

Collective motion in the massive Schwinger model via Tensor Network

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Collaborators: Shile Chen (陈诗乐), Shuzhe Shi (施舒哲)

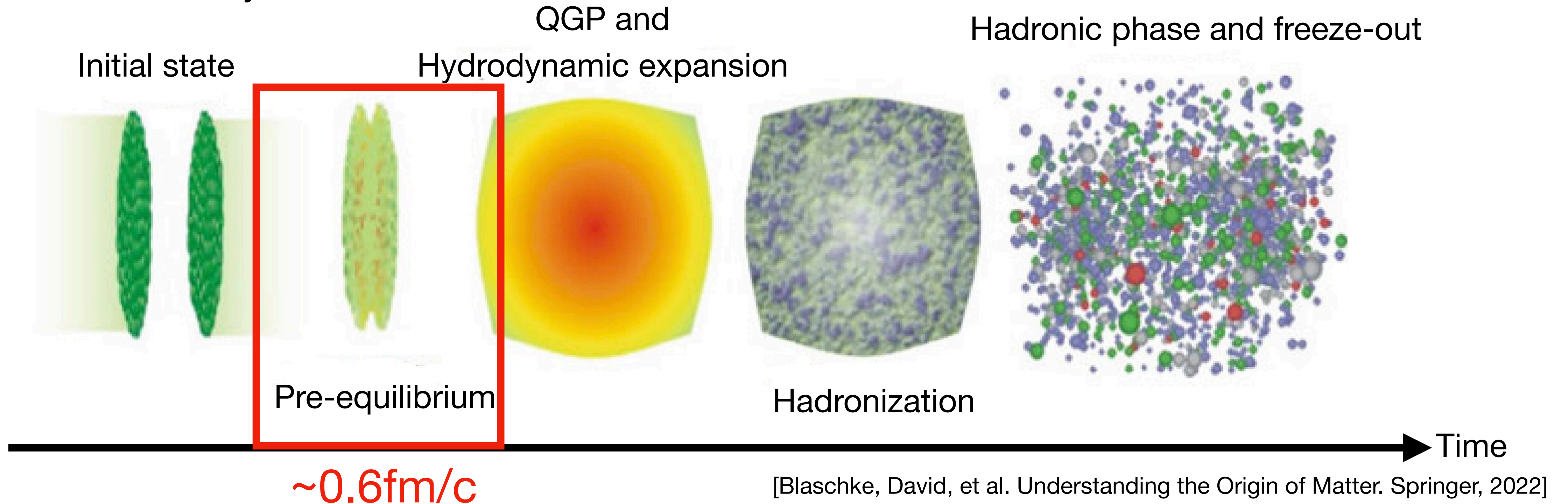


arXiv:2509.10835
arXiv:2509.10855



1.1 Introduction: Big Picture & Motivation

Relativistic Heavy-Ion Collisions



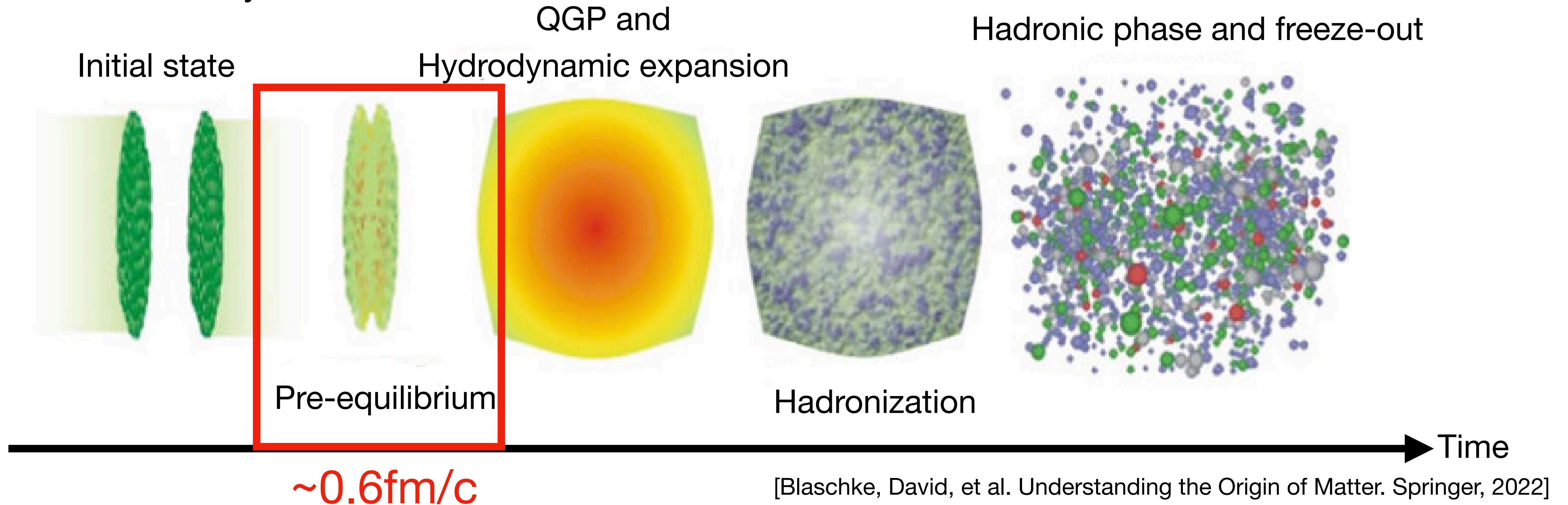
Why does hydrodynamization occur so rapidly in HIC?

Kinetic Theory: Jin Hu's and Xingjian Lu's talks today

Yiyang Peng's and Anping Huang's talks at Monday

1.1 Introduction: Big Picture & Motivation

Relativistic Heavy-Ion Collisions



Does hydrodynamization really exist in QCD systems?

Search for it in a first-principles calculation

1.2 Introduction: Challenge & Our Approach

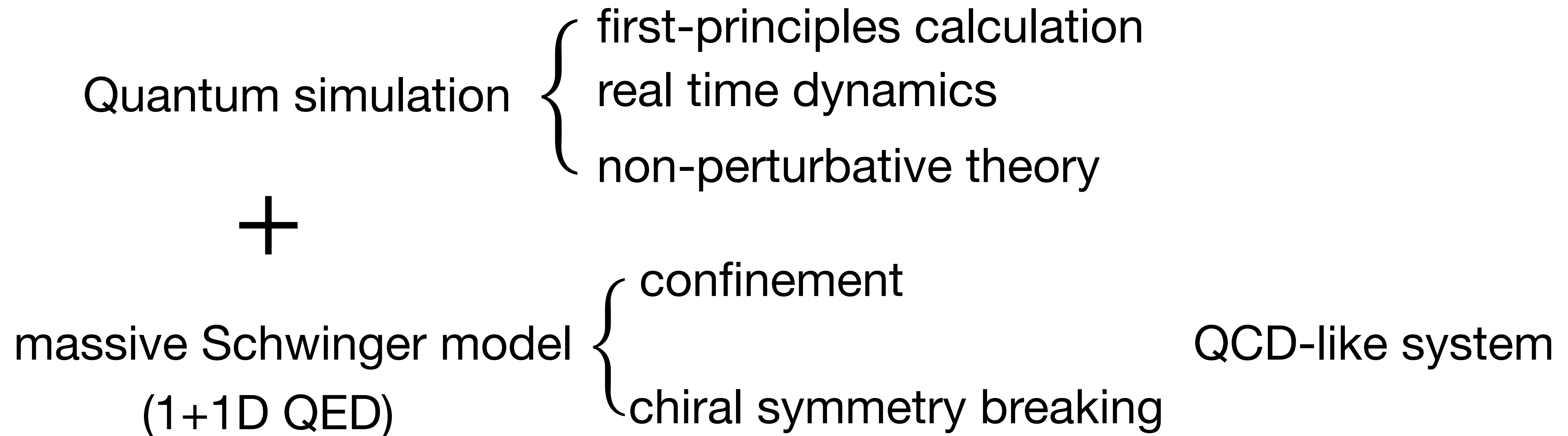
*In a **first-principles** calculation, search for **hydrodynamization***

Quantum simulation { first-principles calculation
real time dynamics
non-perturbative theory

Ideally, we should simulate 3+1 D QCD, but...

1.2 Introduction: Challenge & Our Approach

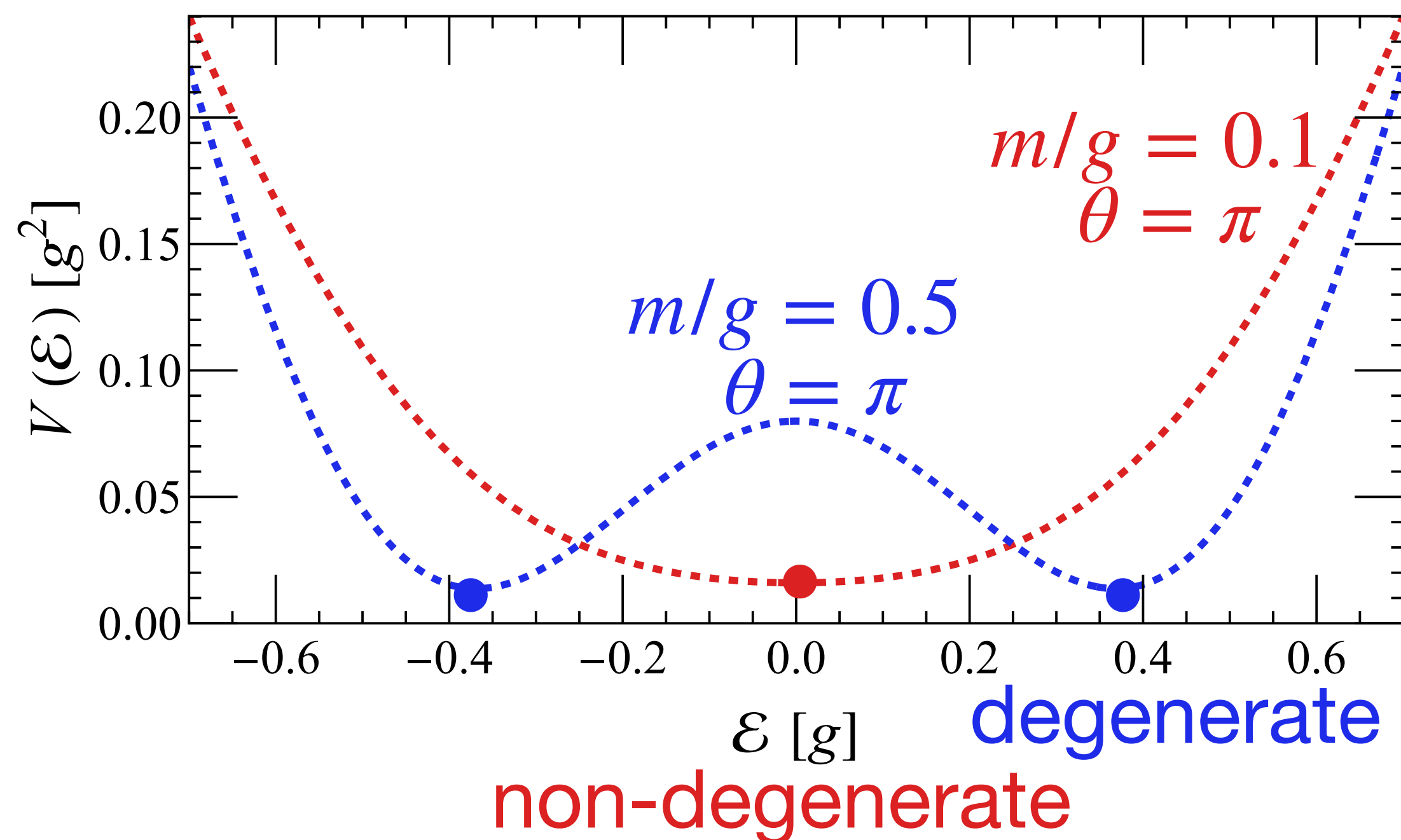
*In a **first-principles** calculation, search for **hydrodynamization***



1.2 Introduction: Challenge & Our Approach

*In a **first-principles** calculation, search for **hydrodynamization***

massive Schwinger model (1+1D QED) $\left\{ \begin{array}{l} \text{confinement} \\ \text{chiral symmetry breaking} \end{array} \right.$ QCD-like system



non-trivial phase structure

Parity-restored $\longleftrightarrow$ Parity-broken

*study **dynamical features** of different phases*

2.1 Model and method: massive Schwinger model

$$H \equiv \int \left(\frac{\mathcal{E}^2}{2} - \frac{i}{2} \bar{\psi} \gamma^1 \partial_z \psi + \frac{i}{2} (\partial_z \bar{\psi}) \gamma^1 \psi + \bar{\psi} (g \gamma^1 A_1 + m e^{i\gamma_5 \theta}) \psi \right) dz$$

$\mathcal{E}$: electric field

A_1 : electric potential

$\psi, \bar{\psi}$: fermion field

$$\frac{g\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu}$$

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Kogut-Susskind

$$\psi(z = a n) \leftrightarrow \frac{1}{\sqrt{a}} \begin{pmatrix} \chi_{2n} \\ \chi_{2n-1} \end{pmatrix}$$

Jordan-Wigner

$$\chi_n = \frac{X_n - iY_n}{2} \prod_{m=1}^{n-1} (-iZ_m)$$

$$L_n \equiv g^{-1} \mathcal{E}(z_n)$$

$$\phi_n \equiv a g A_1(z_n)$$

$$[\phi_n, L_m] = a [A(z_n), \mathcal{E}(z_m)] = i \delta_{nm}$$

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$$\phi_n \equiv agA_1(z_n)$$

$$[\phi_n, L_m] = a [A(z_n), \mathcal{E}(z_m)] = i \delta_{nm}$$

$$H = \frac{ag^2}{2} \sum_{n=1}^{N-1} L_n^2 + \frac{1}{2a} \sum_{n=1}^{N-1} (S_n^+ e^{i\phi_n} S_{n+1}^- + S_{n+1}^+ e^{-i\phi_n} S_n^-)$$

$$+ \frac{m \sin \theta}{2} \sum_{n=1}^{N-1} (-1)^{n+1} (S_n^+ e^{i\phi_n} S_{n+1}^- + S_{n+1}^+ e^{-i\phi_n} S_n^-) + \frac{m \cos \theta}{2} \sum_{n=1}^N (-1)^n Z_n$$

$$N = 100$$

$$a = 0.5/g$$

2.2 Model and method: Tensor Network Method

Matrix Product State (MPS) Representations

$$|\Phi\rangle = \sum_{\{s\}} \Phi_{s_1 s_2 \dots s_{N-1} s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_{N-1}\rangle \otimes |s_N\rangle$$

