清華大學
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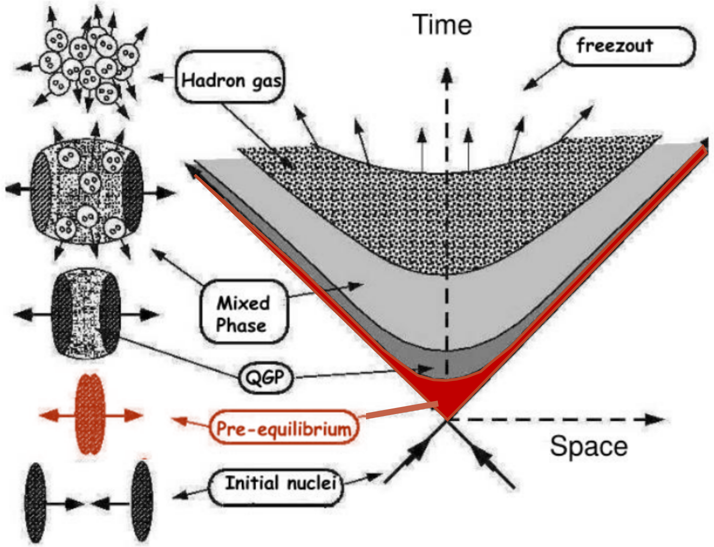
Early Thermalization in kenitic theory via spectral BBGKY(sBBGKY)



New theoretical tool :

(nonlinear) Spectral BBGKY hierarchy

--Fill the Gap in kinetic theory: Correlation, Nonlinear

Physical Insight:

How to distinguish Hydronamization from Thermalization

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arXiv: 2507.14243, 2509.23978, XJ L, Shuzhe Shi

QPT2025, 24-28 Oct., 2025, Guilin, China

Motivation - Early Thermalization Puzzle

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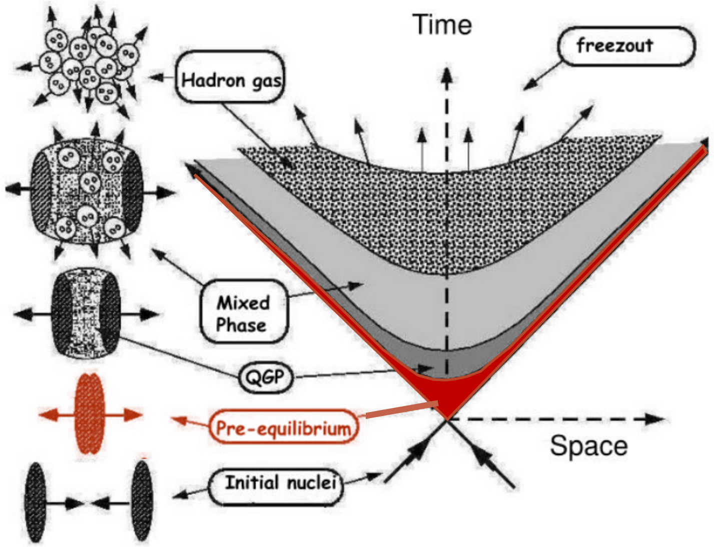
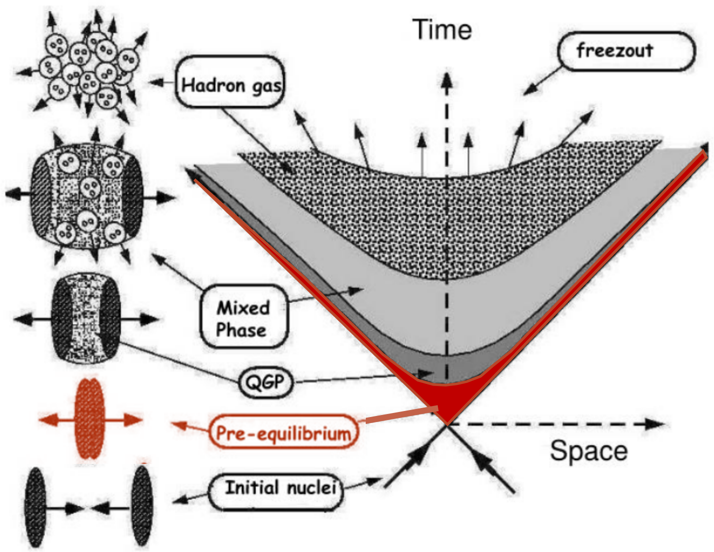


Motivation - Early Thermalization Puzzle



Key Question!

