

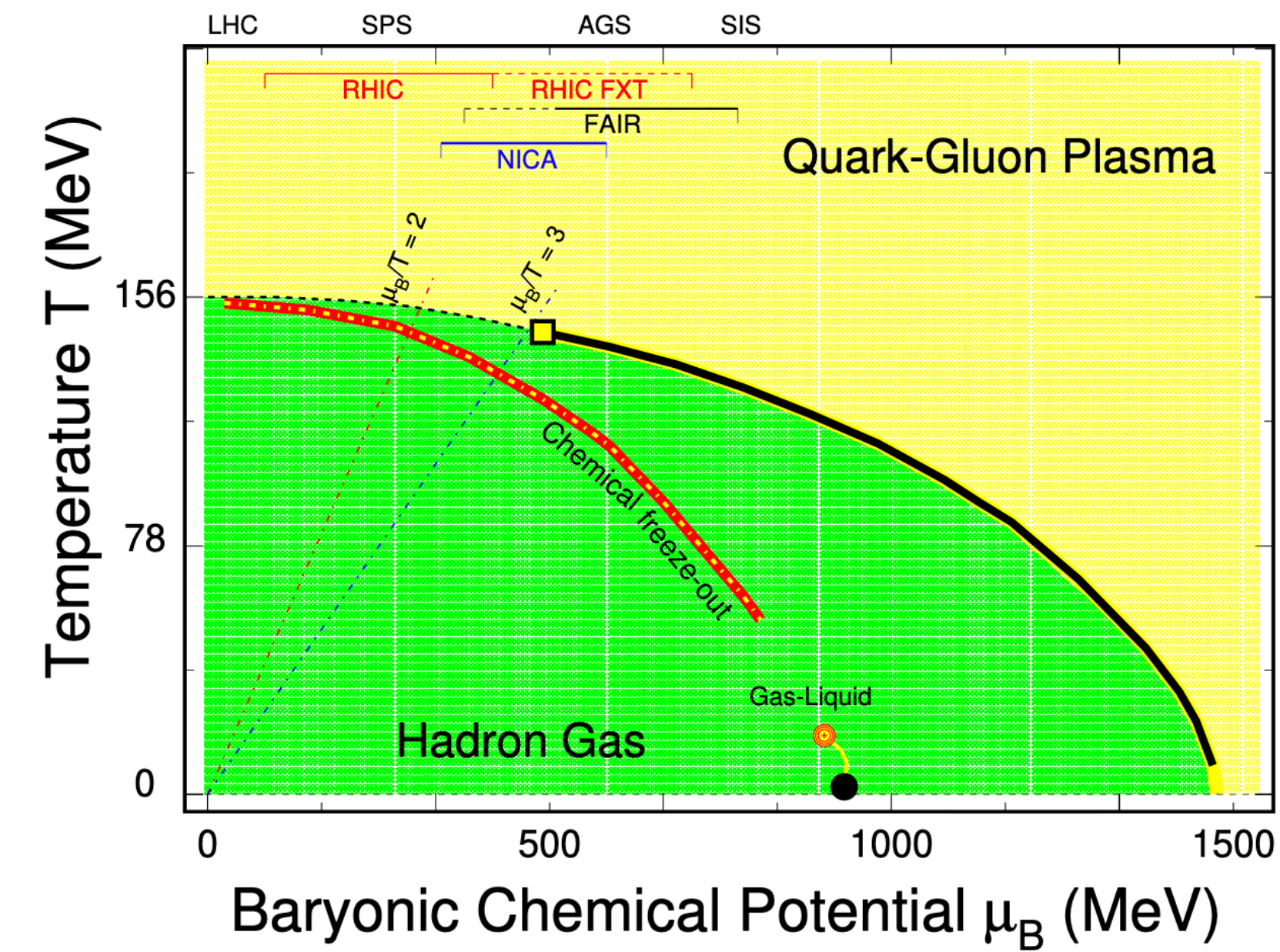
Tracing QCD criticality on the Lattice

Heng-Tong Ding (丁亨通)
Central China Normal University

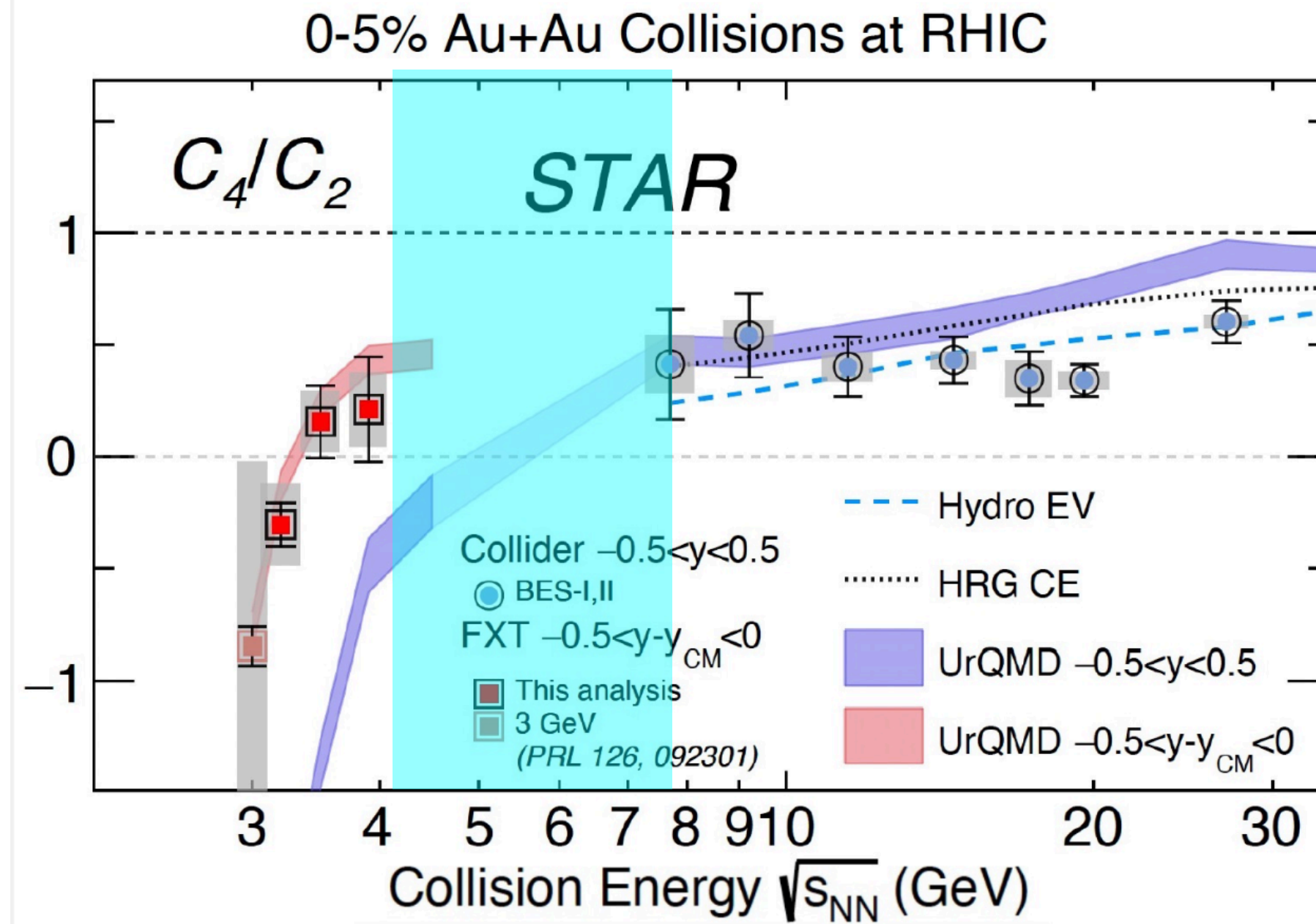
The QCD Phase Diagram: From Theory to Experimental Signatures

Oct. 8-11, 2025 @ Dalian

Search for criticalities

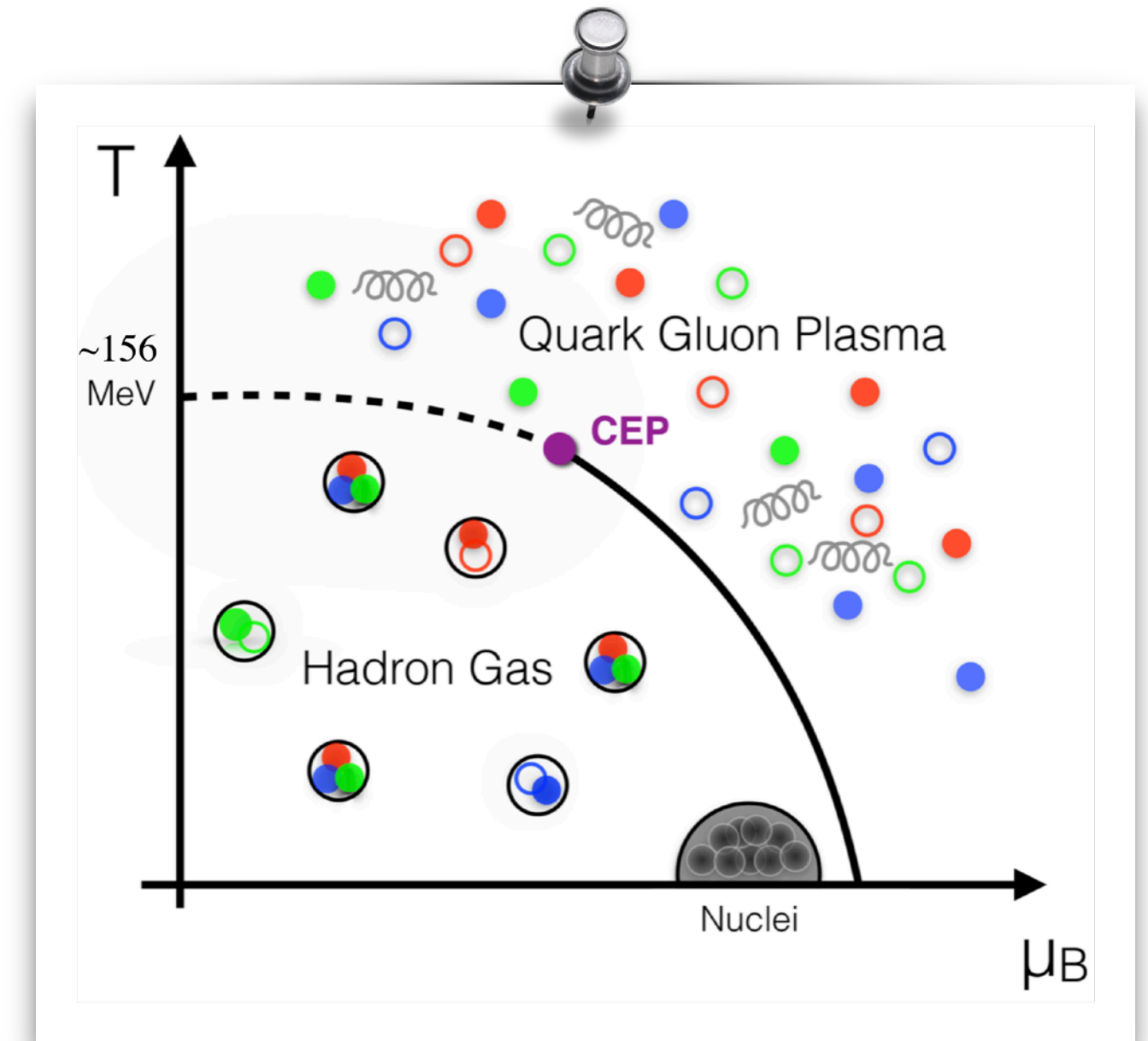


See talk by Nu Xu on Wed



STAR 7.7-27 GeV results from arXiv: 2504.00817, PRL 25'
New results at 3.2, 3.5 and 3.9 GeV from Zachary Sweger (QM2025)

if the CEP exists, most likely, it maybe found in
 $\sqrt{s_{NN}} \in (3.9, 7.7) \text{ GeV}$



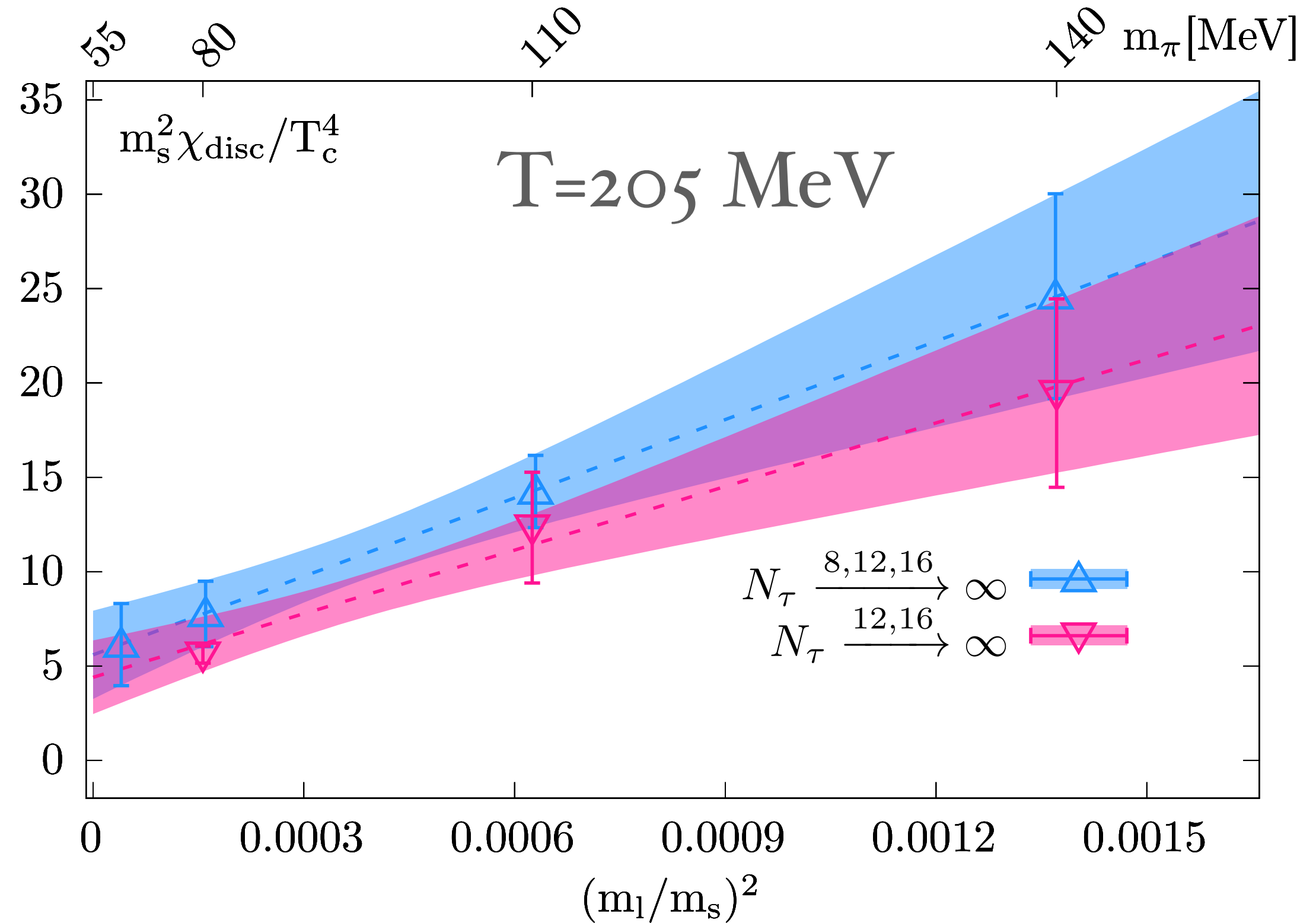
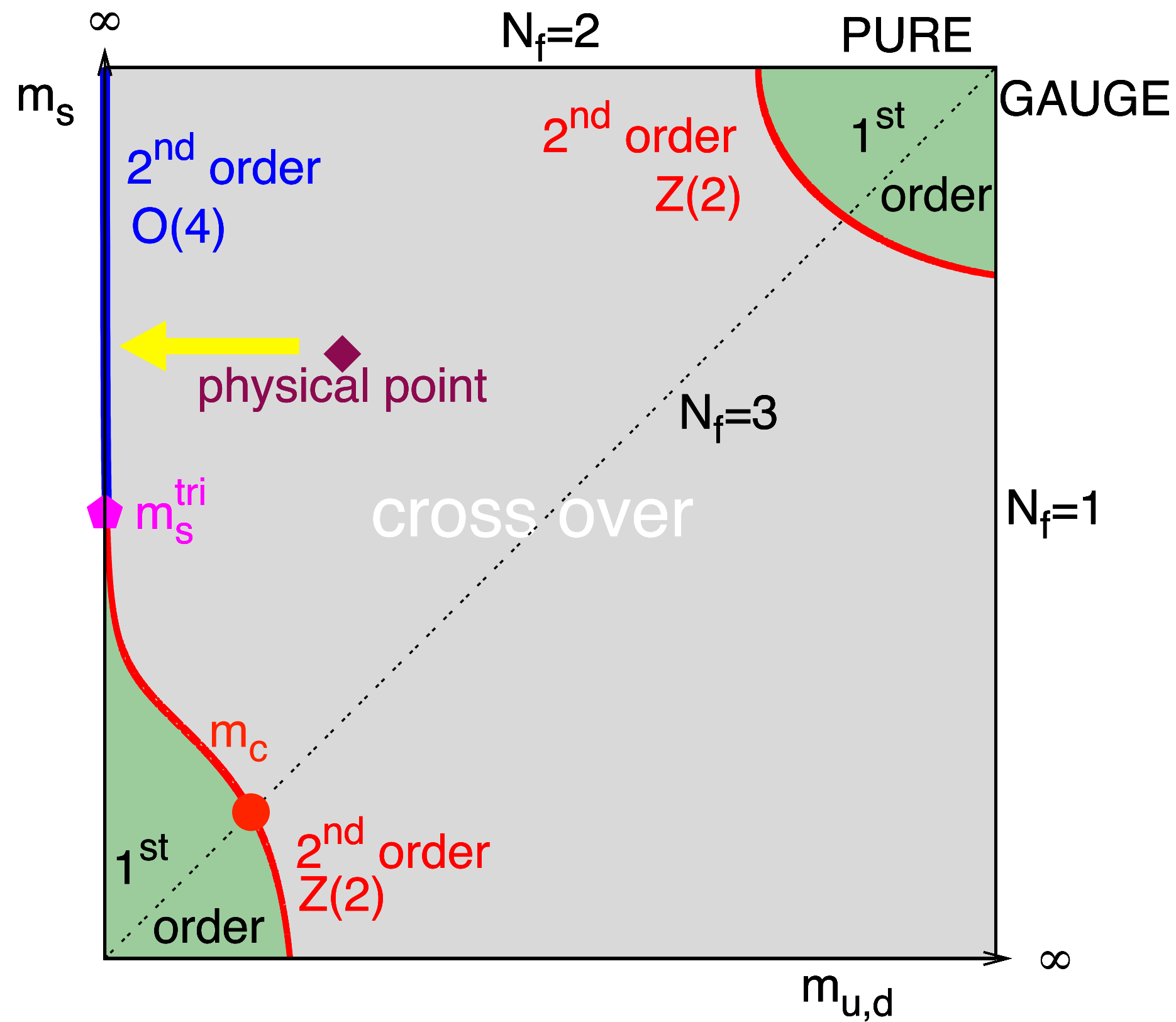
HTD, F. Karsch, S. Mukherjee,
arXiv:1504.05274