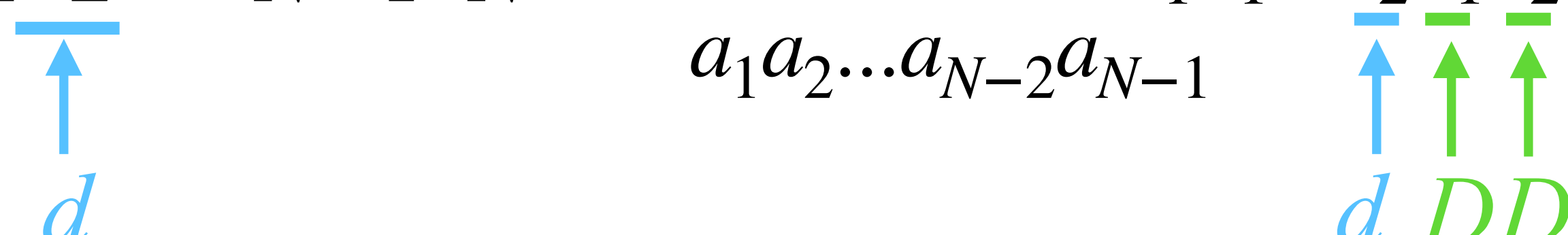
$$\Phi_{s_1 s_2 \dots s_{N-1} s_N}$$


exact representation
complete Hilbert space: d^N

2.2 Model and method: Tensor Network Method

Matrix Product State (MPS) Representations

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$$\Phi_{s_1 s_2 \dots s_{N-1} s_N} \approx \sum_{a_1 a_2 \dots a_{N-2} a_{N-1}} A_{s_1 a_1}^{(1)} A_{s_2 a_1 a_2}^{(2)} \dots A_{s_{N-1} a_{N-2} a_{N-1}}^{(N-1)} A_{s_N a_{N-1}}^{(N)}$$


exact representation
complete Hilbert space: d^N

MPS representation
approximate Hilbert space: NdD^2

2.2 Model and method: Tensor Network Method

Matrix Product State (MPS) Representations

complete Hilbert space: d^N

MPS Hilbert space: NdD^2

When is this representation effective?

Quantum states with **finite entanglement entropy**, such as the ground states (and low-energy excited states) of 1D local Hamiltonians.

Bipartite entanglement entropy of MPS : $S_{EE} = -\text{Tr}(\rho_{\text{half}} \ln(\rho_{\text{half}})) \leq \ln(D)$

Orús, Román. "A practical introduction to tensor networks: Matrix product states and projected entangled pair states." *Annals of physics* 349 (2014): 117-158.

Reliable results require a converged bond dimension D

2.2 Model and method: Tensor Network Method

Time Evolution

$$|\Phi(0)\rangle \quad \text{---} \quad \text{[Orange Box]} \quad \text{---}$$

$$|\Phi(t)\rangle = e^{-it\hat{H}} |\Phi(0)\rangle$$

- Measure the observables:

$$\text{electric field} \quad \mathcal{E} = g \langle \Phi(t) | \hat{L} | \Phi(t) \rangle$$

$$\text{charge density} \quad j^t = \langle \Phi(t) | \hat{j}^t[\chi_n, \chi_n^\dagger] | \Phi(t) \rangle$$

$$\text{stress tensor} \quad T^{\mu\nu} = \langle \Phi(t) | \hat{T}^{\mu\nu}[\chi_n, \chi_n^\dagger, \phi_n, L_n] | \Phi(t) \rangle$$

- Decompose the tensor into:

$$T^{\mu\nu} = (\varepsilon + P + \Pi)u^\mu u^\nu - (P + \Pi)g^{\mu\nu} + \pi^{\mu\nu}$$

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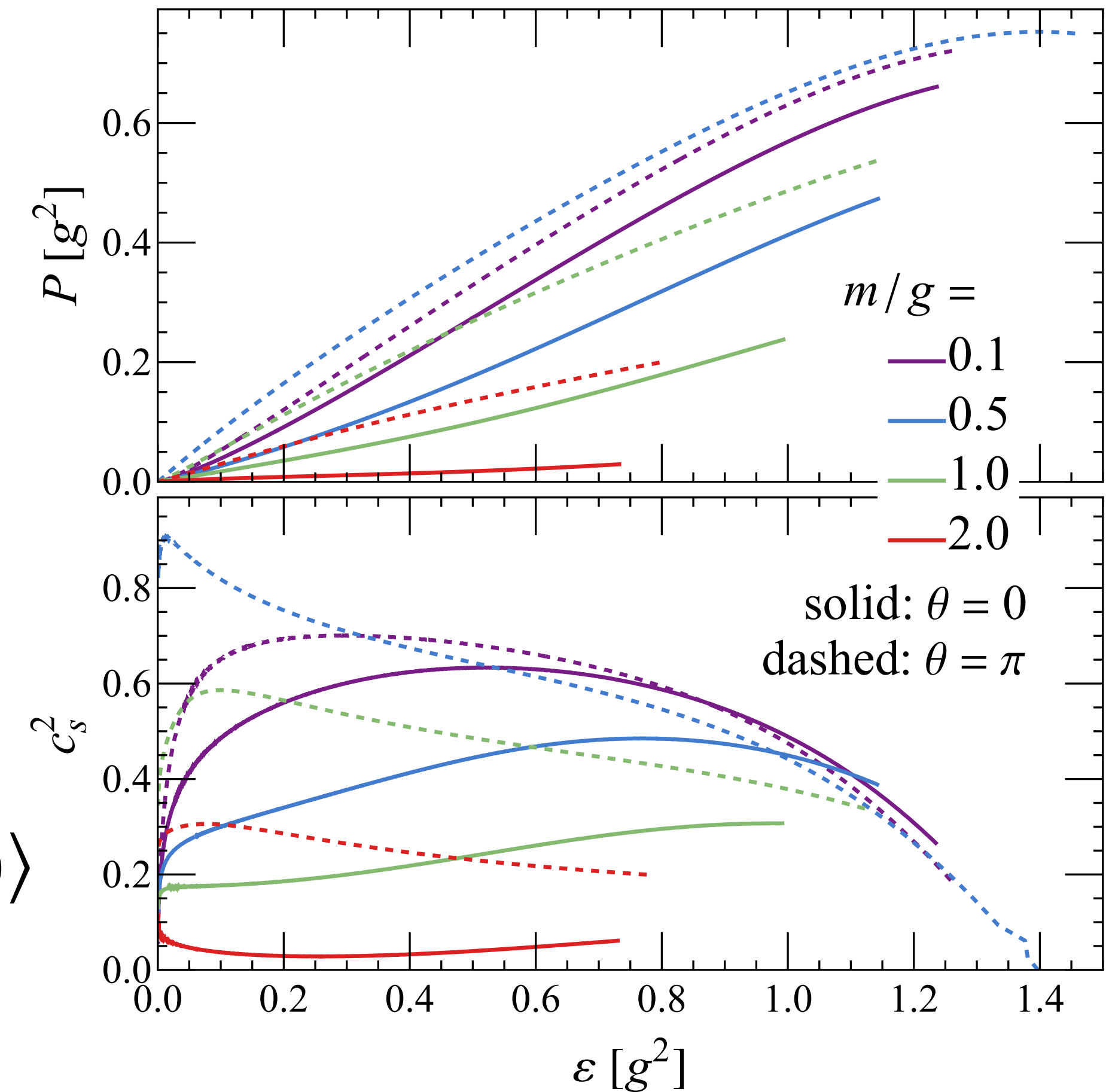
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$$T^{\mu\nu} = (\varepsilon + P + \Pi)u^\mu u^\nu - (P + \Pi)g^{\mu\nu} + \pi^{\mu\nu}$$

$$P \equiv P(\varepsilon)$$



3. Hydrodynamic behavior

small mass $m = 0.1g$ $m = 0.5g$ $m = 1g$ $m = 2g$ large mass

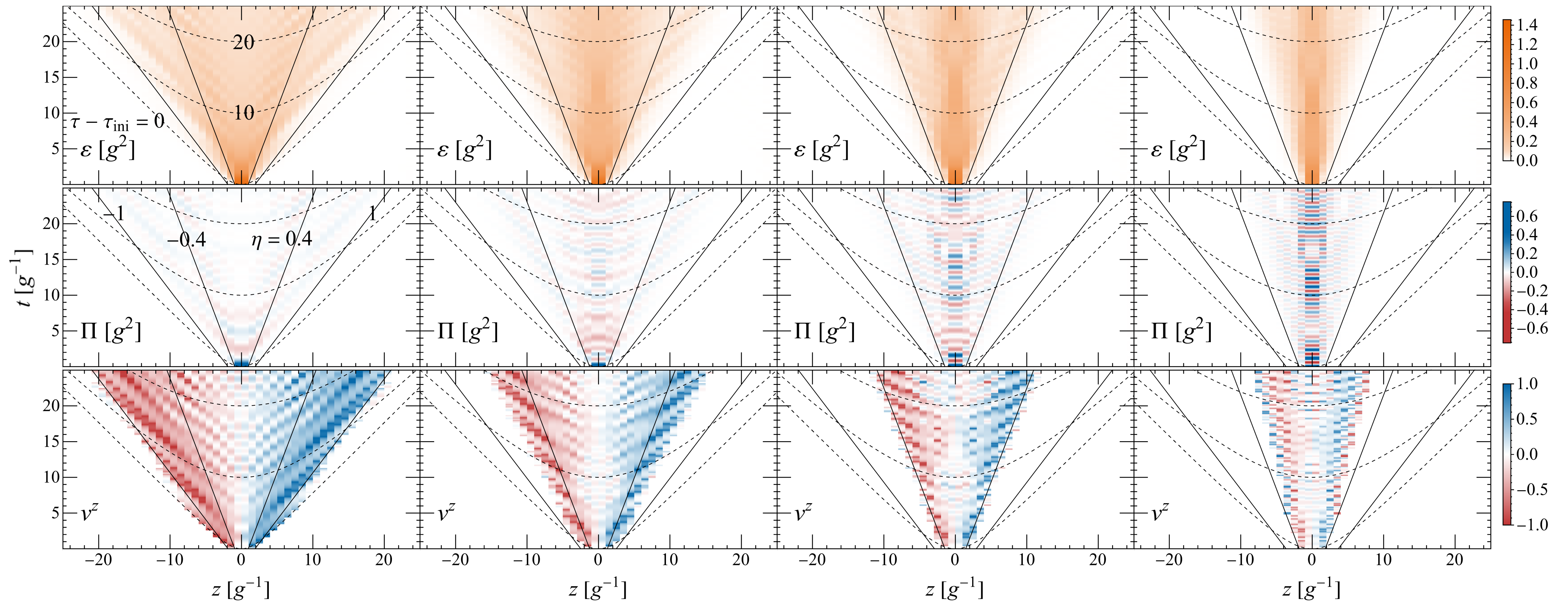
strong coupling $\theta = 0$ weak coupling

