

## 2. BKW solution

R. S. Krupp, M.Sc. thesis, MIT (1967)

A. V. Bobylev, Sov. Phys. Dokl. 20:820, 822 (1976)

1. M. Krook and T. T. Wu, Phys. Rev. Lett. 36:1107 (1976)

D. Bazow etc, Phys. Rev. Lett. 116, 022301 (2016)

So many so far !



$$\varphi_{\text{BKW}}(x, t) = \varphi_M(x) \exp(b_0 x e^{-\lambda t}) (1 - b_0 x e^{-\lambda t}). \quad (4.10.8)$$

几乎与 Bobylev 同时, Krook 和 Wu<sup>[32]</sup> 用不同的方法也找到了这一特解, 因此这个解被称为BKW模. 在 1872 年 Boltzmann 方程提出后的一百年中, 人们所知道的无外场情况下的精确解只是平衡态的 Maxwell 分布. 直到近百年后, 才找到了另一个精确解BKW 模<sup>①</sup>. 这是一个依赖时间的特解, 又是用初等函数表示出来的, 因此十分引人注目.

记  $b = b_0 e^{-\lambda t}$ , 则 (4.10.8) 式可以写为

$$\varphi_{\text{BKW}}(x, t) = \varphi_M(x) e^{bx} (1 - bx), \quad (4.10.9)$$

经逆 Fourier 变换后, 可得 BKW 模的分布函数

$$f_{\text{BKW}}(v, t) = \underbrace{f_M(v)}_{4.9.8} \exp \left[ \frac{-bv^2}{2(1-b)} \right] \cdot \frac{1}{(1-b)^{3/2}} \times \left[ 1 - \frac{3b}{2(1-b)} + \frac{bv^2}{2(1-b)^2} \right]. \quad (4.10.10)$$

输运理论 p209, 黄祖洽, 丁鄂江

# Introduction

Though life (physics) is hard, and always like this,

we have to face Boltzmann eq

$$p^\alpha \partial_\alpha f(x, p) = C[f]$$
$$C[f] \equiv \int dK dP_f dK_f \left( f(x, p_f^\mu) f(x, k_f^\mu) \right. \\ \left. - f(x, p^\mu) f(x, k^\mu) \right) W_{p, k \rightarrow p_f, k_f}$$



Hopefully, we can pull through and find something useful.

# Analytical Solution to Boltzmann Equation

$$p^\alpha \partial_\alpha f(x, p) = C[f]$$

Focus on a homogeneous massless system

$$\hat{p}^0 \partial_{\hat{t}} f(\hat{t}, \hat{p}) = C[f] \quad \hat{t} \equiv t/T, \quad \hat{p} \equiv p/T$$

We remove hat later if nothing confusing occurs.

Try to seek a solution of the following isotropic form

$$f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} \sum_{i=0}^n A_i(t) \cdot (p^0)^i \quad \Rightarrow \quad \text{ODE in powers of } p^0 \quad \Rightarrow \quad \begin{matrix} \alpha(t) \\ A_i(t) \end{matrix}$$

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

# Analytical Solution to Boltzmann Equation

$$p^0 \partial_t f(t, p^0) = C[f]$$

$$f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} \sum_{i=0}^n A_i(t) \cdot (p^0)^i \quad \Rightarrow \quad \text{ODEs in powers of } p^0 \quad \Rightarrow \quad \begin{matrix} \alpha(t) \\ A_i(t) \end{matrix}$$

ODEs should be independent of  $p^0$  as prescribed.