Sign Problem at $\mu_B \neq 0$

Taylor Expansion

Imaginary μ_B

QCD criticality at $\mu_B=0$: relevance to CEP



HTD et al., PRL 126 (2021) 082001

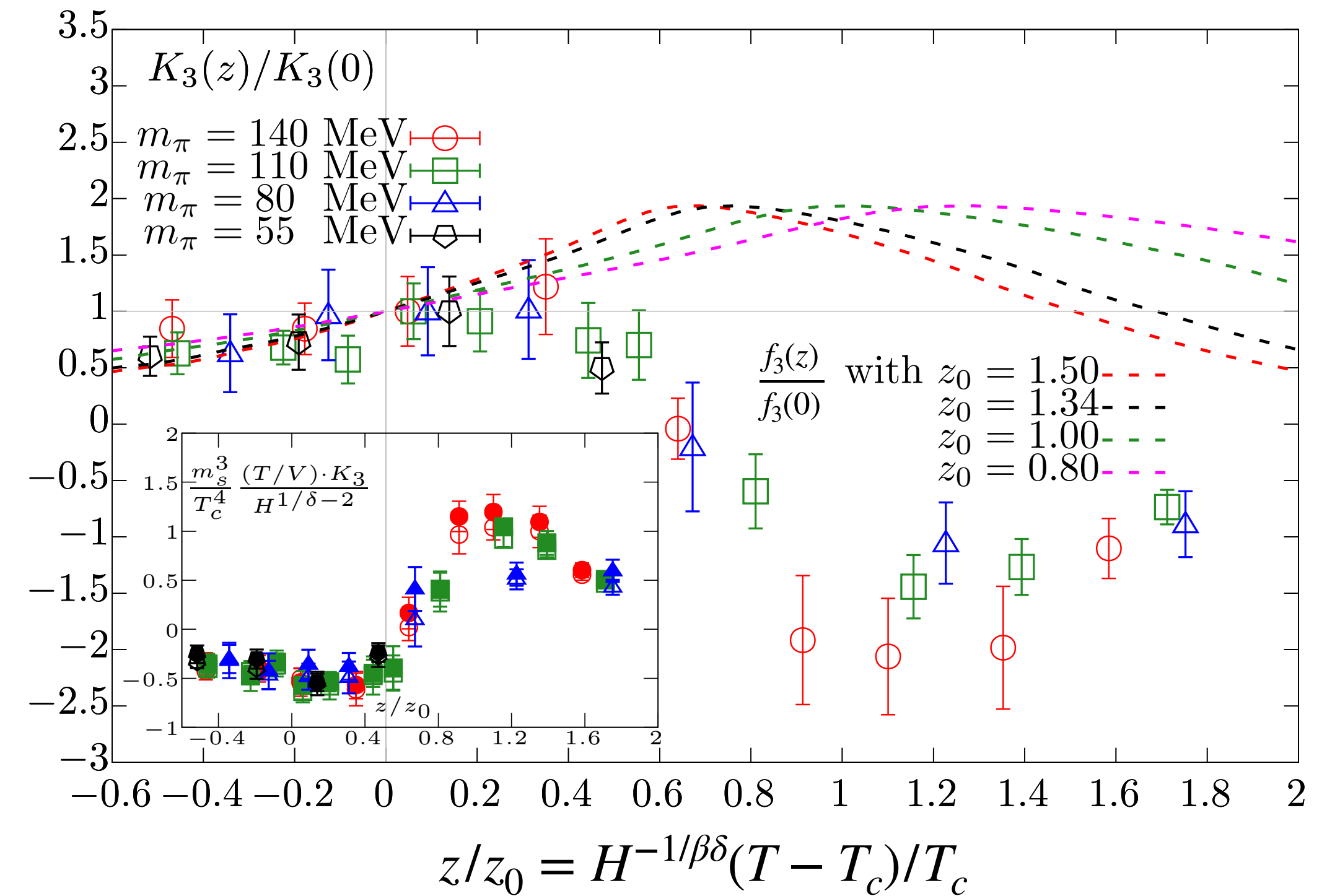
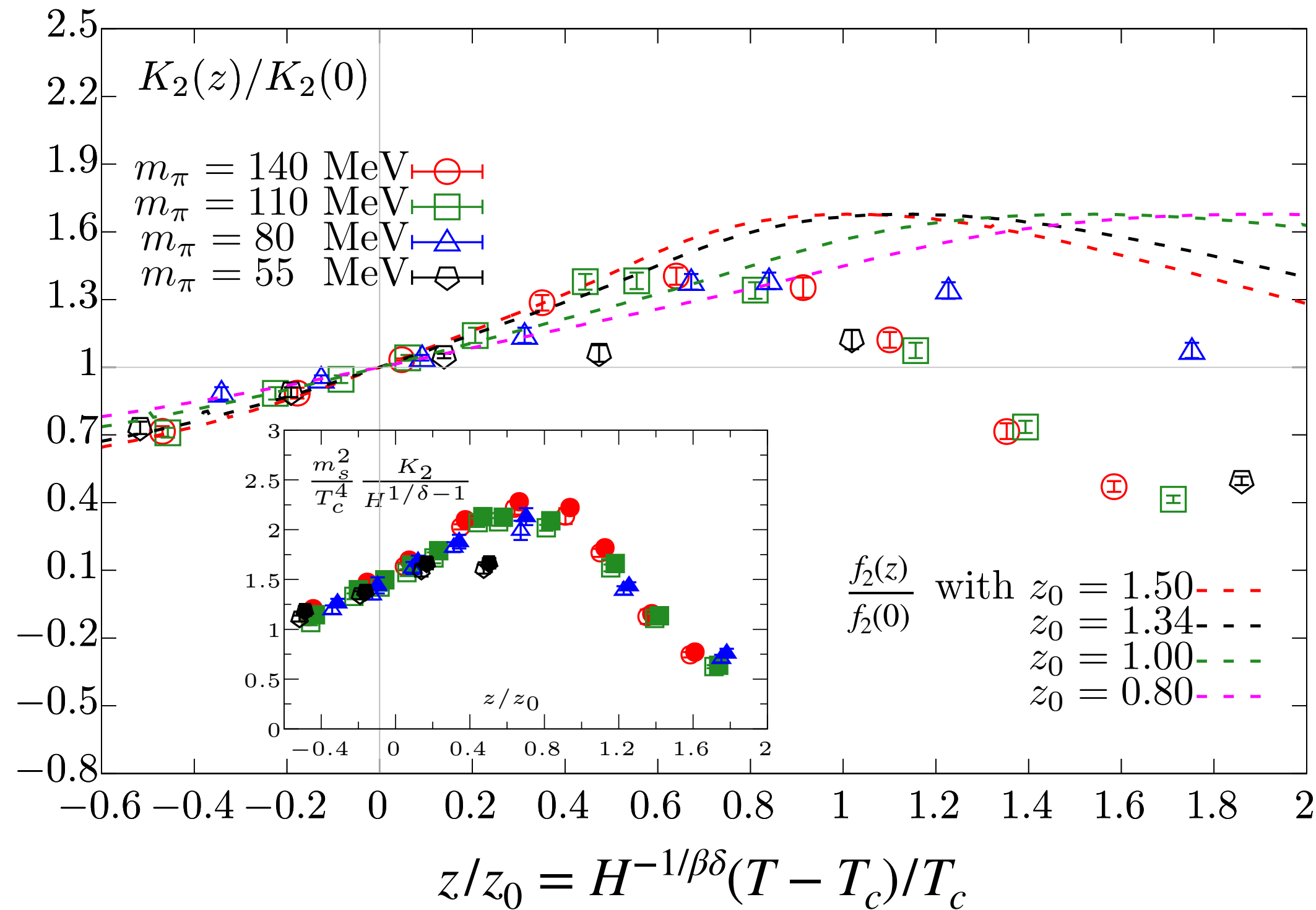
Relevant: 2nd order chiral phase transition belonging to $O(4)$ universality class as axial $U(1)$ symmetry is manifested even at $T=205$ MeV ($\chi_{disc} \neq 0$)

$O(N)$ scaling behavior of $\mathbb{K}_n(\bar{\psi}\psi)$ at $\mu_B = 0$

HISQ, Nt=8, pion mass ranging from 140 to 55 MeV

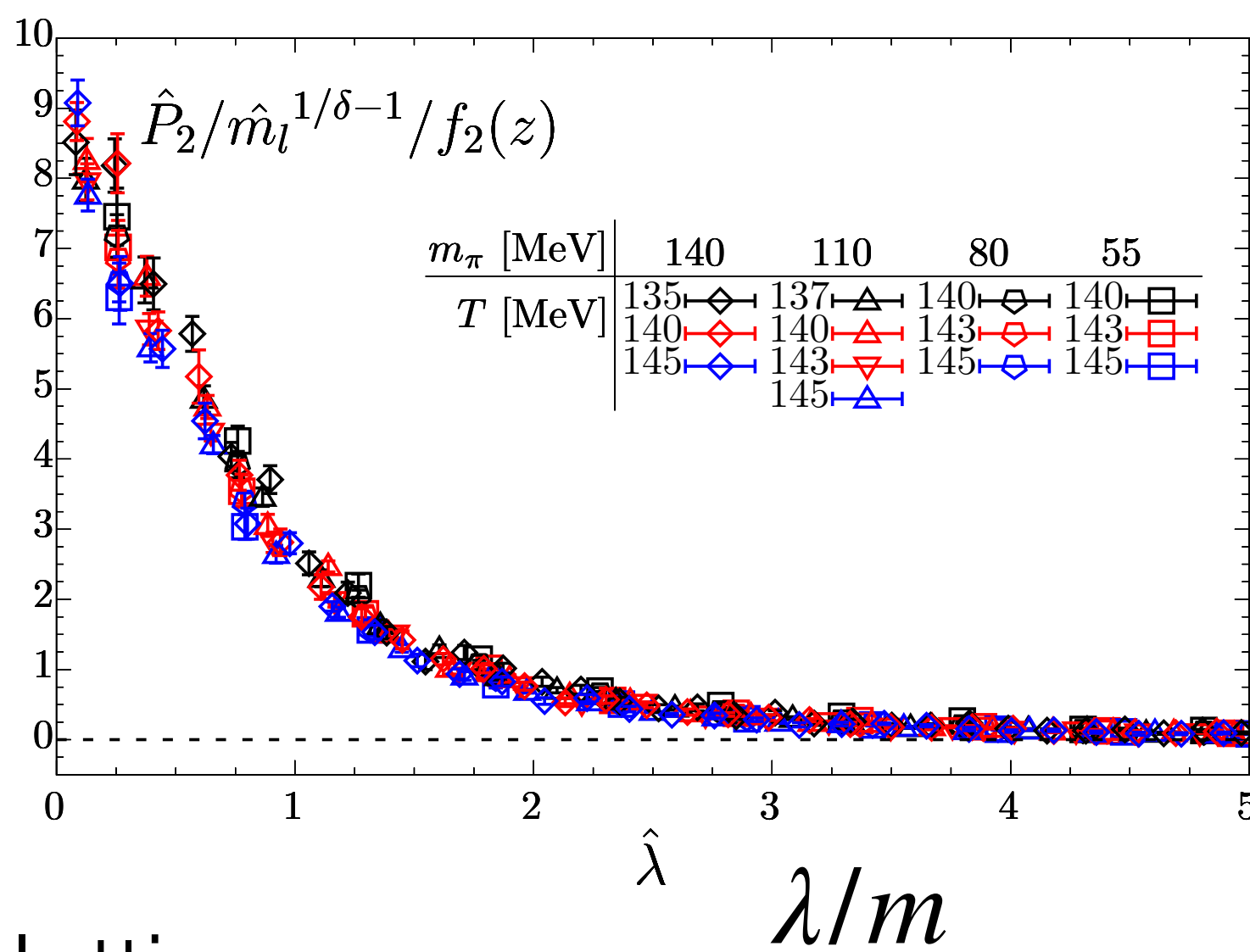
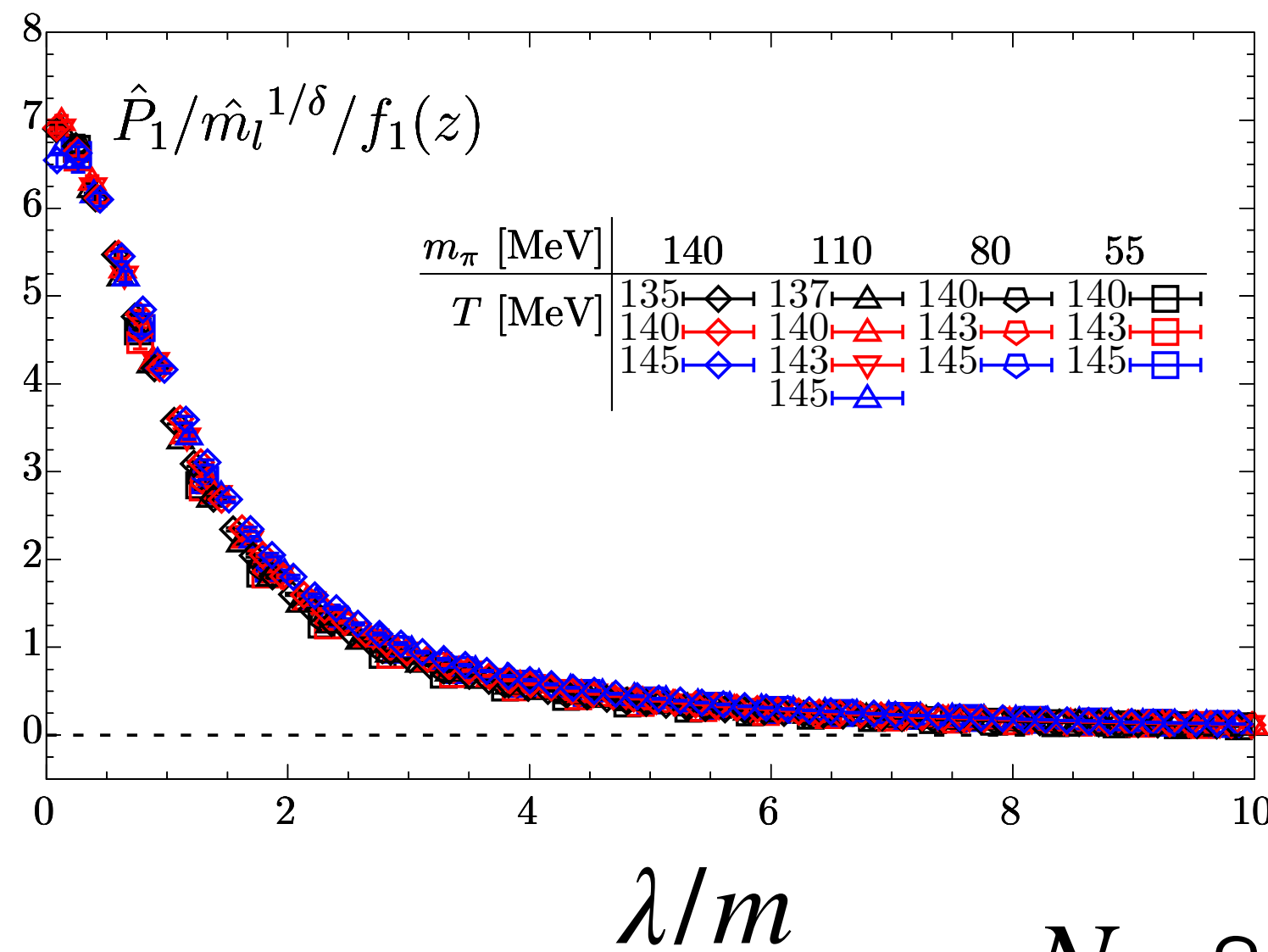
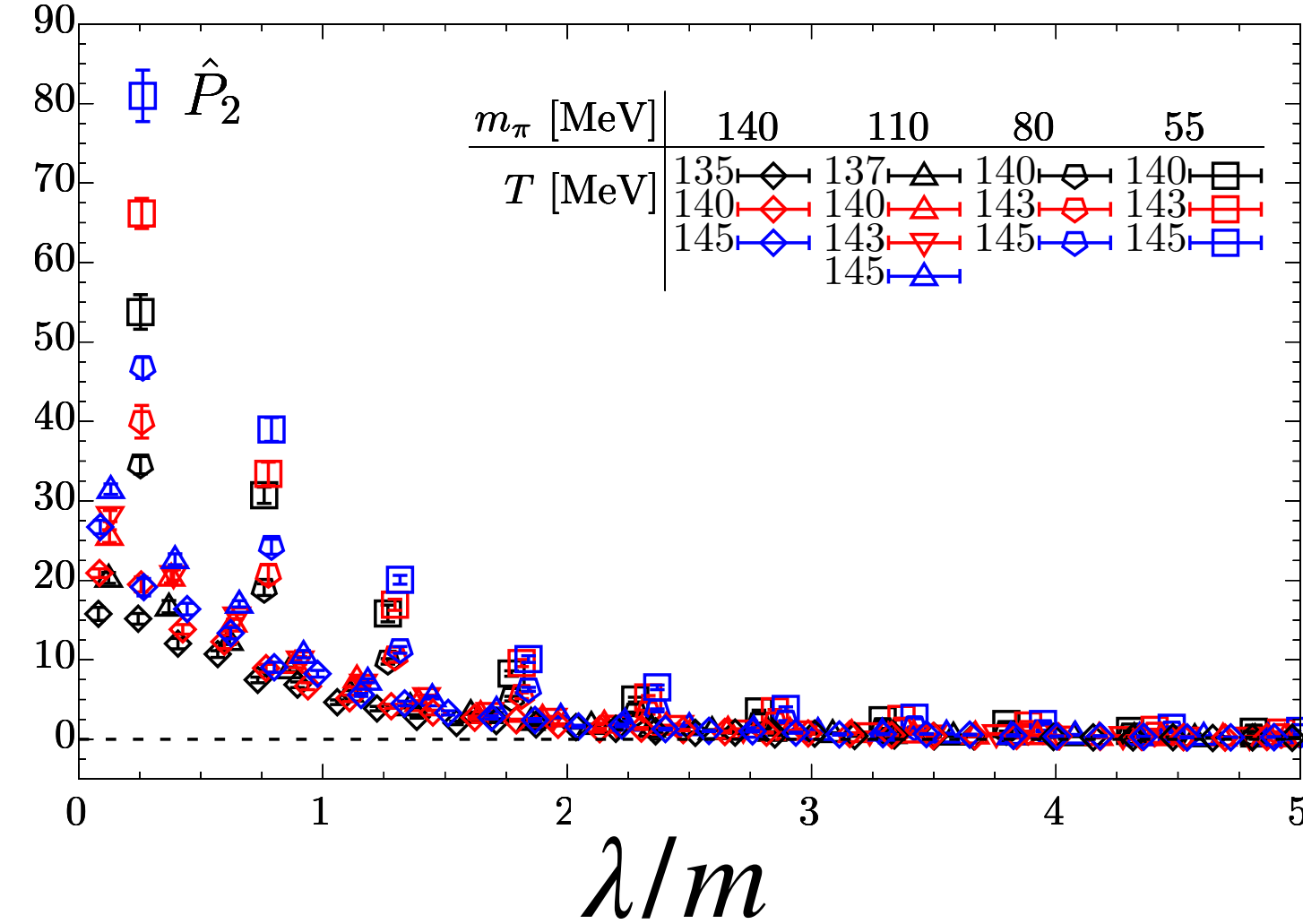
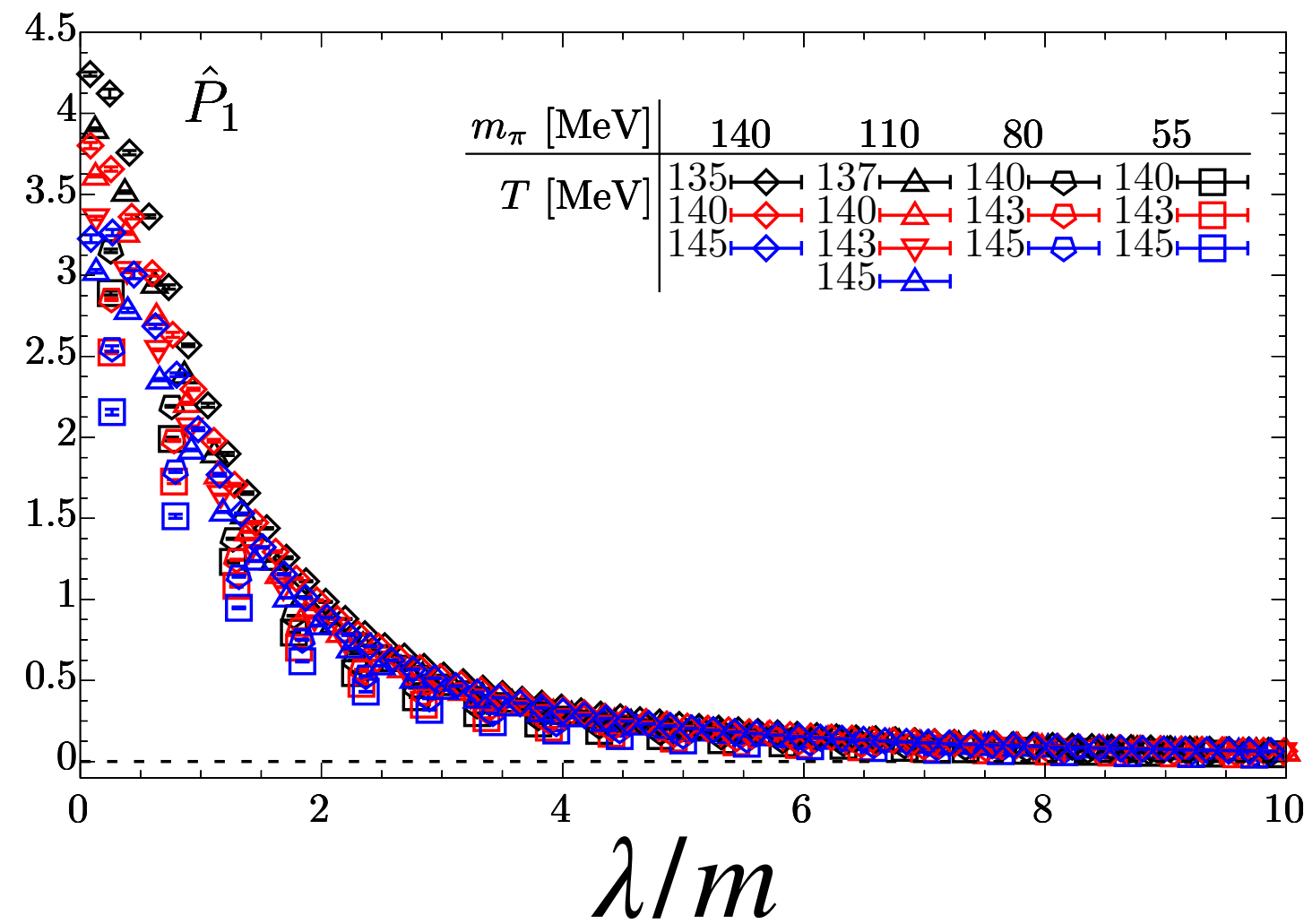
$$\mathbb{K}_2(z) \sim m_l^{1/\delta-1} f_2(z), \quad \mathbb{K}_2(z)/\mathbb{K}_2(z=0) \sim f_2(z)/f_2(0)$$