Many talks from different perspective!

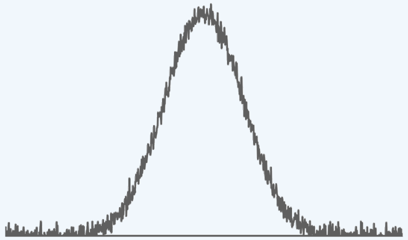
- *Navid Abbasi,* Hydrodynamization Time Hierarchies
- *Jin Hu(进 胡), nonlinear Boltzmann*
- *Anping Huang(安平 黄),* attractors
- *Shuzhe Shi(舒哲 施),* classical & quantum
- *Chenxi Liang(晨曦 梁), entropy in isolated quantum systems*
- *Yiyang Peng(亦扬 彭), ahydro*
- *Haiyang Shao(海洋 邵),* quantum many-body calculation

Motivation - Early Thermalization Puzzle

Perspective:
Kinetic Theory

Motivation - Early Thermalization Puzzle

- **Linear Boltzmann** Enough? ***No!***
  - **Assume** close to equilibrium
  - **But** QGP is created far-from-equilibrium
  - Need consider **Nonlinear effect**



Perspective:
Kinetic Theory

Motivation - Early Thermalization Puzzle

Perspective: Kinetic Theory

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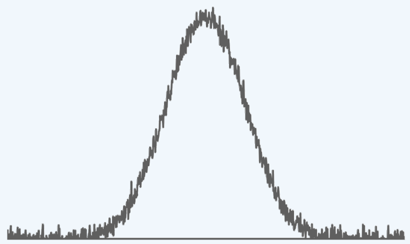


- **Nonlinear Boltzmann** Enough? ***No!***
  - **Assume** molecular chaos(no correlation)
  - **But** QGP is correlated due to momentum conservation
  - Need consider **Correlation**

Map: What we do in the building of nonequilibrium statistical mechanics?

Hamilton Equation/
Schrödinger Equation

Hamilton Equation

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}$$

Schrödinger equation

$$\hat{H}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Map: What we do in the building of nonequilibrium statistical mechanics?

Hamilton Equation/
Schrödinger Equation

↕

Liouville Equation

Liouville's theorem

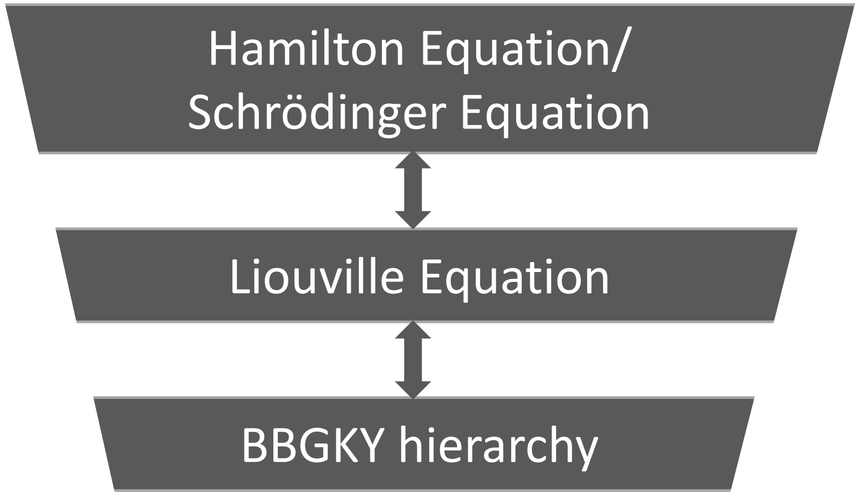
The distribution function is **constant** along any trajectory **in phase space**.

Liouville equation

$$\frac{\partial P_{(N)}}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial P_{(N)}}{\partial q_i}\dot{q}_i + \frac{\partial P_{(N)}}{\partial p_i}\dot{p}_i\right) = 0$$

Full distribution function

Map: What we do in the building of nonequilibrium statistical mechanics?



