Relativistic second-order spin hydrodynamics from Zubarev's non-equilibrium statistical operator

Duan She (佘端)

Institute of Physics, Henan Academy of Sciences

The 16th Workshop on QCD Phase Transition and Relativistic Heavy-Ion Physics (QPT 2025)

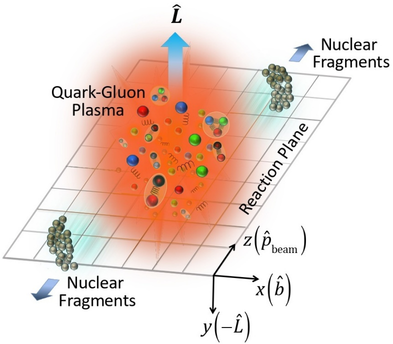
Based on: Duan She, Anping Huang, Defu Hou, Jinfeng Liao, Sci.Bull. 67 (2022) 2265-2268
Duan She, Yi-Wei Qiu, Defu Hou, Phys.Rev.D 111 (2025) 3, 036027
Duan She, Yi-Wei Qiu, Ze-Fang Jiang, Defu Hou, arXiv:2505.01814 (accepted by PRD)

2025 年 10 月 26 日

Outline

Introduction

Global angular momentum in heavy ion collisions

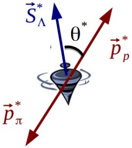


parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P}_{\mathbf{\Lambda}} \cdot \mathbf{p}_{\mathbf{p}}^*)$$

α: Λ decay parameter (=0.642±0.013)
P_Λ: Λ polarization
p_p^*: proton momentum in Λ rest frame



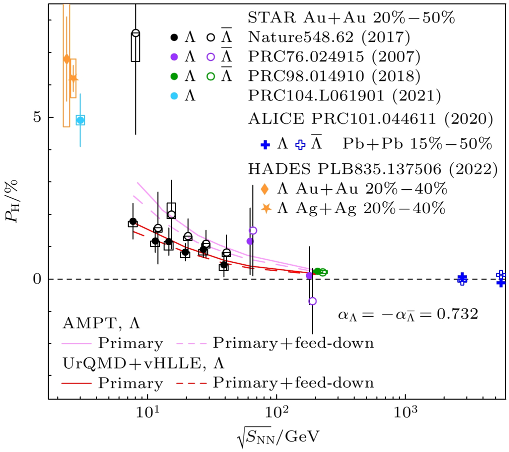
$\Lambda \to p + \pi^+$
(BR: 63.9%, cτ~7.9 cm)

$$L \approx \frac{A\sqrt{s_{NN}}}{2} b\hbar \sim 10^6 \hbar \quad \text{RHIC Au-Au } 200\text{GeV}, b = 10\text{fm}$$

Large initial orbital angular momentum leads to the polarizations of Λ hyperons and spin alignment of vector mesons through spin-orbital coupling.

Zuo-Tang Liang, Xin-Nian Wang, Phys.Rev.Lett. 94 (2005) 102301; Phys.Lett.B 629 (2005) 20-26

Global polarization of Λ and $\bar{\Lambda}$



Sun Xu et al., Acta Phys. Sin. Vol. 72, No. 7(2023) 072401

- The lower energy, the stronger polarization effects.
- Estimation given by Becattini, Karpenko, Lisa, Upsal, Voloshin, PRC95, 054902(2017)

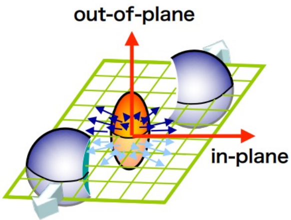
$$\boldsymbol{P}_{\Lambda(\bar{\Lambda})} \simeq \frac{\boldsymbol{\omega}}{2T} \pm \frac{\mu_\Lambda \boldsymbol{B}}{T}$$

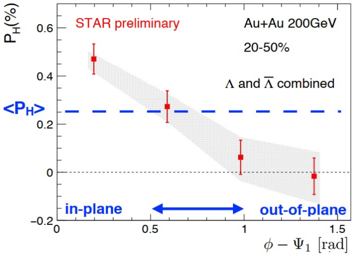
- $\omega = (9 \pm 1) \times 10^{21}/\mathrm{s}$, greater than previously observed in any system.
- Polarization at mid-rapidity decreases with collision energy in good agreement with hydro and transport models.

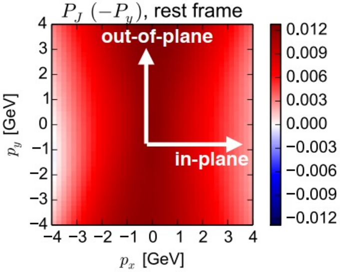
Local polarization

The global polarization reflects the total amount of angular momentum retained in the mid-rapidity region. How is it distributed in different ϕ?

Spin harmonic flow: $\frac{dP_{y,z}}{d\phi} = \frac{1}{2\pi}\left[P_{y,z} + 2f_{2y,z}\sin(2\phi) + 2g_{2y,z}\cos(2\phi) + \cdots\right]$

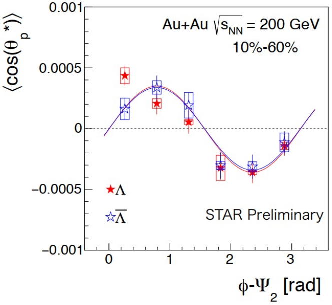


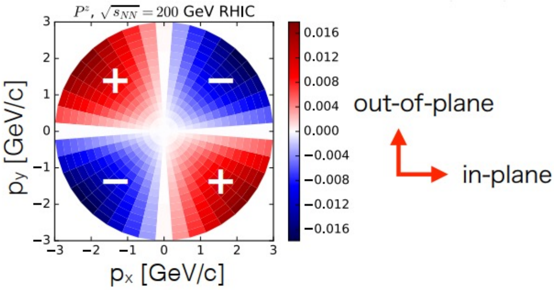




- Exp data: P_H in-plane $>$ P_H out-of-plane, $g_{2y}^{\text{exp}} > 0$
- Simulations: P_H out-of-plane $>$ P_H in-plane, $g_{2y}^{\text{ther}} < 0$

Local polarization alone beam direction





Becattini, Karpenko, PRL (2018)

- Exp data: $f_{2z}^{\text{exp}} > 0$
- Simulations: $f_{2z}^{\text{ther}} < 0$

We have a spin "sign problem"! $\Longrightarrow$ A dynamic framework for spin: Relativistic spin hydrodynamics

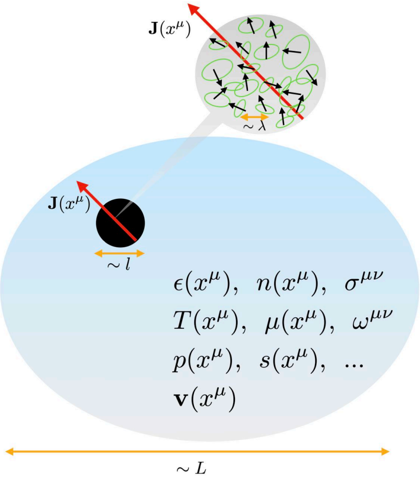
Zubarev's non-equilibrium statistical operators formalism

Coarse-Graining process

The coarse-graining process is the fundamental procedure that bridges microscopic physics with the macroscopic, continuous description of the fluid dynamics.

- The Core Idea (Separation of scales): Hydrodynamics is an effective theory that works because a system's degrees of freedom can be separated into two distinct categories based on their relaxation times:
  1. Fast Modes: These are the microscopic, non-hydrodynamic modes, such as particle-particle collisions or the relaxation of internal quantum numbers. They have a very short relaxation time, $\tau = 1/\Gamma$.
  2. Slow Modes: These are the macroscopic, hydrodynamic modes. They correspond to the system's conserved quantities (e.g., energy, momentum, charge). Because of their conservation, these modes relax very slowly at long wavelengths (their relaxation time $\tau_{\text{hydro}} \to \infty$ as the wave number $k \to 0$).
- The Key Assumption (Local Thermodynamic Equilibrium): This means we don't track individual particles. Instead, we look at a "fluid cell"-a region small on a macroscopic scale but large on a microscopic one. We assume that at any given spacetime point x, this cell is in a state of thermodynamic equilibrium.

Conservation laws



Coarse-graining process: $\lambda \ll l \ll L$

- Charge currents conservation

$$\partial_\mu \hat{N}^\mu_a = 0, \quad a = 1, \ldots, l.$$

- Energy-momentum conservation

$$\partial_\mu \hat{T}^{\mu\nu} = 0.$$

- Angular momentum (AM) conservation

$$\partial_\lambda \hat{J}^{\lambda\mu\nu} = \partial_\lambda \hat{S}^{\lambda\mu\nu} + 2\,\hat{T}^{[\mu\nu]} = 0,$$
$$\hat{J}^{\lambda\mu\nu} = \underbrace{x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu}}_{\text{orbital AM}} + \underbrace{\hat{S}^{\lambda\mu\nu}}_{\text{spin AM}}.$$

Pseudo-gauge ambiguity in spin current

Pseudo-gauge transformation: Transformations that preserve the conservation laws of total energy-momentum tensor and AM

$$
\begin{aligned}
\hat{T}'^{\mu\nu} =& \hat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\hat{\Phi}^{\alpha\mu\nu} - \hat{\Phi}^{\mu\alpha\nu} - \hat{\Phi}^{\nu\alpha\mu}\right), \\
\hat{S}'^{\mu\nu\rho} =& \hat{S}^{\mu\nu\rho} - \hat{\Phi}^{\mu\nu\rho} + \partial_\alpha \hat{Z}^{\mu\nu\rho\alpha}, \\
\hat{\Phi}^{\mu\nu\rho} =& -\hat{\Phi}^{\mu\rho\nu}, \qquad \hat{Z}^{\mu\nu\rho\sigma} = -\hat{Z}^{\nu\mu\rho\sigma} = -\hat{Z}^{\mu\nu\sigma\rho}.
\end{aligned}
$$

Examples (Dirac field):

- Canonical forms (directly obtained from Noether's Theorem):

$$\hat{S}_C^{\lambda\mu\nu} = \frac{1}{2}\overline{\psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\psi, \quad \Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

  It is dual to the axial current: $\hat{S}_C^{\lambda\mu\nu} = \epsilon^{\lambda\mu\nu\rho}\hat{j}_{A\rho}$

- Belinfante-Rosenberg version: $\hat{\Phi}^{\lambda\mu\nu} = \hat{S}^{\lambda\mu\nu}, \hat{Z}^{\mu\nu\rho\alpha} = 0$
- Groot-Leeuwen-Weert and Hilgevoord-Wouthuysen version: $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}, \partial_\lambda \hat{S}^{\lambda\mu\nu} = 0$

In the following, we will adopt the anti-symmetric gauge.