$$\mathbb{K}_3(z) \sim m_l^{1/\delta-2} f_3(z), \quad \mathbb{K}_3(z)/\mathbb{K}_3(z=0) \sim f_3(z)/f_3(0)$$



In the proximity of T_c : $\mathbb{K}_n(\bar{\psi}\psi) \sim m^{1/\delta-n+1} f_n(z)$!

Microscopic encoding of Macroscopic criticality



$N_\tau=8$ lattices

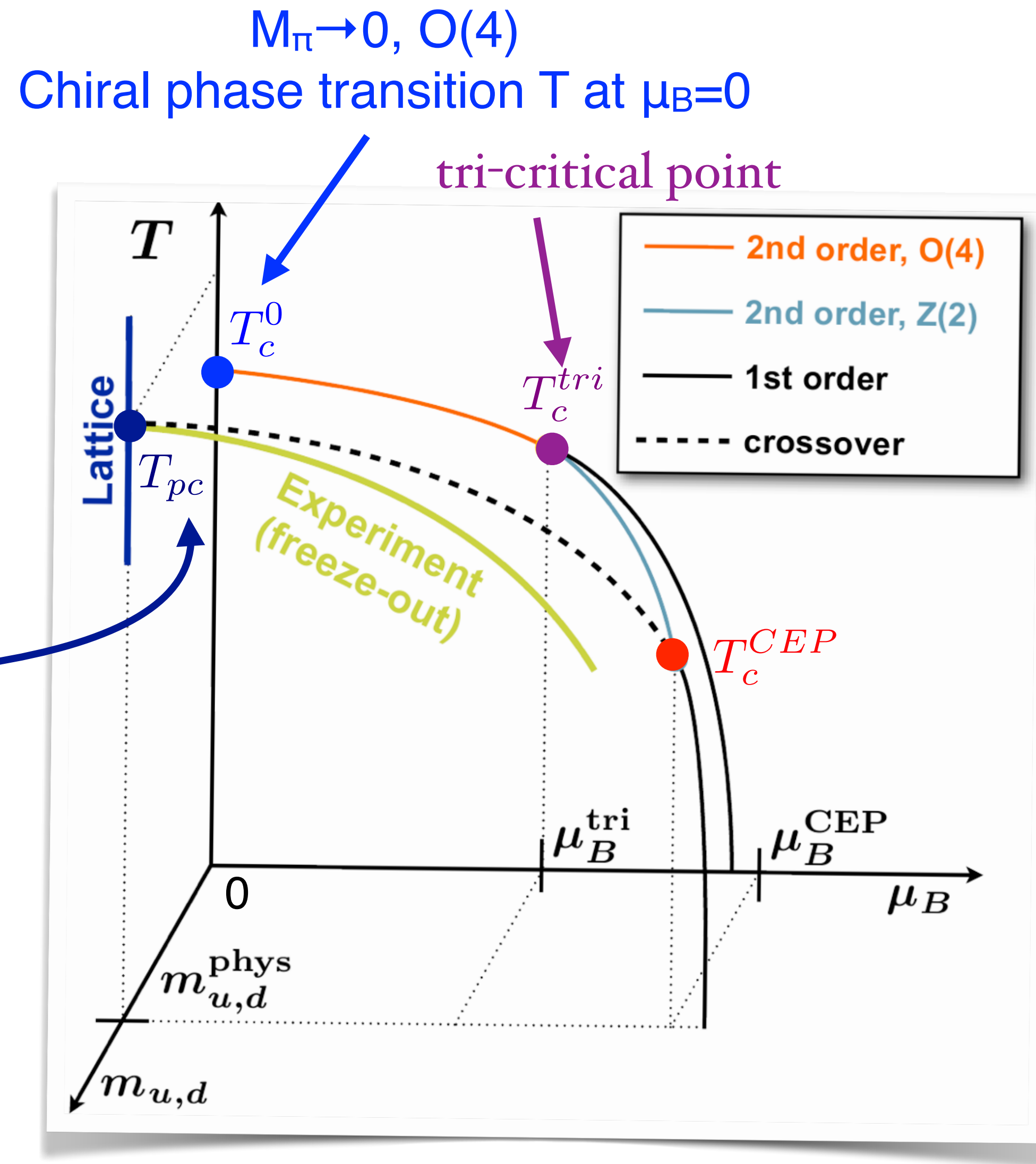
Marco: $\mathbb{K}_n(\bar{\psi}\psi) \sim m^{1/\delta-n+1} f_n(z)$

$$\mathbb{K}_n(\bar{\psi}\psi) = \int_0^\infty P_n(\lambda) d\lambda$$

$$P_n(\lambda) = \int_0^\infty K_1[P_U(\lambda; m_l), P_U(\lambda_2; m_l), \dots, P_U(\lambda_n; m_l)] \prod_{i=2}^n d\lambda_i$$

Mirco: $P_n(\lambda) = m^{1/\delta-n+1} f_n(z) g(\lambda/m)$

QCD phase diagram in 3D: quark mass, μ_B , T



$T_c^0(\mu_B) = (1 - \kappa_2^B \hat{\mu}_B^2) T_c^0(0)$ decreases as μ_B increases from LQCD

$$N_\tau = 8 \text{ lattice: } \kappa_2^B = 0.015(1)$$

HTD et al., PRD 109 (2024) 114516

P. Hegde & HTD, PoS LATTICE2015 (2016) 141

O. Kaczmarek et al., PRD83 (2011) 014504

Tri-critical mean-field:

$$T_c^{tri} - T_c^{CEP}(m_q) \propto m_q^{2/5}$$

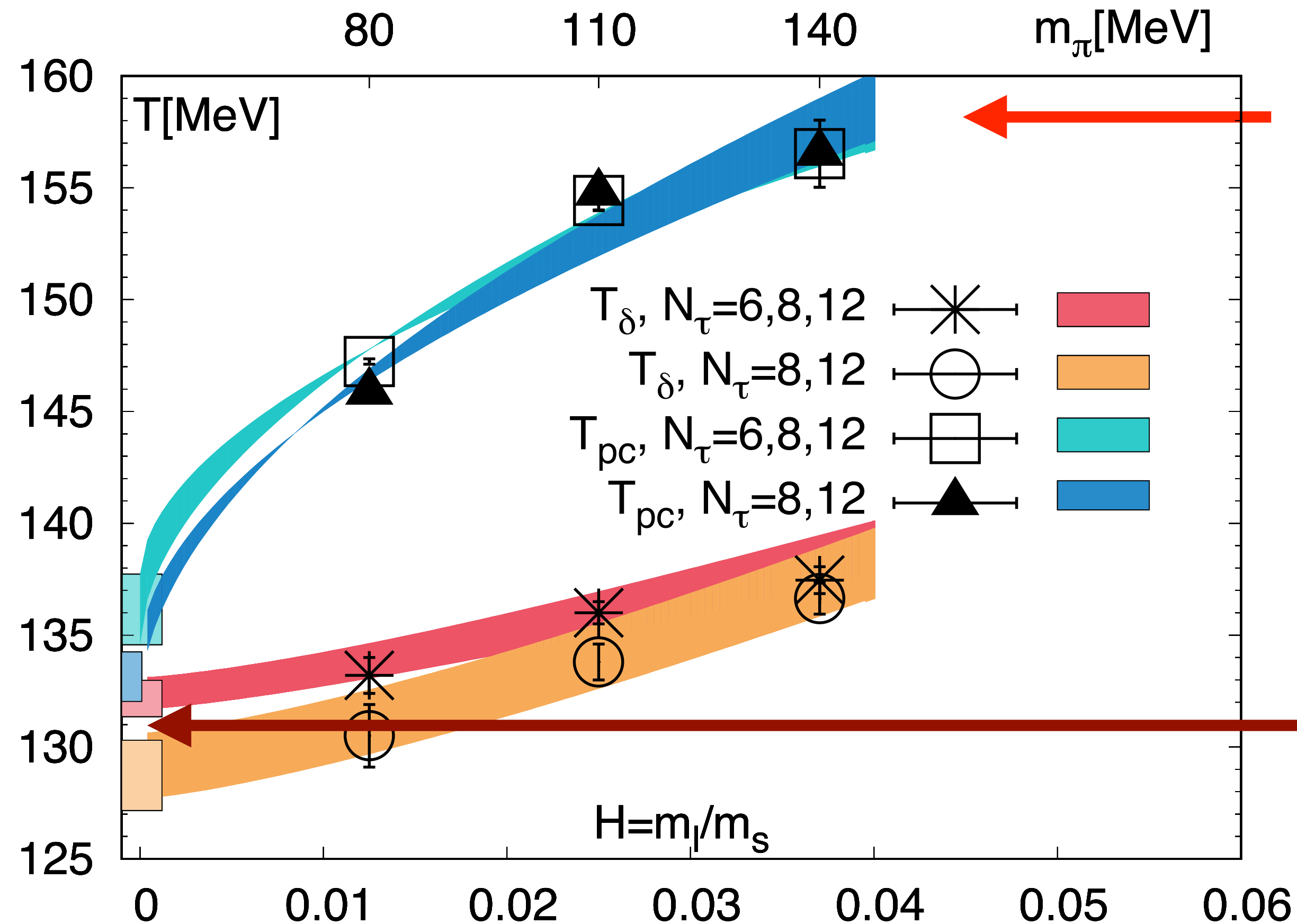
M. A. Halasz et al, PRD 58 (1998) 096007

M. Buballa, S. Carignano, PLB791(2019)361

Y. Hatta & T. Ikeda, PRD67 (2003) 014028

► **Indications:**

$$T_{pc}^{phys} > T_c^0 > T_c^{tri} > T_c^{CEP}$$



Chiral crossover transition temperature

$$T_{pc}^{\text{phys}} = 155 - 158.6 \text{ MeV} \quad \begin{array}{l} \text{HotQCD, PLB 19'} \\ \text{WB, PRL 20'} \end{array}$$

$$T_{pc}(H) = T_c^0 \left(1 + \frac{z_p}{z_0} H^{\frac{1}{\beta\delta}} \right)$$

Chiral phase transition T

$$T_c^0 = 132_{-6}^{+3} \text{ MeV}$$

HTD, P. Hegde, O. Kaczmarek et al.[HotQCD], PRL 123 (2019) 062002

HTD, arXiv:2002.11957

See also in QCD-inspired model calculations:

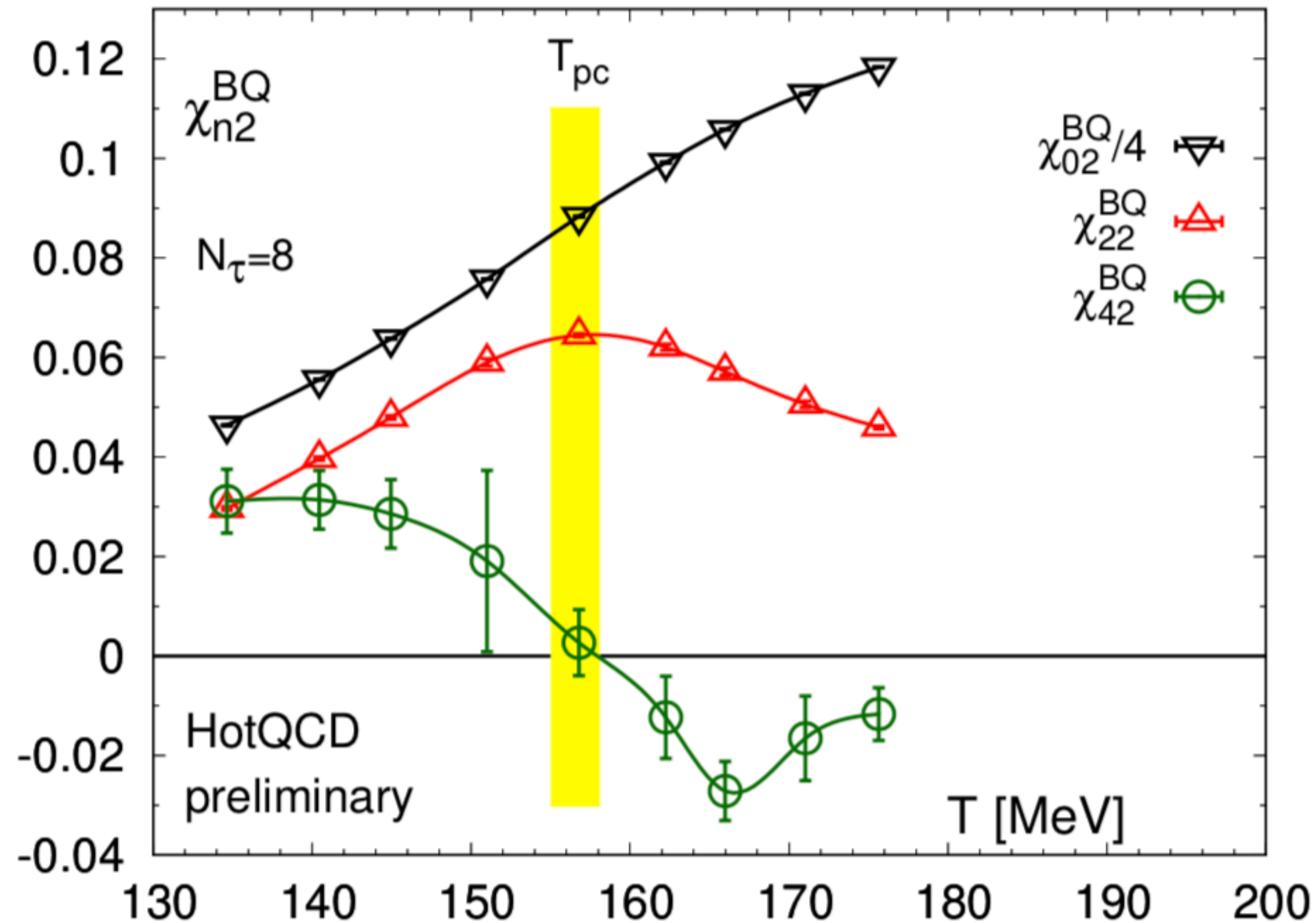
e.g. J. Berges, D. U. Jungnickel and C. Wetterich, Phys. Rev.D59, 034010 (1999)