Liouville equation

$$d_t P_{(N)} = C[\int, P_{(N)}]$$

Full distribution function

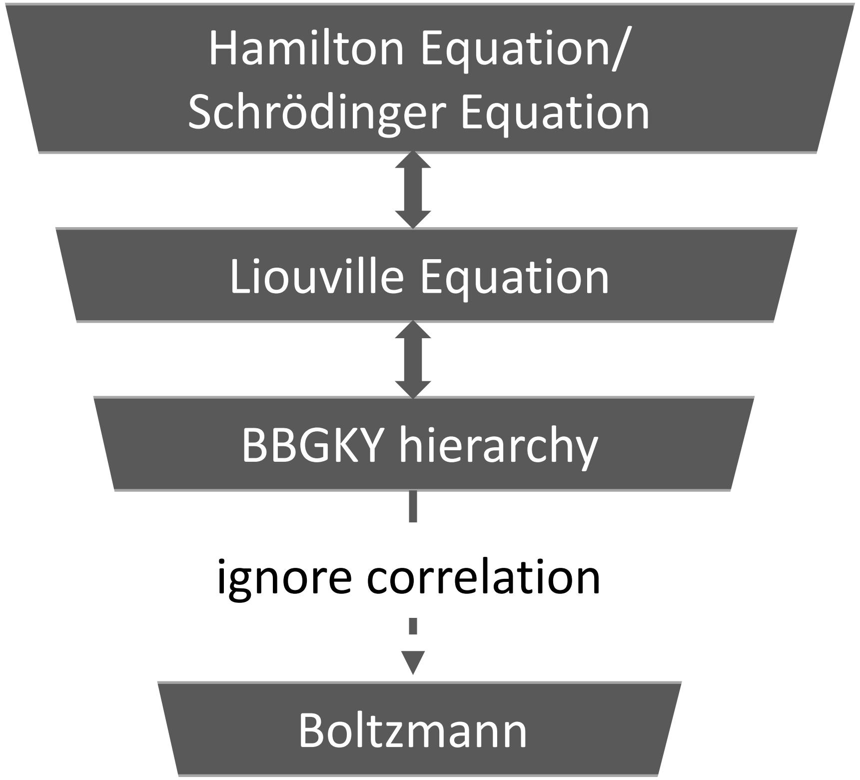
BBGKY hierarchy

(Bogoliubov–Born–Green–Kirkwood–Yvon)

$$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}],$$
$$n = 1,2, \dots, N$$

Reduced distribution function

Map: What we do in the building of nonequilibrium statistical mechanics?



BBGKY hierarchy

(Bogoliubov–Born–Green–Kirkwood–Yvon)