$\Rightarrow$  RHS and LHS should match in the order of  $p^0$

Luckily, we find  $\left\{ \begin{array}{ll} n = 0, & f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} \quad \text{Eq like} \\ n = 1, & f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t)p^0) \quad \text{BKW} \end{array} \right.$

By this way, we know **why BKW solution stands out !**

# Analytical Solution to Boltzmann Equation

$$p^0 \partial_t f(t, p^0) = C[f]$$

Case I: hard sphere

Now we provide a more efficient way to construct relativistic BKW solution

(versus moment method in [D. Bazow etc, Phys. Rev. Lett.116, 022301 \(2016\)](#) )

$$f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t)p^0)$$

The nonlinear collision kernel can be worked out **analytically!** [see Yi Wang's poster.](#)

$$C[f] = e^{-\frac{p^0}{\alpha(t)}} \left( \frac{2A_1^2(t)\alpha^5(t)}{\pi^2} p^0 - \frac{4A_1^2(t)\alpha^4(t)}{3\pi^2} (p^0)^2 + \frac{A_1^2(t)\alpha^3(t)}{6\pi^2} (p^0)^3 \right)$$

# Analytical Solution to Boltzmann Equation

$$p^0 \partial_t f(t, p^0) = C[f]$$

Case I: hard sphere

$$A'_0(t) = \frac{2A_1^2(t)\alpha^5(t)}{\pi^2}$$

$$A'_1(t) + \frac{A_0(t)\alpha'(t)}{\alpha^2(t)} = -\frac{4A_1^2(t)\alpha^4(t)}{3\pi^2}$$

$$\alpha'(t) = \frac{A_1(t)\alpha^5(t)}{6\pi^2}$$

Counting:

3 first order ODEs demand  
3 initial data.

We have:

2 conservation laws

$e_0, n_0$  and 1 free data.

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# Analytical Solution to Boltzmann Equation

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

Case I: hard sphere

$$f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t)p^0)$$

with

$$A_0(t) = \frac{\pi^2(-e_0 + 4n_0\alpha(t))}{\alpha^4(t)} \quad A_1(t) = \frac{\pi^2(e_0 - 3n_0\alpha(t))}{3\alpha^5(t)}$$

$$\alpha(t) = \frac{e_0}{3n_0} + ce^{-\frac{n_0 t}{6}}$$

c is an integration constant.

c = 3/4 reproduces D. Bazow etc, Phys. Rev. Lett.116, 022301 (2016)

# Analytical Solution to Boltzmann Equation

JHEP 2025 (2025) 07, 066, Jin Hu

Case II: anisotropic scattering

$\sigma = \kappa \chi(\Theta)$   $\kappa$  is approximated as momentum-independent.

In this case, the collision kernel is hard to be worked out, so we define

$$\partial_t \rho_n(t) = C^{(n)}(t) \quad f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t) p^0)$$

$$\rho_n \equiv \int dP (p^0)^{n+1} f(t, p^0) \quad C^{(n)}(t) \equiv \int dP (p^0)^n C[f]$$

# Analytical Solution to Boltzmann Equation

JHEP 2025 (2025) 07, 066, Jin Hu

Case II: anisotropic scattering

Using the conservation of  $e_0, n_0$

$$\Rightarrow \left| \begin{aligned} A_0(t) &= \frac{\pi^2(4n_0\alpha(t) - e_0)}{\alpha(t)^4}, & A_1(t) &= \frac{\pi^2(e_0 - 3n_0\alpha(t))}{3\alpha(t)^5} \end{aligned} \right.$$

$$\partial_t \rho_n(t) = C^{(n)}(t) \Rightarrow \alpha'(t) = -\frac{1}{90}(2\pi\sigma(0,2) - 5)(e_0 - 3n_0\alpha(t))$$

$$\text{with } \sigma(f, g) \equiv \frac{2g+1}{2} \int_{-1}^1 dx x^f P_g(x) \chi(x).$$