J. Braun, B. Klein, H.-J. Pirner and A. H. Rezaeian, Phys. Rev. D73, 074010 (2006)

And consistent results using Wilson fermions from Kotov et al, Phys.Lett. B 823(2021) 136749

Indication of $T_{\text{CEP}} \lesssim 135 \text{ MeV}$

Critical behavior from higher order cumulants at $\mu_B = 0$



F. Karsch, PoS CORFU2018 (2019) 163

$$\chi_{ijk}^{BQS}(T) = \left. \frac{\partial^{i+j+k} P(T, \hat{\mu}) / T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k} \right|_{\hat{\mu}=0}$$

$$t \sim \frac{T - T_c^0}{T_c^0} + \kappa_2^{B,0} \left(\frac{\mu_B}{T} \right)^2$$

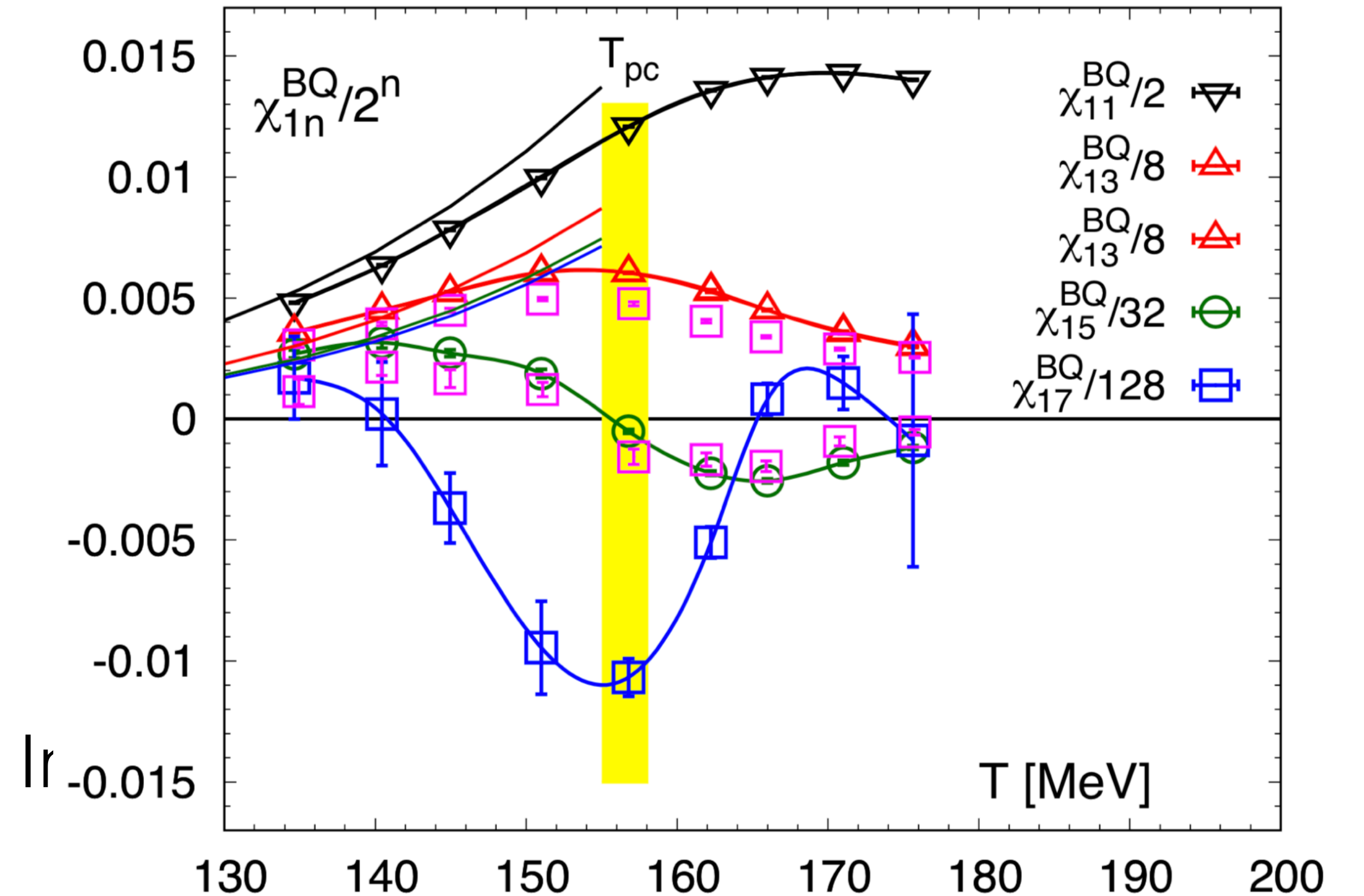
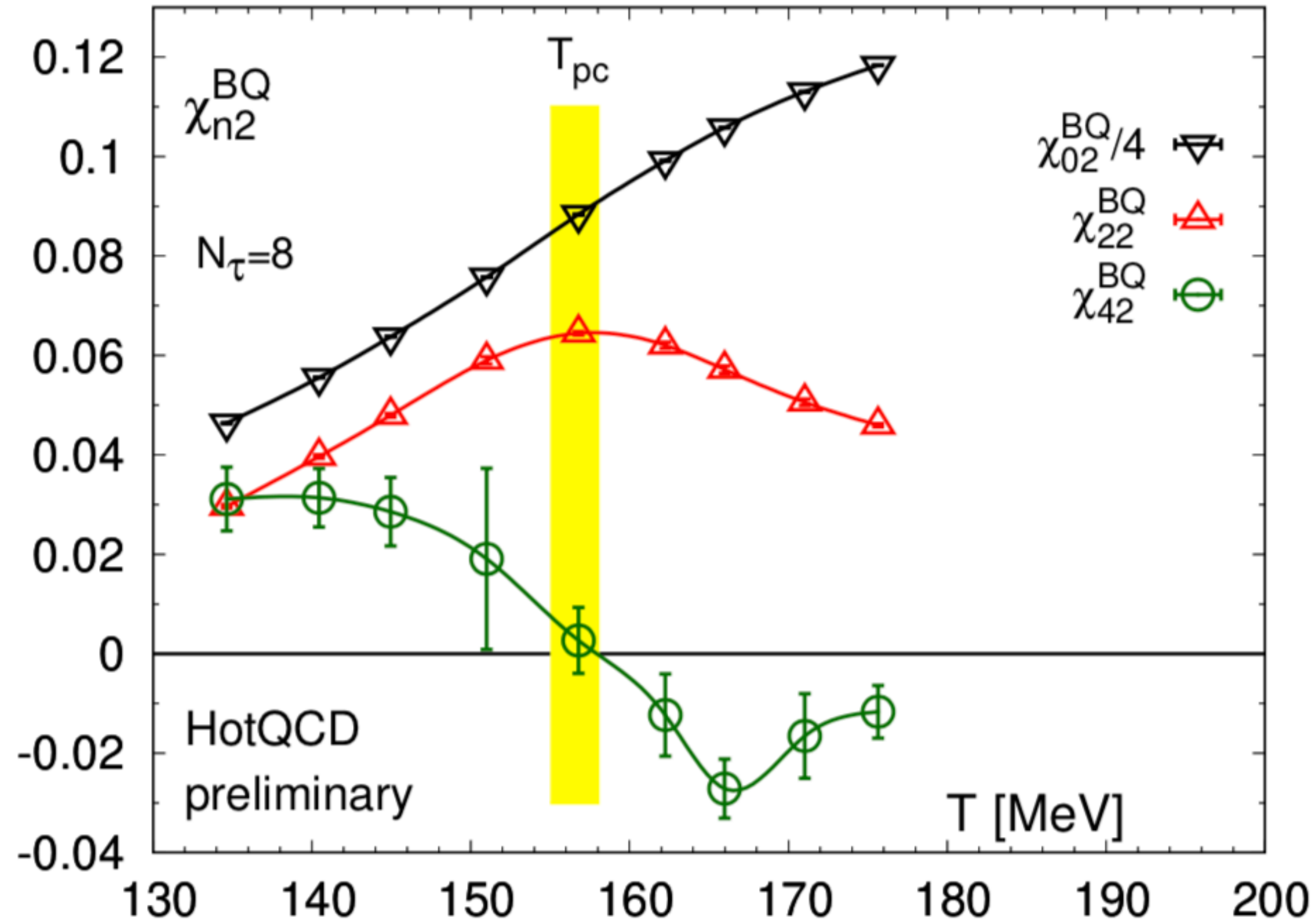
In the scaling regime:

$$\frac{\partial}{\partial T} \simeq \frac{\partial^2}{\partial \mu_B^2}$$

Sign change seen at $T > T_{pc}$ in χ_{42}^{BQ}

Sign change expected at $T \gtrsim 135$ MeV in χ_{62}^{BQ}

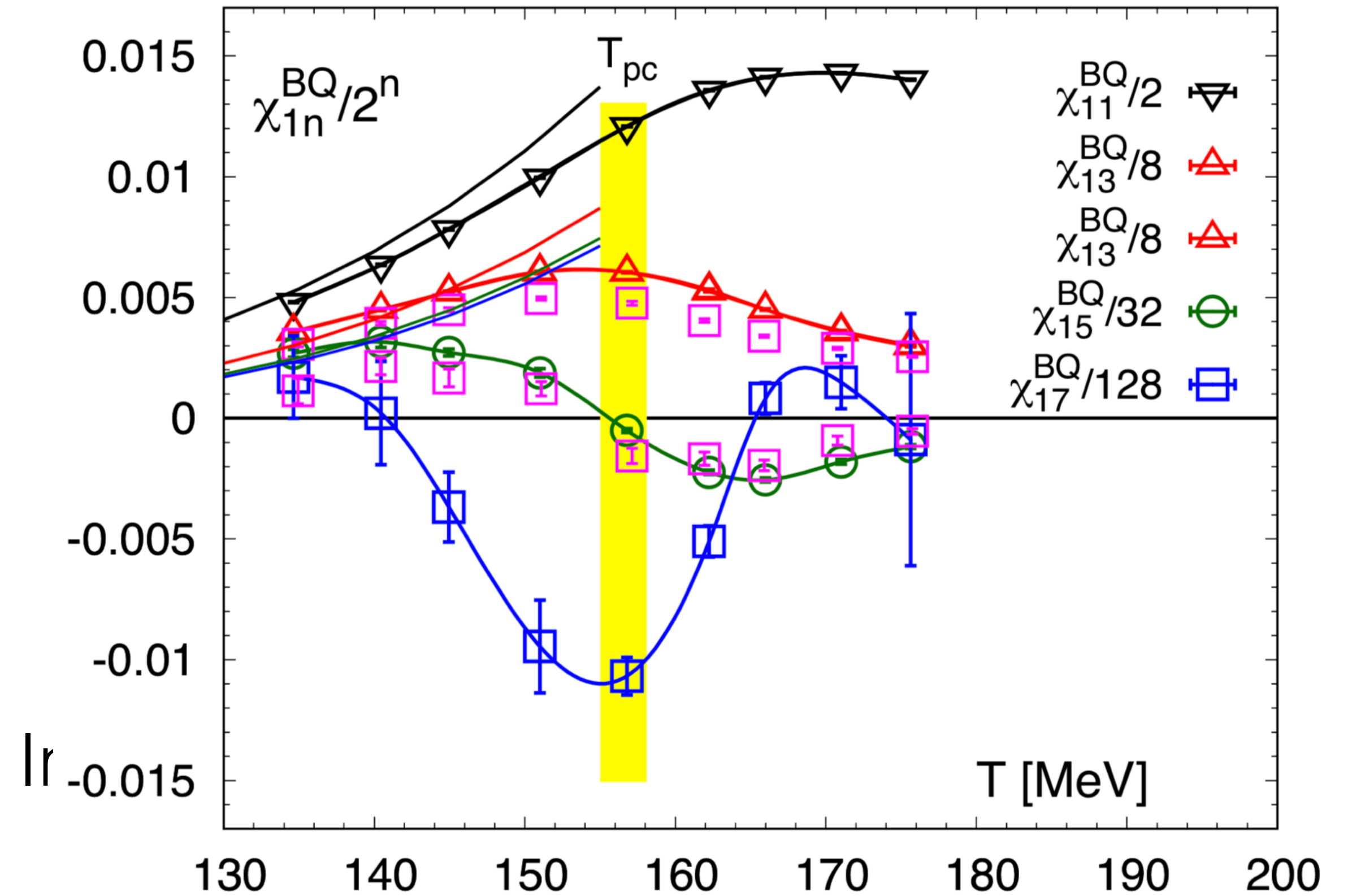
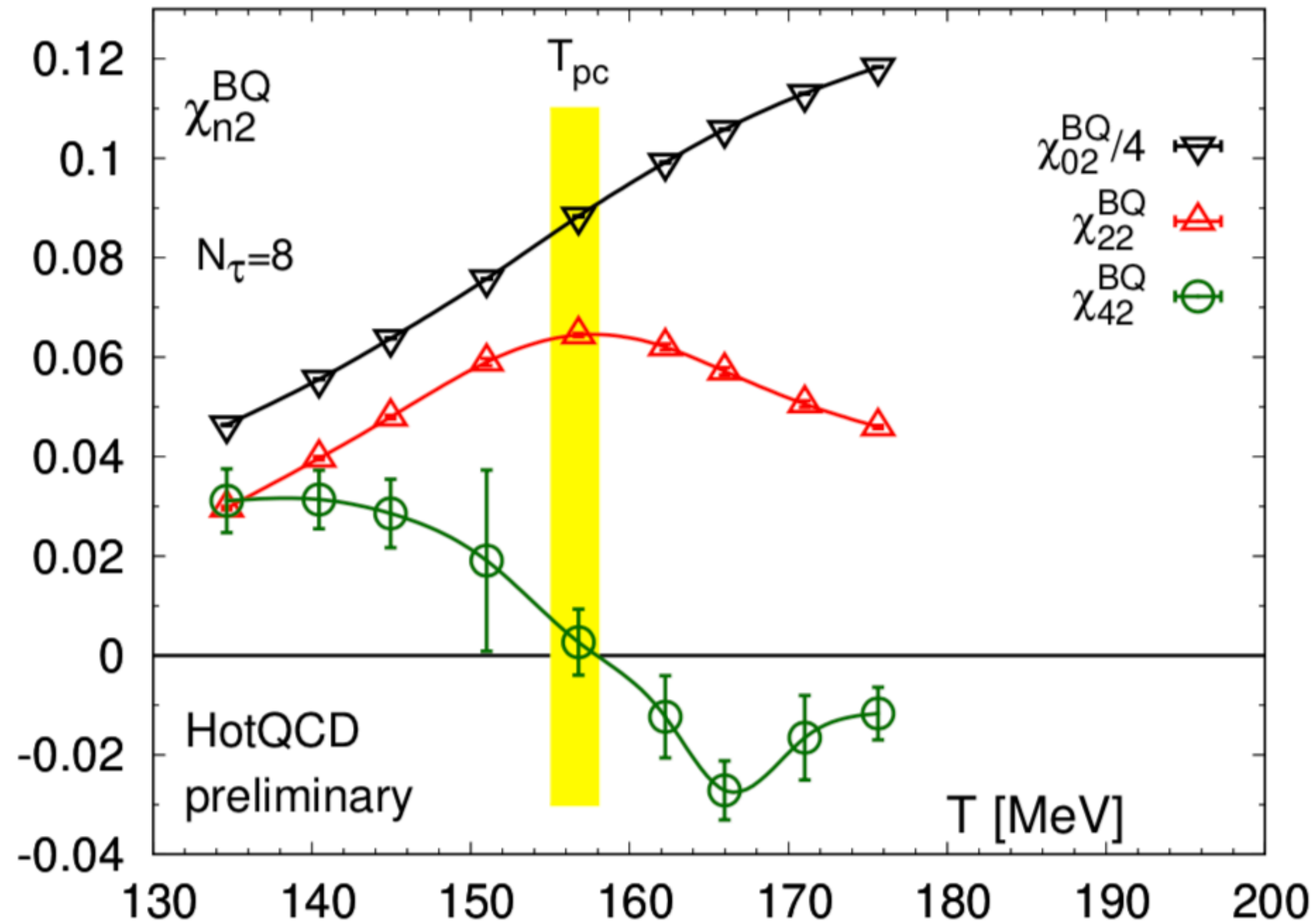
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Many 8th order fluctuations turn to be negative at $T \gtrsim 135$ -140 MeV