$g = 10m$

$g = 2m$

$g = 1m$

$g = 0.5m$



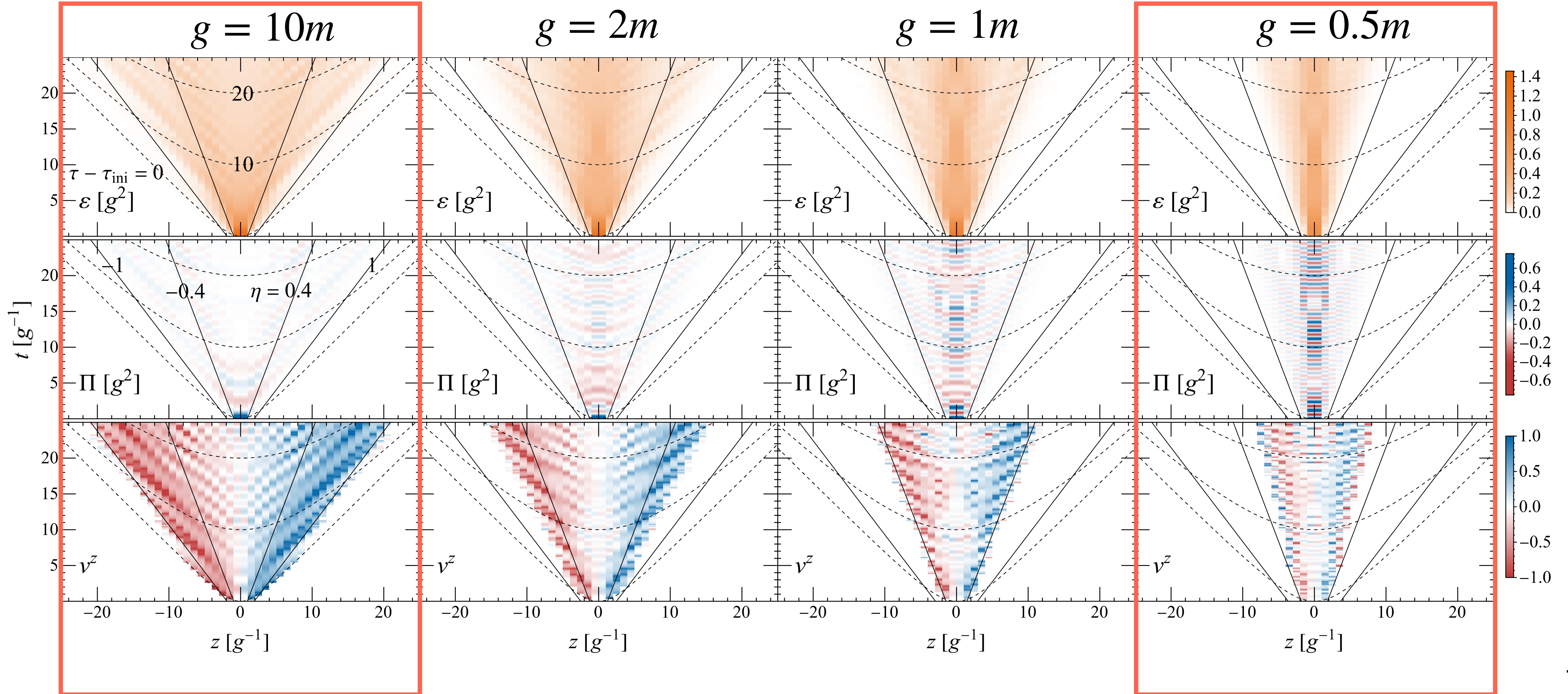
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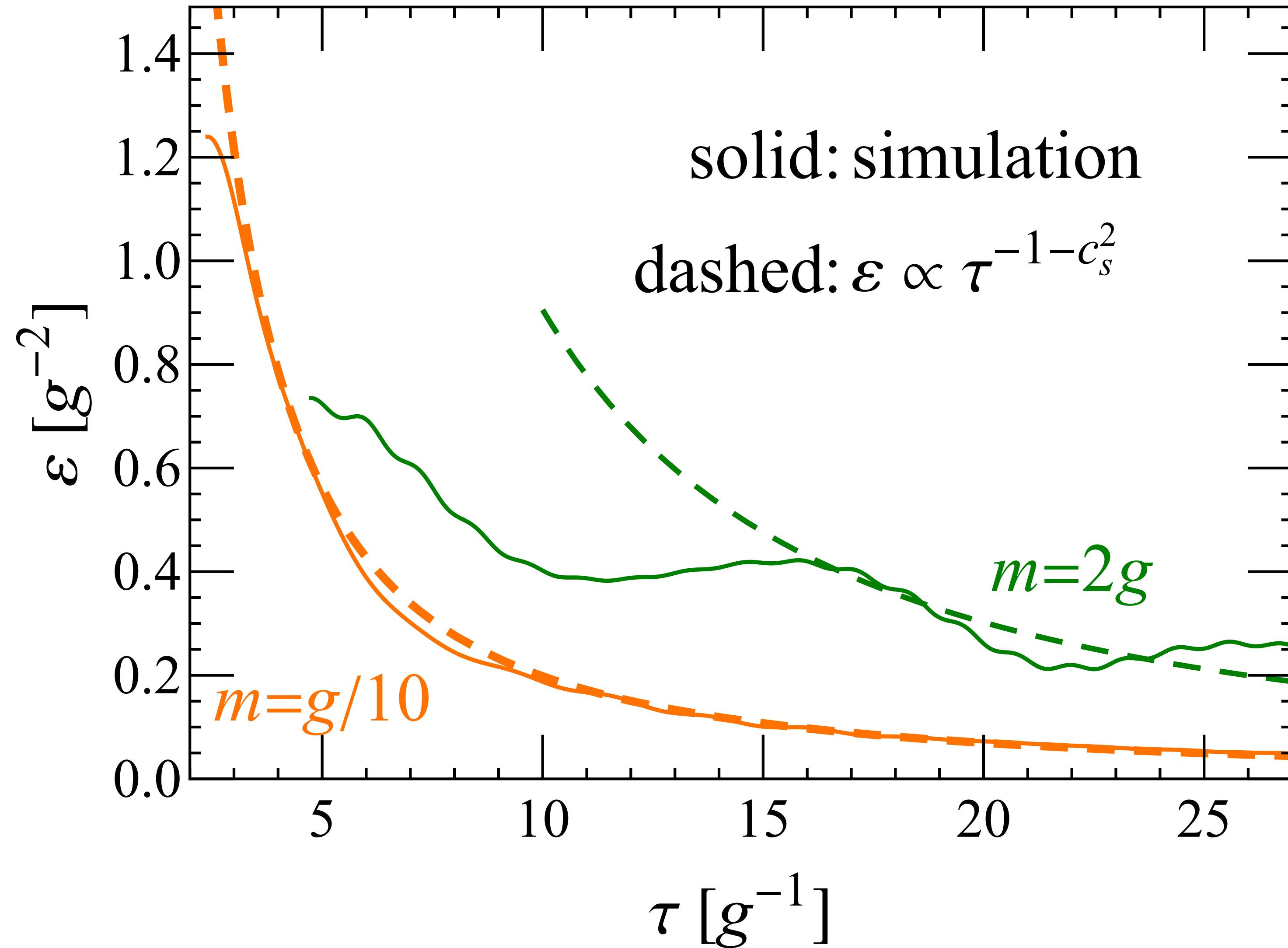
strong coupling

$\theta = 0$

weak coupling

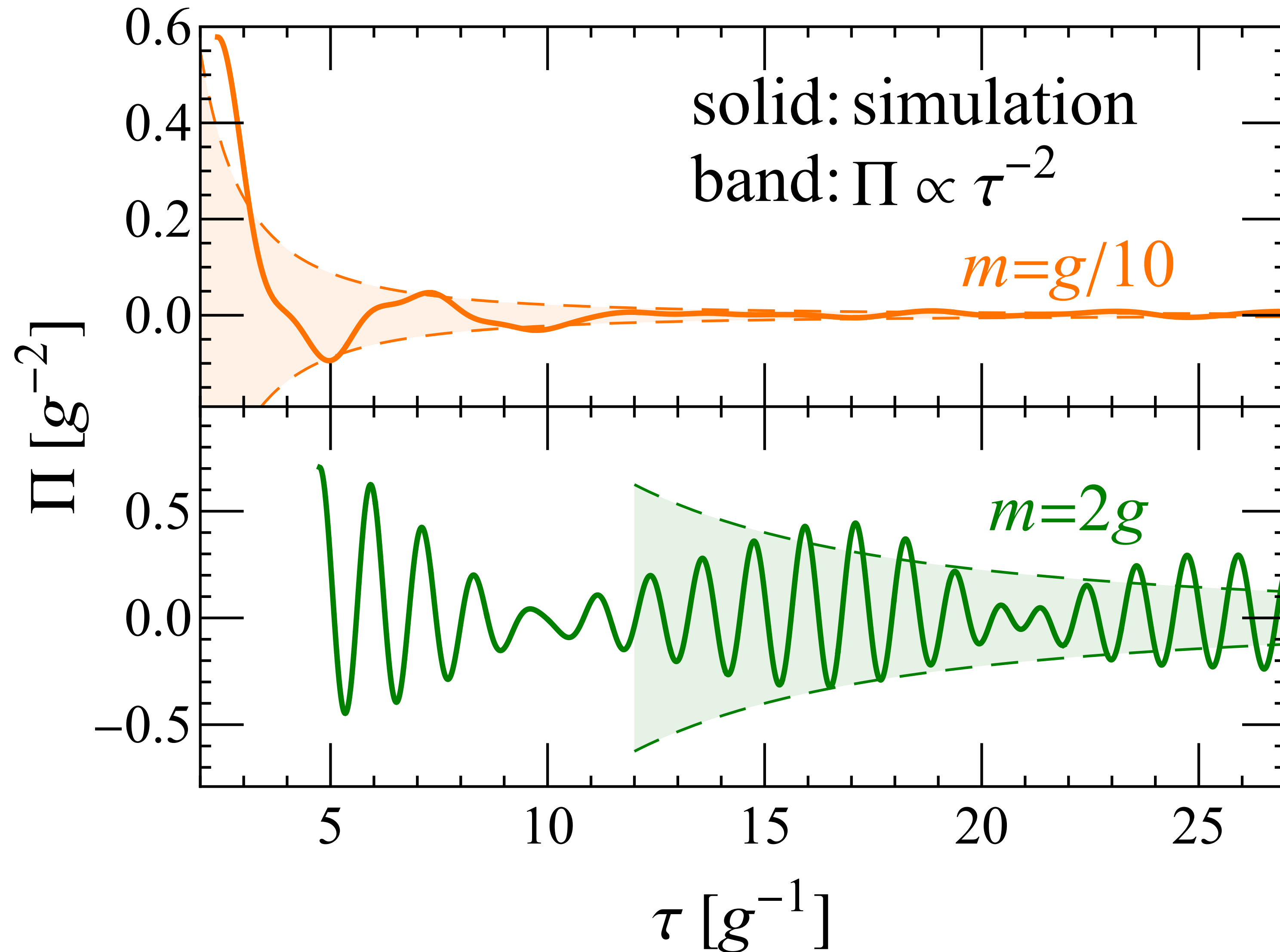


3. Hydrodynamic behavior



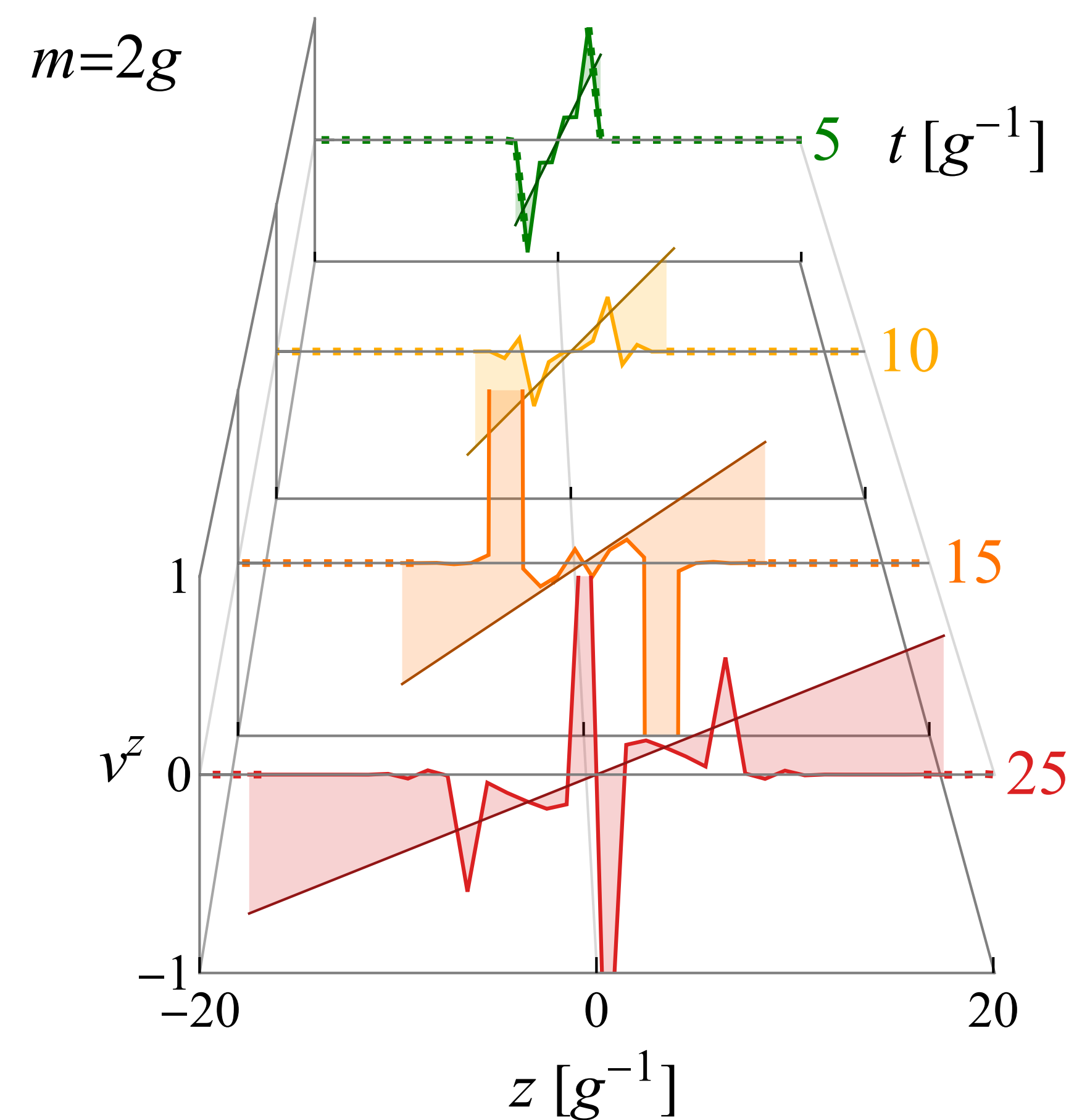
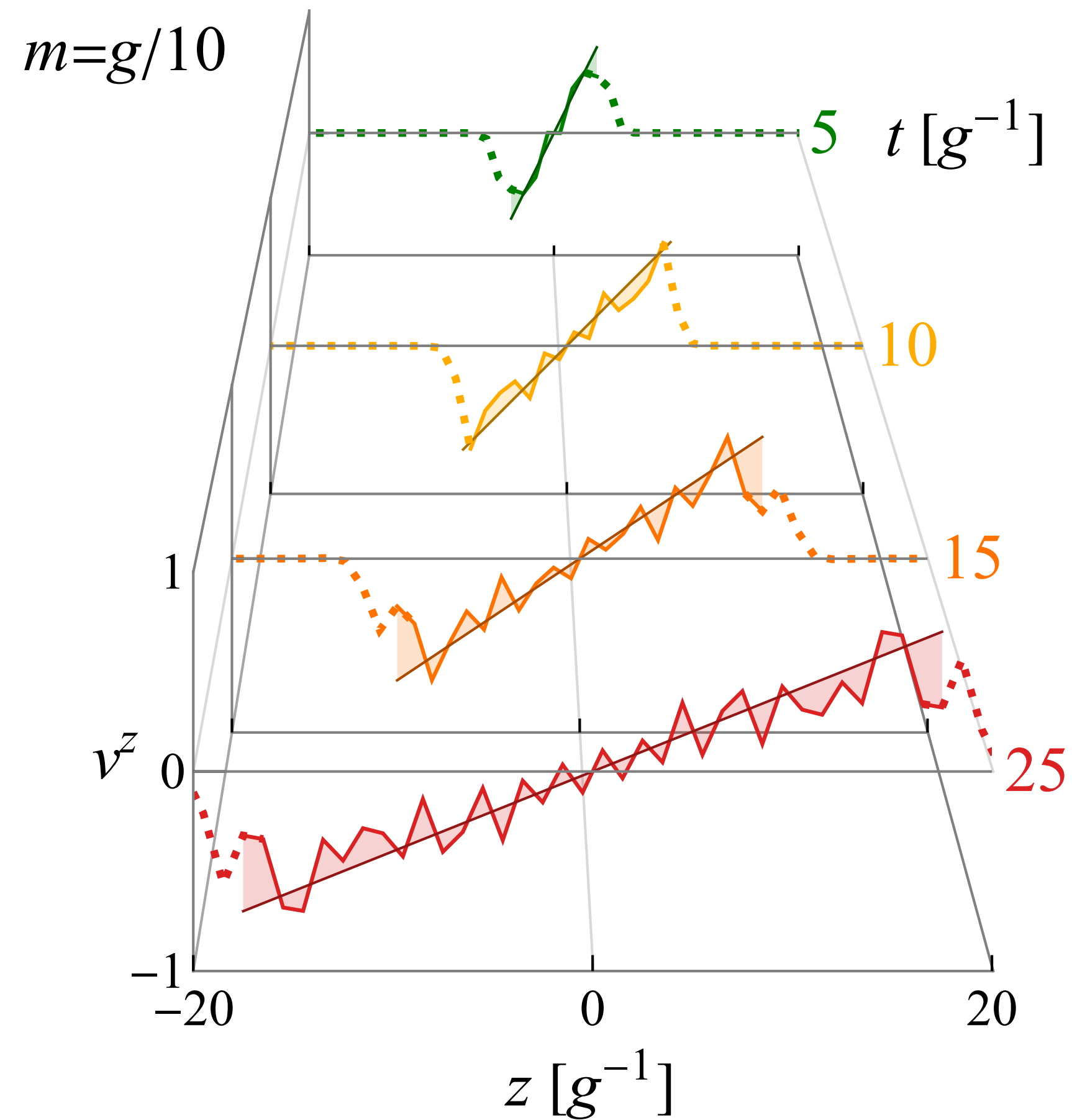
Only for strong coupling,
for large enough τ ,
in the central rapidity region
 $\eta = 0$: $\epsilon_{\eta=0} \propto \tau^{-(1+c_s^2)}$