**Independent of n !**

# Analytical Solution to Boltzmann Equation

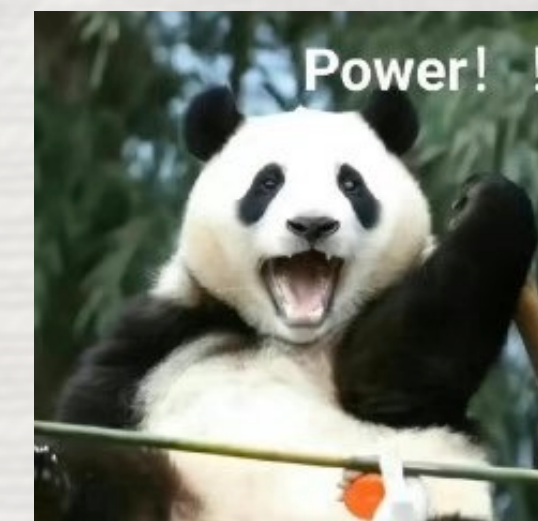
JHEP 2025 (2025) 07, 066, Jin Hu

Case II: anisotropic scattering

Collecting all,  $f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t)p^0)$

with  $A_0(t) = \frac{\pi^2(4n_0\alpha(t) - e_0)}{\alpha(t)^4}, \quad A_1(t) = \frac{\pi^2(e_0 - 3n_0\alpha(t))}{3\alpha(t)^5}$

$$\alpha(t) = (c - \frac{e_0}{3n_0})e^{\frac{1}{90}t(6\pi n_0\sigma(0,2)-15n_0)} + \frac{e_0}{3n_0}.$$



the **first** analytical solution to the relativistic BE with anisotropic scattering

# Analytical Solution to Boltzmann Equation

JHEP 2025 (2025) 07, 066, Jin Hu

Discussion 1:

Even though our cross sections describe anisotropic scatterings,  
The distribution could be isotropic in momentum space.

Discussion 2:  $f \geq 0$

$$\Rightarrow \sigma(0,2) < \frac{5}{2\pi}, \quad \frac{e_0}{4n_0} \leq c \leq \frac{e_0}{3n_0}, \quad \text{where } \sigma(f,g) \equiv \frac{2g+1}{2} \int_{-1}^1 dx x^f P_g(x) \chi(x).$$

Discussion 3:  $\alpha'(t) = -1/90(2\pi\sigma(0,2) - 5)(e_0 - 3n_0\alpha(t))$

$\alpha = e_0/3n_0$  is the only stable fixed point (eq solution)!

# Lie group structure of BE

Consider a general transformation

$$x, p, f \quad \longrightarrow \quad x' \equiv e^{\theta \hat{L}} x, p' \equiv e^{\theta \hat{L}} p, f' \equiv e^{\theta \hat{L}} f$$

$$p \cdot \partial f(x, p) = C[f] \quad (1)$$

If  $f(x, p)$  solves (1), and  $f'(x', p')$  also solves (1),

$$\text{i.e., } p' \cdot \partial' f'(x', p') = C'[f']$$

$$\text{then } \hat{L} = \Omega(x^\mu, p^\mu) \frac{\partial}{\partial f} + X^\mu(x^\mu, p^\mu) \frac{\partial}{\partial x^\mu} + P^\mu(x^\mu, p^\mu) \frac{\partial}{\partial p^\mu}$$

forms a Lie algebra to BE.

# Lie group structure of BE

The PDEs determining these transformation are challenging to solve.

It is not sensible to do it. Changing strategies.

We need to combine physical insights, or it turns into a mathematical game.

$$p^\alpha \partial_\alpha f(x, p) = C[f]$$

First, we know that BE is constructed respecting Poincare group. Then

$$\begin{aligned} \hat{L}_{ij}^{(0)} &= (x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}) + (p_i \frac{\partial}{\partial p^j} - p_j \frac{\partial}{\partial p^i}) & \hat{L}_i^{(1)} &= \frac{\partial}{\partial x^i} \\ \hat{L}_i^{(2)} &= (x_0 \frac{\partial}{\partial x^i} + x_i \frac{\partial}{\partial x^0}) + (p_0 \frac{\partial}{\partial p^i} + p_i \frac{\partial}{\partial p^0}) & \hat{L}_i^{(3)} &= \frac{\partial}{\partial x^0} \end{aligned}$$

These are necessary but trivial because they are close.