Local-equilibrium statistical operator

The thermodynamic state of a macroscopic quantum system is described by the statistical operator (rest frame)

Spin chemical potential

$$\hat{\rho}_{\rm eq} = e^{\Omega - \beta\left(\hat{H} - \sum_a \mu_a \hat{\mathcal{N}}_a - \frac{1}{2}\omega_{\alpha\beta}\hat{\mathcal{S}}^{\alpha\beta}\right)}, \quad e^{-\Omega} = \mathrm{Tr}\, e^{-\beta\left(\hat{H} - \sum_a \mu_a \hat{\mathcal{N}}_a - \frac{1}{2}\omega_{\alpha\beta}\hat{\mathcal{S}}^{\alpha\beta}\right)}.$$

To generalize this equilibrium distribution to an arbitrary reference frame, we perform a Lorentz transformation of the Hamiltonian: $\hat{H} \to \hat{\mathcal{P}}^{\mu} u_{\mu}$.

$$\hat{\mathcal{P}}^{\nu} = \int d^3x \hat{T}^{0\nu}(x), \quad \hat{\mathcal{N}}_a = \int d^3x \hat{N}_a^0(x), \quad \hat{\mathcal{S}}^{\alpha\beta} = \int d^3x \hat{S}^{0\alpha\beta}(x).$$

Local equilibrium statistical operator: $\beta^{\nu} = \beta u^{\nu}, \alpha_a = \beta\mu_a, \Omega_{\alpha\beta} = \beta\omega_{\alpha\beta}$

$$\hat{\rho}_l(t) = \exp\left\{\Omega_l(t) - \int d^3x \left[\beta_{\nu}(x)\,\hat{T}^{0\nu}(x) - \sum_a \alpha_a(x)\hat{N}_a^0(x) - \frac{1}{2}\Omega_{\alpha\beta}(x)\,\hat{S}^{0\alpha\beta}(x)\right]\right\},$$

$$e^{-\Omega_l(t)} = \mathrm{Tr}\exp\left\{-\int d^3x \left[\beta_{\nu}(x)\,\hat{T}^{0\nu}(x) - \sum_a \alpha_a(x)\hat{N}_a^0(x) - \frac{1}{2}\Omega_{\alpha\beta}(x)\hat{S}^{0\alpha\beta}(x)\right]\right\}.$$

Local-equilibrium statistical operator

Local-equilibrium statistical operator (local rest frame): $d^3\widetilde{x} = u^0\left(x\right)d^3x$

$$\hat{\rho}_l(t) = \exp\left\{\Omega_l(t) - \int d^3\widetilde{x}\beta(x)\left[\hat{\epsilon}(x) - \sum_a \mu_a(x)\hat{n}_a(x) - \frac{1}{2}\omega_{\alpha\beta}\left(x\right)\hat{S}^{\alpha\beta}\left(x\right)\right]\right\},$$

$$e^{-\Omega_l(t)} = \mathrm{Tr}\exp\left\{-\int d^3\widetilde{x}\beta(x)\left[\hat{\epsilon}(x) - \sum_a \mu_a(x)\hat{n}_a(x) - \frac{1}{2}\omega_{\alpha\beta}\left(x\right)\hat{S}^{\alpha\beta}\left(x\right)\right]\right\},$$

Thermodynamic relations (entropy operator $\hat{S} = -\ln\hat{\rho}_l$):

$$Tds = d\epsilon - \sum_a \mu_a dn_a - \frac{1}{2}\omega_{\alpha\beta}dS^{\alpha\beta},$$

$$\epsilon + p = Ts + \sum_a \mu_a n_a + \frac{1}{2}\omega_{\alpha\beta}S^{\alpha\beta},$$

$$dp = sdT + \sum_a n_a d\mu_a + \frac{1}{2}S^{\alpha\beta}d\omega_{\alpha\beta}.$$

Non-equilibrium statistical operator

Modify $\hat{\rho}_l$ to incorporate the irreversibility of the thermodynamic processes

$$\hat{\rho}_\varepsilon(t) \equiv \exp\left[-\varepsilon \int_{-\infty}^{t} dt' \, e^{\varepsilon(t'-t)} \hat{S}(t')\right].$$

Complete statistical operator:

Local equilibrium

Dissipative corrections

$$\hat{\rho}(t) = Q^{-1} e^{-\hat{A}+\hat{B}}, \qquad Q = \mathrm{Tr} e^{-\hat{A}+\hat{B}},$$

$$\hat{A}(t) = \int d^3x \left[\beta_\nu(\boldsymbol{x},t)\hat{T}^{0\nu}(\boldsymbol{x},t) - \sum_a \alpha_a(\boldsymbol{x},t)\hat{N}_a^0(\boldsymbol{x},t) - \frac{1}{2}\Omega_{\alpha\beta}(\boldsymbol{x},t)\hat{S}^{0\alpha\beta}(\boldsymbol{x},t)\right],$$

$$\hat{B}(t) = \int d^3x \int_{-\infty}^{t} dt_1 \, e^{\varepsilon(t_1-t)} \hat{C}(\boldsymbol{x},t_1),$$

$$\begin{aligned}\hat{C}(\boldsymbol{x},t) =& \hat{T}^{\mu\nu}(\boldsymbol{x},t)\partial_\mu\beta_\nu(\boldsymbol{x},t) - \sum_a \hat{N}_a^\mu(\boldsymbol{x},t)\partial_\mu\alpha_a(\boldsymbol{x},t) - \frac{1}{2}\hat{S}^{\lambda\alpha\beta}(\boldsymbol{x},t)\,\partial_\lambda\Omega_{\alpha\beta}(\boldsymbol{x},t)\\ &+ \Omega_{\alpha\beta}(\boldsymbol{x},t)\,\hat{T}^{[\alpha\beta]}(\boldsymbol{x},t).\end{aligned}$$

Second-order expansion of the statistical operator

Keeping the second-order terms in the Taylor expansion of $\hat{\rho}(t)$ with respect to the operator $\hat{B}(t)$ using the Kubo identity:

$$e^{-\hat{A}+\hat{B}} = e^{-\hat{A}} + \int_0^1 d\tau e^{\tau\left(-\hat{A}+\hat{B}\right)} \hat{B} e^{\tau\hat{A}} e^{-\hat{A}}.$$

Second-order expansion of the non-equilibrium statistical operator

$$\begin{aligned}
\hat{\rho} =& \hat{\rho}_I + \hat{\rho}_1 + \hat{\rho}_2, \\
\hat{\rho}_1 =& \int d^4x_1 \int_0^1 d\tau \left[\hat{C}_\tau(x_1) - \langle\hat{C}_\tau(x_1)\rangle_I\right]\hat{\rho}_I, \quad \hat{X}_\tau = e^{-\tau\hat{A}}\hat{X}e^{\tau\hat{A}}, \quad \int d^4x_1 \equiv \int d^3x_1 \int_{-\infty}^t dt_1 e^{\varepsilon(t_1-t)}, \\
\hat{\rho}_2 =& \frac{1}{2}\int d^4x_1 d^4x_2 \int_0^1 d\tau \int_0^1 d\lambda \Big[\widetilde{T}\{\hat{C}_\lambda(x_1)\,\hat{C}_\tau(x_2)\} - \langle\widetilde{T}\{\hat{C}_\lambda(x_1)\,\hat{C}_\tau(x_2)\}\rangle_I \\
& - \langle\hat{C}_\lambda(x_1)\rangle_I\hat{C}_\tau(x_2) - \hat{C}_\lambda(x_1)\,\langle\hat{C}_\tau(x_2)\rangle_I + 2\langle\hat{C}_\lambda(x_1)\rangle_I\langle\hat{C}_\tau(x_2)\rangle_I\Big]\hat{\rho}_I.
\end{aligned}$$

$\widetilde{T}$ is the anti-chronological time-ordering operator that rearranges the operators in chronological order with later times to the right.

Second-order expansion of the statistical operator

Statistical average of an arbitrary operator $\langle \hat{X}(x) \rangle$:

$$\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_l + \int d^4x_1 (\hat{X}(x), \hat{C}(x_1)) + \int d^4x_1 \int d^4x_2 (\hat{X}(x), \hat{C}(x_1), \hat{C}(x_2)),$$

Two-point correlation function and the three-point correlation function:

$$(\hat{X}(x), \hat{Y}(x_1)) = \int_0^1 d\tau \Big\langle \hat{X}(x) [\hat{Y}_\tau(x_1) - \langle \hat{Y}_\tau(x_1) \rangle_l] \Big\rangle_l ,$$

$$\Big(\hat{X}(x), \hat{Y}(x_1), \hat{Z}(x_2)\Big) = \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \Big\langle \widetilde{T} \Big\{ \hat{X}(x) \Big[\hat{Y}_\lambda(x_1) \hat{Z}_\tau(x_2) - \langle \widetilde{T} \hat{Y}_\lambda(x_1) \hat{Z}_\tau(x_2) \rangle_l$$
$$- \langle \hat{Y}_\lambda(x_1) \rangle_l \hat{Z}_\tau(x_2) - \hat{Y}_\lambda(x_1) \langle \hat{Z}_\tau(x_2) \rangle_l + 2 \langle \hat{Y}_\lambda(x_1) \rangle_l \langle \hat{Z}_\tau(x_2) \rangle_l \Big] \Big\} \Big\rangle_l .$$

Symmetry relation:

$$\int d^4x_1 d^4x_2 \Big(\hat{X}(x), \hat{Y}(x_1), \hat{Z}(x_2)\Big) = \int d^4x_1 d^4x_2 \Big(\hat{X}(x), \hat{Z}(x_1), \hat{Y}(x_2)\Big) .$$

Spin hydrodynamic equations

Tensor decomposition

$$\hat{N}_a^\mu = \hat{n}_a u^\mu + \hat{j}_a^\mu,$$
$$\hat{S}^{\lambda\mu\nu} = u^\lambda \hat{S}^{\mu\nu} + u^\mu \hat{S}^{\nu\lambda} + u^\nu \hat{S}^{\lambda\mu} + \hat{\varpi}^{\lambda\mu\nu},$$
$$\hat{T}^{\mu\nu} = \hat{\epsilon} u^\mu u^\nu - \hat{p}\Delta^{\mu\nu} + \hat{h}^\mu u^\nu + \hat{h}^\nu u^\mu + \hat{\pi}^{\mu\nu} + \hat{q}^\mu u^\nu - \hat{q}^\nu u^\mu + \hat{\phi}^{\mu\nu}.$$

$$\Pi = \langle \hat{p} \rangle - p\left(\epsilon, n_a, S^{\alpha\beta}\right)$$

Power counting schemes

$$\hat{n}_a \sim \hat{S}^{\mu\nu} \sim \hat{\epsilon} \sim u^\mu \sim \mathcal{O}\left(\partial^0\right), \quad \frac{\partial \Omega_{\alpha\beta}}{\partial S^{\rho\sigma}} \sim \mathcal{O}\left(\partial^2\right),$$
$$\hat{h}^\mu \sim \hat{\pi}^{\mu\nu} \sim \hat{q}^\mu \sim \hat{\phi}^{\mu\nu} \sim \hat{\varpi}^{\lambda\mu\nu} \sim \omega^{\mu\nu} \sim \frac{\partial p}{\partial S^{\alpha\beta}} \sim \frac{\partial \beta}{\partial S^{\alpha\beta}} \sim \frac{\partial \alpha_a}{\partial S^{\alpha\beta}} \sim \mathcal{O}\left(\partial^1\right).$$

$$\hat{\epsilon} = u_\mu u_\nu \hat{T}^{\mu\nu}, \quad \hat{p} = -\frac{1}{3}\Delta_{\mu\nu}\hat{T}^{\mu\nu}, \quad \hat{h}^\mu = \Delta^\mu_{(\alpha} u_{\beta)} \hat{T}^{\alpha\beta}, \quad \hat{\pi}^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\hat{T}^{\alpha\beta}, \quad \hat{q}^\mu = \Delta^\mu_{[\alpha} u_{\beta]} \hat{T}^{\alpha\beta},$$
$$\hat{\phi}^{\mu\nu} = \cancel{\Delta}^{\mu\nu}_{\alpha\beta}\hat{T}^{\alpha\beta}, \quad \hat{n}_a = u_\mu \hat{N}_a^\mu, \quad \hat{j}_a^\nu = \Delta^\nu_\mu \hat{N}_a^\mu, \quad \hat{S}^{\mu\nu} = u_\lambda \hat{S}^{\lambda\mu\nu}, \quad \hat{\varpi}^{\lambda\mu\nu} = \cancel{\Delta}^{\lambda\mu\nu}_{\rho\sigma\delta}\hat{S}^{\rho\sigma\delta}.$$