3. Hydrodynamic behavior



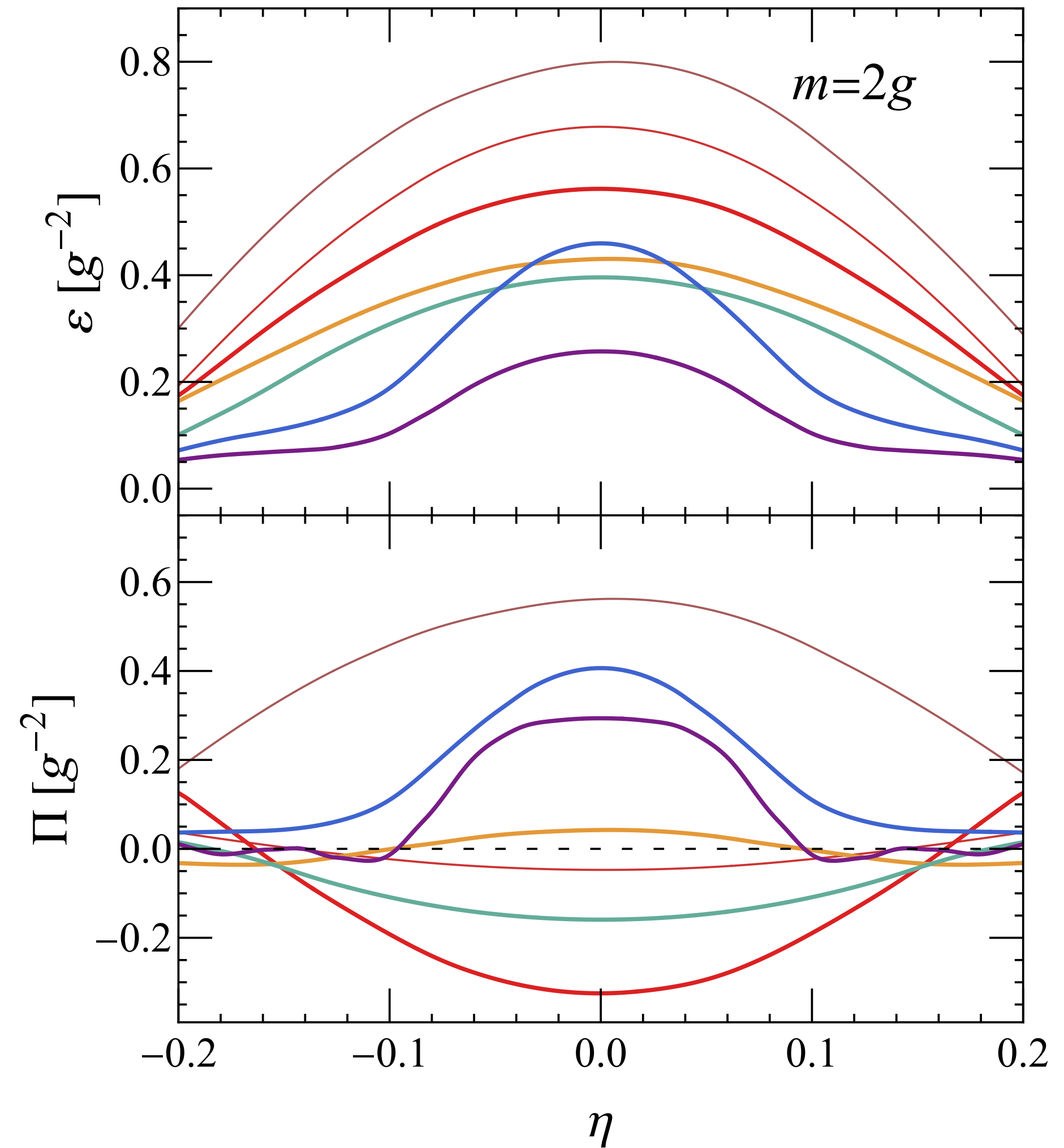
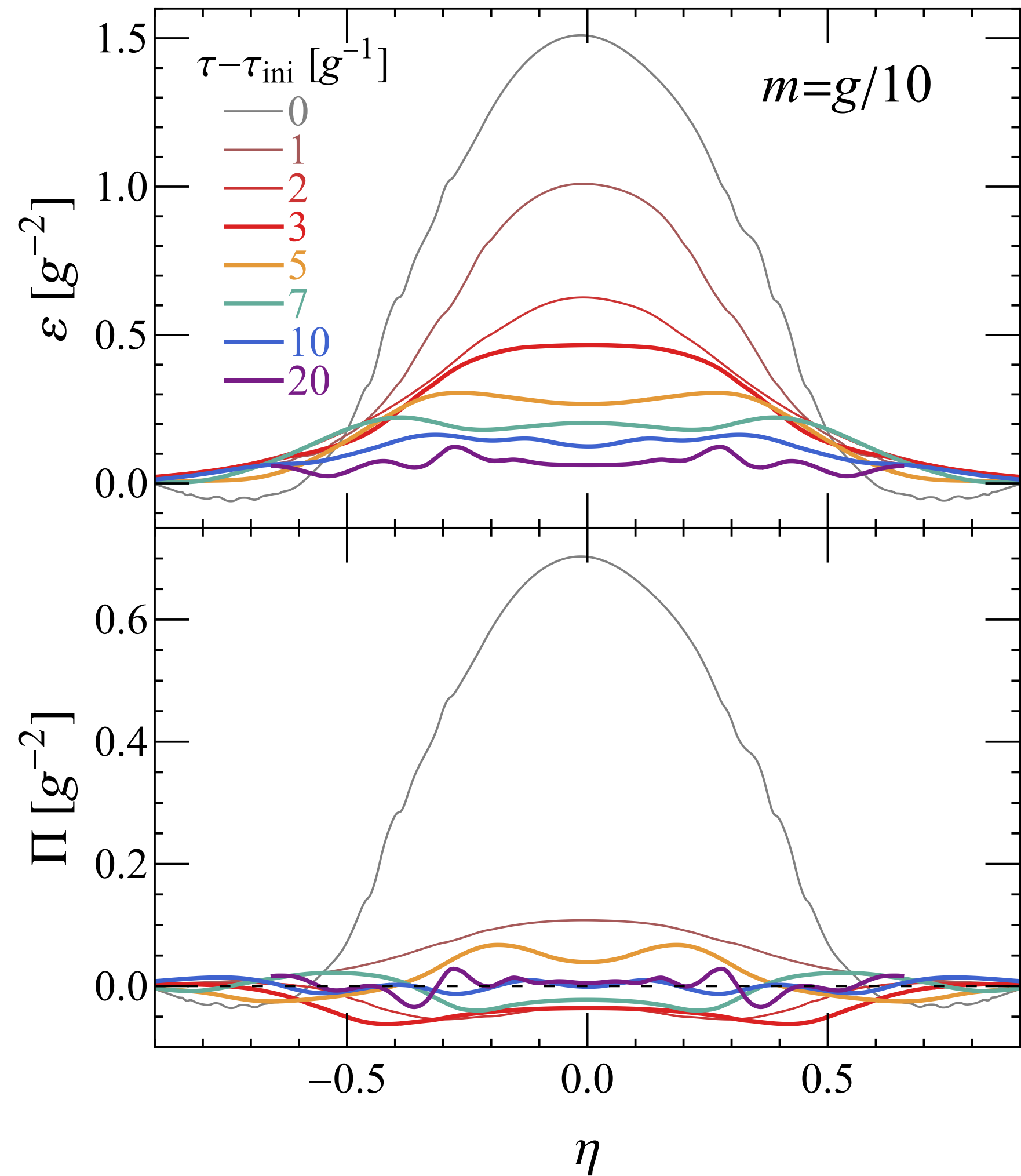
Only for strong coupling,
for large enough τ ,
in the central rapidity region
 $\eta = 0$: $\Pi_{\eta=0} \approx \propto \tau^{-2}$

3. Hydrodynamic behavior



Only for strong coupling, flow velocity satisfies boost invariant: $v^z \approx \frac{z}{t}$

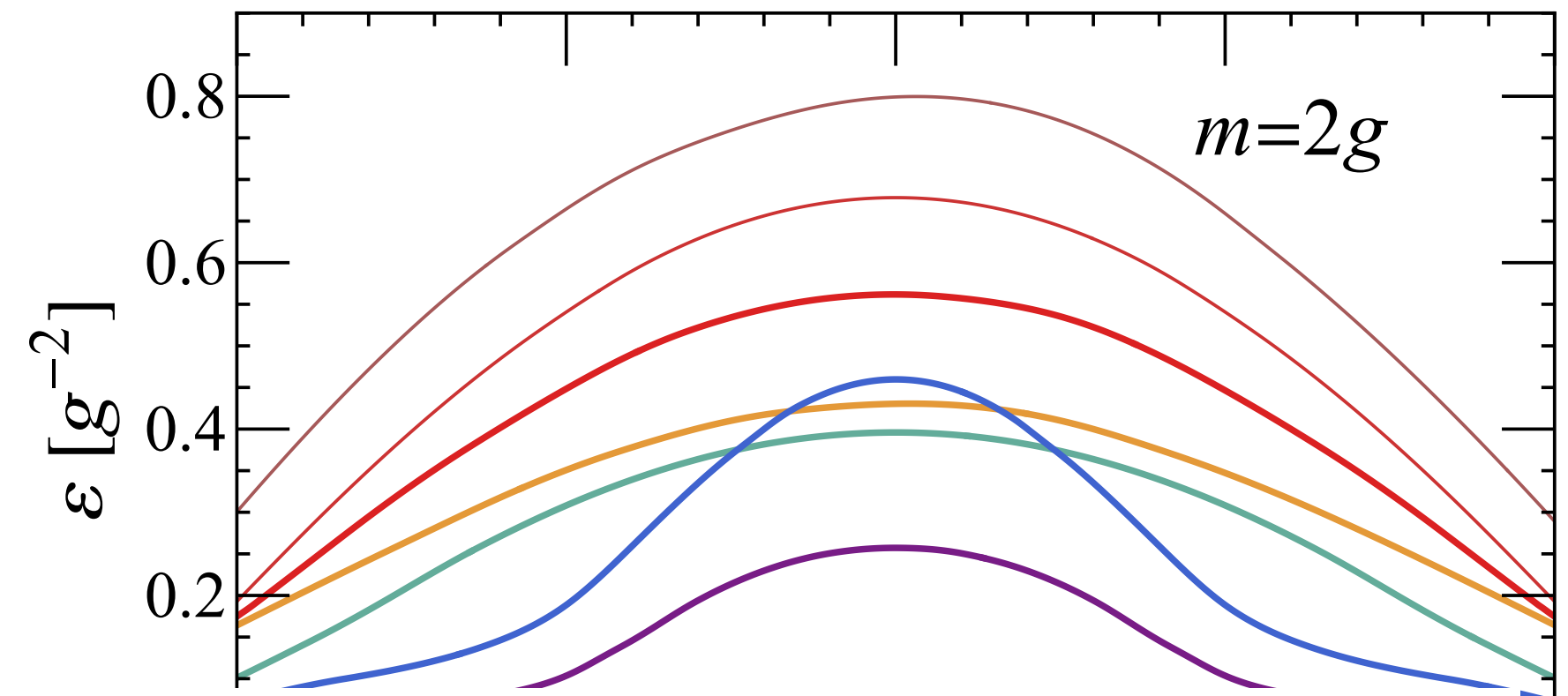
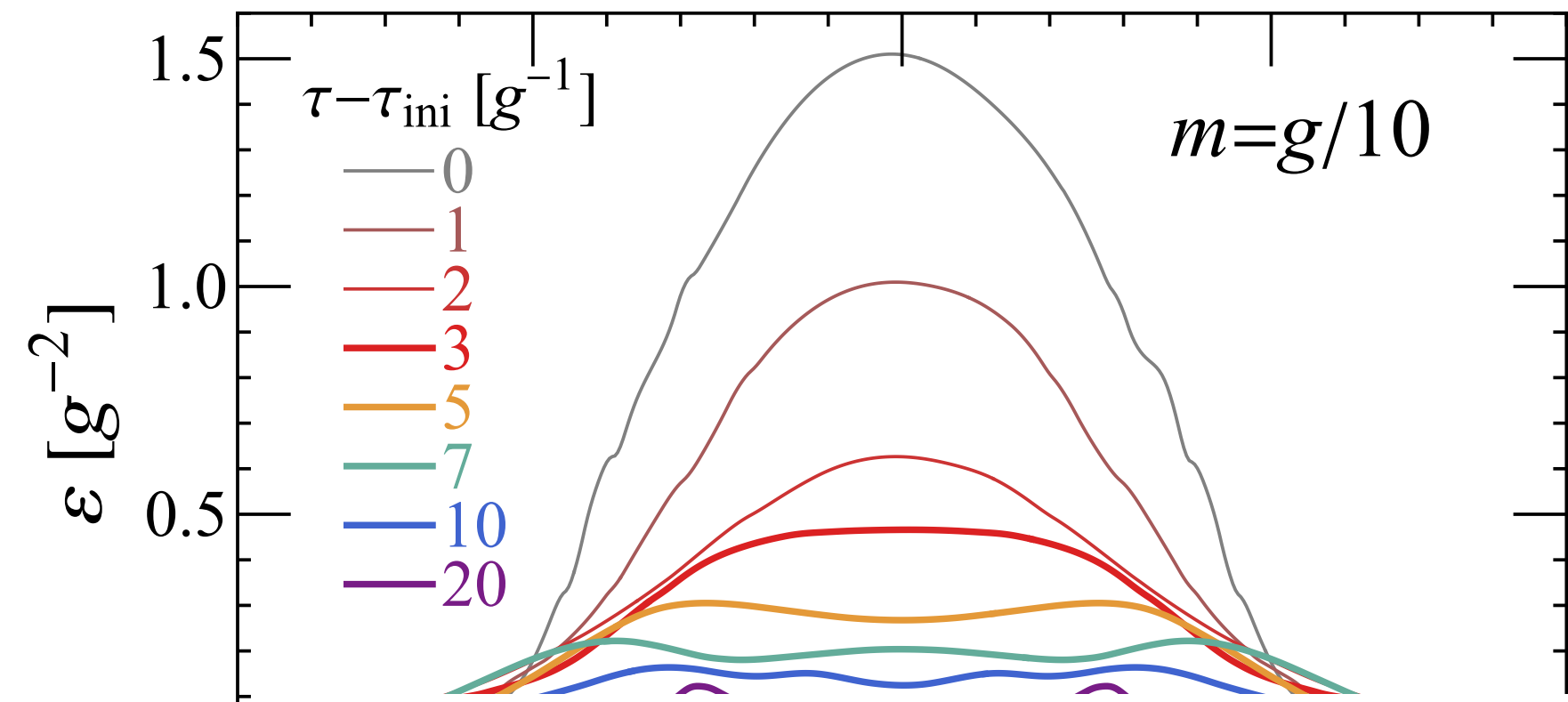
3. Hydrodynamic behavior