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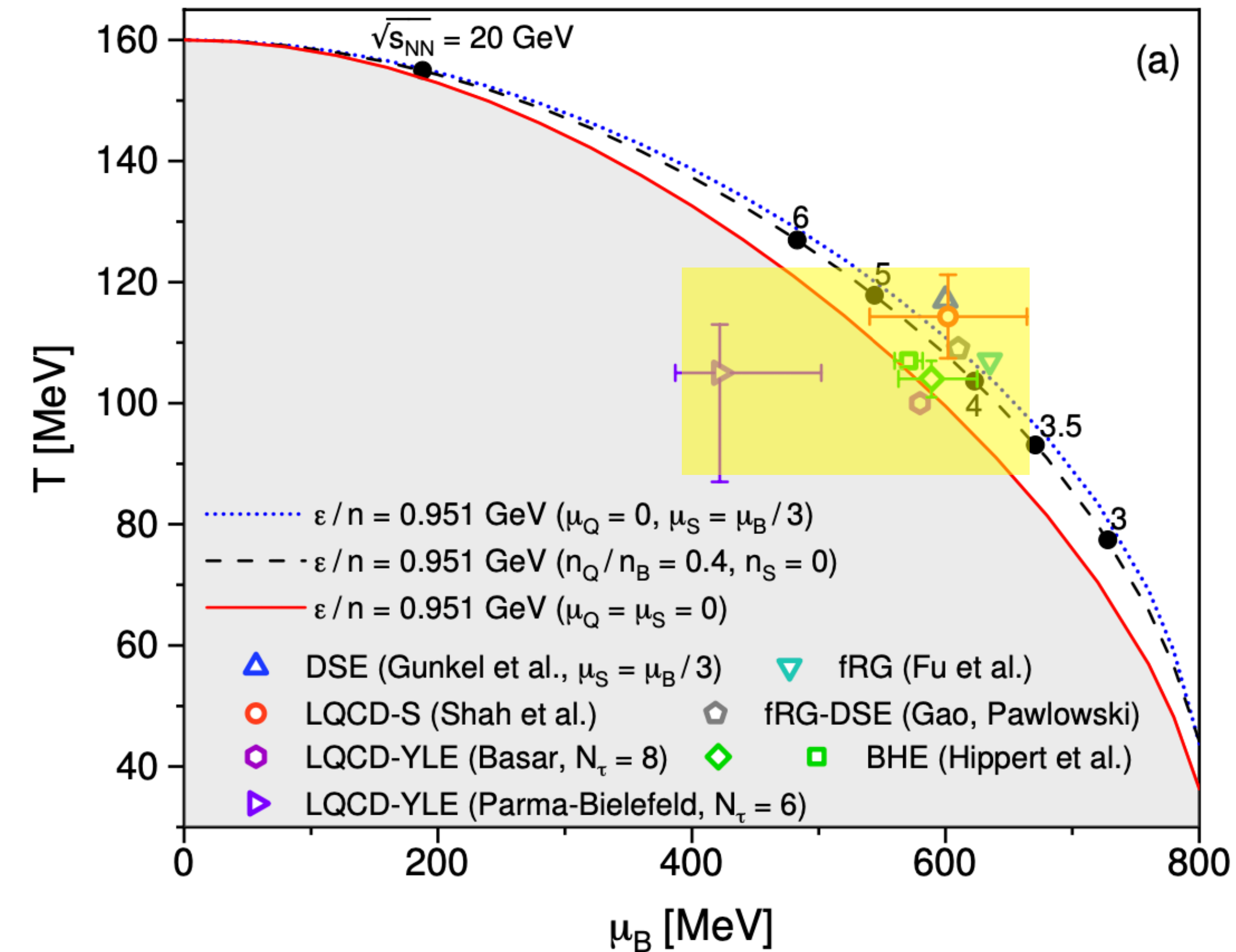
F. Karsch, PoS CORFU2018 (2019) 163

Many 8th order fluctuations turn to be negative at $T \gtrsim 135$ -140 MeV

Suggests singularity in the complex plane at $T > 135$ -140 MeV

More support for $T_{CEP} < T_c^0$

Recent theoretical predictions/estimates of CEP



Recent predictions/estimates:

$$\begin{aligned} T_c^{\text{CEP}} &\approx 83 - 120 \text{ MeV} \\ \mu_{B,c}^{\text{CEP}} &\approx 387 - 664 \text{ MeV} \end{aligned}$$

or CEP does not exist

The region hope to find to CEP in EXP

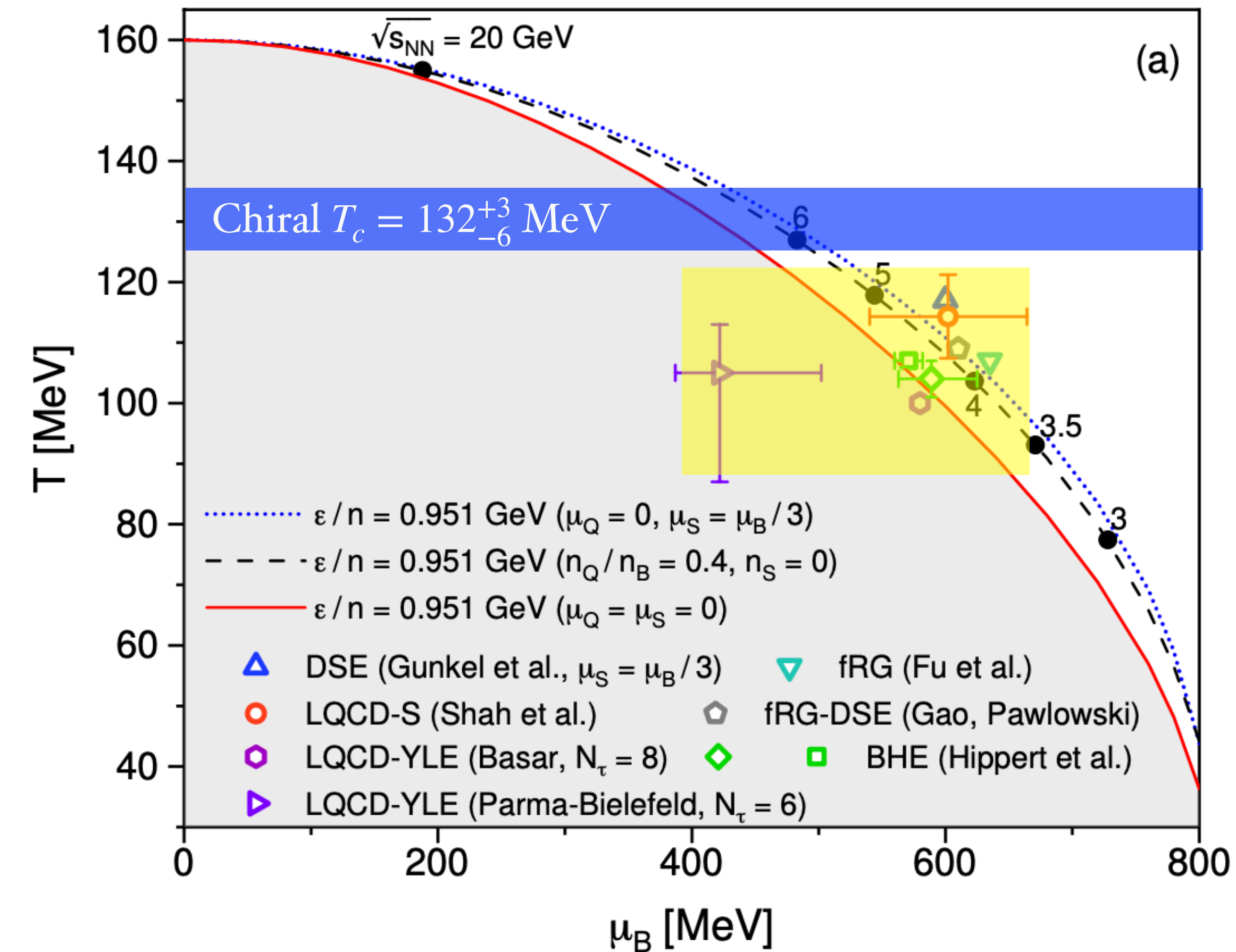
$$\sqrt{s_{NN}} \approx 3.9 - 7 \text{ GeV}$$

$$\mu_{B,c}^{\text{CEP}} \approx 420 - 632 \text{ MeV}$$

Lysenko et al., arXiv:2408.06473, *Phys.Rev.C* 111 (2025) 5, 054903

See Jan Pawlowski's talk on FRG today

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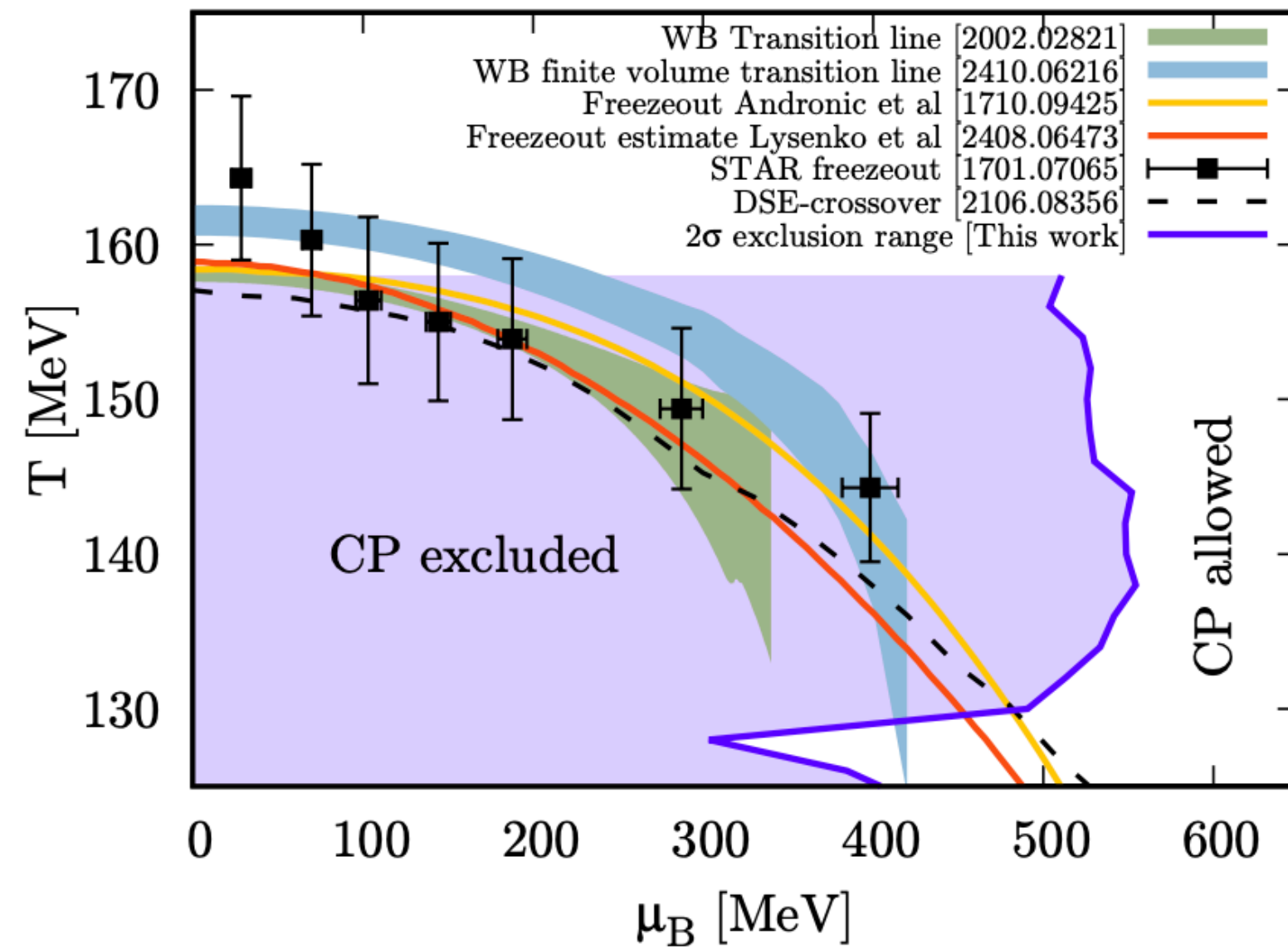
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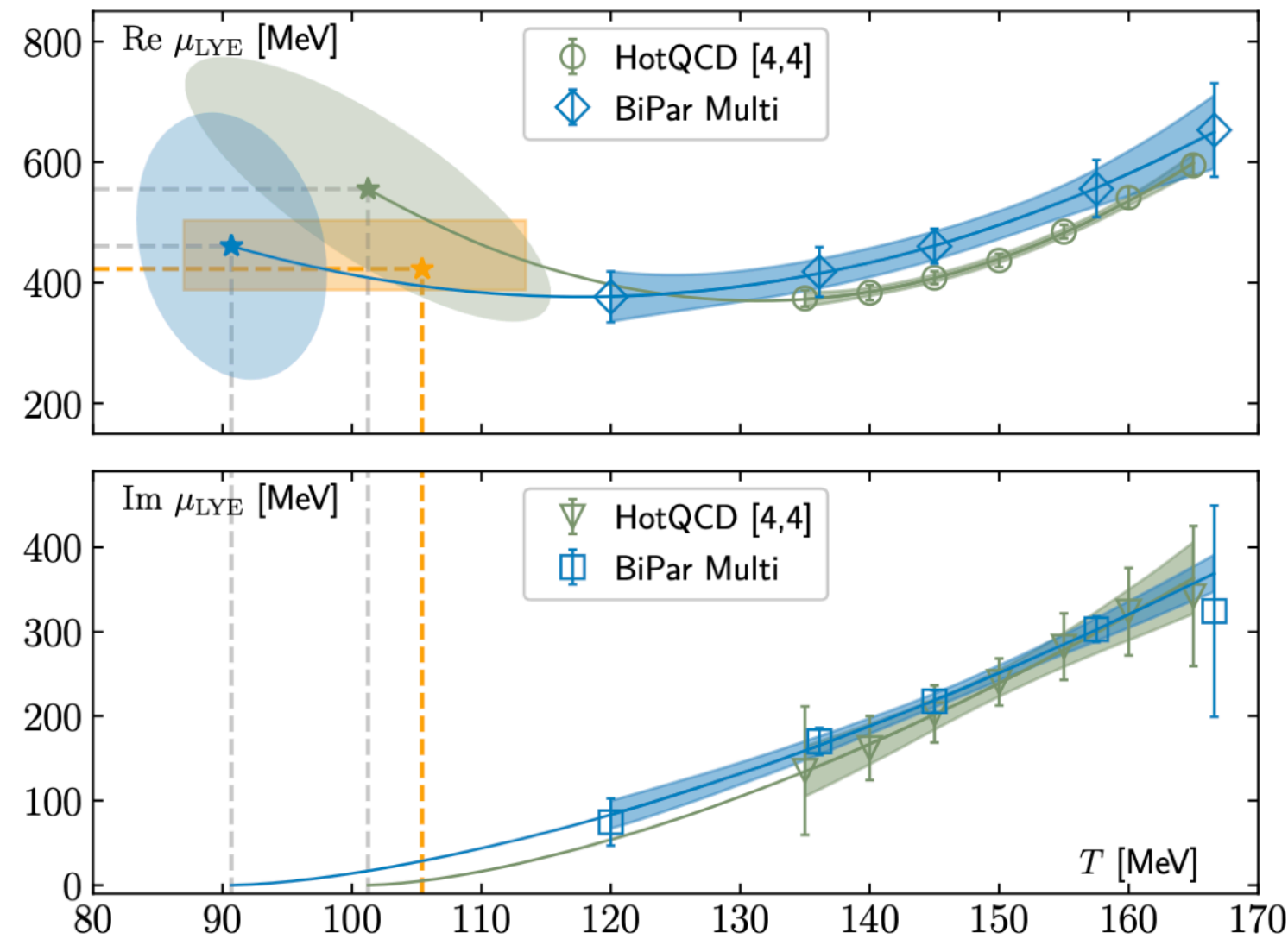
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Recent results on the CEP search in Lattice QCD