Spin hydrodynamic equations

Dissipative spin hydrodynamics: $\partial_\mu u_\nu = u_\mu D u_\nu + \frac{1}{3}\theta\Delta_{\mu\nu} + \sigma_{\mu\nu} + \nabla_{[\mu}u_{\nu]}$

$$
\begin{aligned}
&Dn_a + n_a\theta + \partial_\mu j_a^\mu = 0,\\
&D\epsilon + (\epsilon + p + \Pi)\theta + \partial_\mu h^\mu - h^\mu Du_\mu - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu q^\mu + q^\mu Du_\mu - \phi^{\mu\nu}\partial_\mu u_\nu = 0,\\
&(\epsilon + p + \Pi)Du_\alpha - \nabla_\alpha(p + \Pi) + \Delta_{\alpha\mu}Dh^\mu + h^\mu\partial_\mu u_\alpha + h_\alpha\theta + \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}\\
&+ q^\mu\partial_\mu u_\alpha - q_\alpha\theta - \Delta_{\alpha\nu}Dq^\nu + \Delta_{\alpha\nu}\partial_\mu\phi^{\mu\nu} = 0,\\
&DS^{\mu\nu} + \theta S^{\mu\nu} + u^\mu\partial_\lambda S^{\nu\lambda} + S^{\nu\lambda}\partial_\lambda u^\mu + u^\nu\partial_\lambda S^{\lambda\mu} + S^{\lambda\mu}\partial_\lambda u^\nu + \partial_\lambda\varpi^{\lambda\mu\nu}\\
&+ 2q^\mu u^\nu - 2q^\nu u^\mu + 2\phi^{\mu\nu} = 0.
\end{aligned}
$$

Ideal spin hydrodynamics: $\langle\hat{h}^\mu\rangle_l = 0, \langle\hat{\pi}^{\mu\nu}\rangle_l = 0, \langle\hat{q}^\mu\rangle_l = 0, \langle\hat{\phi}^{\mu\nu}\rangle_l = 0, \langle\hat{j}_a^\mu\rangle_l = 0, \langle\hat{\varpi}^{\lambda\mu\nu}\rangle_l = 0$

$$
\begin{aligned}
&Dn_a + n_a\theta = 0,\\
&D\epsilon + (\epsilon + p)\,\theta = 0,\\
&(\epsilon + p)\,Du_\alpha = \nabla_\alpha p,\\
&DS^{\mu\nu} + \theta S^{\mu\nu} + u^\mu\partial_\lambda S^{\nu\lambda} + S^{\nu\lambda}\partial_\lambda u^\mu + u^\nu\partial_\lambda S^{\lambda\mu} + S^{\lambda\mu}\partial_\lambda u^\nu = 0.
\end{aligned}
$$

Decomposition into different dissipative processes

$$\hat{C} = \hat{\epsilon} D\beta - \hat{p}\beta\theta - \sum_a \hat{n}_a D\alpha_a - \frac{1}{2}\hat{S}^{\alpha\beta} D\Omega_{\alpha\beta} + \hat{h}^\mu \left(\beta D u_\mu + \partial_\mu \beta\right) - \sum_a \hat{j}^\mu_a \partial_\mu \alpha_a$$
$$+ \hat{q}^\mu \left(\partial_\mu \beta - \beta D u_\mu\right) + \beta \hat{\pi}^{\mu\nu} \partial_\mu u_\nu - u^\alpha \hat{S}^{\beta\lambda} \partial_\lambda \Omega_{\alpha\beta} + \hat{\phi}^{\mu\nu} \left(\beta \partial_\mu u_\nu + \Omega_{\mu\nu}\right) - \frac{1}{2} \hat{\varpi}^{\lambda\alpha\beta} \partial_\lambda \Omega_{\alpha\beta}.$$

Final expression for the operator $\hat{C}$ accurate to second order in gradients:

$$\hat{C}(x) = \hat{C}_1(x) + \hat{C}_2(x),$$
$$\hat{C}_1(x) = -\beta\theta\hat{p}^* - \sum_a \hat{\mathscr{J}}^\sigma_a \nabla_\sigma \alpha_a + \beta \hat{q}^\mu M_\mu + \beta \hat{\pi}^{\mu\nu} \sigma_{\mu\nu} + \beta \hat{\phi}^{\mu\nu} \xi_{\mu\nu},$$
$$\hat{C}_2(x) = \beta \hat{S}^{\alpha\beta} \mathcal{R}_{\alpha\beta} + \beta \sum_i \Big[\left(\hat{\mathfrak{D}}_i \partial^i_{\epsilon n} \beta\right) \mathcal{X} + \sum_a \left(\hat{\mathfrak{D}}_i \partial^i_{\epsilon n} \alpha_a\right) \mathcal{Y}_a + \left(\hat{\mathfrak{D}}_i \partial^i_{\epsilon n} \Omega_{\mu\nu}\right) \mathcal{Z}^{\mu\nu} \Big]$$
$$+ \beta \hat{h}^\sigma \mathcal{H}_\sigma + \beta \hat{q}^\mu \mathcal{Q}_\mu + \hat{\varpi}^{\lambda\alpha\beta} \Xi_{\lambda\alpha\beta},$$

$$
\begin{aligned}
\hat{p}^* =& \hat{p} - \Gamma\hat{\epsilon} - \sum_a \delta_a \hat{n}_a, \Gamma \equiv \left.\frac{\partial p}{\partial \epsilon}\right|_{n_a, S^{\alpha\beta}}, \delta_c \equiv \left.\frac{\partial p}{\partial n_c}\right|_{\epsilon, n_b \neq n_c, S^{\alpha\beta}}, \mathcal{K}_{\alpha\beta} \equiv \left.\frac{\partial p}{\partial S^{\alpha\beta}}\right|_{\epsilon, n_a}, \hat{\mathscr{J}}_a^{\sigma} = \hat{j}_a^{\sigma} - \frac{n_a}{\epsilon + p}\hat{h}^{\sigma}, \\
M_\mu =& -\left(h^{-1}\nabla_\mu p + \beta \nabla_\mu T\right), \xi_{\mu\nu} = \Delta_{\mu\nu}^{\lambda\delta}\left(\partial_\lambda u_\delta + \beta^{-1}\Omega_{\lambda\delta}\right), \sigma_{\mu\nu} = \Delta_{\mu\nu}^{\lambda\delta}\partial_\lambda u_\delta, \\
\mathcal{Z}^{\alpha\beta} =& \beta^{-1}\left(u^{[\alpha}\partial_\lambda S^{\beta]\lambda} + S^{[\beta\lambda}\partial_\lambda u^{\alpha]} + q^\alpha u^\beta - q^\beta u^\alpha + \phi^{\alpha\beta}\right), \mathcal{W}_{\beta\lambda} = -\beta^{-1}\Delta_{\beta\lambda}^{\rho\sigma} u^\alpha \partial_\sigma \Omega_{\alpha\rho}, \\
\Xi_{\lambda\alpha\beta} =& -\frac{1}{2}\Delta_{\lambda\alpha\beta\rho\sigma\delta}\partial^\rho \Omega^{\sigma\delta}, \mathcal{Y}_a = \beta^{-1}\partial_\mu j_a^\mu, \mathcal{G}^{\alpha\beta} = \frac{1}{2}\beta^{-1}\partial_\lambda \varpi^{\lambda\alpha\beta}, \\
\mathcal{X} =& -\beta^{-1}\left(\Pi\theta + \partial_\mu h^\mu - h^\mu D u_\mu - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu q^\mu + q^\mu D u_\mu - \phi^{\mu\nu}\partial_\mu u_\nu\right), \\
\mathcal{H}_\sigma =& -h^{-1}\Big(-\frac{1}{2}\beta^{-1}S^{\alpha\beta}\nabla_\sigma \Omega_{\alpha\beta} - \nabla_\sigma \Pi + \Pi D u_\sigma + \Delta_{\sigma\nu} D h^\nu + h^\mu \partial_\mu u_\sigma + h_\sigma \theta + \Delta_{\sigma\nu}\partial_\mu \pi^{\mu\nu} + q^\mu \partial_\mu u_\sigma \\
& - q_\sigma \theta - \Delta_{\sigma\nu} D q^\nu + \Delta_{\sigma\nu}\partial_\mu \phi^{\mu\nu}\Big), \\
\mathcal{Q}_\mu =& h^{-1}\left(\Pi D u_\mu - \nabla_\mu \Pi + \Delta_{\mu\nu} D h^\nu + h^\nu \partial_\nu u_\mu + h_\mu \theta + \Delta_{\mu\nu}\partial_\rho \pi^{\rho\nu} + q^\nu \partial_\nu u^\mu - q_\mu \theta - \Delta_{\mu\nu} D q^\nu + \Delta_{\mu\nu}\partial_\rho \phi^{\rho\nu}\right), \\
\mathcal{R}_{\alpha\beta} =& \theta \mathcal{K}_{\alpha\beta} + \mathcal{W}_{\alpha\beta}, \mathcal{D}_{\alpha\beta} = \left.\frac{\partial \beta}{\partial S^{\alpha\beta}}\right|_{\epsilon, n_b}, \mathcal{E}_{\alpha\beta}^a = \left.\frac{\partial \alpha_a}{\partial S^{\alpha\beta}}\right|_{\epsilon, n_b}, \mathcal{F}_{\alpha\beta\delta\rho} = \left.\frac{\partial \Omega_{\alpha\beta}}{\partial S^{\delta\rho}}\right|_{\epsilon, n_b, S^{\alpha\beta} \neq S^{\delta\rho}}, \\
\hat{\mathfrak{D}}_i =& \left(\hat{\epsilon}, \hat{n}_1, \hat{n}_2, \cdots, \hat{n}_l\right), \partial_{\epsilon n}^i = \left(\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial n_1}, \frac{\partial}{\partial n_2}, \cdots, \frac{\partial}{\partial n_l}\right), \sum_i = \sum_{i=0}^{l} = \sum_{i=0} + \sum_{a=1}^{l}.
\end{aligned}
$$

Statistical average up to the second order

Statistical average of operator $\hat{X}(x)$:

$$\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_l + \langle \hat{X}(x) \rangle_1 + \langle \hat{X}(x) \rangle_2 .$$

First-order correction: $\langle \hat{\pi}_{\mu\nu}(x) \rangle_1 = \int d^4 x_1 \left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1) \right) \beta(x_1) \sigma^{\rho\sigma}(x_1)$

$$\langle \hat{X}(x) \rangle_1 = \int d^4 x_1 \left(\hat{X}(x), \hat{C}_1(x_1) \right) \Big|_{\text{loc}},$$

Second-order correction: $\langle \hat{X}(x) \rangle_2 = \langle \hat{X}(x) \rangle_2^1 + \langle \hat{X}(x) \rangle_2^2 + \langle \hat{X}(x) \rangle_2^3$

$$\langle \hat{X}(x) \rangle_2^1 = \int d^4 x_1 \left(\hat{X}(x), \hat{C}_1(x_1) \right) - \langle \hat{X}(x) \rangle_1, \longleftarrow \text{Non-locality effect}$$

$$\langle \hat{X}(x) \rangle_2^2 = \int d^4 x_1 \left(\hat{X}(x), \hat{C}_2(x_1) \right), \longleftarrow \text{Generalized thermodynamic forces}$$

$$\langle \hat{X}(x) \rangle_2^3 = \int d^4 x_1 d^4 x_2 \left(\hat{X}(x), \hat{C}_1(x_1), \hat{C}_1(x_2) \right) \longleftarrow \text{Nonlinear corrections}$$

First-order spin hydrodynamics

First-order spin hydrodynamics

Curie's theorem: in an isotropic medium the correlations between operators of different rank or parity vanish.