$$d_t \boxed{P_{(n)}} = C[\int, P_{(n)}, P_{(n+1)}]$$

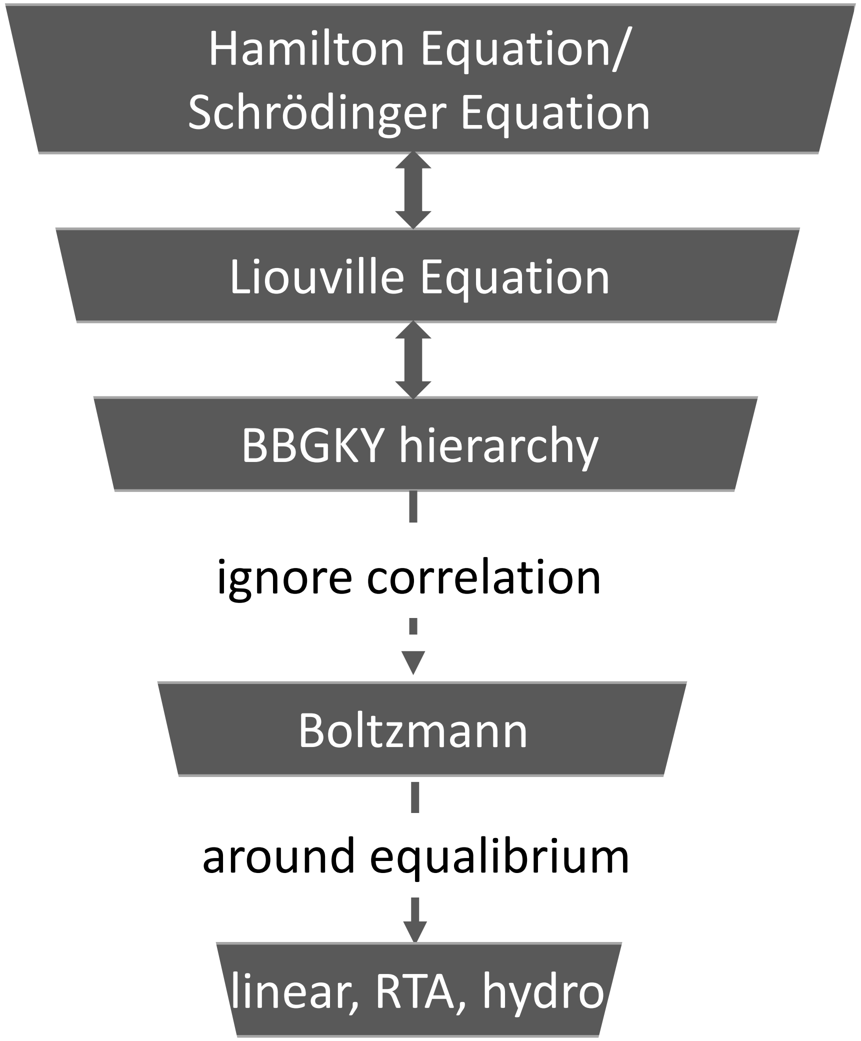
Reduced distribution function

Assume $P_2(\phi_a, \phi_b) - P_1(\phi_a) P_1(\phi_b) = 0$

Boltzmann Equation

$$d_t P_{(1)} = C[\int, P_{(1)}]$$

Map: What we do in the building of nonequilibrium statistical mechanics?



Boltzmann Equation

$$d_t f_1 \propto (f_3 f_4 - f_1 f_2)$$

Around equalibrium

Linear Boltzmann Equation

$$d_t f_1 \propto (f_3 + f_4 - f_1 - f_2)$$

RTA Boltzmann Equation

$$d_t f_1 \propto (f_1 - f_{eq})/\tau$$

Hydrodynamics

$$\nabla_\mu T^{\mu\nu} = 0$$

Map: What we do in the building of nonequilibrium statistical mechanics?