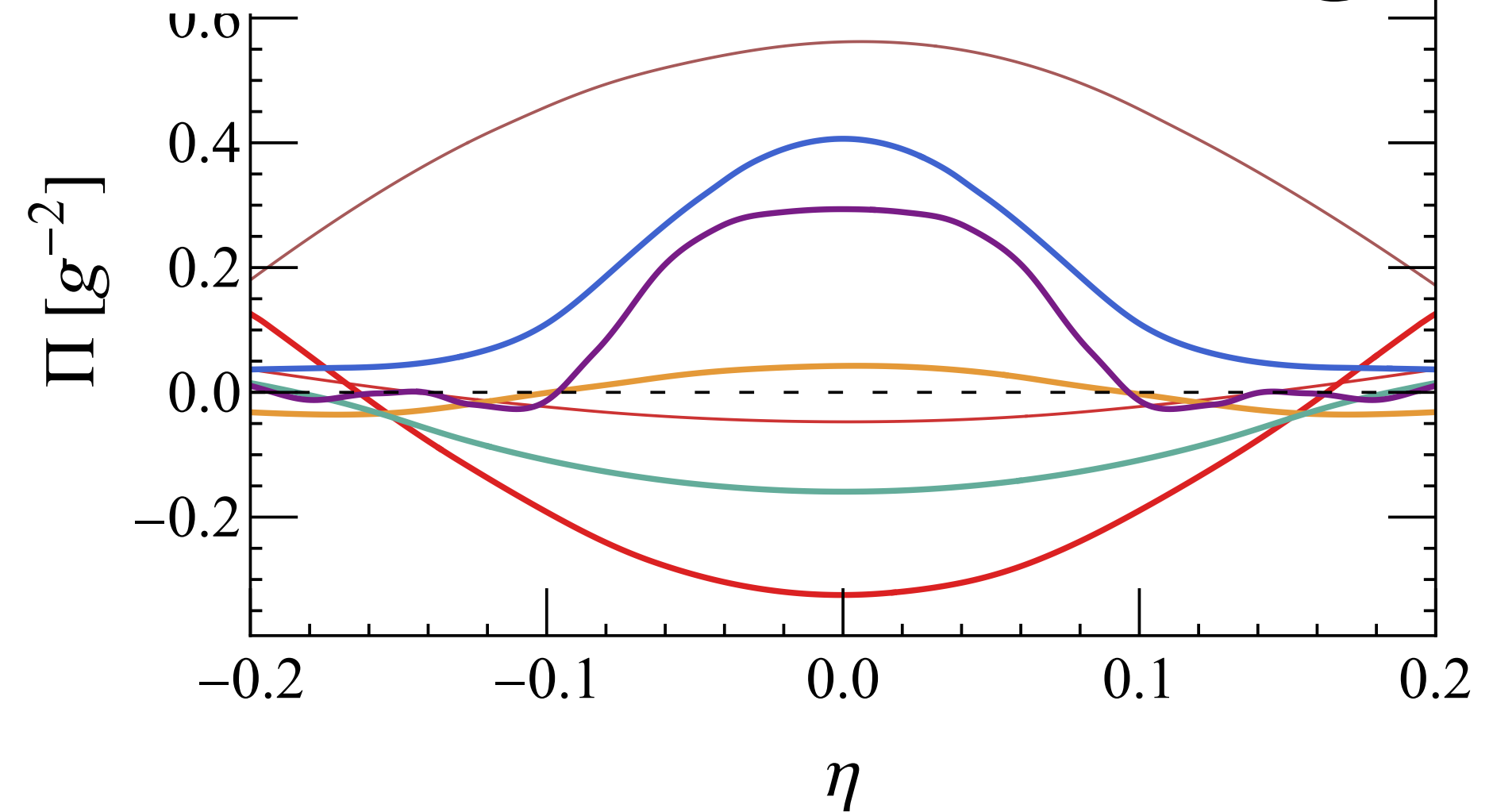
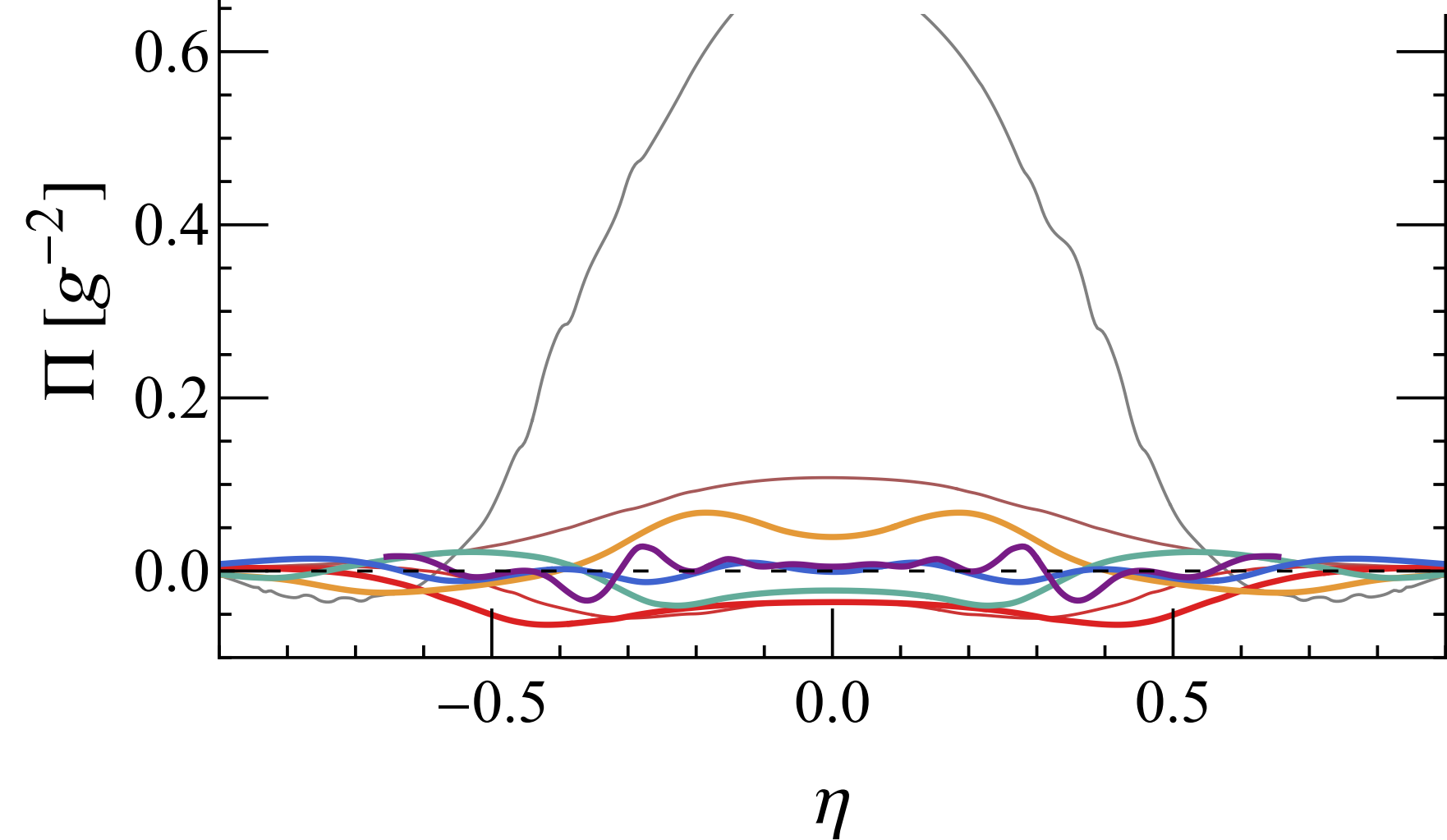
Only for strong coupling, for large enough τ :

rapidity plateau in ϵ and Π

3. Hydrodynamic behavior



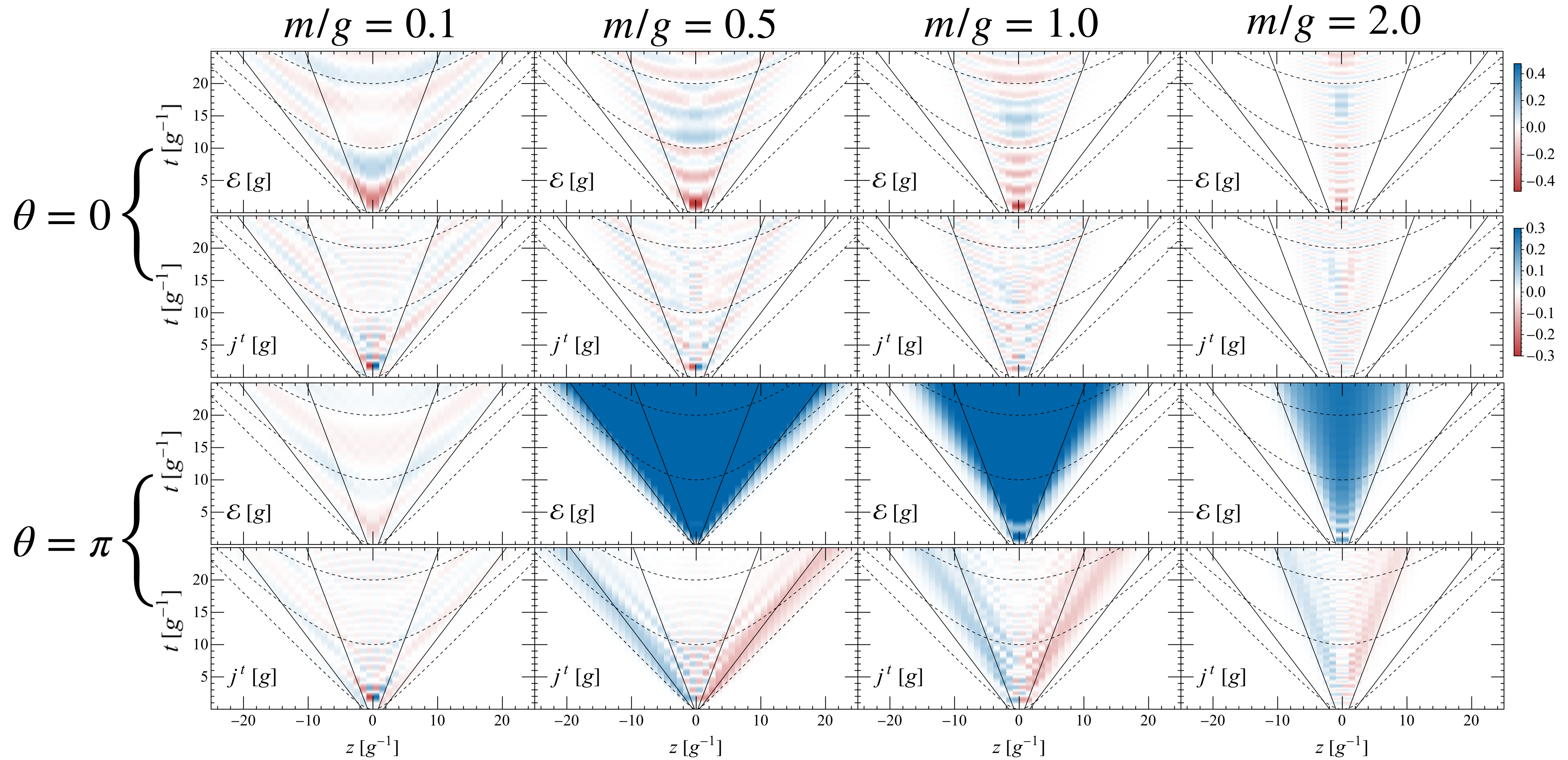
Bjorken-like flow occurs only at small m/g