# Lie group structure of BE

Relativistic BE admits the scaling solution (NTFP)

$$f(t, p^0) = t^\alpha f_s(t^\beta p^0)$$

e.g., Phys. Rev. Lett. 101, 041603 (2008), J. Berges etc

$$t \rightarrow \lambda t, \quad p^0 \rightarrow p^0 \lambda^{-\beta}, \quad f \rightarrow \lambda^\alpha f$$

It is natural to think about scale symmetry.

We do have scale invariance  $\hat{L}^{(4)} = x^\mu \frac{\partial}{\partial x^\mu} - f \frac{\partial}{\partial f}$

Take-home message:

1. momentum doesn't scale (generally not)
2. Full BE with Bose enhancement or Pauli blocking effects could violate it.
3. Unluckily,  $\hat{L}^{(4)}$  together with  $\hat{L}^{(0)}$  to  $\hat{L}^{(3)}$  forms a **closed** algebra.

# Lie group structure of BE

When can momentum also scale ?

Massless, scaled cross-sections  $p^\mu \rightarrow p^\mu \lambda, \sigma \rightarrow \lambda^{2(l-1)} \sigma$

$$\hat{L}^{(5)} = v x^\mu \frac{\partial}{\partial x^\mu} - \frac{1}{2l+1} p^\mu \frac{\partial}{\partial p^\mu} + (1-v) f \frac{\partial}{\partial f}$$

Take-home message:

1. only for scaled cross-sections  $p^\mu \rightarrow p^\mu \lambda, \sigma \rightarrow \lambda^{2(l-1)} \sigma$
2. Full BE with Bose enhancement or Pauli blocking effects cannot admit scaling solution (why over-occupied condition used ).
3. Unluckily,  $\hat{L}^{(5)}$  together with  $\hat{L}^{(0)}$  to  $\hat{L}^{(4)}$  still forms a closed algebra. Nothing new appears

# Lie group structure of BE

$$\hat{L}_{ij}^{(0)} = (x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}) + (p_i \frac{\partial}{\partial p^j} - p_j \frac{\partial}{\partial p^i})$$

$$\hat{L}_i^{(1)} = \frac{\partial}{\partial x^i}$$

$$\hat{L}^{(4)} = x^\mu \frac{\partial}{\partial x^\mu} - f \frac{\partial}{\partial f}$$

$$\hat{L}_i^{(2)} = (x_0 \frac{\partial}{\partial x^i} + x_i \frac{\partial}{\partial x^0}) + (p_0 \frac{\partial}{\partial p^i} + p_i \frac{\partial}{\partial p^0})$$

$$\hat{L}_i^{(3)} = \frac{\partial}{\partial x^0}$$

$$\hat{L}^{(5)} = v x^\mu \frac{\partial}{\partial x^\mu} - \frac{1}{2l+1} p^\mu \frac{\partial}{\partial p^\mu} + (1-v) f \frac{\partial}{\partial f}, \quad p^\mu \rightarrow p^\mu \lambda, \quad \sigma \rightarrow \lambda^{2(l-1)} \sigma$$



We give, for the **first** time, a closed invariant Lie algebra to the relativistic BE.