Constitutive relations:

$$
\begin{aligned}
\pi_{\mu\nu} &= 2\underbrace{\eta}_{\text{Shear viscosity}}\sigma_{\mu\nu}, \\
\Pi &= -\underbrace{\zeta}_{\text{Bulk viscosity}}\theta, \\
\phi^{\mu\nu} &= 2\underbrace{\gamma}_{\text{Rotational viscosity}}\xi^{\mu\nu}, \\
\mathscr{J}^{\mu}_{a} &= \sum_{b} \underbrace{\chi_{ab}}_{\text{Diffusion coefficients}}\nabla^{\mu}\alpha_{b} + \chi_{\mathscr{J}_a q} M^{\mu}, \\
q^{\mu} &= -\underbrace{\lambda}_{\text{Boost heat conductivity}} M^{\mu} + \sum_{a} \lambda_{q\mathscr{J}_a}\nabla^{\mu}\alpha_{a},
\end{aligned}
$$

This theory is parabolic and suffers from acausality and instabilities.

First-order transport coefficients

$$\eta = \frac{\beta}{10}\int d^4x_1 \left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}^{\mu\nu}(x_1)\right) = -\frac{1}{10}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\Big|_{\omega=0},$$

$$\zeta = \beta\int d^4x_1 \left(\hat{p}^*(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{p}^*\hat{p}^*}(\omega)\Big|_{\omega=0},$$

$$\gamma = \frac{1}{6}\beta\int d^4x_1 \left(\hat{\phi}^{\lambda\eta}(x), \hat{\phi}_{\lambda\eta}(x_1)\right) = -\frac{1}{6}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{\phi}^{\lambda\eta}\hat{\phi}_{\lambda\eta}}(\omega)\Big|_{\omega=0},$$

$$\chi_{ab} = -\frac{1}{3}\int d^4x_1 \left(\hat{\mathscr{J}}^{\lambda}_a(x), \hat{\mathscr{J}}_{b\lambda}(x_1)\right) = \frac{T}{3}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{\mathscr{J}}^{\lambda}_a\hat{\mathscr{J}}_{b\lambda}}(\omega)\Big|_{\omega=0},$$

$$\chi_{\mathscr{J}_a q} = \frac{1}{3}\beta\int d^4x_1 \left(\hat{\mathscr{J}}^{\lambda}_a(x), \hat{q}_{\lambda}(x_1)\right) = -\frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{\mathscr{J}}^{\lambda}_a\hat{q}_{\lambda}}(\omega)\Big|_{\omega=0},$$

$$\lambda = -\frac{1}{3}\beta\int d^4x_1 \left(\hat{q}^{\lambda}(x), \hat{q}_{\lambda}(x_1)\right) = \frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{q}^{\lambda}\hat{q}_{\lambda}}(\omega)\Big|_{\omega=0},$$

$$\lambda_{q\mathscr{J}_a} = -\frac{1}{3}\int d^4x_1 \left(\hat{q}^{\lambda}(x), \hat{\mathscr{J}}_{a\lambda}(x_1)\right) = \frac{T}{3}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{q}^{\lambda}\hat{\mathscr{J}}_{a\lambda}}(\omega)\Big|_{\omega=0}.$$

$G^R_{\hat{X}\hat{Y}}(\omega) = -i\int_0^{\infty} dt e^{i\omega t}\int d^3x \left\langle\left[\hat{X}(\boldsymbol{x},t), \hat{Y}(\boldsymbol{0},0)\right]\right\rangle_l$

$G^R_{\hat{Y}\hat{X}}(\omega) = \eta_X\eta_Y G^R_{\hat{X}\hat{Y}}(\omega)$, $\eta_{X,Y} = \pm 1$ for even/odd parity under time reversal

Second-order spin hydrodynamics

Relaxation equation for the shear stress tensor

$$\tau_\pi \Delta_{\mu\nu\rho\sigma} D\pi^{\rho\sigma} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \widetilde{\eta}_\pi \theta \pi_{\mu\nu} + 2\widetilde{\eta}\theta\Gamma\sigma_{\mu\nu}$$
$$+2\eta_{\pi p \pi}\theta\sigma_{\mu\nu} + \sum_{ab}\eta_{\pi \mathscr{J}_a \mathscr{J}_b}\nabla_{\langle\mu}\alpha_a\nabla_{\nu\rangle}\alpha_b + 2\sum_a \eta_{\pi \mathscr{J}_a q}\nabla_{\langle\mu}\alpha_a M_{\nu\rangle}$$
$$+\eta_{\pi qq} M_{\langle\mu}M_{\nu\rangle} + \eta_{\pi\pi\pi}\sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}^{\ \ \alpha} + 2\eta_{\pi\pi\phi}\sigma_{\alpha\langle\mu}\xi_{\nu\rangle}^{\ \ \alpha} + \eta_{\pi\phi\phi}\xi_{\alpha\langle\mu}\xi_{\nu\rangle}^{\ \ \alpha},$$

$$\tau_\pi = -\widetilde{\eta}\eta^{-1}, \widetilde{\eta} = i\frac{d}{d\omega}\eta(\omega)\Big|_{\omega=0} = -\frac{1}{20}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\Big|_{\omega=0}, \widetilde{\eta}_\pi = \tau_\pi\eta^{-1}\beta\left(\frac{\partial\eta}{\partial\beta}\Gamma - \sum_a \frac{\partial\eta}{\partial\alpha_a}\delta_a\right),$$

$$\eta_{\pi p\pi} = -\frac{1}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{\pi}_{\gamma\delta}(x), \hat{p}^*(x_1), \hat{\pi}^{\gamma\delta}(x_2)\right), \eta_{\pi\mathscr{J}_a\mathscr{J}_b} = \frac{1}{5}\int d^4x_1 d^4x_2\left(\hat{\pi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^\gamma(x_1), \hat{\mathscr{J}}_b^\delta(x_2)\right),$$

$$\eta_{\pi\mathscr{J}_a q} = -\frac{1}{5}\beta\int d^4x_1 d^4x_2\left(\hat{\pi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^\gamma(x_1), \hat{q}^\delta(x_2)\right), \eta_{\pi qq} = \frac{1}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{\pi}_{\gamma\delta}(x), \hat{q}^\gamma(x_1), \hat{q}^\delta(x_2)\right),$$

$$\eta_{\pi\pi\pi} = \frac{12}{35}\beta^2\int d^4x_1 d^4x_2\left(\hat{\pi}_\gamma^{\ \delta}(x), \hat{\pi}_\delta^{\ \lambda}(x_1), \hat{\pi}_\lambda^{\ \gamma}(x_2)\right), \eta_{\pi\pi\phi} = -\frac{4}{15}\beta^2\int d^4x_1 d^4x_2\left(\hat{\pi}_\gamma^{\ \delta}(x), \hat{\pi}_\delta^{\ \lambda}(x_1), \hat{\phi}_\lambda^{\ \gamma}(x_2)\right),$$

$$\eta_{\pi\phi\phi} = \frac{4}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{\pi}_\gamma^{\ \delta}(x), \hat{\phi}_\delta^{\ \lambda}(x_1), \hat{\phi}_\lambda^{\ \gamma}(x_2)\right), \eta(\omega) = \frac{\beta}{10}\int d^3x_1\int_{-\infty}^{t} e^{i\omega(t-t_1)}\left(\hat{\pi}_{\mu\nu}(\boldsymbol{x},t), \hat{\pi}^{\mu\nu}(\boldsymbol{x}_1,t_1)\right).$$

Relaxation equation for the bulk viscous pressure

$$\tau_\Pi D\Pi + \boxed{\Pi = \; -\zeta\theta} + \widetilde{\zeta}_\Pi\theta\Pi + \left[\frac{1}{2}\frac{\partial^2 p}{\partial\epsilon^2}\zeta_{\epsilon p}^2 + \frac{1}{2}\sum_{ab}\frac{\partial^2 p}{\partial n_a\partial n_b}\zeta_{n_a p}\zeta_{n_b p} + \sum_a \frac{\partial^2 p}{\partial\epsilon\partial n_a}\zeta_{\epsilon p}\zeta_{n_a p}\right]\theta^2 + \zeta_{S\phi}\mathcal{K}_{\alpha\beta}\xi^{\alpha\beta}$$

$$-\left[\Gamma\widetilde{\zeta} + \widetilde{\Gamma}\widetilde{\zeta}_{p\epsilon} + \sum_a \widetilde{\zeta}_{pn_a}\widetilde{\delta}_a\right]\theta^2 + \sum_i \zeta_{p\mathfrak{D}_i}\left[\left(\partial^i_{\epsilon n}\beta\right)\mathcal{X} + \sum_a\left(\partial^i_{\epsilon n}\alpha_a\right)\mathcal{Y}_a + \left(\partial^i_{\epsilon n}\Omega_{\alpha\beta}\right)\mathcal{Z}^{\alpha\beta}\right]$$

$$+\zeta_{ppp}\theta^2 + \sum_{ab}\zeta_{p\mathscr{J}_a\mathscr{J}_b}\nabla_\alpha\alpha_a\nabla^\alpha\alpha_b + 2\sum_a\zeta_{p\mathscr{J}_a q}\nabla^\sigma\alpha_a M_\sigma + \zeta_{pqq}M^\sigma M_\sigma + \zeta_{p\pi\pi}\sigma^{\rho\sigma}\sigma_{\rho\sigma} + \zeta_{p\phi\phi}\xi^{\rho\sigma}\xi_{\rho\sigma}.$$

$$\tau_\Pi = -\widetilde{\zeta}\zeta^{-1}, \widetilde{\zeta} = i\frac{d}{d\omega}\zeta(\omega)\Big|_{\omega=0} = -\frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{p}^*\hat{p}^*}(\omega)\Big|_{\omega=0}, \widetilde{\zeta}_\Pi = \tau_\Pi\zeta^{-1}\beta\left(\frac{\partial\zeta}{\partial\beta}\Gamma - \sum_a\frac{\partial\zeta}{\partial\alpha_a}\delta_a\right),$$

$$\zeta_{\epsilon p} = \beta\int d^4x_1\left(\hat{\epsilon}(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\epsilon}\hat{p}^*}(\omega)\Big|_{\omega=0},$$

$$\zeta_{n_a p} = \beta\int d^4x_1\left(\hat{n}_a(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{n}_a\hat{p}^*}(\omega)\Big|_{\omega=0},$$

$$\zeta_{S\phi} = -\frac{1}{3}\beta\int d^4x_1\left(\hat{S}^{\lambda\eta}(x),\hat{\phi}_{\lambda\eta}(x_1)\right) = \frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{S}^{\lambda\eta}\hat{\phi}_{\lambda\eta}}(\omega)\bigg|_{\omega=0},$$