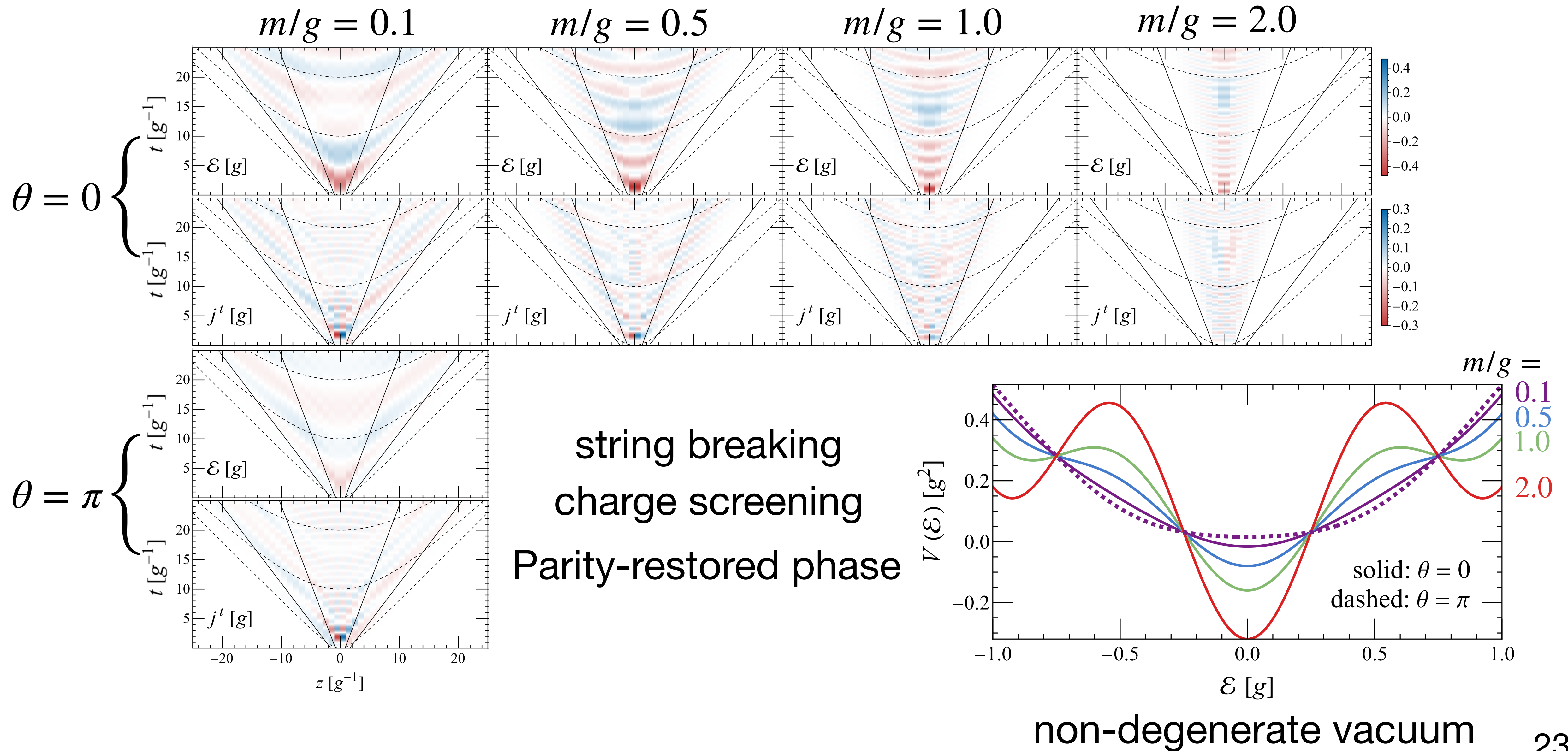
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4. Dynamical signatures of different phases



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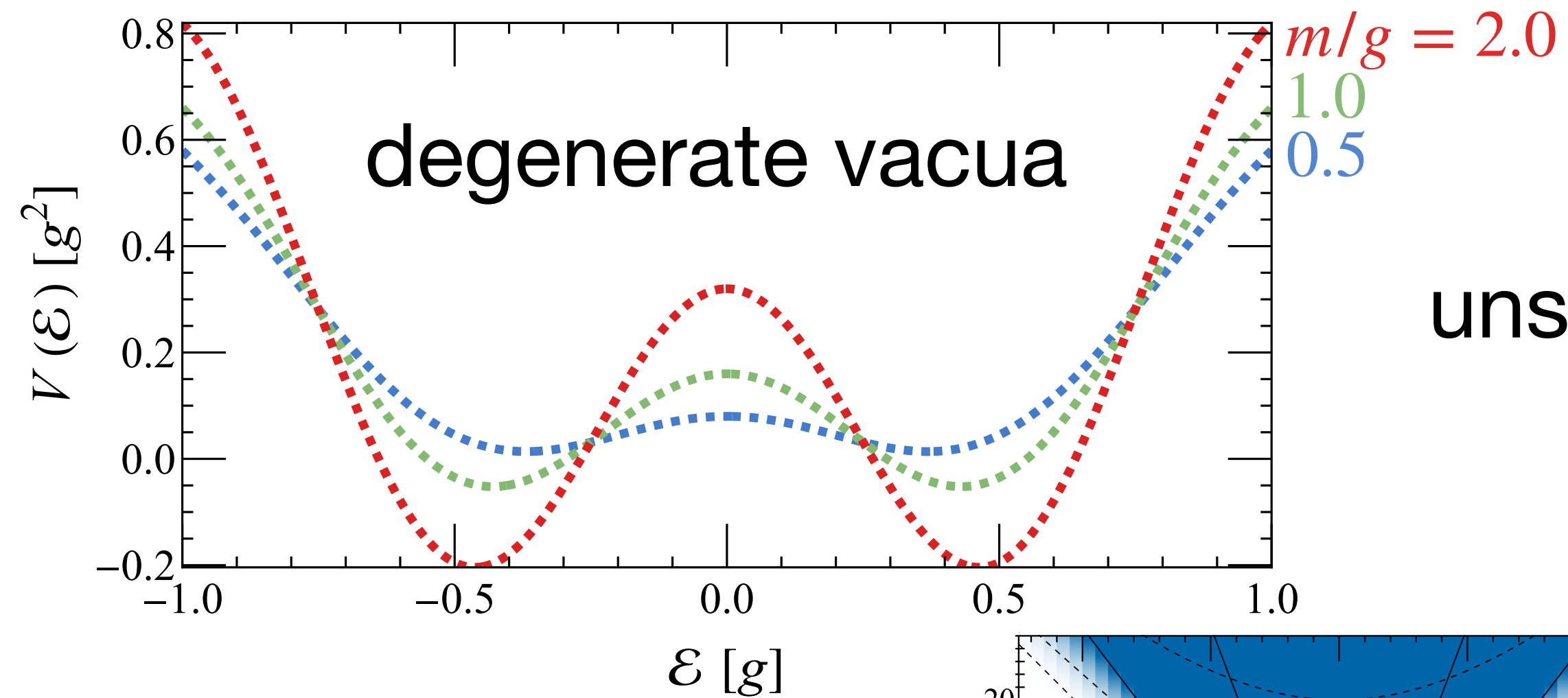
4. Dynamical signatures of different phases

$$m/g = 0.1$$

$$m/g = 0.5$$

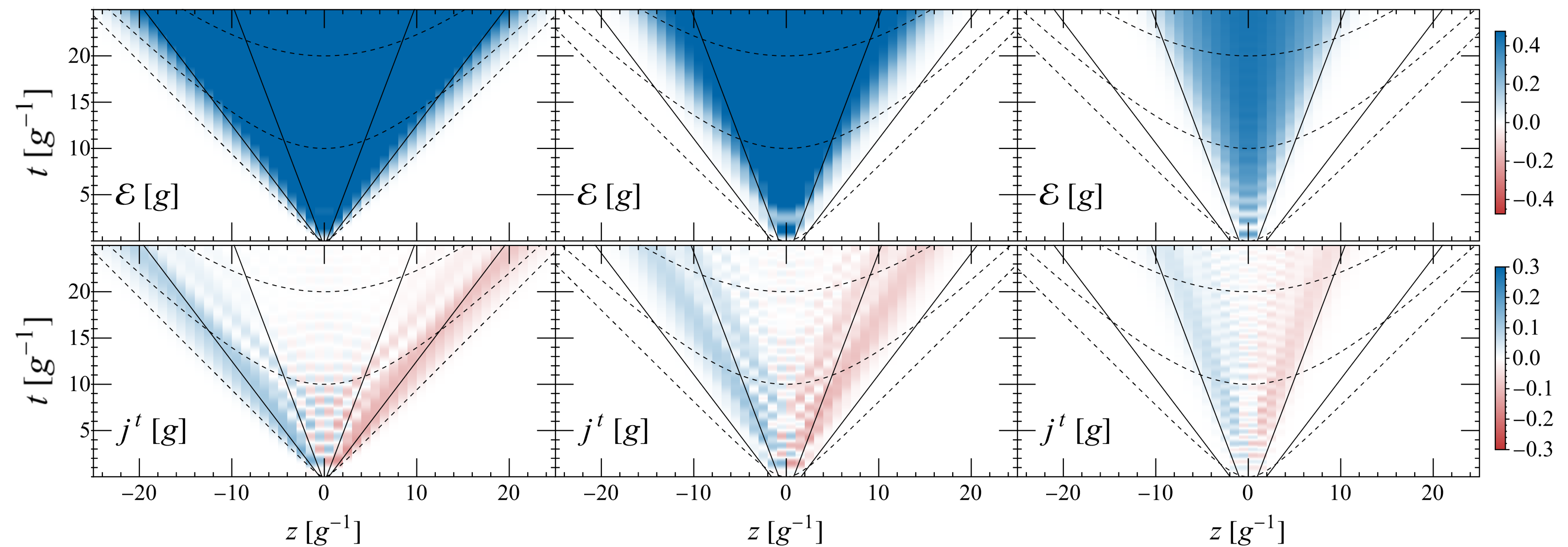
$$m/g = 1.0$$

$$m/g = 2.0$$

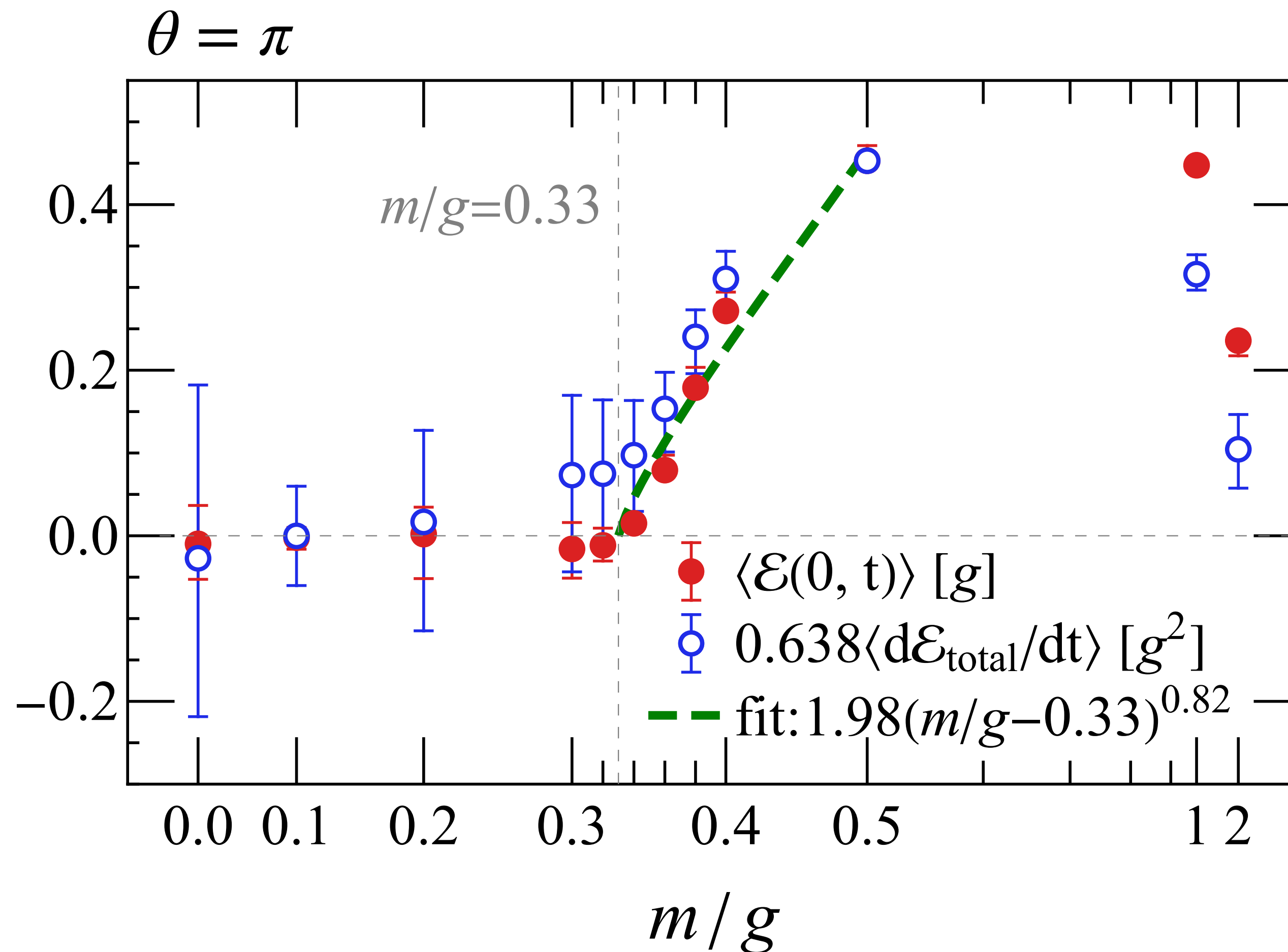


unscreened, long-range electric fields
Parity-broken phase

$$\theta = \pi \left\{ \begin{array}{l} \\ \end{array} \right.$$



4. Dynamical signatures of different phases



Dynamical order parameters

$$\langle \mathcal{E}(0, t) \rangle \quad \langle d\mathcal{E}_{\text{total}}/dt \rangle$$

$$\langle A(t) \rangle = \frac{g}{15} \int_{10g^{-1}}^{25g^{-1}} A(t) dt$$

Critical point $m/g = 0.33$ coincides with the known ground-state phase transition

Byrnes, T. M. R., Sriganesh, P., Bursill, R. J., & Hamer, C. J. (2002). Density matrix renormalization group approach to the massive Schwinger model. *Physical Review D*, 66(1), 013002.