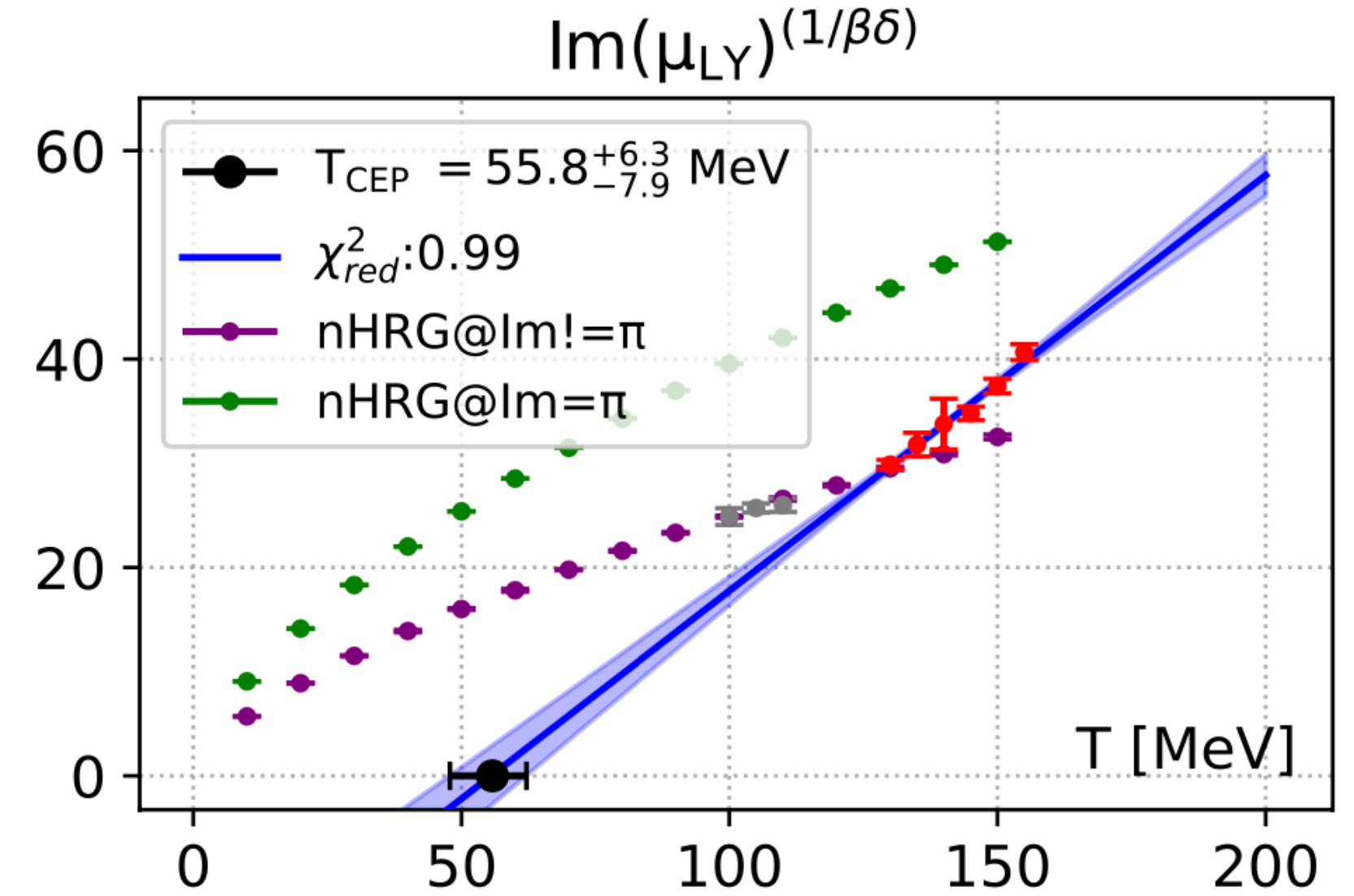
Contour of constant entropy



Lee-Yang edge singularity



Lee-Yang edge singularity



No CEP at $\mu_B < 450$ MeV at the 2σ level

Borsányi et al., arXiv:2502.10267

$$T^{\text{CEP}} = 114.3 \pm 6.9 \text{ MeV}$$

$$\mu_B^{\text{CEP}} = 602.1 \pm 62.1 \text{ MeV}$$

Shah et al., arXiv:2410.16206

See Szabolcs Borsanyi's talk today

$$T^{\text{CEP}} = 105^{+8}_{-18} \text{ MeV}$$

$$\mu_B^{\text{CEP}} = 422^{+80}_{-35} \text{ MeV}$$

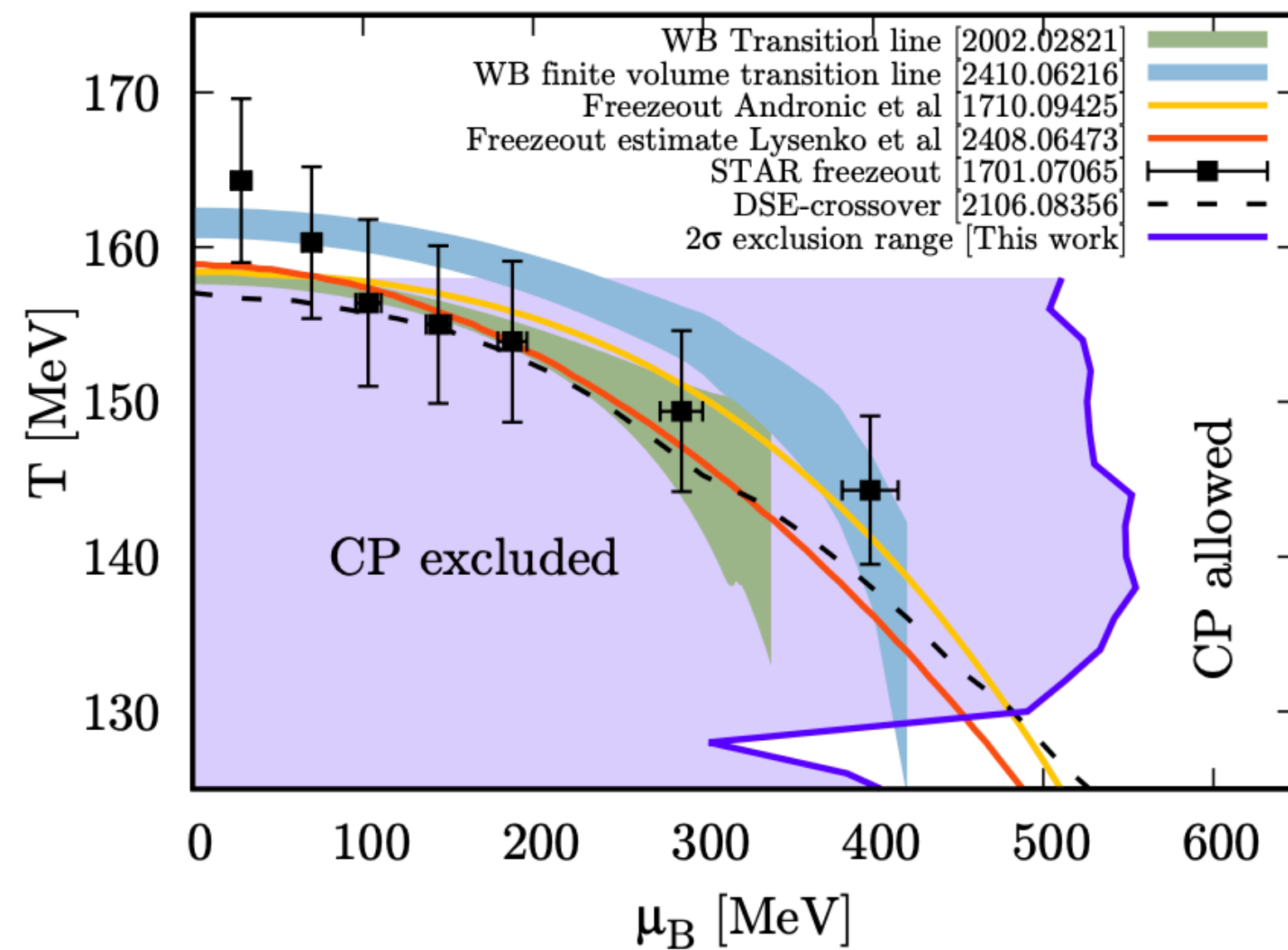
Bielefeld-Parma, arXiv:2405.10196

$T^{\text{CEP}} < 103 \text{ MeV}$
with 84% probability or
CEP does not exist

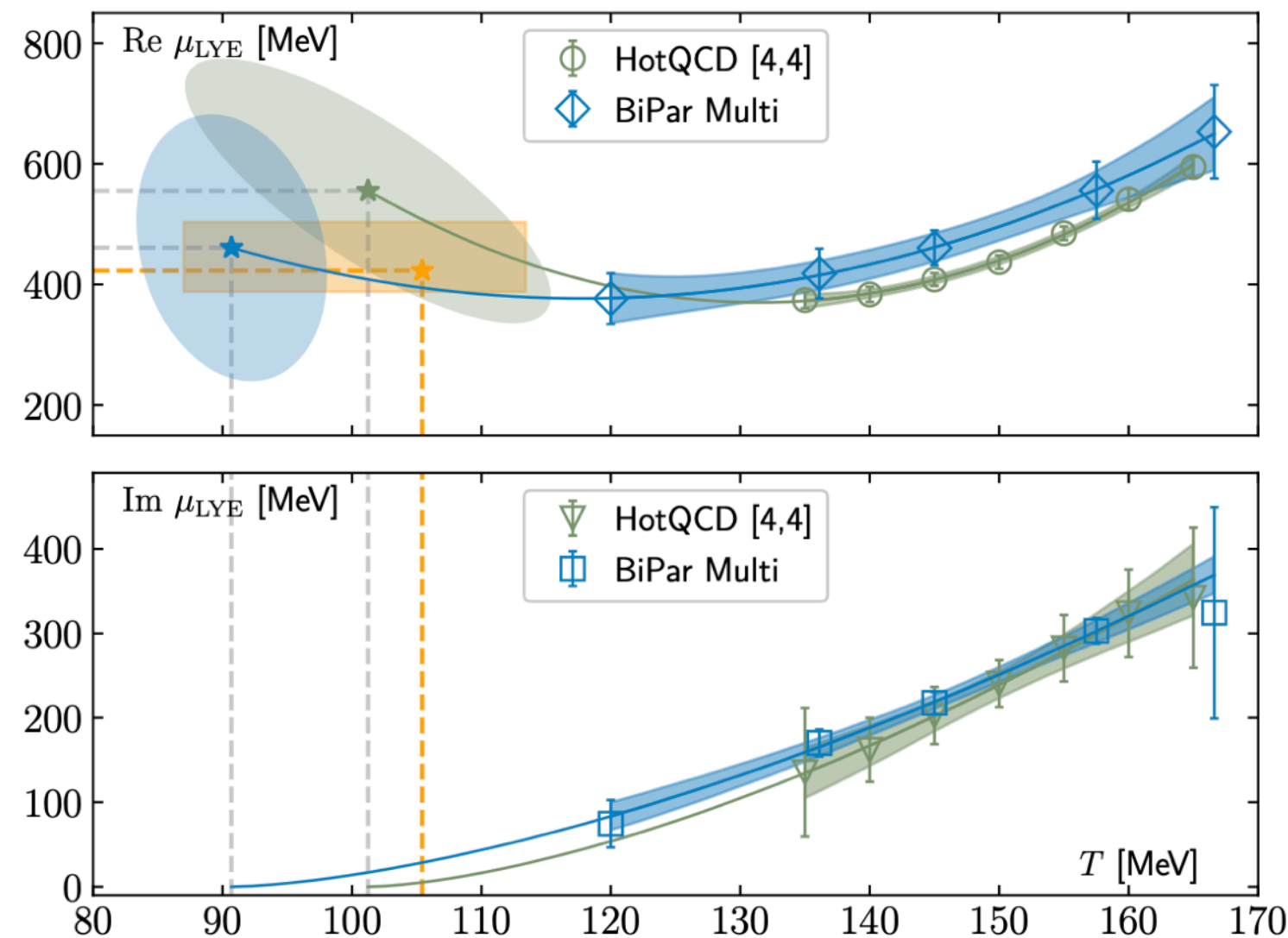
Adam et al., arXiv:2507.13254

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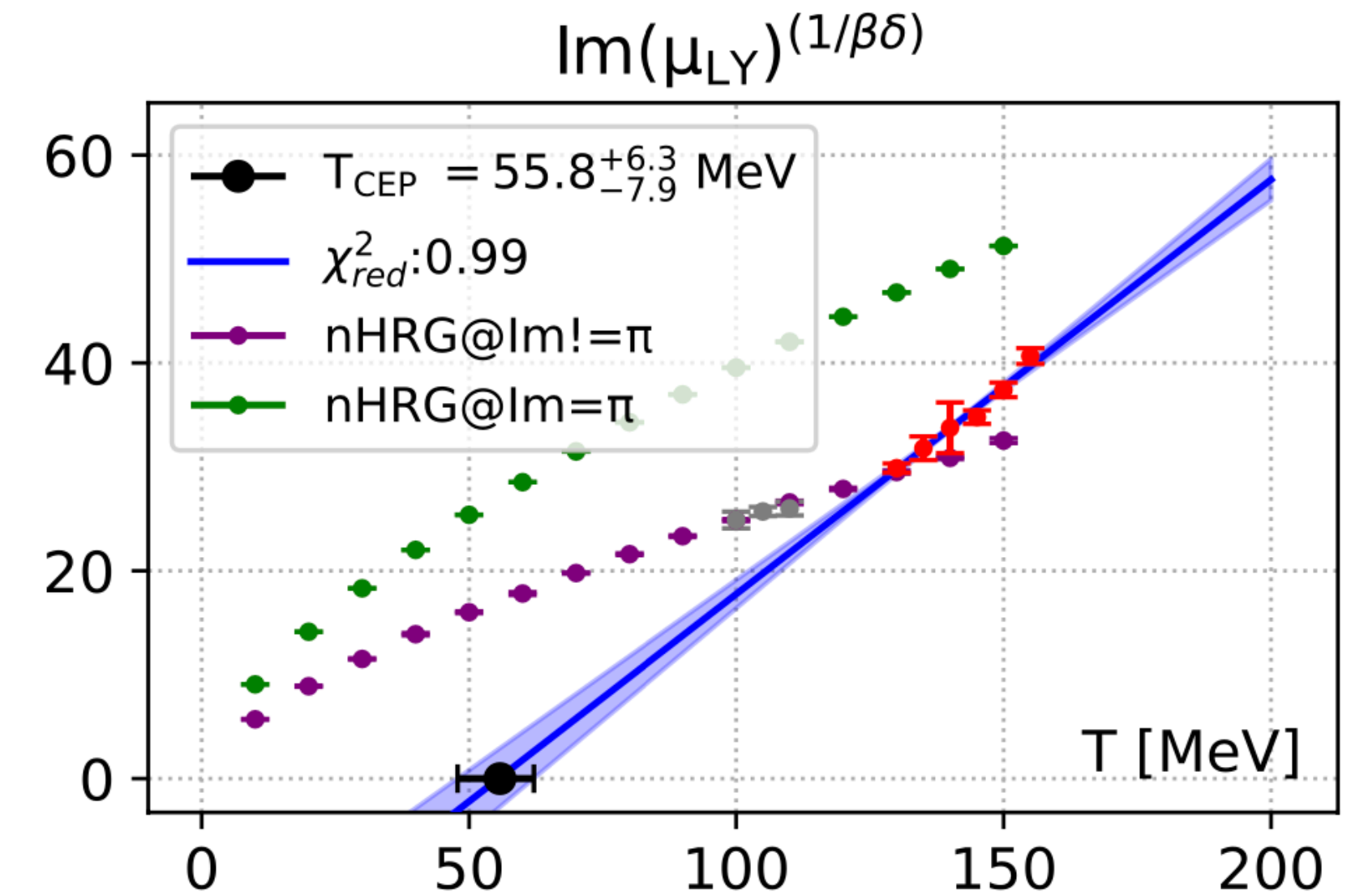
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See Szabo

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Bielefeld-Parma, arXiv:2405.10196

$$T^{\text{CEP}} < 103 \text{ MeV}$$

with 84% probability or
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Adam et al., arXiv:2507.13254

All these analyses are done based on extrapolations
from lattice QCD results at $T \gtrsim 120 - 130$ MeV

Recent theoretical predictions/estimates of CEP

- ◆ $T^{\text{CEP}} = 114.3 \pm 6.9 \text{ MeV}$
- ◆ $T^{\text{CEP}} = 105^{+8}_{-18} \text{ MeV}$
- ◆ $T^{\text{CEP}} < 103 \text{ MeV}$ w. 84% prob. or No CEP

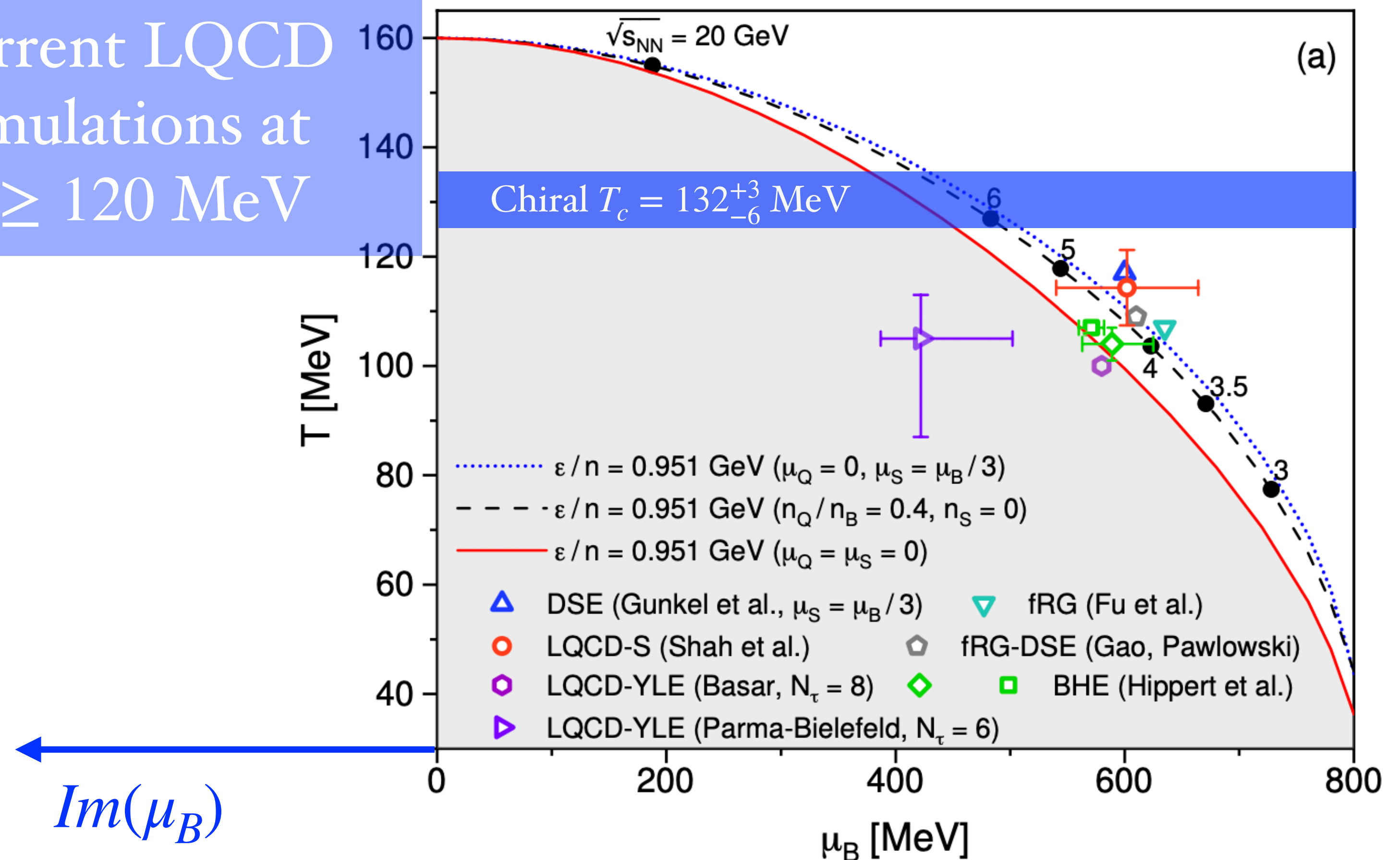
Current LQCD simulations at $T \geq 120 \text{ MeV}$

➤ Recall recent estimates/predictions:

$$T_c^{\text{CEP}} \approx 83 - 120 \text{ MeV}$$

$$\mu_{B,c}^{\text{CEP}} \approx 387 - 664 \text{ MeV}$$

➤ How about perform a direct LQCD simulations in the currently predicted T_c^{CEP} region? E.g. 108 MeV