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

# Lie algebra of BE

| $\hat{L}_\beta \backslash \hat{L}_\alpha$ | $\hat{L}_{ij}^{(0)}$                                                                                        | $\hat{L}_i^{(1)}$      | $\hat{L}_i^{(2)}$    | $\hat{L}^{(3)}$  | $\hat{L}^{(4)}$ | $\hat{L}^{(5)}$ |
|-------------------------------------------|-------------------------------------------------------------------------------------------------------------|------------------------|----------------------|------------------|-----------------|-----------------|
| $\hat{L}_{mn}^{(0)}$                      | $g_{jm}\hat{L}_{in}^{(0)} + g_{in}\hat{L}_{jm}^{(0)} + g_{jn}\hat{L}_{mi}^{(0)} + g_{im}\hat{L}_{nj}^{(0)}$ | $\backslash$           | $\backslash$         | $\backslash$     | $\backslash$    | $\backslash$    |
| $\hat{L}_m^{(1)}$                         | $g_{jm}\hat{L}_i^{(1)} - g_{im}\hat{L}_j^{(1)}$                                                             | 0                      | $\backslash$         | $\backslash$     | $\backslash$    | $\backslash$    |
| $\hat{L}_m^{(2)}$                         | $g_{jm}\hat{L}_i^{(2)} - g_{im}\hat{L}_j^{(2)}$                                                             | $-g_{im}\hat{L}^{(3)}$ | $\hat{L}_{mi}^{(0)}$ | $\backslash$     | $\backslash$    | $\backslash$    |
| $\hat{L}^{(3)}$                           | 0                                                                                                           | 0                      | $-\hat{L}_i^{(1)}$   | 0                | $\backslash$    | $\backslash$    |
| $\hat{L}^{(4)}$                           | 0                                                                                                           | $\hat{L}_i^{(1)}$      | 0                    | $\hat{L}^{(3)}$  | 0               | $\backslash$    |
| $\hat{L}^{(5)}$                           | 0                                                                                                           | $v\hat{L}_i^{(1)}$     | 0                    | $v\hat{L}^{(3)}$ | 0               | 0               |

**Table 1.** The commutators between the Lie algebra generators are presented in the form of  $[\hat{L}_\alpha, \hat{L}_\beta]$ . For commutators not explicitly listed, it suffices to compare them with the corresponding ones already provided: simply include a minus sign and interchange the indices.

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

# Summary

- Provide a novel simple method to construct the relativistic BKW solution
- The **first** analytical solution to the relativistic BE with anisotropic scattering
- We **first** provide a closed invariant lie algebra to the relativistic BE.

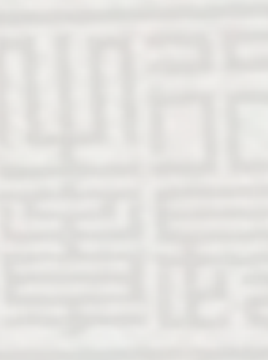
# Outlook

- Other particular solutions
- (non)linear response around BKW
- Relation with NTFP
- Generalized hydro from BKW (done!) **Jin Hu**, to appear
- First-principle determination of Lie algebra
- Multicomponents (done!) Lin-yuan Wei, Yi Wang, Bao-yi Chen, **Jin Hu**, to appear
- Inelastic collisions and application to pQCD

# Back Up

20  
YUAN

2005年



# Analytical Solution in FLRW spacetime

FLRW spacetime:

$$ds^2 = -dt^2 + a^2(t) \frac{dx^2 + dy^2 + dz^2}{1 - K(x^2 + y^2 + z^2)}$$

In maximal symmetric spacetime,  $K$  is a constant.

$$K = \begin{cases} -1 & \textbf{Hyperbolic} \text{ (open, infinite)} \\ 1 & \textbf{Spherical} \text{ (closed, finite)} \\ 0 & \textbf{Flat} \text{ (infinite)} \end{cases}$$

Modern cosmological observational data strongly support that the universe is spatially flat on large scales.  $K \simeq 0$

D. Bazow etc, Phys. Rev. Lett.116, 022301 (2016) also takes this choice.

# Analytical Solution in FLRW spacetime

hard sphere case

$$f(t, p^0) = e^{-\frac{p^0}{\alpha(t)}} (A_0(t) + A_1(t)p^0)$$

$$\text{FLRW: } ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

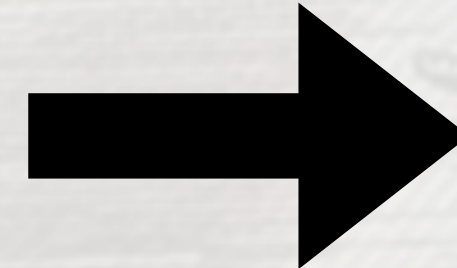
with

$$A_0(t) = \frac{\pi^2(-e_0 + 4n_0\alpha(t))}{\alpha^4(t)}$$

$$A_1(t) = \frac{\pi^2(e_0 - 3n_0\alpha(t))}{3\alpha^5(t)}$$

$$\alpha(t) = \frac{e_0}{3n_0} + ce^{-\frac{n_0 t}{6}}$$

how ?