5. Summary and Outlook

Summary

arXiv:2509.10835

arXiv:2509.10855

- Hydrodynamic behavior:
 - *Bjorken-like flow occurs at small m/g , but breaks down as m/g increases.*
- Dynamical signatures of different phases:

<i>Parity-restored phase</i>	<i>Parity-broken phase</i>
<i>string breaking, charge screening</i>	<i>unscreened, long-range electric fields</i>

- *Order parameters identify a critical point coincides with the known ground-state phase transition*

5. Summary and Outlook

Outlook

- What is essential for hydrodynamization?

Other models, e.g. Nambu–Jona-Lasinio model

Advantage: non-zero chemical potential

Thanks

Back Up Slides

Back Up

1. Paramaters and excitation operator

Number of lattice sites: $N = 100$

lattice spacing: $a = 0.5/g$

Truncation of the gauge field: $L_n \in [-3,3]$

the maximum bond dimension: $D_{max} = 500$

the time evolution step : $dt = 0.004g^{-1}$

Gauss' law: $G_n \left| \psi_{\text{phys}} \right\rangle = 0$

$$G_n = L_n - L_{n-1} - \frac{1}{2}[Z_n + (-1)^n]$$

$$\hat{j}_n^t = \frac{Z_n + (-1)^n}{2a}$$

Excitation operator:

$$\left| \Phi(t=0) \right\rangle = e^{-i\pi(\hat{O}_{50} + \hat{O}_{51})} \left| \Phi_{GS} \right\rangle \quad \hat{O}_n = (-1)^n \frac{1}{2}(1 + Z_n)$$

Spin representation of : $\bar{\psi}\psi$

Back Up

2. Energy-momentum tensor

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L}$$

$$T_S^{\mu\nu} \equiv \frac{i}{2} \left(\bar{\psi} (\gamma^\mu \partial^\nu + \gamma^\nu \partial^\mu) \psi - (\partial^\nu \bar{\psi}) \gamma^\mu \psi - (\partial^\mu \bar{\psi}) \gamma^\nu \psi \right)$$

$$- \frac{g}{2} \bar{\psi} (\gamma^\mu A^\nu + \gamma^\nu A^\mu) \psi - g^{\mu\nu} \bar{\psi} (i \gamma^\alpha \mathcal{D}_\alpha - m) \psi + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\alpha} F^\nu{}_\alpha,$$

$$\left\{ \begin{array}{l} T_{S,n}^{tt} = \frac{1}{4a^2} (S_n^+ e^{i\theta_n} S_{n+1}^- + S_{n-1}^+ e^{i\theta_{n-1}} S_n^- + h.c.) + \frac{g^2}{4} (L_{n-1}^2 + L_n^2) + \frac{(-1)^n}{2a} m(1 + Z_n) \\ T_{S,n}^{zz} = \frac{1}{4a^2} (S_n^+ e^{i\theta_n} S_{n+1}^- + S_{n-1}^+ e^{i\theta_{n-1}} S_n^- + h.c.) - \frac{g^2}{4} (L_{n-1}^2 + L_n^2) \\ T_{S,n}^{tz} = T_{F,n}^{zt} = \frac{-i}{4a^2} (S_{n-1}^+ Z_n S_{n+1}^- e^{i\theta_n} e^{i\theta_{n-1}} - h.c.) \end{array} \right.$$

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2. Energy-momentum tensor

- Decompose the stress tensor into:

$$T^{\mu\nu} = (\varepsilon + P_{\Pi})u^{\mu}u^{\nu} - P_{\Pi}g^{\mu\nu} + \pi^{\mu\nu} \quad P_{\Pi} \equiv P(\varepsilon) + \Pi$$

$$\pi^{\mu\nu} = \pi^{\nu\mu}, \quad g_{\mu\nu}\pi^{\mu\nu} = 0, \quad \pi^{\mu\nu}u_{\mu} = 0 \implies 1+1\text{D: } \pi^{\mu\nu} = 0$$

- In the Landau-Lifshitz frame: $T^{\mu}_{\nu}u^{\nu} = \varepsilon u^{\mu}$

$$\begin{aligned} T_{+} &\equiv \frac{T^{tt} + T^{zz}}{2} & T_{-} &\equiv \frac{T^{tt} - T^{zz}}{2} & R &\equiv \sqrt{(T_{+})^2 - (T^{tz})^2} \\ \varepsilon &= T_{-} + R & v^z &\equiv \frac{u^z}{u^t} = \frac{T^{tz}}{T_{+} + R} & P_{\Pi} &= \varepsilon - g_{\mu\nu}T^{\mu\nu} = R - T_{-} \end{aligned}$$

Back Up

3. Schwinger model in the bosonized description

$$\psi_R \sim: e^{i2\sqrt{\pi}\phi_R} : \quad \psi_L \sim: e^{-i2\sqrt{\pi}\phi_L} : \quad \phi_{R,L} = \frac{1}{2} \left(\phi \mp \int_{-\infty}^x \dot{\phi} dx' \right)$$

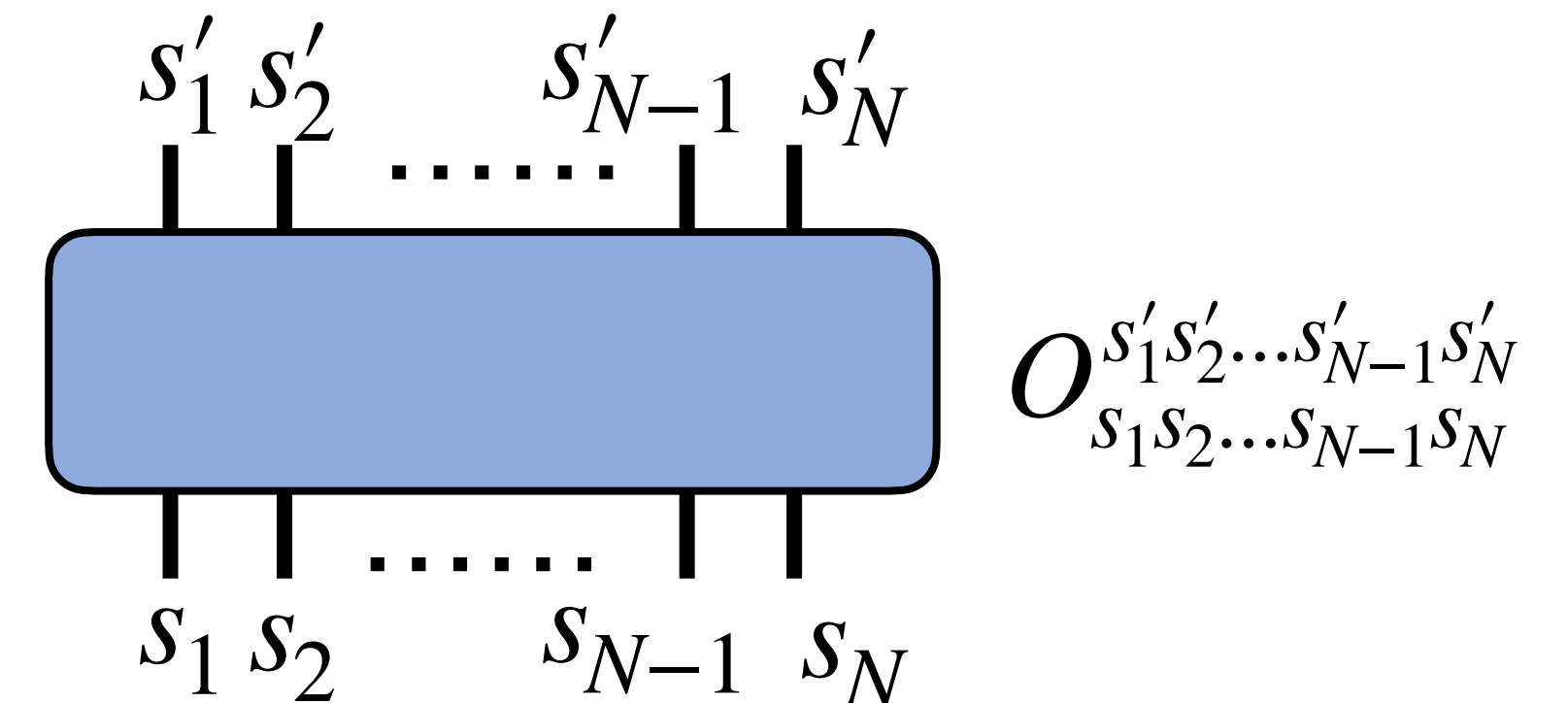
$$\bar{\psi}\psi \sim \cos(\sqrt{4\pi}\phi)$$

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \left(\pi_{\phi} \right)^2 + \frac{1}{2} \left(\phi' \right)^2 + \frac{g^2}{2\pi} \left(\phi + \frac{\theta}{2\sqrt{\pi}} \right)^2 - \frac{e^{\gamma} mg}{2\pi^{3/2}} \cos(2\sqrt{\pi}\phi) \\ \mathcal{E} &= -\frac{g}{\sqrt{\pi}} \left(\phi + \frac{\theta}{2\sqrt{\pi}} \right) \quad V = \frac{g^2}{2\pi} \left(\phi + \frac{\theta}{2\sqrt{\pi}} \right)^2 - \frac{e^{\gamma} mg}{2\pi^{3/2}} \cos(2\sqrt{\pi}\phi) \\ &= \frac{1}{2} \mathcal{E}^2 - \frac{e^{\gamma} mg}{2\pi^{3/2}} \cos \left(\frac{2\pi\mathcal{E}}{g} + \theta \right) \end{aligned}$$

Back Up

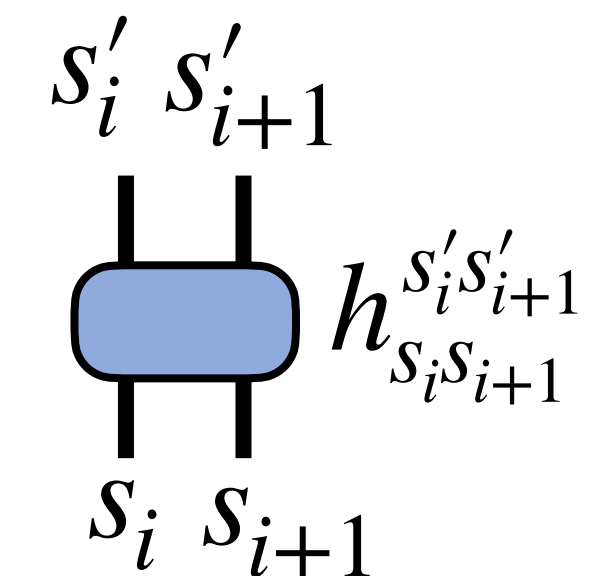
4.1 Matrix Product State (MPS) Representations

$$\hat{O} = \sum_{\{s,s'\}} O_{s_1 s_2 \dots s_{N-1} s_N}^{s'_1 s'_2 \dots s'_{N-1} s'_N} |s'_1 s'_2 \dots s'_{N-1} s'_N\rangle \otimes \langle s_1 s_2 \dots s_{N-1} s_N|$$



For localized interaction system:

$$\hat{H} = \sum_i \hat{h}_{i,i+1} \quad \hat{h}_{i,i+1} = \sum_{\{s,s'\}} h_{s_i s_{i+1}}^{s'_i s'_{i+1}} |s'_i s'_{i+1}\rangle \otimes \langle s_i s_{i+1}|$$