Lysenko et al., arXiv:2408.06473, *Phys.Rev.C* 111 (2025) 5, 054903

Lattice setup at $T \approx 108$ MeV

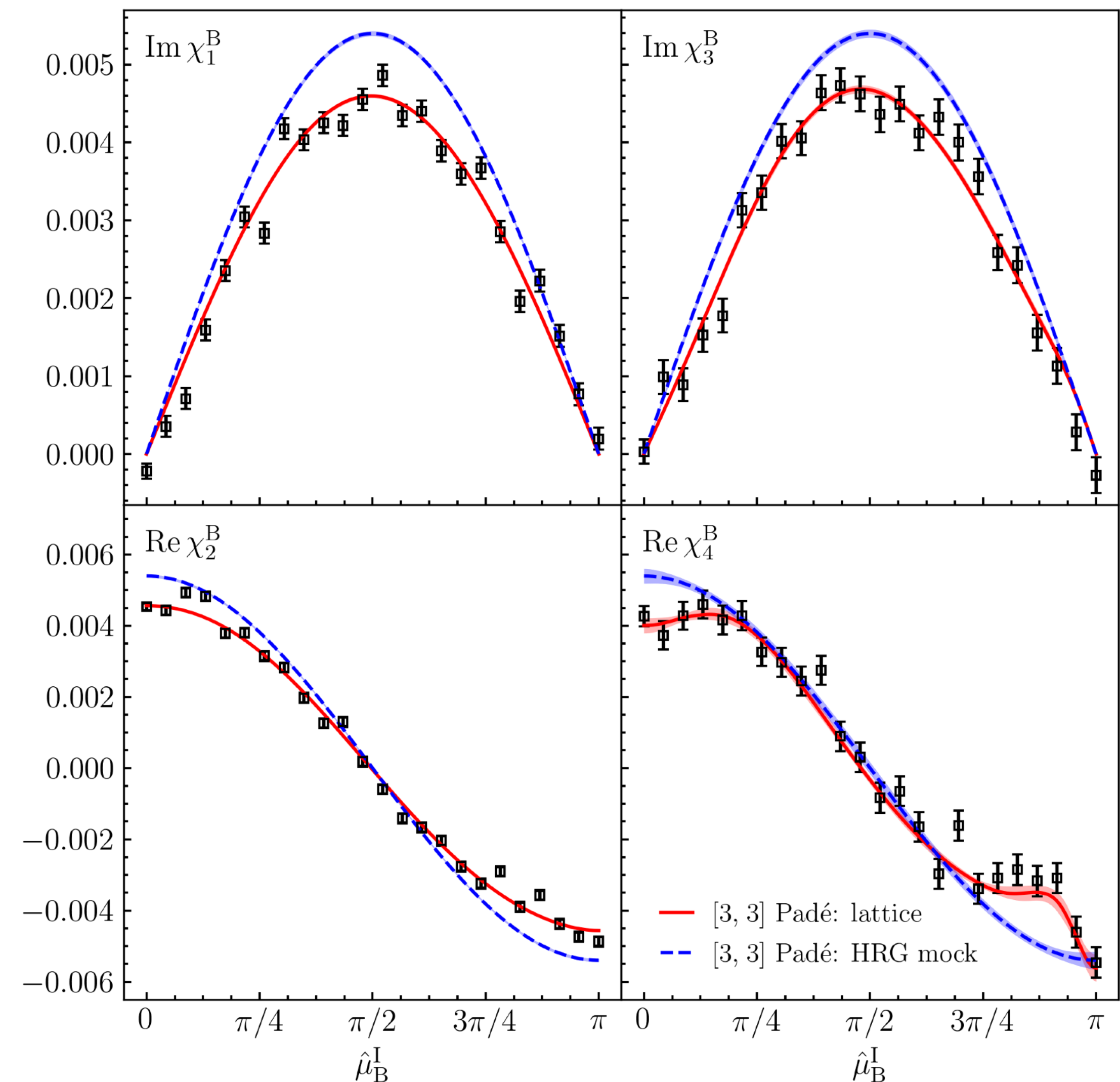
- 📌 $N_f=2+1$ QCD on $20^3 \times 10$ lattices with at $T=107.7$ MeV using HISQ fermions at the physical point with $\mu_u = \mu_d = \mu_s = \mu_B/3$
- 📌 24 values of $i\mu_B \in [0, i\pi]$ each for χ_{1-4}^B
- 📌 400k configurations for zero chemical potential, 200k for others
- 📌 Additional $32^3 \times 10$ lattices generated to check the volume dependence

Multi-point Padé fits

$$P(x) \equiv R_k^m(x) = \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^k b_j x^j}, x = \cosh(\mu_B) - 1$$

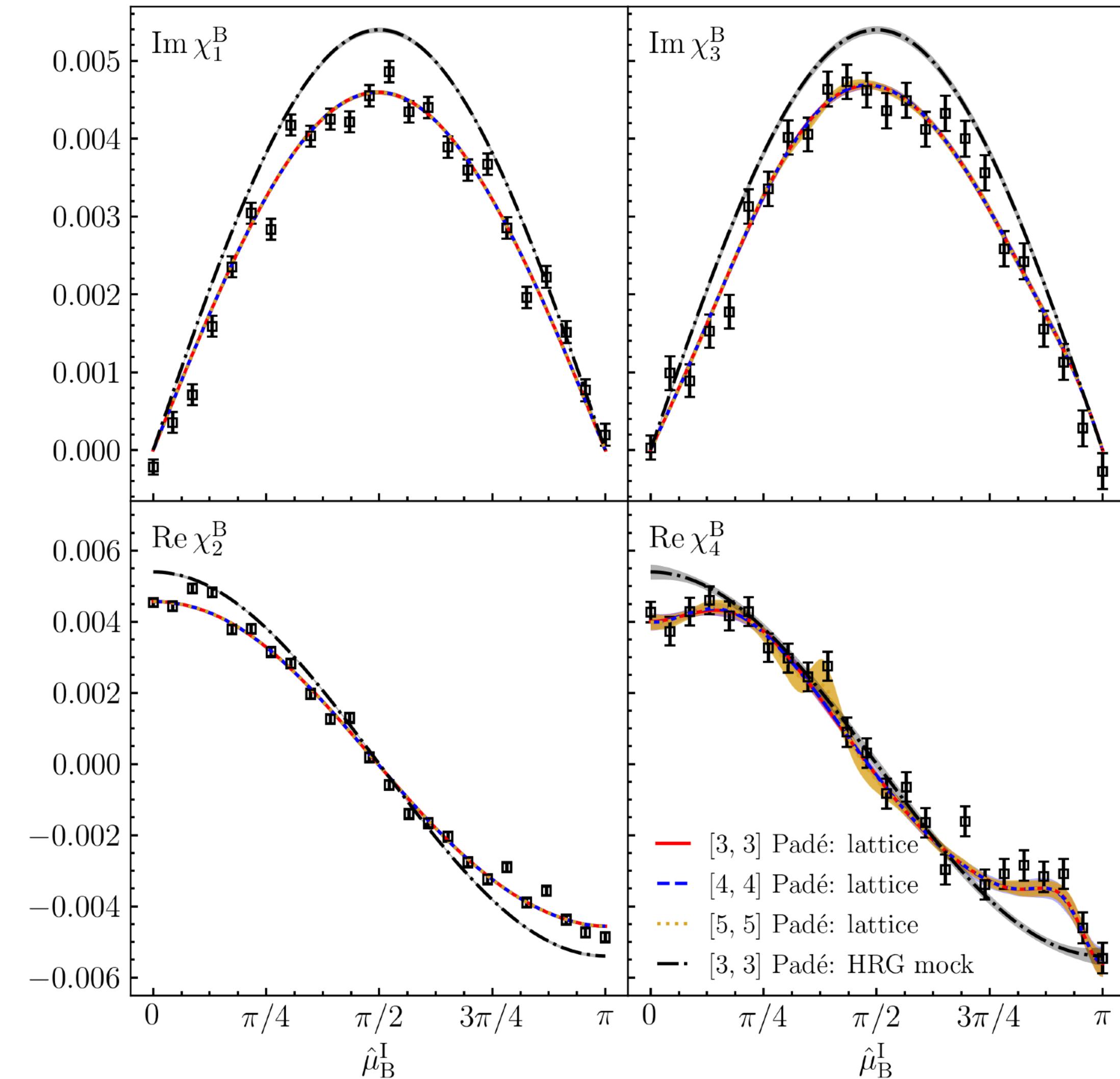
- Generate N_{boot} bootstrap samples by resampling $N_{conf,i}$ configurations allowing replacement for each one of 24 discrete values of $\mu_{B,i}$, and compute mean value of χ_{1-4}^B at $\mu_{B,i}$ within each bootstrap sample
- Joint fit to χ_n^B with $\partial^n P(x)/\partial \mu_B^n$ with $n=1-4$ in each bootstrap sample with covariance matrix obtained from the original data sets
- Final uncertainties from 68% confidence interval over bootstrap fits

[3,3]



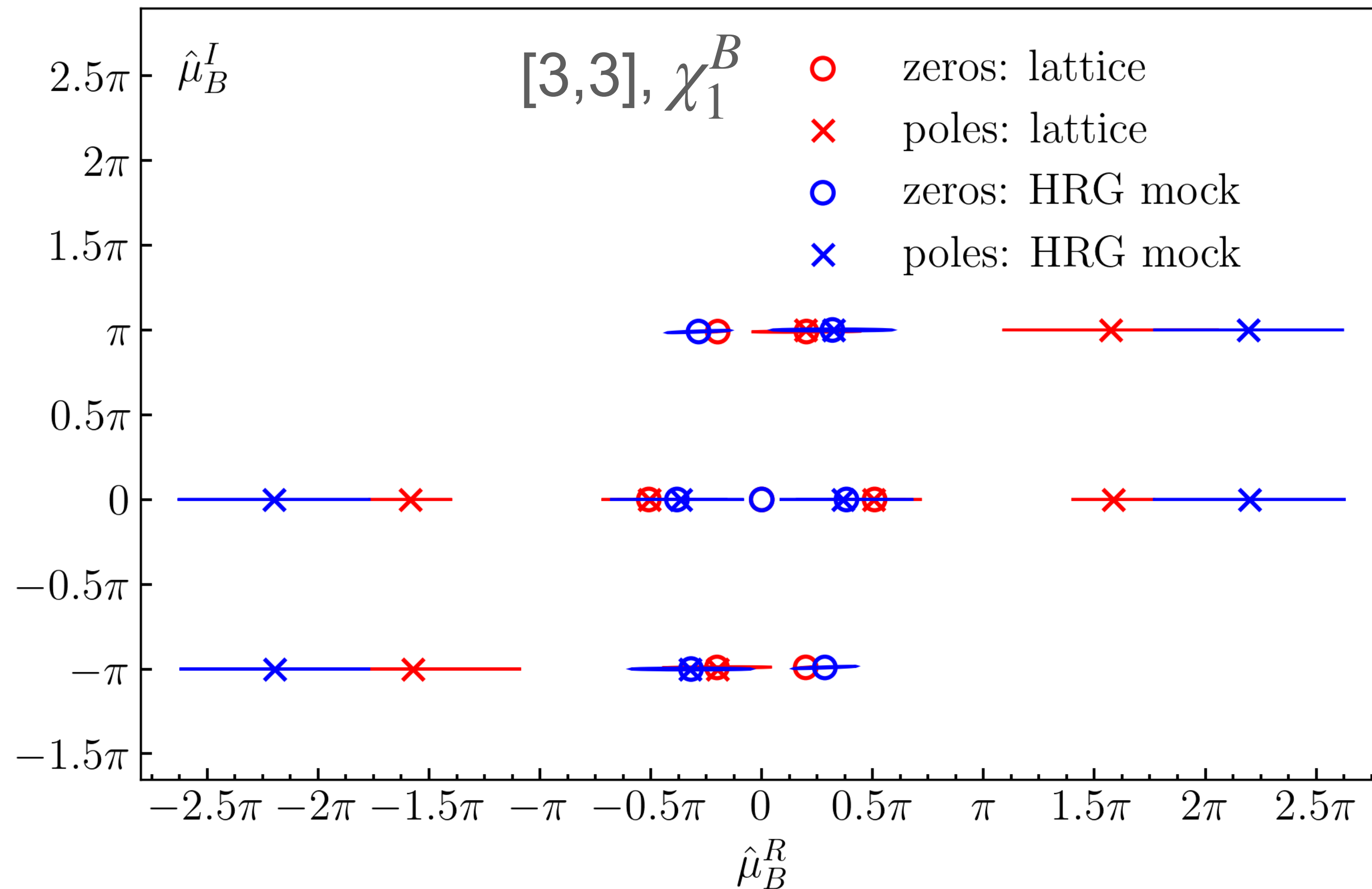
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$$P(x) \equiv R_n^m(x) = \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^n b_j x^j}, x = \cosh(\mu_B) - 1$$



		Padé [3,3]		Padé [4,4]		Padé [5,5]	
		All 24 points	Random 12 points	All 24 points	Random 12 points	All 24 points	Random 12 points
$\chi^2/\text{d.o.f.}$	lattice	4.00(55)	3.39(75)	4.06(57)	4.45(90)	4.03(57)	3.99(85)
	HRG mock	0.97(21)	0.95(30)	0.97(21)	0.94(31)	0.96(21)	0.93(31)
AICc	lattice	373(50)	156(31)	375(50)	198(36)	370(49)	178(32)
	HRG mock	101(19)	54(13)	103(18)	57(12)	105(18)	61(12)

Zero/poles extracted from Padé [3,3] analyses



$$P(x) = \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^n b_j x^j}$$

$$\chi_1^B = \partial P(x) / \partial \mu_B$$

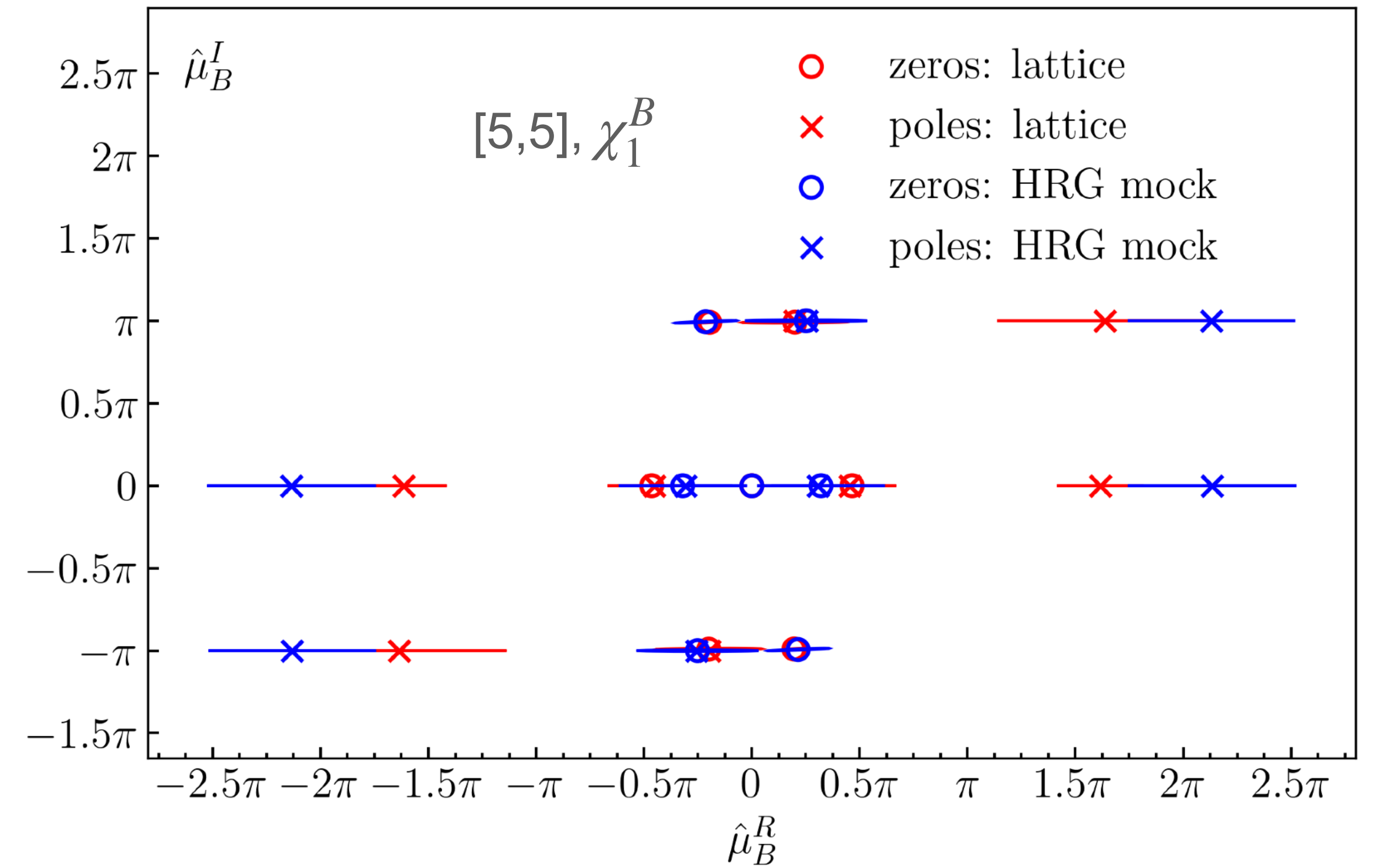
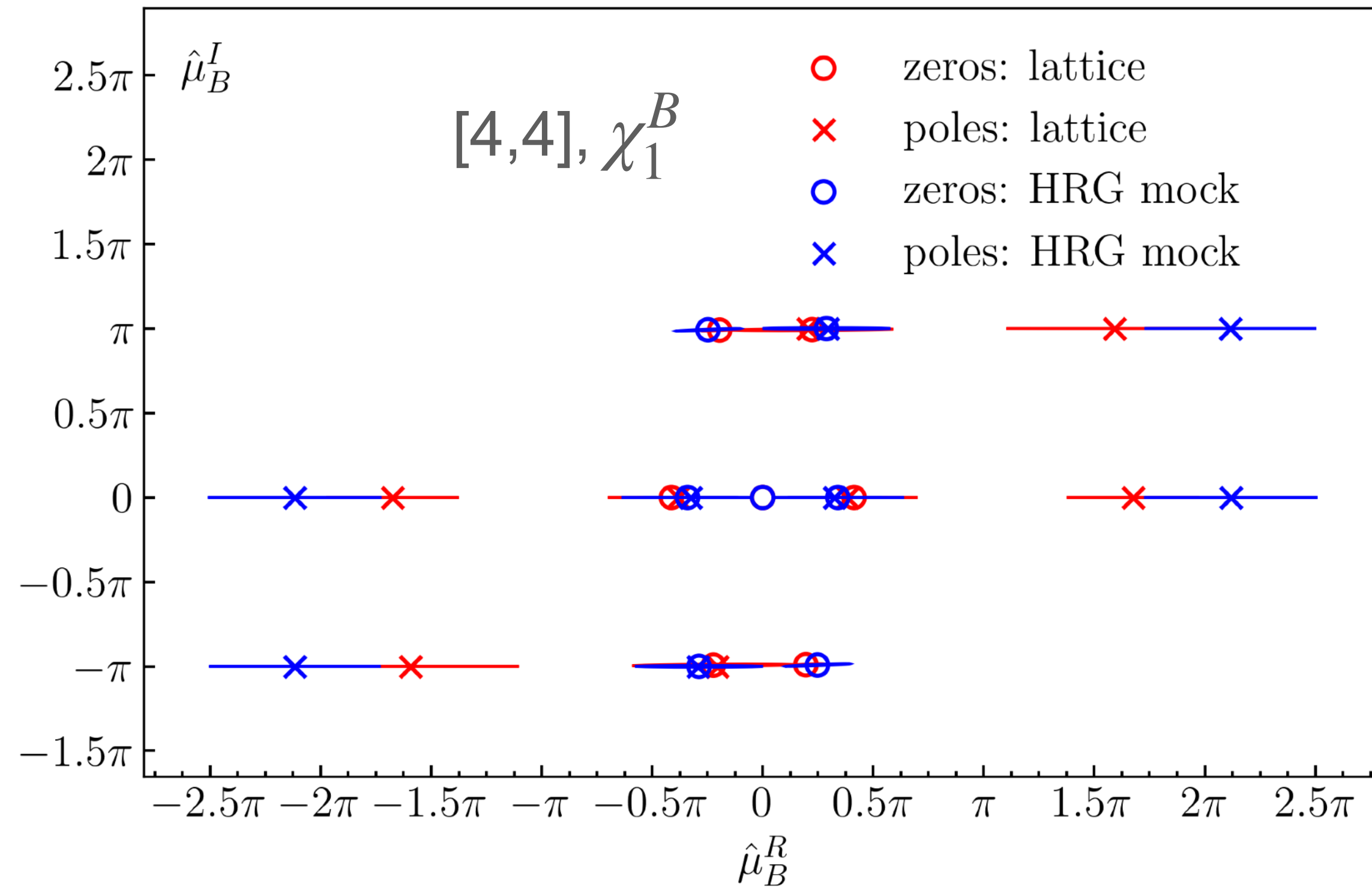
Zeros: roots of numerator