Hamilton Equation/
Schrödinger Equation

$\updownarrow$

Liouville Equation

$\updownarrow$

BBGKY hierarchy

ignore correlation

$\downarrow$

Boltzmann

around equalibrium

$\downarrow$

linear, RTA, hydro

Time-reversal symmetry, Cornerstone

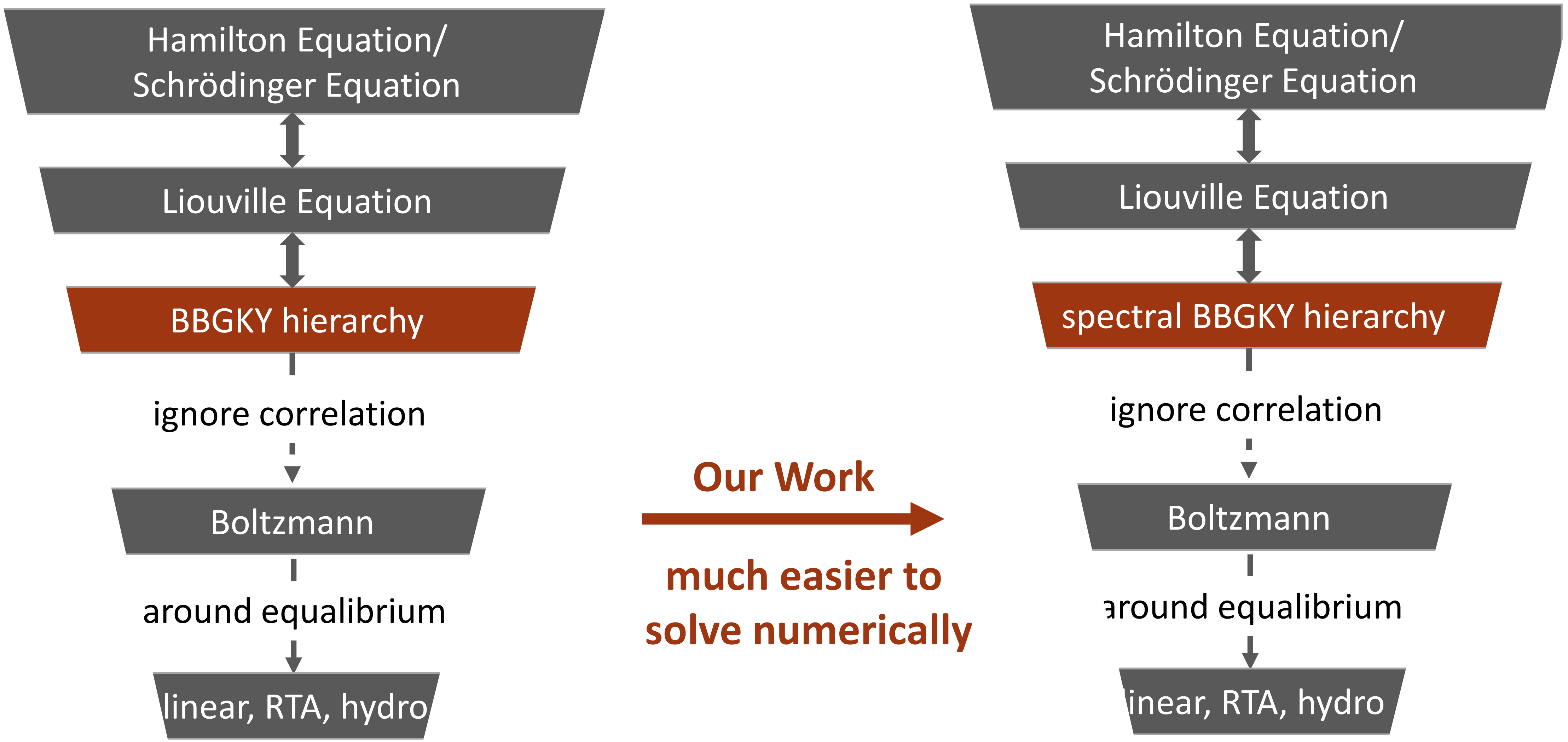
Gap: correlation

Tractable proxy

Gap: nonlinear

Workhorse

Map: What we do in the building of nonequilibrium statistical mechanics?



New theoretical tool: Spectral BBGKY Hierarchy

BBGKY Hierarchy

$$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}]$$

$\int$ means high dimentional-integral

New theoretical tool: Spectral BBGKY Hierarchy

BBGKY Hierarchy
$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}]$

$\int$ **means high dimentional-integral**

Difficult to solve

- **Until now**, almost all simulations are based on **lowest-order truncation**, *the Boltzmann equation*.
- Because **huge memory demands** N^{6n} to discretize $6n$ D
  - $n = 1$, discretize 6 D
  - $n = 2$, discretize 12 D
  - for ref, Lattice QCD discretize $3 + 1$ D, ~30 GB memory

New theoretical tool: Spectral BBGKY Hierarchy

Spectrum Method

$$P_{(n)}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n) = \boldsymbol{P}^{i_1 \cdots i_n} \underbrace{\mathcal{P}_{i_1}(p_1) \cdots \mathcal{P}_{i_n}(p_n)}_{\text{basis}}$$

BBGKY Hierarchy

$$d_t P_{(n)} = C[\textstyle\int, P_{(n)}, P_{(n+1)}]$$

New theoretical tool: Spectral BBGKY Hierarchy

Spectrum Method

$$P_{(n)}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n) = \boldsymbol{P}^{i_1 \cdots i_n} \underbrace{\mathcal{P}_{i_1}(p_1) \cdots \mathcal{P}_{i_n}(p_n)}_{\textbf{basis}}$$

BBGKY Hierarchy

$$d_t P_{(n)} = C[\textstyle\int, P_{(n)}, P_{(n+1)}]$$

- not a new idea. But we should **choose basis carefully.**
- We hope

  first few basis functions

  already capture the important physics

New theoretical tool: Spectral BBGKY Hierarchy

Spectrum Method

$$P_{(n)}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n) = \boldsymbol{P}^{i_1 \cdots i_n} \mathcal{P}_{i_1}(p_1) \cdots \mathcal{P}_{i_n}(p_n)$$