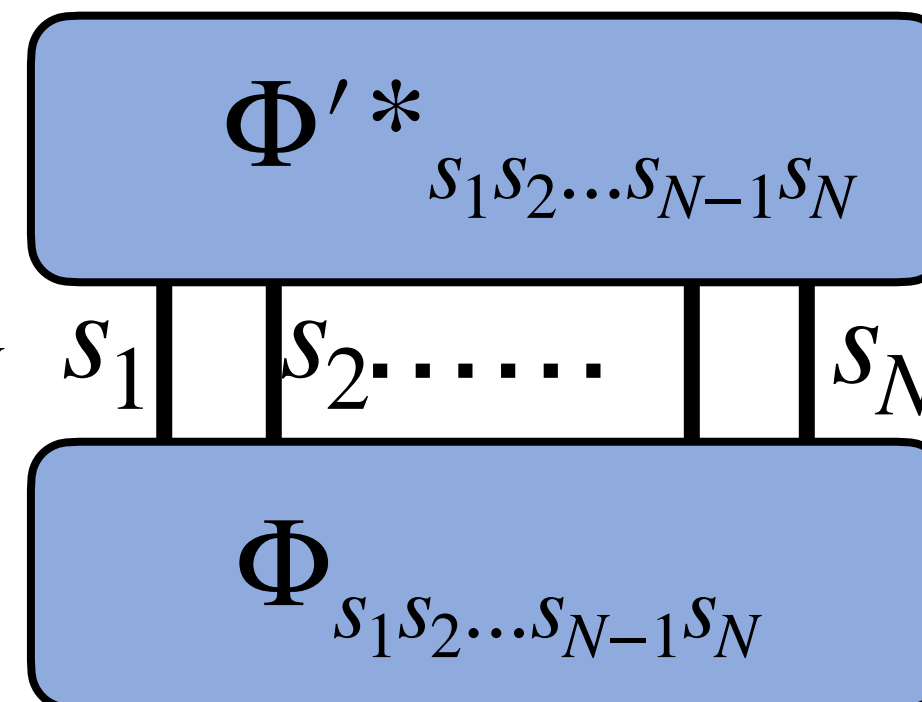


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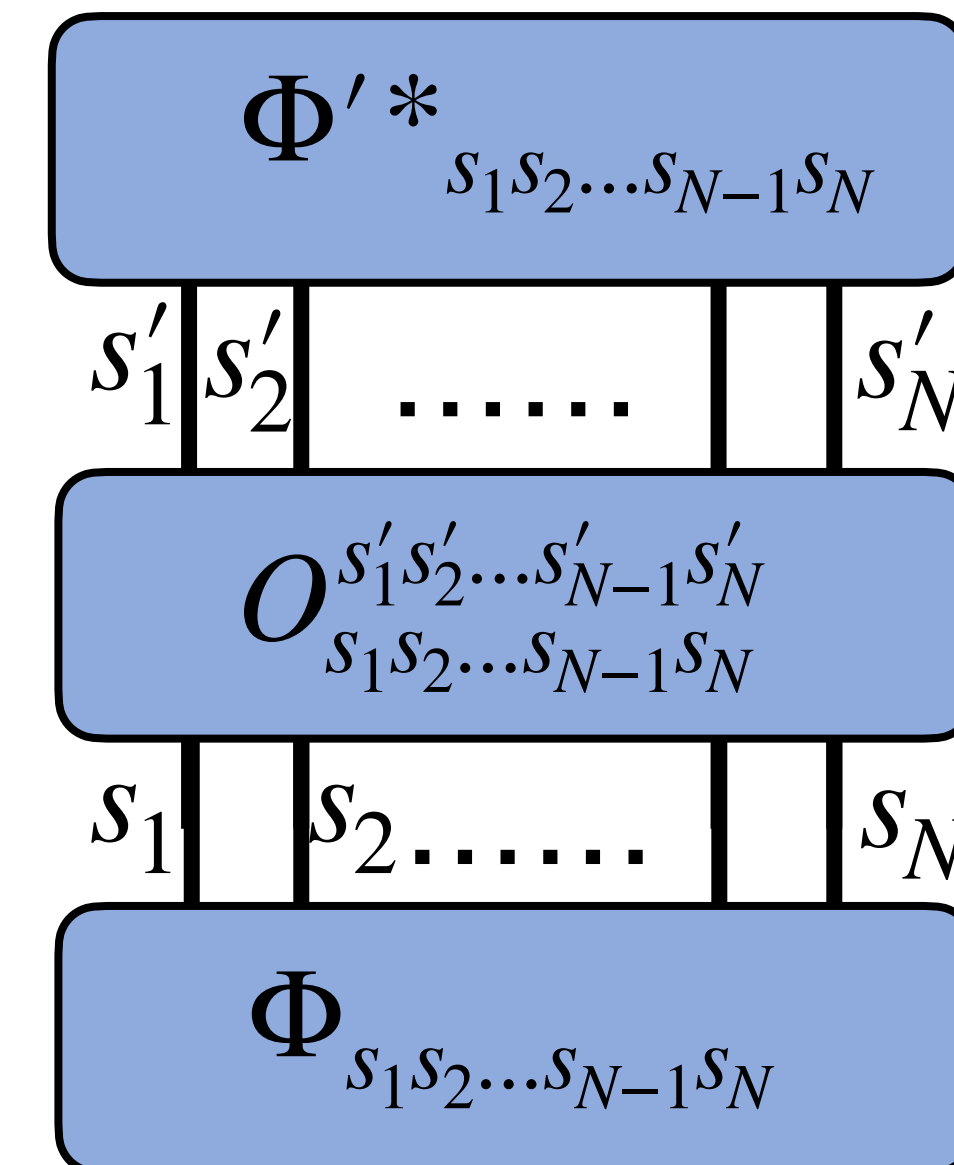
- state overlap

$$\langle \Phi' | \psi \rangle = \sum_{\{s\}} \Phi'^*_{s_1 s_2 \dots s_{N-1} s_N} \Phi_{s_1 s_2 \dots s_{N-1} s_N}$$



- operator expected value

$$\langle \Phi | \hat{O} | \Phi \rangle = \sum_{\{s, s'\}} \Phi^*_{s'_1 s'_2 \dots s'_{N-1} s'_N} O_{s_1 s_2 \dots s_{N-1} s_N}^{s'_1 s'_2 \dots s'_{N-1} s'_N} \Phi_{s_1 s_2 \dots s_{N-1} s_N}$$



Back Up

4.2 Time-Evolving Block Decimation (TEBD)

$$|GS\rangle = \frac{1}{C} \lim_{\beta \rightarrow \infty} e^{-\beta \hat{H}} |\Phi\rangle \quad |\Phi(t)\rangle = e^{-it\hat{H}} |\Phi(0)\rangle$$

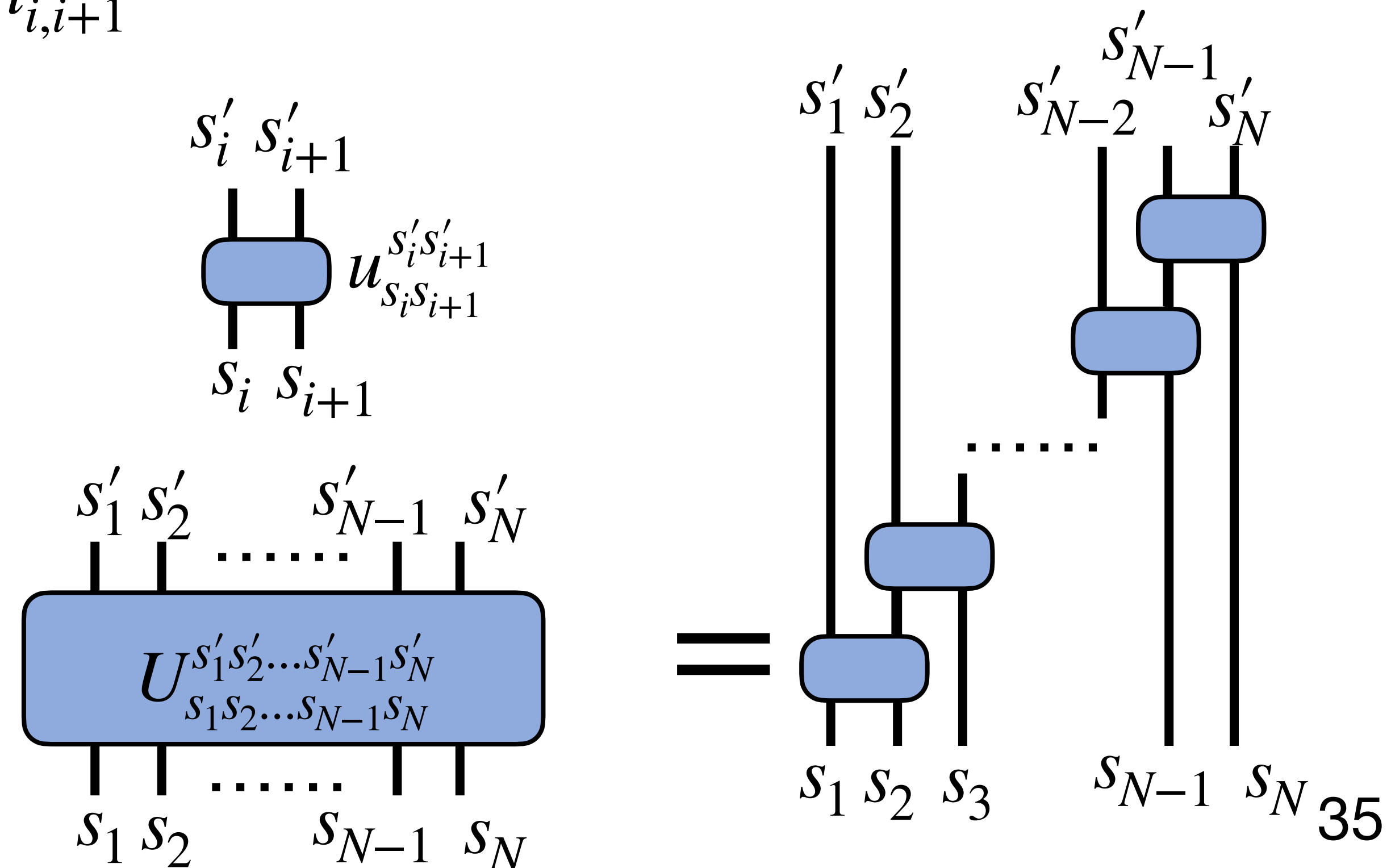
For localized interaction system: $\hat{H} = \sum_{i=1}^{N-1} \hat{h}_{i,i+1}$

Trotter-Suzuki decomposition:

$$[\hat{A}, \hat{B}] \neq 0$$

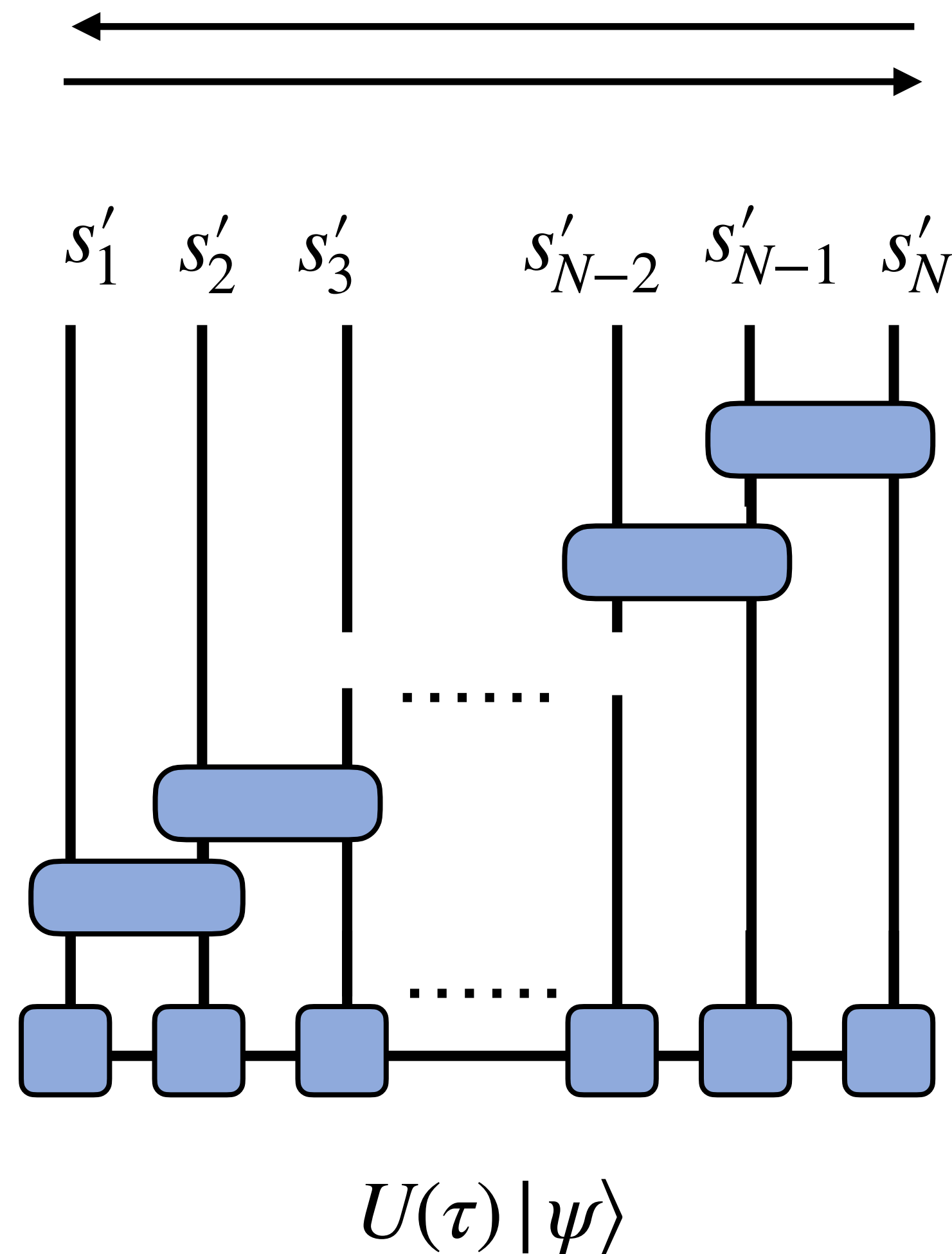
$$e^{t(\hat{A}+\hat{B})} = e^{t\hat{A}}e^{t\hat{B}} + O(t^2) = e^{\frac{t}{2}\hat{A}}e^{t\hat{B}}e^{\frac{t}{2}\hat{A}} + O(t^3)$$

$$U(\tau) = e^{-\tau\hat{H}} = e^{-\tau\sum_{i=1}^{N-1}\hat{h}_{ij}} = \prod_{i=1}^{N-1} e^{-\tau\hat{h}_{ij}} + O(\tau^2)$$

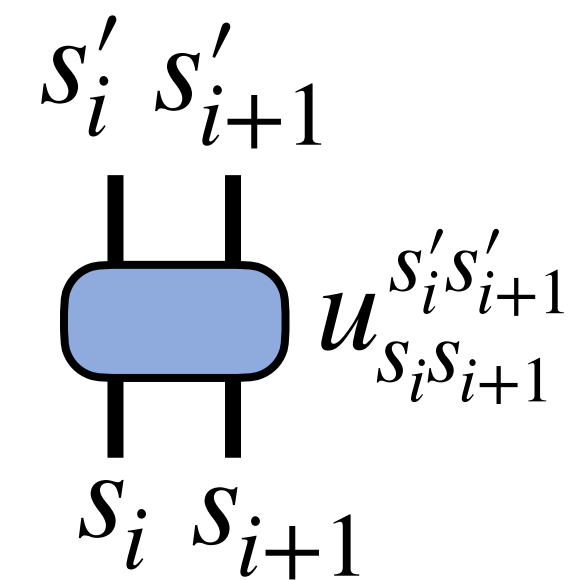


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4.2 Time-Evolving Block Decimation (TEBD)



$$e^{-\tau \hat{h}_{ij}} = \sum_{\{s, s'\}} u_{s_i s_{i+1}}^{s'_i s'_{i+1}} |s'_i s'_{i+1}\rangle \otimes \langle s_i s_{i+1}|$$



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5. Convergence of Trotter-Suzuki decomposition

Trotter-Suzuki decomposition:

$$[\hat{A}, \hat{B}] \neq 0 \quad e^{t(\hat{A}+\hat{B})} = e^{t\hat{A}}e^{t\hat{B}} + O(t^2) = e^{\frac{t}{2}\hat{A}}e^{t\hat{B}}e^{\frac{t}{2}\hat{A}} + O(t^3)$$

$$\begin{aligned} e^{t(\hat{A}+\hat{B})} &= \left[e^{\frac{t}{n}\hat{A}}e^{\frac{t}{n}\hat{B}} + O\left(\frac{t^2}{n^2}\right) \right]^n \\ &= \left[e^{\frac{t}{n}\hat{A}}e^{\frac{t}{n}\hat{B}} \right]^n + O\left(\frac{t^2}{n}\right) \\ &= \left[e^{\frac{t}{n}\hat{A}}e^{\frac{t}{n}\hat{B}} \right]^n \quad (n \rightarrow \infty) \end{aligned}$$