$$f(\tau, p^0) = \lambda \exp\left(-\frac{p^0}{\mathcal{K}(\tau)T_0}\right)$$

$$\times \left[ \frac{4\mathcal{K}(\tau) - 3}{\mathcal{K}^4(\tau)} + \frac{p^0}{T_0} \left( \frac{1 - \mathcal{K}(\tau)}{\mathcal{K}^5(\tau)} \right) \right].$$

$$\text{with } \mathcal{K}(\tau) \equiv 1 - \frac{1}{4}e^{-\tau/6}, \quad \tau(t) = \int_{t_0}^t \frac{1}{a^3(s)} ds$$

Yi Wang, Xuan Zhao, Zhe Xu, **Jin Hu**, to appear

D. Bazow etc, Phys. Rev. Lett.116, 022301 (2016)

# Introduction

In general cases, we linearize it (we partly suppress x-dependence below)

$$p^\alpha \partial_\alpha f(x, p) = C[f] \quad f(x, p) = f_0(x, p)(1 + \chi(x, p))$$

$$-f_0(p) \mathcal{L}_1[\chi] \equiv f_0(p) \int dK dP_f dK_f f_0(k) W(\chi(p_f) + \chi(k_f) - \chi(p) - \chi(k))$$

Mathematical aspects:

S.R.DeGroot et al, Relativistic kinetic theory

1.  $\mathcal{L}_1$  is self-adjoint, positive semidefinite.
2.  $\mathcal{L}_1[\psi] = 0$ ,  $\psi = a + b \cdot p$  collision invariants

Though linearized, it is still intractable. e.g. Chapman-Enskog, moment expansion

# Introduction

Also in rare cases, the eigen spectrum can be solved analytically.

$$-f_0(p)\mathcal{L}_1[\chi] \equiv f_0(p) \int dK dP_f dK_f f_0(k) W \\ \times (\chi(p_f) + \chi(k_f) - \chi(p) - \chi(k))$$



1. Non-relativistic monatomic gases given by C.S.Wang and G.E.Uhlenbeck

Studies in the Statistical Mechanics (1970), J.De Boer and G.E.Uhlenbeck, Chap IV

2. Relativistic massless  $\lambda\phi^4$  theory ( for  $\mathcal{L}_1$  )

G.S.Denicol, J.Noronha, *Phys.Lett.B* 850 (2024) 138487

So many so far !

# Lie group structure of BE

Relativistic BE admits the scaling solution (NTFP)  $f(t, p^0) = t^\alpha f_s(t^\beta p^0)$

$$\hat{L}^{(5)} = ax^\mu \frac{\partial}{\partial x^\mu} - \frac{1}{2l+1} p^\mu \frac{\partial}{\partial p^\mu} + (1-a)$$

Why full BE cannot admit scaling solution ?

$$t \rightarrow \lambda t, \quad p^0 \rightarrow p^0 \lambda^{-\beta}, \quad f \rightarrow \lambda^\alpha f. \quad \text{Note time evolves = time scales !}$$

$\lambda^\alpha f$  is still the solution of BE

Choose  $a(2l+1) = 1/\beta$ , unless  $a = 1$  for given  $\beta, l$

Full BE with quantum effects cannot admit scaling solution !

# Lie group structure of BE

$$p^\alpha \partial_\alpha f(x, p) = C[f]$$

Relativistic BE admits the scaling solution (NTFP)  $f(t, p^0) = t^\alpha f_s(t^\beta p^0)$

Phys. Rev. Lett. 101, 041603 (2008), J. Berges etc

We do have scale invariance  $\hat{L}^{(4)} = x^\mu \frac{\partial}{\partial x^\mu} + 1$

But it is trivial:

1. momentum doesn't scale (generally not)
2. Unluckily,  $\hat{L}^{(4)}$  together with  $\hat{L}^{(0)}$  to  $\hat{L}^{(3)}$  forms a **closed** algebra.