BBGKY Hierarchy

$$d_t P_{(n)} = C[\textstyle\int, P_{(n)}, P_{(n+1)}]$$

Our basis

$$\mathcal{P}_{n,\ell,m}(p_\mu) = e^{-p_\mu u^\mu/\Lambda} \left(\frac{p_\mu u^\mu}{\Lambda}\right)^\ell \underbrace{Y_{\ell,m}(\theta,\phi)}_{\text{Angular}} \underbrace{L_n^{(2\ell+2)}\left(\frac{p_\mu u^\mu}{\Lambda}\right)}_{\text{Radial}}$$

The physics

- equilibrium
- particle number, energy-momentum conservation
- nonlinear behavior

mainly depends on the first few basis.

New theoretical tool: Spectral BBGKY Hierarchy

$$n = \frac{4\sqrt{\pi}}{(2\pi)^3} T^3 f^{(0,0,0)}$$

$$\langle p^0 \rangle = \frac{12\sqrt{\pi}}{(2\pi)^3} T^4 \left(f^{(0,0,0)} - f^{(1,0,0)} \right)$$

$$\langle p^1 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,1)}$$

$$\langle p^2 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,-1)}$$

$$\langle p^3 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,0)}$$

Spectrum Method

$$P_{(n)}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n) = \boldsymbol{P}^{i_1 \cdots i_n} \mathcal{P}_{i_1}(p_1) \cdots \mathcal{P}_{i_n}(p_n)$$

BBGKY Hierarchy

$$d_t P_{(n)} = C[\textstyle\int, P_{(n)}, P_{(n+1)}]$$

Our basis

$$\mathcal{P}_{n,\ell,m}(p_\mu) = e^{-p_\mu u^\mu/\Lambda} \left(\frac{p_\mu u^\mu}{\Lambda} \right)^\ell Y_{\ell,m}(\theta, \phi) L_n^{(2\ell+2)} \left(\frac{p_\mu u^\mu}{\Lambda} \right)$$

Angular Radial

The physics

- equilibrium
- particle number, energy-momentum conservation
- nonlinear behavior

mainly depends on the first few basis.

New theoretical tool: Spectral BBGKY Hierarchy

Spectrum Method

$$P_{(n)}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n) = \boldsymbol{P}^{i_1 \cdots i_n} \mathcal{P}_{i_1}(p_1) \cdots \mathcal{P}_{i_n}(p_n)$$

BBGKY Hierarchy

$$d_t P_{(n)} = C[\textstyle\int, P_{(n)}, P_{(n+1)}]$$

Spectral BBGKY Hierarchy

$$d_t \boldsymbol{P}_{(n)} = \boldsymbol{C} \cdot \boldsymbol{P}_{(n)} + \boldsymbol{A} \cdot \boldsymbol{P}_{(n+1)}$$

$\boldsymbol{A}$ & $\boldsymbol{C}$: Collision Integral

New theoretical tool: Spectral BBGKY Hierarchy

No any approximation but three advantages

1

- **Decouple** collision integral from time evolution
- **Analytically** calculate Collision Integral

$$\boxed{Cost_{nonlin} = Cost_{lin}}$$

Spectral BBGKY Hierarchy

$$d_t P_{(n)} = C \cdot P_{(n)} + A \cdot P_{(n+1)}$$

A & C: Collision Integral

New theoretical tool: Spectral BBGKY Hierarchy

No any approximation but three advantages

2

- **before:** discretize 6D (3coorediate,3momentum)
- **now:** discretize 3D(3coorediate)

$$\boxed{\boldsymbol{Cost}_{\text{now}} = \sqrt{\boldsymbol{Cost}_{\text{before}}}}$$

Spectral BBGKY Hierarchy

$$d_t\boldsymbol{P}_{(n)} = \boldsymbol{C} \cdot \boldsymbol{P}_{(n)} + \boldsymbol{A} \cdot \boldsymbol{P}_{(n+1)}$$

$\boldsymbol{A}$ & $\boldsymbol{C}$: Collision Integral

New theoretical tool: Spectral BBGKY Hierarchy

No any approximation but three advantages

3

- **From Single Simulation to Ensemble Average**
- conventional cascade method, ensemble average over many simulations
- spectral method: One simulation per initial state