$$\zeta_{p\mathfrak{D}_i} = \beta\int d^4x_1\left(\hat{p}^*(x),\hat{\mathfrak{D}}_i(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{p}^*\hat{\mathfrak{D}}_i}(\omega)\bigg|_{\omega=0},$$

$$\widetilde{\zeta}_{p\epsilon} = i\frac{d}{d\omega}\zeta_{p\epsilon}(\omega)\bigg|_{\omega=0} = -\frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{\epsilon}}(\omega)\bigg|_{\omega=0},\widetilde{\zeta}_{pn_a} = i\frac{d}{d\omega}\zeta_{pn_a}(\omega)\bigg|_{\omega=0} = -\frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{n}_a}(\omega)\bigg|_{\omega=0},$$

$$\widetilde{\Gamma} = \frac{\partial\Gamma}{\partial\epsilon}h + \sum_a\frac{\partial\Gamma}{\partial n_a}n_a + \frac{\partial\Gamma}{\partial S^{\alpha\beta}}\left[S^{\alpha\beta} + \theta^{-1}\left(u^\alpha\partial_\lambda S^{\beta\lambda} + S^{\beta\lambda}\partial_\lambda u^\alpha + u^\beta\partial_\lambda S^{\lambda\alpha} + S^{\lambda\alpha}\partial_\lambda u^\beta\right)\right],$$

$$\widetilde{\delta}_a = \frac{\partial\delta_a}{\partial\epsilon}h + \sum_b\frac{\partial\delta_a}{\partial n_b}n_b + \frac{\partial\delta_a}{\partial S^{\alpha\beta}}\left[S^{\alpha\beta} + \theta^{-1}\left(u^\alpha\partial_\lambda S^{\beta\lambda} + S^{\beta\lambda}\partial_\lambda u^\alpha + u^\beta\partial_\lambda S^{\lambda\alpha} + S^{\lambda\alpha}\partial_\lambda u^\beta\right)\right],$$

$$\zeta_{ppp} = \beta^2\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{p}^*(x_1),\hat{p}^*(x_2)\right),\zeta_{p\mathscr{J}_a\mathscr{J}_b} = \frac{1}{3}\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{\mathscr{J}}_{a\gamma}(x_1),\hat{\mathscr{J}}_b^{\gamma}(x_2)\right),$$

$$\zeta_{p\mathscr{J}_a q} = -\frac{1}{3}\beta\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{\mathscr{J}}_{a\gamma}(x_1),\hat{q}^\gamma(x_2)\right),\zeta_{pqq} = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{q}_\gamma(x_1),\hat{q}^\gamma(x_2)\right),$$

$$\zeta_{p\pi\pi} = \frac{1}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{\pi}_{\gamma\delta}(x_1),\hat{\pi}^{\gamma\delta}(x_2)\right),\zeta_{p\phi\phi} = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{p}^*(x),\hat{\phi}_{\gamma\delta}(x_1),\hat{\phi}^{\gamma\delta}(x_2)\right).$$

Relaxation equation for the charge diffusion currents

$$
\begin{aligned}
\sum_b \tau_{\mathscr{J}}^{cb} \Delta_{\mu\beta} D \mathscr{J}_b^{\beta} + \boxed{\mathscr{J}_{c\mu} = \textstyle\sum_b \chi_{cb} \nabla_\mu \alpha_b + \chi_{\mathscr{J}_{cq}} M_\mu} &+ \widetilde{\chi}_{\mathscr{J}_{ch}} \sum_a n_a h^{-2} \Big(\Gamma h + \sum_a \delta_a n_a\Big) \theta \nabla_\mu \alpha_a \\
&+ \overline{\chi}^c \theta M_\mu + \sum_b \tau_{\mathscr{J}}^{cb} \chi_{\mathscr{J}_{bq}} \Delta_{\mu\beta} D M^\beta + \sum_b \widetilde{\chi}_{\mathscr{J}}^{cb} \theta \mathscr{J}_{b\mu} - \sum_b \widetilde{\chi}_{\mathscr{J}}^{cb} \chi_{\mathscr{J}_{bq}} \theta M_\mu \\
&+ \widetilde{\chi}_{\mathscr{J}_{cq}} \theta \Gamma M_\mu + \widetilde{\chi}_{\mathscr{J}_{cq}} \Delta_{\mu\beta} D M^\beta + \beta \chi_{\mathscr{J}_{ch}} \mathcal{H}_\mu + \chi_{\mathscr{J}_{cq}} \mathcal{Q}_\mu + 2 \sum_a \chi_{\mathscr{J}_{cp} \mathscr{J}_a} \theta \nabla_\mu \alpha_a \\
&+ 2 \chi_{\mathscr{J}_{cpq}} \theta M_\mu + 2 \sum_a \chi_{\mathscr{J}_c \mathscr{J}_{a\pi}} \nabla^\nu \alpha_a \sigma_{\mu\nu} + 2 \sum_a \chi_{\mathscr{J}_c \mathscr{J}_{a\phi}} \nabla^\nu \alpha_a \xi_{\mu\nu} \\
&+ 2 \chi_{\mathscr{J}_{cq\pi}} M^\nu \sigma_{\mu\nu} + 2 \chi_{\mathscr{J}_{cq\phi}} M^\nu \xi_{\mu\nu},
\end{aligned}
$$

$$
\begin{aligned}
\tau^{cb}_{\mathscr{J}} =& -(\widetilde{\chi}\chi^{-1})_{cb} = -\sum_a \widetilde{\chi}_{ca}(\chi^{-1})_{ab}, \widetilde{\chi}_{ac} = i\frac{d}{d\omega}\chi_{ac}(\omega)\Big|_{\omega=0} = \frac{T}{6}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{\mathscr{J}}_a^{\lambda}\,\hat{\mathscr{J}}_{c\lambda}}(\omega)\Big|_{\omega=0},\\
\widetilde{\chi}_{\mathscr{J}_c h} =& i\frac{d}{d\omega}\chi_{\mathscr{J}_c h}(\omega)\Big|_{\omega=0} = -\frac{T}{6}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{\mathscr{J}}_c^{\lambda}\hat{h}_\lambda}(\omega)\Big|_{\omega=0}, \widetilde{\chi}_{\mathscr{J}_c q} = i\frac{d}{d\omega}\chi_{\mathscr{J}_c q}(\omega)\Big|_{\omega=0} = -\frac{1}{6}\frac{d^2}{d\omega^2}\mathrm{Re}G^R_{\hat{\mathscr{J}}_c^{\lambda}\hat{q}_\lambda}(\omega)\Big|_{\omega=0},\\
\chi_{\mathscr{J}_a h} =& \frac{1}{3}\int d^4x_1\left(\hat{\mathscr{J}}_a^{\lambda}(x), \hat{h}_\lambda(x_1)\right) = -\frac{T}{3}\frac{d}{d\omega}\mathrm{Im}G^R_{\hat{\mathscr{J}}_a^{\lambda}\hat{h}_\lambda}(\omega)\Big|_{\omega=0},\\
\widetilde{\chi}^{cb}_{\mathscr{J}} =& \beta\sum_a \widetilde{\chi}_{ca}\Big(\Gamma\frac{\partial(\chi^{-1})_{ab}}{\partial\beta} - \sum_d \delta_d\frac{\partial(\chi^{-1})_{ab}}{\partial\alpha_d}\Big), \overline{\chi}^c = \beta\sum_b \tau^{cb}_{\mathscr{J}}\Big(\Gamma\frac{\partial\chi_{\mathscr{J}_b q}}{\partial\beta} - \sum_d\delta_d\frac{\partial\chi_{\mathscr{J}_b q}}{\partial\alpha_d}\Big),\\
\chi_{\mathscr{J}_c p\mathscr{J}_a} =& \frac{1}{3}\beta\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\beta}(x), \hat{p}^*(x_1), \hat{\mathscr{J}}_a^{\beta}(x_2)\Big), \chi_{\mathscr{J}_c pq} = -\frac{1}{3}\beta^2\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\beta}(x), \hat{p}^*(x_1), \hat{q}^{\beta}(x_2)\Big),\\
\chi_{\mathscr{J}_c\mathscr{J}_a\pi} =& -\frac{1}{5}\beta\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\lambda}(x), \hat{\mathscr{J}}_{a\delta}(x_1), \hat{\pi}^{\lambda\delta}(x_2)\Big), \chi_{\mathscr{J}_c\mathscr{J}_a\phi} = -\frac{1}{3}\beta\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\lambda}(x), \hat{\mathscr{J}}_{a\delta}(x_1), \hat{\phi}^{\lambda\delta}(x_2)\Big),\\
\chi_{\mathscr{J}_c q\pi} =& \frac{1}{5}\beta^2\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\lambda}(x), \hat{q}_\delta(x_1), \hat{\pi}^{\lambda\delta}(x_2)\Big), \chi_{\mathscr{J}_c q\phi} = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2\Big(\hat{\mathscr{J}}_{c\lambda}(x), \hat{q}_\delta(x_1), \hat{\phi}^{\lambda\delta}(x_2)\Big).
\end{aligned}
$$

Relaxation equation for the rotational stress tensor

$$\tau_\phi \Delta_{\mu\nu\rho\sigma} D\phi^{\rho\sigma} + \phi_{\mu\nu} = 2\gamma\xi_{\mu\nu} + \tilde{\gamma}_\phi \theta\phi_{\mu\nu} + 2\tilde{\gamma}\theta\Gamma\xi_{\mu\nu} + \gamma_{\phi s}\mathcal{R}_{\langle\mu\rangle\langle\nu\rangle}$$
$$+2\gamma_{\phi p\phi}\theta\xi_{\mu\nu} + \sum_{ab}\gamma_{\phi\mathscr{J}_a\mathscr{J}_b}\nabla_{[\mu}\alpha_a\nabla_{\nu]}\alpha_b + 2\sum_a \gamma_{\phi\mathscr{J}_a q}\nabla_{[\mu}\alpha_a M_{\nu]}$$
$$+\gamma_{\phi qq}M_{[\mu}M_{\nu]} + \gamma_{\phi\pi\pi}\sigma_{\alpha[\mu}\sigma_{\nu]}{}^{\alpha} + 2\gamma_{\phi\pi\phi}\sigma_{\alpha[\mu}\xi_{\nu]}{}^{\alpha} + \gamma_{\phi\phi\phi}\xi_{\alpha[\mu}\xi_{\nu]}{}^{\alpha}.$$

$$\tau_\phi = -\tilde{\gamma}\gamma^{-1}, \tilde{\gamma} = i\frac{d}{d\omega}\gamma(\omega)\Big|_{\omega=0} = -\frac{1}{12}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{\phi}_{\mu\nu}\hat{\phi}^{\mu\nu}}(\omega)\Big|_{\omega=0},$$

$$\tilde{\gamma}_\phi = \tau_\phi\gamma^{-1}\beta\left(\frac{\partial\gamma}{\partial\beta}\Gamma - \sum_a \frac{\partial\gamma}{\partial\alpha_a}\delta_a\right), \gamma_{\phi s} = \frac{1}{3}\beta\int d^4x_1\left(\hat{\phi}^{\lambda\eta}(x), \hat{S}_{\lambda\eta}(x_1)\right) = -\frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\phi}^{\lambda\eta}\hat{S}_{\lambda\eta}}(\omega)\Big|_{\omega=0},$$

$$\gamma_{\phi p\phi} = -\frac{1}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{\phi}_{\gamma\delta}(x), \hat{p}^*(x_1), \hat{\phi}^{\gamma\delta}(x_2)\right), \gamma_{\phi\mathscr{J}_a\mathscr{J}_b} = \frac{1}{3}\int d^4x_1 d^4x_2\left(\hat{\phi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^{\gamma}(x_1), \hat{\mathscr{J}}_b^{\delta}(x_2)\right),$$

$$\gamma_{\phi\mathscr{J}_a q} = -\frac{1}{3}\beta\int d^4x_1 d^4x_2\left(\hat{\phi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^{\gamma}(x_1), \hat{q}^{\delta}(x_2)\right), \gamma_{\phi qq} = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{\phi}_{\gamma\delta}(x), \hat{q}^{\gamma}(x_1), \hat{q}^{\delta}(x_2)\right),$$

$$\gamma_{\phi\pi\pi} = -\frac{4}{15}\beta^2\int d^4x_1 d^4x_2\left(\hat{\phi}_\lambda{}^{\delta}(x), \hat{\pi}_\delta{}^{\eta}(x_1), \hat{\pi}_\eta{}^{\lambda}(x_2)\right), \gamma_{\phi\pi\phi} = \frac{4}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{\phi}_\lambda{}^{\delta}(x), \hat{\pi}_\delta{}^{\eta}(x_1), \hat{\phi}_\eta{}^{\lambda}(x_2)\right),$$

$$\gamma_{\phi\phi\phi} = -\frac{4}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{\phi}_\lambda{}^{\delta}(x), \hat{\phi}_\delta{}^{\eta}(x_1), \hat{\phi}_\eta{}^{\lambda}(x_2)\right).$$