Spectral BBGKY Hierarchy

$$d_t \boldsymbol{P}_{(n)} = \boldsymbol{C} \cdot \boldsymbol{P}_{(n)} + \boldsymbol{A} \cdot \boldsymbol{P}_{(n+1)}$$

$\boldsymbol{A}$ & $\boldsymbol{C}$: Collision Integral

Open the door

of pre-equilibrium stage evolution

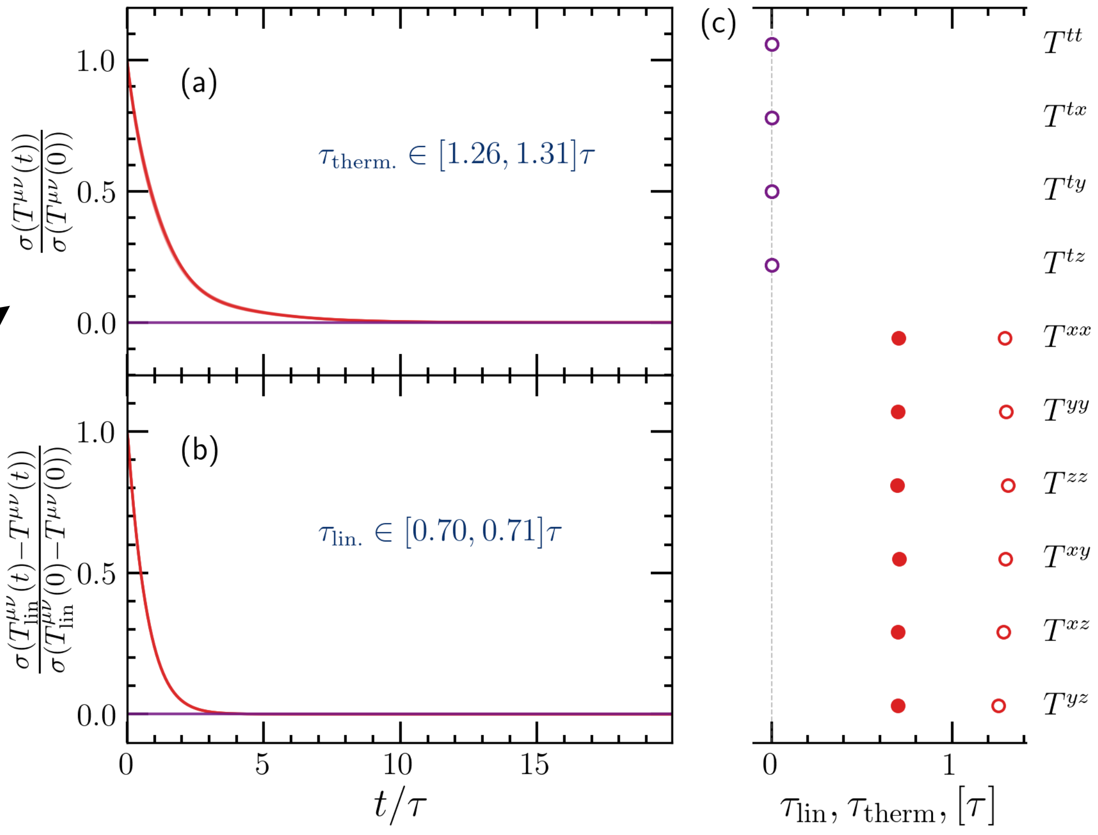
Correlation, Nonlinear

Decoupling hydrodynamization from thermalization via spectral nonlinear Boltzmann equation

People before
hydrodynamization
= **equilibrium**

We need
hydrodynamization
= **near equilibrium**

We find:
$\boldsymbol{\tau_{lin}} \approx \boldsymbol{\tau_{therm}}/2$



An explanation for the early applicability of hydrodynamics

arXiv:2509.23978

Summary

- New theoretical tool: **Spectral BBGKY Hierarchy**(arXiv:2507.14243)
  - **Fill the Gap: Correlation, Nonlinear**
- Physical Insight: Decoupling hydrodynamization from thermalization via **nonlinear** Boltzmann equation(arXiv:2509.23978)

$$\tau_{lin} \approx \tau_{therm}/2$$

Outlook

- The thermalization time and linearization time in an expanding system
- The correlation contribution to thermalization process
- $gg \leftrightarrow ggg$
- Bose Enhancement, Pauli Blocking