Relaxation equation for the boost heat vector

$$
\begin{aligned}
\tau_q \Delta_{\mu\nu} D q^\nu + \boxed{q_\mu = \sum_a \lambda_{q\mathscr{J}_a} \nabla_\mu \alpha_a - \lambda M_\mu} &+ \sum_a \widetilde{\lambda}_{qh} n_a h^{-2} \Big(\Gamma h + \sum_a \delta_a n_a \Big) \theta \nabla_\mu \alpha_a + \sum_a \widetilde{\lambda}_{q\mathscr{J}_a} \Delta_{\mu\gamma} D (\nabla^\gamma \alpha_a) \\
&- \widetilde{\lambda} \theta \Gamma M_\mu + \tau_q \Delta_{\mu\gamma} \sum_a \lambda_{q\mathscr{J}_a} D (\nabla^\gamma \alpha_a) + \sum_a \overline{\lambda}^a \theta \nabla_\mu \alpha_a + \widetilde{\lambda}_q \theta q_\mu - \sum_a \widetilde{\lambda}_q \lambda_{q\mathscr{J}_a} \theta \nabla_\mu \alpha_a \\
&+ \lambda_{qh} \mathcal{H}_\mu - \lambda \mathcal{Q}_\mu + 2 \sum_a \lambda_{qp\mathscr{J}_a} \theta \nabla_\mu \alpha_a + 2 \lambda_{qpq} \theta M_\mu + 2 \sum_a \lambda_{q\mathscr{J}_a \pi} \nabla^\nu \alpha_a \sigma_{\mu\nu} \\
&+ 2 \sum_a \lambda_{q\mathscr{J}_a \phi} \nabla^\nu \alpha_a \xi_{\mu\nu} + 2 \lambda_{qq\pi} M^\nu \sigma_{\mu\nu} + 2 \lambda_{qq\phi} M^\nu \xi_{\mu\nu} .
\end{aligned}
$$

$$
\tau_q = -\widetilde{\lambda} \lambda^{-1}, \widetilde{\lambda} = i \frac{d}{d\omega} \lambda(\omega) \Big|_{\omega=0} = \frac{1}{6} \frac{d^2}{d\omega^2} \mathrm{Re} G^R_{\hat{q}^\lambda \hat{q}_\lambda}(\omega) \Big|_{\omega=0},
$$

$$
\widetilde{\lambda}_{qh} = i \frac{d}{d\omega} \lambda_{qh}(\omega) \Big|_{\omega=0} = -\frac{1}{6} \frac{d^2}{d\omega^2} \mathrm{Re} G^R_{\hat{q}^\lambda \hat{h}_\lambda}(\omega) \Big|_{\omega=0}, \widetilde{\lambda}_{q\mathscr{J}_a} = i \frac{d}{d\omega} \lambda_{qa}(\omega) \Big|_{\omega=0} = \frac{T}{6} \frac{d^2}{d\omega^2} \mathrm{Re} G^R_{\hat{q}^\lambda \hat{\mathscr{J}}_{a\lambda}}(\omega) \Big|_{\omega=0},
$$

$$
\begin{aligned}
\lambda_{qh} =& \frac{1}{3}\beta \int d^4x_1 \left(\hat{q}^{\lambda}(x), \hat{h}_{\lambda}(x_1)\right) = -\frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\, G^{R}_{\hat{q}^{\lambda}\hat{h}_{\lambda}}(\omega)\Big|_{\omega=0},\\
\widetilde{\lambda}_q =& \beta\tau_q\lambda^{-1}\Big(\frac{\partial\lambda}{\partial\beta}\Gamma - \sum_d \frac{\partial\lambda}{\partial\alpha_d}\delta_d\Big), \overline{\lambda}^{a} = \beta\tau_q\Big(\frac{\partial\lambda_{q\mathscr{J}_a}}{\partial\beta}\Gamma - \sum_d \frac{\partial\lambda_{q\mathscr{J}_a}}{\partial\alpha_d}\delta_d\Big),\\
\lambda_{qp\mathscr{J}_a} =& \frac{1}{3}\beta\int d^4x_1 d^4x_2 \Big(\hat{q}_{\beta}(x), \hat{p}^{*}(x_1), \hat{\mathscr{J}}_a^{\beta}(x_2)\Big), \lambda_{qpq} = -\frac{1}{3}\beta^2\int d^4x_1 d^4x_2\Big(\hat{q}_{\beta}(x), \hat{p}^{*}(x_1), \hat{q}^{\beta}(x_2)\Big),\\
\lambda_{q\mathscr{J}_a\pi} =& -\frac{1}{5}\beta\int d^4x_1 d^4x_2\Big(\hat{q}_{\lambda}(x), \hat{\mathscr{J}}_{a\delta}(x_1), \hat{\pi}^{\lambda\delta}(x_2)\Big), \lambda_{q\mathscr{J}_a\phi} = -\frac{1}{3}\beta\int d^4x_1 d^4x_2\Big(\hat{q}_{\lambda}(x), \hat{\mathscr{J}}_{a\delta}(x_1), \hat{\phi}^{\lambda\delta}(x_2)\Big),\\
\lambda_{qq\pi} =& \frac{1}{5}\beta^2\int d^4x_1 d^4x_2\Big(\hat{q}_{\lambda}(x), \hat{q}_{\delta}(x_1), \hat{\pi}^{\lambda\delta}(x_2)\Big), \lambda_{qq\phi} = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2\Big(\hat{q}_{\lambda}(x), \hat{q}_{\delta}(x_1), \hat{\phi}^{\lambda\delta}(x_2)\Big).
\end{aligned}
$$

Dissipative flux from spin tensor

$$\varpi^{\lambda\mu\nu} = \varphi \Xi^{\lambda\mu\nu} + 2\sum_a \varphi_{\varpi \mathscr{J}_a \phi} \nabla_\rho \alpha_a \xi_{\sigma\delta} \Delta^{\lambda\mu\nu\rho\sigma\delta} + 2\varphi_{\varpi q \phi} M_\rho \xi_{\sigma\delta} \Delta^{\lambda\mu\nu\rho\sigma\delta}.$$

$$\varphi = \int d^4 x_1 \left(\hat{\varpi}^{\rho\sigma\delta}(x), \hat{\varpi}_{\rho\sigma\delta}(x_1) \right) = -T \frac{d}{d\omega} \mathrm{Im}\, G^R_{\hat{\varpi}^{\rho\sigma\delta} \hat{\varpi}_{\rho\sigma\delta}}(\omega) \Big|_{\omega=0},$$

$$\varphi_{\varpi \mathscr{J}_a \phi} = -\beta \int d^4 x_1 d^4 x_2 \left(\hat{\varpi}^{\gamma\varepsilon\zeta}(x), \hat{\mathscr{J}}_{a\gamma}(x_1), \hat{\phi}_{\varepsilon\zeta}(x_2) \right), \varphi_{\varpi q \phi} = \beta^2 \int d^4 x_1 d^4 x_2 \left(\hat{\varpi}^{\gamma\varepsilon\zeta}(x), \hat{q}_\gamma(x_1), \hat{\phi}_{\varepsilon\zeta}(x_2) \right).$$

Generalized relativistic second-order spin hydrodynamics

The additional second-order contributions arising from nonlocal corrections due to two-point correlation functions of tensors of different ranks at distinct spacetime points.

$$\hat{C}(x_1) = \hat{C}_1(x_1)_x + \hat{C}_2(x_1)_x + \partial_\tau \hat{C}(x_1)_x (x_1 - x)^\tau$$

$$\begin{aligned}
\partial_\tau \hat{C} = & -\hat{p}^* \partial_\tau (\beta\theta) + \beta\theta \left[\hat{\epsilon}(\partial_\tau \Gamma) + \sum_a \hat{n}_a (\partial_\tau \delta_a)\right] - (\partial_\tau u_\rho) \sum_a (\nabla^\rho \alpha_a) \left[\frac{n_a}{h}(\hat{\epsilon} + \hat{p}) - \hat{n}_a\right] \\
& + \hat{h}_\rho \left[-2\beta\sigma^{\rho\sigma}(\partial_\tau u_\sigma) - 2\left(\frac{1}{3} - \Gamma\right)\beta\theta(\partial_\tau u^\rho) + \sum_a \partial_\tau (n_a h^{-1}) \nabla^\rho \alpha_a + \beta\theta(\partial_\tau u^\rho) h^{-1} \sum_a n_a \delta_a\right] \\
& + \hat{q}_\rho \left[\partial_\tau (\beta M^\rho) - 2\beta\xi^{\rho\sigma}(\partial_\tau u_\sigma)\right] + \sum_a \hat{\mathscr{J}}_a^\rho \left[\beta\theta(\partial_\tau u_\rho)\delta_a - \partial_\tau (\nabla_\rho \alpha_a)\right] \\
& + \hat{\pi}_{\rho\sigma} \left[\partial_\tau (\beta\sigma^{\rho\sigma}) + (\partial_\tau u^\rho) \sum_a \frac{n_a}{h} \nabla^\sigma \alpha_a\right] + \hat{\phi}_{\rho\sigma} \left[\partial_\tau (\beta\xi^{\rho\sigma}) + (\partial_\tau u^\rho) \sum_a \frac{n_a}{h} \nabla^\sigma \alpha_a\right].
\end{aligned}$$

Arus Harutyunyan, Armen Sedrakian, Annals Phys. 481 (2025) 170159

Relaxation equation for the shear-stress tensor

$$
\begin{aligned}
\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} =& 2\eta\sigma_{\mu\nu} + \widetilde{\eta}_\pi \theta \pi_{\mu\nu} + 2\widetilde{\eta}\theta\Gamma\sigma_{\mu\nu} + 2\widetilde{\eta} T h^{-1} \textstyle\sum_a n_a \dot{u}_{\langle\mu} \nabla_{\nu\rangle} \alpha_a \\
&+ 2\eta_{\pi p\pi}\theta\sigma_{\mu\nu} + \sum_{ab} \eta_{\pi \mathscr{J}_a \mathscr{J}_b} \nabla_{\langle\mu}\alpha_a \nabla_{\nu\rangle}\alpha_b + 2\sum_a \eta_{\pi \mathscr{J}_a q} \nabla_{\langle\mu}\alpha_a M_{\nu\rangle} \\
&+ \eta_{\pi qq} M_{\langle\mu} M_{\nu\rangle} + \eta_{\pi\pi\pi} \sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^{\alpha} + 2\eta_{\pi\pi\phi}\sigma_{\alpha\langle\mu}\xi_{\nu\rangle}{}^{\alpha} + \eta_{\pi\phi\phi}\xi_{\alpha\langle\mu}\xi_{\nu\rangle}{}^{\alpha},
\end{aligned}
$$

$$
\eta_{\pi p\pi} = \frac{1}{10}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma\delta}\hat{p}^*\hat{\pi}^{\gamma\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \eta_{\pi \mathscr{J}_a \mathscr{J}_b} = -\frac{T^2}{10}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma\delta}\hat{\mathscr{J}}_a^{\gamma}\hat{\mathscr{J}}_b^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0},
$$

$$
\eta_{\pi \mathscr{J}_a q} = \frac{T}{10}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma\delta}\hat{\mathscr{J}}_a^{\gamma}\hat{q}^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \eta_{\pi qq} = -\frac{1}{10}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma\delta}\hat{q}^{\gamma}\hat{q}^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0},
$$

$$
\eta_{\pi\pi\pi} = -\frac{6}{35}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma}{}^{\delta}\hat{\pi}_{\delta}{}^{\lambda}\hat{\pi}_{\lambda}{}^{\gamma}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \eta_{\pi\pi\phi} = \frac{2}{15}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma}{}^{\delta}\hat{\pi}_{\delta}{}^{\lambda}\hat{\phi}_{\lambda}{}^{\gamma}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0},
$$

$$
\eta_{\pi\phi\phi} = -\frac{2}{5}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\pi}_{\gamma}{}^{\delta}\hat{\phi}_{\delta}{}^{\lambda}\hat{\phi}_{\lambda}{}^{\gamma}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}.
$$

$$
G^R_{\hat{X}\hat{Y}\hat{Z}}(\omega_1,\omega_2) = -\tfrac{1}{2}\int_{-\infty}^{0} dt_1\, e^{-i\omega_1 t_1} \int_{-\infty}^{0} dt_2\, e^{-i\omega_2 t_2} \int d^3x_1 \int d^3x_2 \left\{ \left\langle \left[\left[\hat{X}(\mathbf{0},0), \hat{Y}(\mathbf{x}_1,t_1)\right], \hat{Z}(\mathbf{x}_2,t_2)\right]\right\rangle_l + \left\langle \left[\left[\hat{X}(\mathbf{0},0), \hat{Z}(\mathbf{x}_2,t_2)\right], \hat{Y}(\mathbf{x}_1,t_1)\right]\right\rangle_l \right\}
$$

Relaxation equation for the bulk viscous pressure

$$
\begin{aligned}
\Pi + \tau_\Pi \dot{\Pi} = &- \zeta\theta + \widetilde{\zeta}_\Pi \theta\Pi + \left[\frac{1}{2}\frac{\partial^2 p}{\partial \epsilon^2}\zeta_{\epsilon p}^2 + \frac{1}{2}\sum_{ab}\frac{\partial^2 p}{\partial n_a \partial n_b}\zeta_{n_a p}\zeta_{n_b p} + \sum_a \frac{\partial^2 p}{\partial \epsilon \partial n_a}\zeta_{\epsilon p}\zeta_{n_a p} \right]\theta^2 \\
&+ \zeta_{s\phi}\mathcal{K}_{\alpha\beta}\xi^{\alpha\beta} - \left[\Gamma\widetilde{\zeta} + \widetilde{\Gamma}\widetilde{\zeta}_{p\epsilon} + \sum_a \widetilde{\zeta}_{pn_a}\widetilde{\delta}_a \right]\theta^2 \boxed{- \textstyle\sum_a \bar{\zeta}_a \dot{u}_\rho \nabla^\rho \alpha_a} \\
&+ \sum_i \zeta_{p\mathfrak{D}_i}\left[\left(\partial^i_{\epsilon n}\beta\right)\mathcal{X} + \sum_a \left(\partial^i_{\epsilon n}\alpha_a\right)\mathcal{Y}_a + \left(\partial^i_{\epsilon n}\Omega_{\alpha\beta}\right)\mathcal{Z}^{\alpha\beta} \right] + \zeta_{ppp}\theta^2 \\
&+ \sum_{ab}\zeta_{p\mathscr{J}_a\mathscr{J}_b}\nabla_\alpha \alpha_a \nabla^\alpha \alpha_b + 2\sum_a \zeta_{p\mathscr{J}_a q}\nabla^\sigma \alpha_a M_\sigma + \zeta_{pqq}M^\sigma M_\sigma + \zeta_{p\pi\pi}\sigma^{\rho\sigma}\sigma_{\rho\sigma} + \zeta_{p\phi\phi}\xi^{\rho\sigma}\xi_{\rho\sigma}.
\end{aligned}
$$

$$
\zeta_{ppp} = -\frac{1}{2}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{p}^*\hat{p}^*}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \zeta_{p\mathscr{J}_a\mathscr{J}_b} = -\frac{T^2}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{\mathscr{J}}_{a\gamma}\hat{\mathscr{J}}_b^\gamma}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0},
$$

$$
\zeta_{p\mathscr{J}_a q} = \frac{T}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{\mathscr{J}}_{a\gamma}\hat{q}^\gamma}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \zeta_{pqq} = -\frac{1}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{q}_\gamma\hat{q}^\gamma}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0},
$$

$$
\zeta_{p\pi\pi} = -\frac{1}{10}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{\pi}_{\gamma\delta}\hat{\pi}^{\gamma\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \quad \zeta_{p\phi\phi} = -\frac{1}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{p}^*\hat{\phi}_{\gamma\delta}\hat{\phi}^{\gamma\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}.
$$

Relaxation equation for the charge-diffusion currents

$$
\begin{aligned}
\mathscr{J}_{c\mu} + \sum_b \tau^{cb}_{\mathscr{J}} \dot{\mathscr{J}}_{b\mu} = {} & \sum_b \chi_{cb} \nabla_\mu \alpha_b + \chi_{\mathscr{J}cq} M_\mu + \widetilde{\chi}_{\mathscr{J}ch} h^{-2} \left(\Gamma h + \sum_b \delta_b n_b \right) \theta \sum_a n_a \nabla_\mu \alpha_a + \overline{\chi}^c \theta M_\mu \\
& + \sum_b \tau^{cb}_{\mathscr{J}} \chi_{\mathscr{J}bq} \Delta_{\mu\beta} D M^\beta + \sum_b \widetilde{\chi}^{cb}_{\mathscr{J}} \theta \mathscr{J}_{b\mu} - \sum_b \widetilde{\chi}^{cb}_{\mathscr{J}} \chi_{\mathscr{J}bq} \theta M_\mu + \widetilde{\chi}_{\mathscr{J}cq} \theta \Gamma M_\mu \\
& + \widetilde{\chi}_{\mathscr{J}cq} \Delta_{\mu\beta} D M^\beta - \beta\theta \dot{u}_\mu \textstyle\sum_a \delta_a \widetilde{\chi}_{ac} - \beta \widetilde{\chi}_{\mathscr{J}ch} \left(2\sigma_{\mu\nu} \dot{u}^\nu + y \theta \dot{u}_\mu \right) - 2 \widetilde{\chi}_{\mathscr{J}cq} \xi_{\mu\nu} \dot{u}^\nu \\
& + \beta \chi_{\mathscr{J}ch} \mathcal{H}_\mu + \chi_{\mathscr{J}cq} \mathcal{Q}_\mu + 2 \sum_a \chi_{\mathscr{J}cp\mathscr{J}a} \theta \nabla_\mu \alpha_a + 2 \chi_{\mathscr{J}cpq} \theta M_\mu \\
& + 2 \sum_a \chi_{\mathscr{J}c\mathscr{J}a\pi} \nabla^\nu \alpha_a \sigma_{\mu\nu} + 2 \sum_a \chi_{\mathscr{J}c\mathscr{J}a\phi} \nabla^\nu \alpha_a \xi_{\mu\nu} + 2 \chi_{\mathscr{J}cq\pi} M^\nu \sigma_{\mu\nu} + 2 \chi_{\mathscr{J}cq\phi} M^\nu \xi_{\mu\nu} .
\end{aligned}
$$

$y = \frac{2}{3} - 2\Gamma - \sum_a \delta_a n_a h^{-1}$

$$
\chi_{\mathscr{J}cp\mathscr{J}a} = -\frac{T}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\beta} \hat{p}^* \hat{\mathscr{J}}^\beta_a} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0}, \quad \chi_{\mathscr{J}cpq} = \frac{1}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\beta} \hat{p}^* \hat{q}^\beta} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0},
$$

$$
\chi_{\mathscr{J}c\mathscr{J}a\pi} = \frac{T}{10} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\lambda} \hat{\mathscr{J}}_{a\delta} \hat{\pi}^{\lambda\delta}} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0}, \quad \chi_{\mathscr{J}c\mathscr{J}a\phi} = \frac{T}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\lambda} \hat{\mathscr{J}}_{a\delta} \hat{\phi}^{\lambda\delta}} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0},
$$

$$
\chi_{\mathscr{J}cq\pi} = -\frac{1}{10} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\lambda} \hat{q}_\delta \hat{\pi}^{\lambda\delta}} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0}, \quad \chi_{\mathscr{J}cq\phi} = -\frac{1}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re} G^R_{\hat{\mathscr{J}}_{c\lambda} \hat{q}_\delta \hat{\phi}^{\lambda\delta}} (\omega_1, \omega_2) \Big|_{\omega_{1,2}=0}.
$$

Relaxation equation for the rotational stress tensor

$$\begin{aligned}\phi_{\mu\nu} + \tau_\phi \dot{\phi}_{\mu\nu} =& 2\gamma\xi_{\mu\nu} + \widetilde{\gamma}_\phi \theta\phi_{\mu\nu} + 2\widetilde{\gamma}\theta\Gamma\xi_{\mu\nu} + 2\widetilde{\gamma}Th^{-1}\sum_a n_a \dot{u}_{[\mu}\nabla_{\nu]}\alpha_a + \gamma_{\phi S}\mathcal{R}_{\langle\mu\rangle\langle\nu\rangle} \\ &+ 2\gamma_{\phi p\phi}\theta\xi_{\mu\nu} + \sum_{ab}\gamma_{\phi\mathscr{J}_a\mathscr{J}_b}\nabla_{[\mu}\alpha_a\nabla_{\nu]}\alpha_b + 2\sum_a \gamma_{\phi\mathscr{J}_a q}\nabla_{[\mu}\alpha_a M_{\nu]} \\ &+ \gamma_{\phi qq}M_{[\mu}M_{\nu]} + \gamma_{\phi\pi\pi}\sigma_{\alpha[\mu}\sigma_{\nu]}{}^{\alpha} + 2\gamma_{\phi\pi\phi}\sigma_{\alpha[\mu}\xi_{\nu]}{}^{\alpha} + \gamma_{\phi\phi\phi}\xi_{\alpha[\mu}\xi_{\nu]}{}^{\alpha}.\end{aligned}$$

$$\begin{aligned}\gamma_{\phi p\phi} =& \frac{1}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\gamma\delta}\hat{p}^*\hat{\phi}^{\gamma\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \gamma_{\phi\mathscr{J}_a\mathscr{J}_b} = -\frac{T^2}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\gamma\delta}\hat{\mathscr{J}}_a^{\gamma}\hat{\mathscr{J}}_b^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \\ \gamma_{\phi\mathscr{J}_a q} =& \frac{T}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\gamma\delta}\hat{\mathscr{J}}_a^{\gamma}\hat{q}^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \gamma_{\phi qq} = -\frac{1}{6}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\gamma\delta}\hat{q}^{\gamma}\hat{q}^{\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \\ \gamma_{\phi\pi\pi} =& \frac{2}{15}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\lambda}{}^{\delta}\hat{\pi}_{\delta}{}^{\eta}\hat{\pi}_{\eta}{}^{\lambda}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \gamma_{\phi\pi\phi} = -\frac{2}{5}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\lambda}{}^{\delta}\hat{\pi}_{\delta}{}^{\eta}\hat{\phi}_{\eta}{}^{\lambda}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}, \\ \gamma_{\phi\phi\phi} =& \frac{2}{3}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}G^R_{\hat{\phi}_{\lambda}{}^{\delta}\hat{\phi}_{\delta}{}^{\eta}\hat{\phi}_{\eta}{}^{\lambda}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}.\end{aligned}$$

Relaxation equation for the boost heat vector

$$
\begin{aligned}
q_\mu + \tau_q \dot{q}_\mu = & \sum_a \lambda_{q\mathscr{J}_a} \nabla_\mu \alpha_a - \lambda M_\mu + \widetilde{\lambda}_{qh} h^{-2} \left(\Gamma h + \sum_c \delta_c n_c \right) \theta \sum_a n_a \nabla_\mu \alpha_a + \sum_a \widetilde{\lambda}_{q\mathscr{J}_a} \Delta_{\mu\gamma} D(\nabla^\gamma \alpha_a) \\
& - \widetilde{\lambda}\theta\Gamma M_\mu + \tau_q \Delta_{\mu\gamma} \sum_a \lambda_{q\mathscr{J}_a} D(\nabla^\gamma \alpha_a) + \sum_a \overline{\lambda}^a \theta \nabla_\mu \alpha_a + \widetilde{\lambda}_q \theta q_\mu - \sum_a \widetilde{\lambda}_q \lambda_{q\mathscr{J}_a} \theta \nabla_\mu \alpha_a \\
& -2\widetilde{\lambda}_{qh} \beta \sigma_{\mu\nu} \dot{u}^\nu - \widetilde{\lambda}_{qh} y \beta \theta \dot{u}_\mu + 2\widetilde{\lambda} \xi_{\mu\nu} \dot{u}^\nu - \textstyle\sum_a \widetilde{\lambda}_{q\mathscr{J}_a} \beta \theta \dot{u}_\mu \delta_a + \lambda_{qh} \mathcal{H}_\mu - \lambda \mathcal{Q}_\mu \\
& + 2 \sum_a \lambda_{qp\mathscr{J}_a} \theta \nabla_\mu \alpha_a + 2\lambda_{qpq} \theta M_\mu + 2 \sum_a \lambda_{q\mathscr{J}_a\pi} \nabla^\nu \alpha_a \sigma_{\mu\nu} + 2 \sum_a \lambda_{q\mathscr{J}_a\phi} \nabla^\nu \alpha_a \xi_{\mu\nu} \\
& + 2\lambda_{qq\pi} M^\nu \sigma_{\mu\nu} + 2\lambda_{qq\phi} M^\nu \xi_{\mu\nu} .
\end{aligned}
$$

$$
\lambda_{qp\mathscr{J}_a} = -\frac{T}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\beta \hat{p}^* \mathscr{J}_a^\beta}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} , \quad \lambda_{qpq} = \frac{1}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\beta \hat{p}^* \hat{q}^\beta}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} ,
$$

$$
\lambda_{q\mathscr{J}_a\pi} = \frac{T}{10} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\lambda \hat{\mathscr{J}}_{a\delta} \hat{\pi}^{\lambda\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} , \quad \lambda_{q\mathscr{J}_a\phi} = \frac{T}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\lambda \hat{\mathscr{J}}_{a\delta} \hat{\phi}^{\lambda\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} ,
$$

$$
\lambda_{qq\pi} = -\frac{1}{10} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\lambda \hat{q}_\delta \hat{\pi}^{\lambda\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} , \quad \lambda_{qq\phi} = -\frac{1}{6} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{q}_\lambda \hat{q}_\delta \hat{\phi}^{\lambda\delta}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0} .
$$

Dissipative flux from spin tensor

$$\varpi^{\lambda\mu\nu} = \varphi \Xi^{\lambda\mu\nu} + 2\sum_a \varphi_{\varpi \mathscr{J}_a \phi} \Delta^{\lambda\mu\nu\rho\sigma\delta} \xi_{\sigma\delta} \nabla_\rho \alpha_a + 2\varphi_{\varpi q \phi} \Delta^{\lambda\mu\nu\rho\sigma\delta} M_\rho \xi_{\sigma\delta}.$$

$$\varphi_{\varpi \mathscr{J}_a \phi} = \frac{T}{2} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\varpi}^{\gamma\varepsilon\zeta} \hat{\mathscr{J}}_{a\gamma} \hat{\phi}_{\varepsilon\zeta}}(\omega_1, \omega_2)\Big|_{\omega_{1,2}=0},$$

$$\varphi_{\varpi q \phi} = -\frac{1}{2} \frac{\partial}{\partial\omega_1} \frac{\partial}{\partial\omega_2} \mathrm{Re}\, G^R_{\hat{\varpi}^{\gamma\varepsilon\zeta} \hat{q}_{\gamma} \hat{\phi}_{\varepsilon\zeta}}(\omega_1, \omega_2)\Big|_{\omega_{1,2}=0}.$$

Summary and Outlook

Summary and outlook

Summary: We present a new derivation of relativistic canonical-like second-order spin hydrodynamics from Zubarev's non-equilibrium statistical operator. We obtain relaxation equations for dissipative fluxes, including the shear stress tensor, bulk viscous pressure, charge-diffusion currents, rotational stress tensor, boost heat vector, and spin tensor-related dissipative current. Furthermore, we express all transport coefficients represented by two-point or three-point correlations in terms of retarded Green's functions.

Outlook:

- Calculation of transport coefficients in spin hydrodynamics via thermal field theory.
- Research on causality and stability conditions for spin hydrodynamics in the nonlinear regime (Ongoing work).
- Numerical spin hydrodynamics (In collaboration with Hui Zhang from SCNU).

Thank you!

Bulk viscous pressure

$$
\begin{aligned}
\Pi =& \langle \hat{p} \rangle - p\left(\epsilon, n_a, S^{\alpha\beta}\right) \\
=& \langle \hat{p} \rangle_l + \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p\left(\epsilon, n_a, S^{\alpha\beta}\right) \\
=& p\left(\langle \hat{\epsilon} \rangle_l, \langle \hat{n}_a \rangle_l, \langle \hat{S}^{\alpha\beta} \rangle_l\right) + \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p\left(\epsilon, n_a, S^{\alpha\beta}\right) \\
=& p\left(\epsilon - \Delta\epsilon, n_a - \Delta n_a, S^{\alpha\beta} - \Delta S^{\alpha\beta}\right) + \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p\left(\epsilon, n_a, S^{\alpha\beta}\right) \\
=& p\left(\epsilon, n_a, S^{\alpha\beta}\right) - \frac{\partial p}{\partial \epsilon}\Delta\epsilon - \sum_a \frac{\partial p}{\partial n_a}\Delta n_a - \frac{\partial p}{\partial S^{\alpha\beta}}\Delta S^{\alpha\beta} + \frac{1}{2}\frac{\partial^2 p}{\partial \epsilon^2}(\Delta\epsilon)^2 \\
&+ \frac{1}{2} \times 2\sum_a \frac{\partial^2 p}{\partial\epsilon\partial n_a}\Delta\epsilon\Delta n_a + \frac{1}{2}\sum_{ab}\frac{\partial^2 p}{\partial n_a \partial n_b}\Delta n_a \Delta n_b + \frac{1}{2} \times 2\sum_a \frac{\partial^2 p}{\partial n_a \partial S^{\alpha\beta}}\Delta n_a \Delta S^{\alpha\beta} \\
&+ \frac{1}{2}\frac{\partial^2 p}{\partial S^{\alpha\beta}\partial S^{\rho\sigma}}\Delta S^{\alpha\beta}\Delta S^{\rho\sigma} + \frac{1}{2} \times 2 \frac{\partial^2 p}{\partial\epsilon\partial S^{\alpha\beta}}\Delta\epsilon\Delta S^{\alpha\beta} + \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p\left(\epsilon, n_a, S^{\alpha\beta}\right)
\end{aligned}
$$

$$
\Delta\epsilon = \langle \hat{\epsilon} \rangle_1 + \langle \hat{\epsilon} \rangle_2, \Delta n_a = \langle \hat{n}_a \rangle_1 + \langle \hat{n}_a \rangle_2, \Delta S^{\alpha\beta} = \langle \hat{S}^{\alpha\beta} \rangle_1 + \langle \hat{S}^{\alpha\beta} \rangle_2
$$

Correlation functions and Kubo formulas

$$\beta \int d^4x_1 \left(\hat{X}(x)\,,\,\hat{Y}(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{X}\hat{Y}}(\omega)\Big|_{\omega=0}$$

$$\beta \int d^4x_1 \left(\hat{X}(x)\,,\,\hat{Y}(x_1)\right)(x_1-x)^\tau = -\frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{X}\hat{Y}}(\omega)\Big|_{\omega=0} u^\tau$$

$$\beta^2 \int d^4x_1 d^4x_2 \left(\hat{X}(x)\,,\,\hat{Y}(x_1)\,,\,\hat{Z}(x_2)\right) = -\frac{1}{2}\frac{\partial}{\partial\omega_1}\frac{\partial}{\partial\omega_2}\mathrm{Re}\,G^R_{\hat{X}\hat{Y}\hat{Z}}(\omega_1,\omega_2)\Big|_{\omega_{1,2}=0}$$

$G^R_{\hat{Y}\hat{X}}(\omega) = \eta_X\eta_Y G^R_{\hat{X}\hat{Y}}(\omega)\,,\quad \eta_{X,Y} = \pm 1$ for even/odd parity under time reversal

$$G^R_{\hat{X}\hat{Y}}(\omega) = -i\int_0^\infty dt e^{i\omega t}\int d^3x \left\langle\left[\hat{X}(\boldsymbol{x},t)\,,\,\hat{Y}(\boldsymbol{0},0)\right]\right\rangle_l$$

$$G^R_{\hat{X}\hat{Y}\hat{Z}}(\omega_1,\omega_2) = -\frac{1}{2}\int_{-\infty}^0 dt_1 e^{-i\omega_1 t_1}\int_{-\infty}^0 dt_2 e^{-i\omega_2 t_2}\int d^3x_1\int d^3x_2$$
$$\times\left\{\left\langle\left[\left[\hat{X}(\boldsymbol{0},0)\,,\,\hat{Y}(\boldsymbol{x}_1,t_1)\right],\hat{Z}(\boldsymbol{x}_2,t_2)\right]\right\rangle_l + \left\langle\left[\left[\hat{X}(\boldsymbol{0},0)\,,\,\hat{Z}(\boldsymbol{x}_2,t_2)\right],\hat{Y}(\boldsymbol{x}_1,t_1)\right]\right\rangle_l\right\}$$