Poles: roots of denominator

📌 No uncanceled poles

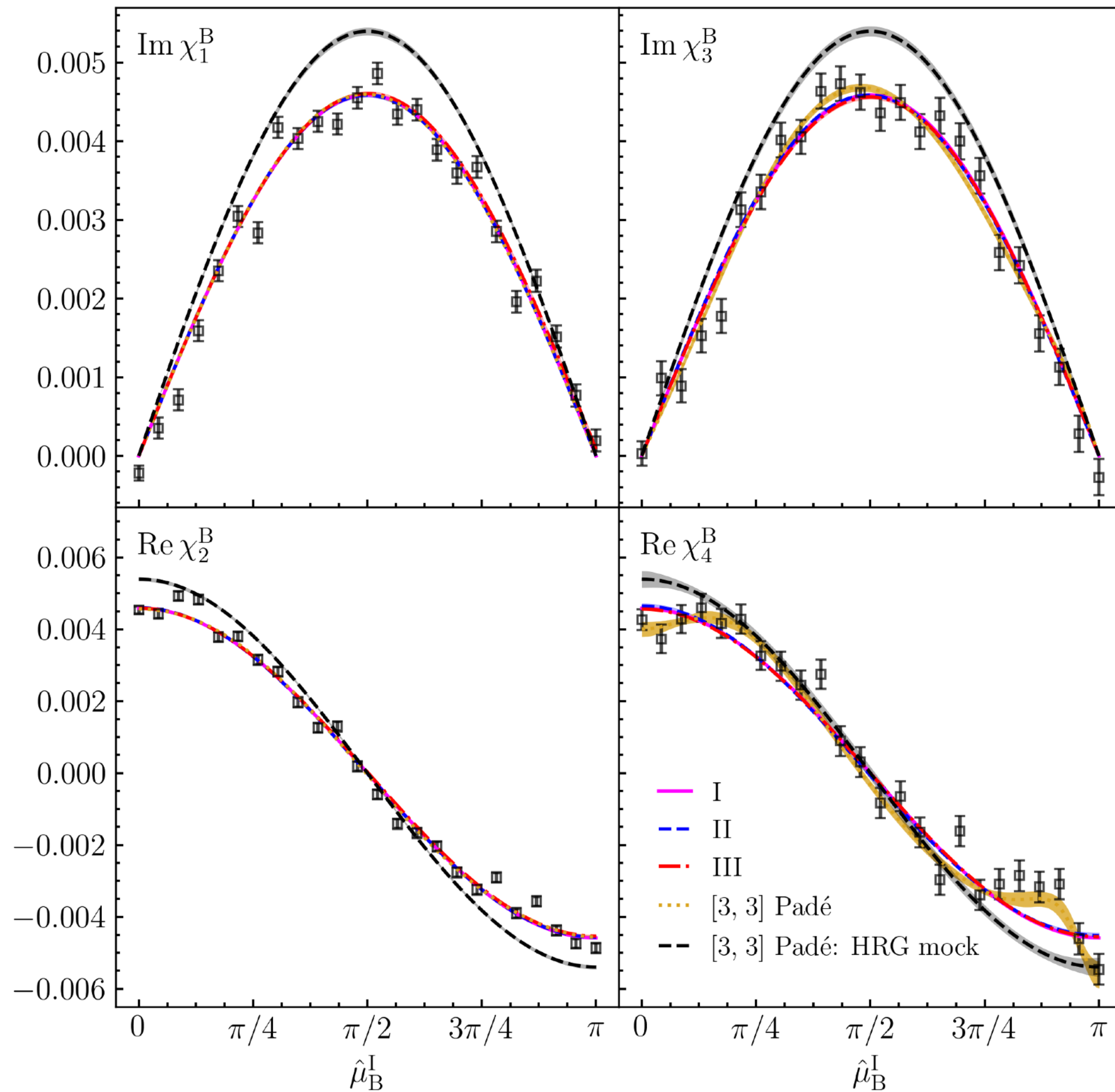
📌 Similar pattern between
Lattice and HRG mock

Zero/poles extracted from higher order Padé analyses



Similar observation as in the case of Padé [3,3]

Fits with physical-motivated non-critical ansatz



Non-critical ansatz for pressure:

Ansatz I: $f(\mu_B) = A \cosh(\mu_B)$

Ansatz II: $f(\mu_B) = A \cosh(\mu_B) + B \cosh(2\mu_B)$

	Lattice QCD Data			
	All 24 points Ansatz I	Ansatz II	Random 12 points Ansatz I	Ansatz II
Parameters				
$A (\times 10^{-3})$	4.584(35)	4.582(34)	4.528(48)	4.662(50)
$B (\times 10^{-5})$	—	0.38(64)	—	0.001(898)
Statistics				
$\chi^2/\text{d.o.f.}$	4.20(55)	4.13(55)	4.45(82)	4.23(81)
AICc	401(52)	401(52)	211(38)	199(37)

Singularities in the real plane?

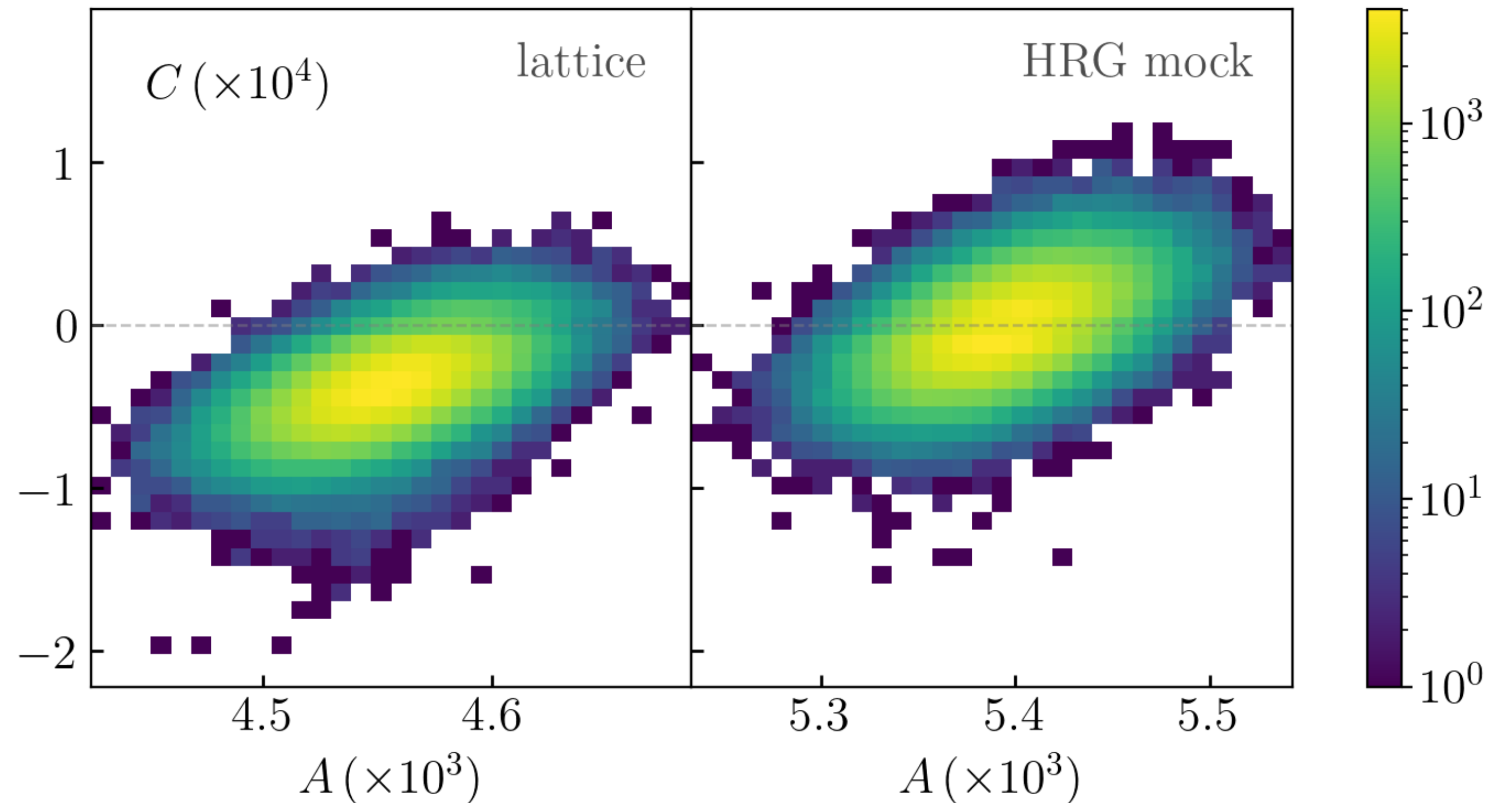
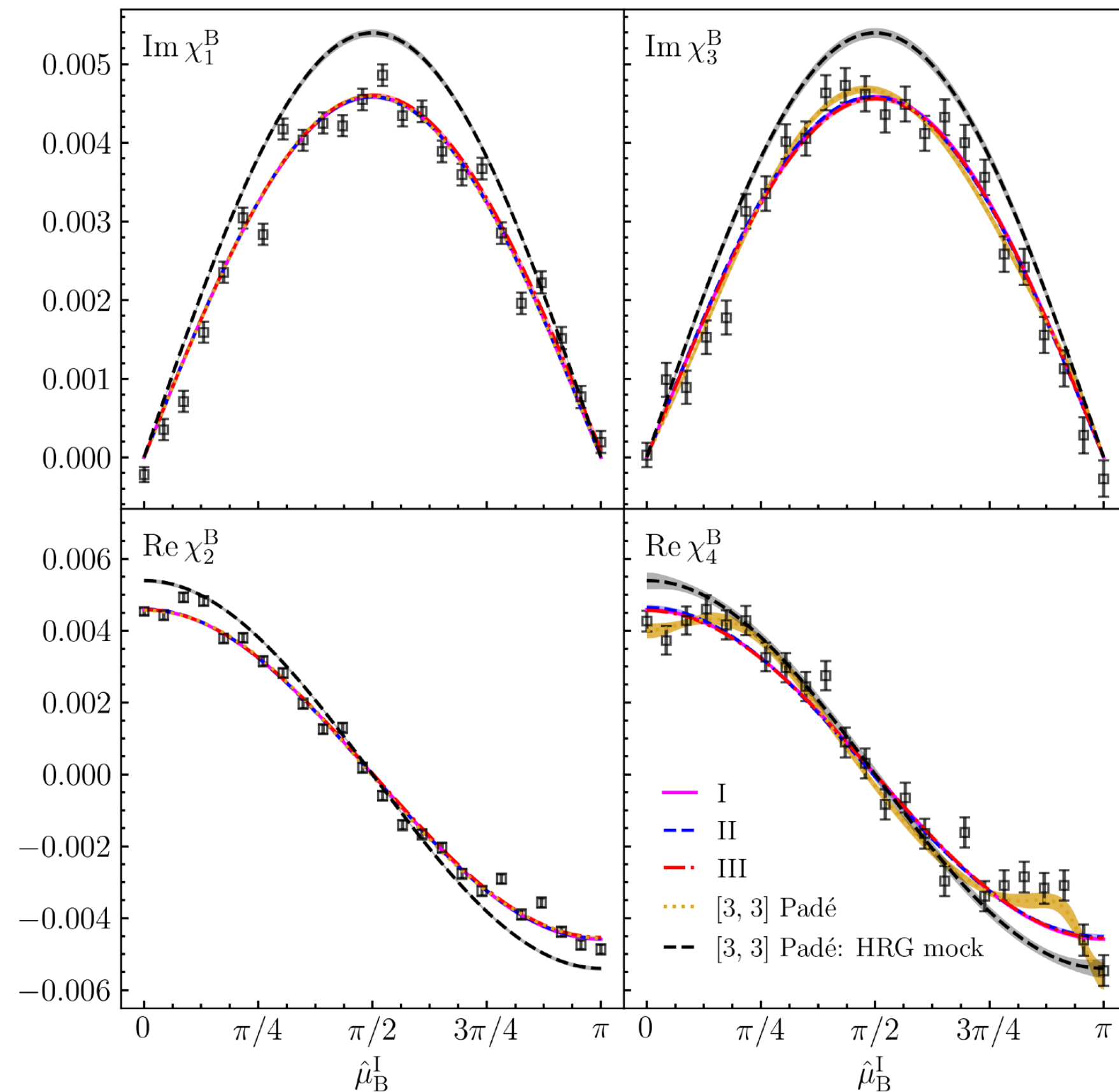
Critical ansatz III: $f(\mu_B) = A \cosh(\mu_B) + C(|\mu_B - \mu_{B,c}|^\alpha + |\mu_B + \mu_{B,c}|^\alpha)$

Impose lower bound of $\mu_{B,c}$: 350, 360, 380, 400 MeV

$$\alpha = 1 + \delta_{\text{ISNG}} = 1.208$$

Hasenbusch , PRB 82(2010)174433

Karch, Schmidt & Singh, PRD109(2024)014508



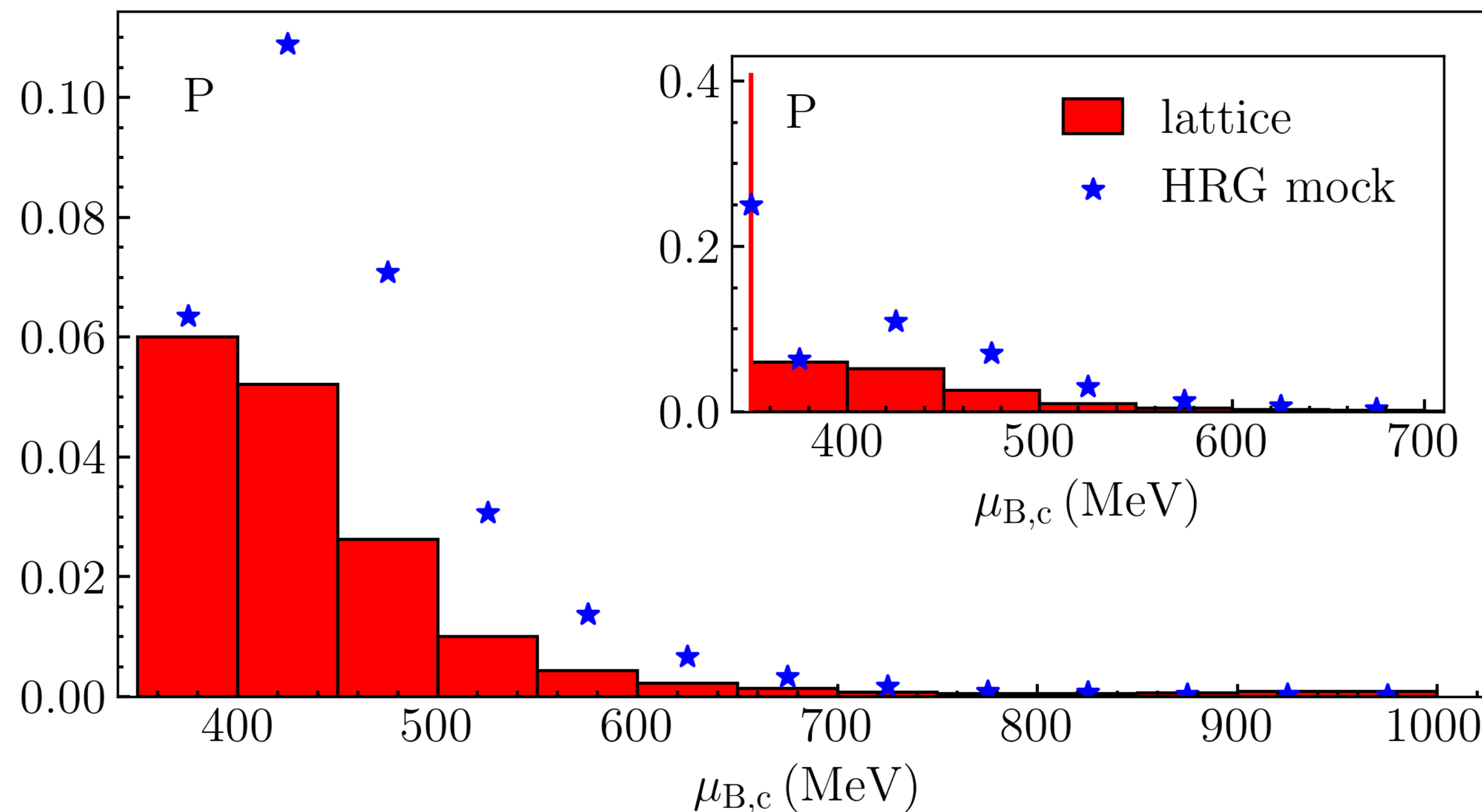
Diagnostic measure of $\mu_{B,c}$ in a certain region $\mathcal{R}$

Define a Diagnostic measure P for
 $\mu_{B,c}$ in a region $\mathcal{R}$:

Lower bound of $\mu_{B,c}$ = 350 MeV

$$P = \frac{N_{\mathcal{R}}}{N_{total}}$$

$N_{\mathcal{R}}$: # of bootstrap samples
yield $\mu_{B,c}$ in the region $\mathcal{R}$
 N_{total} : total # of bootstrap
samples used in the fit



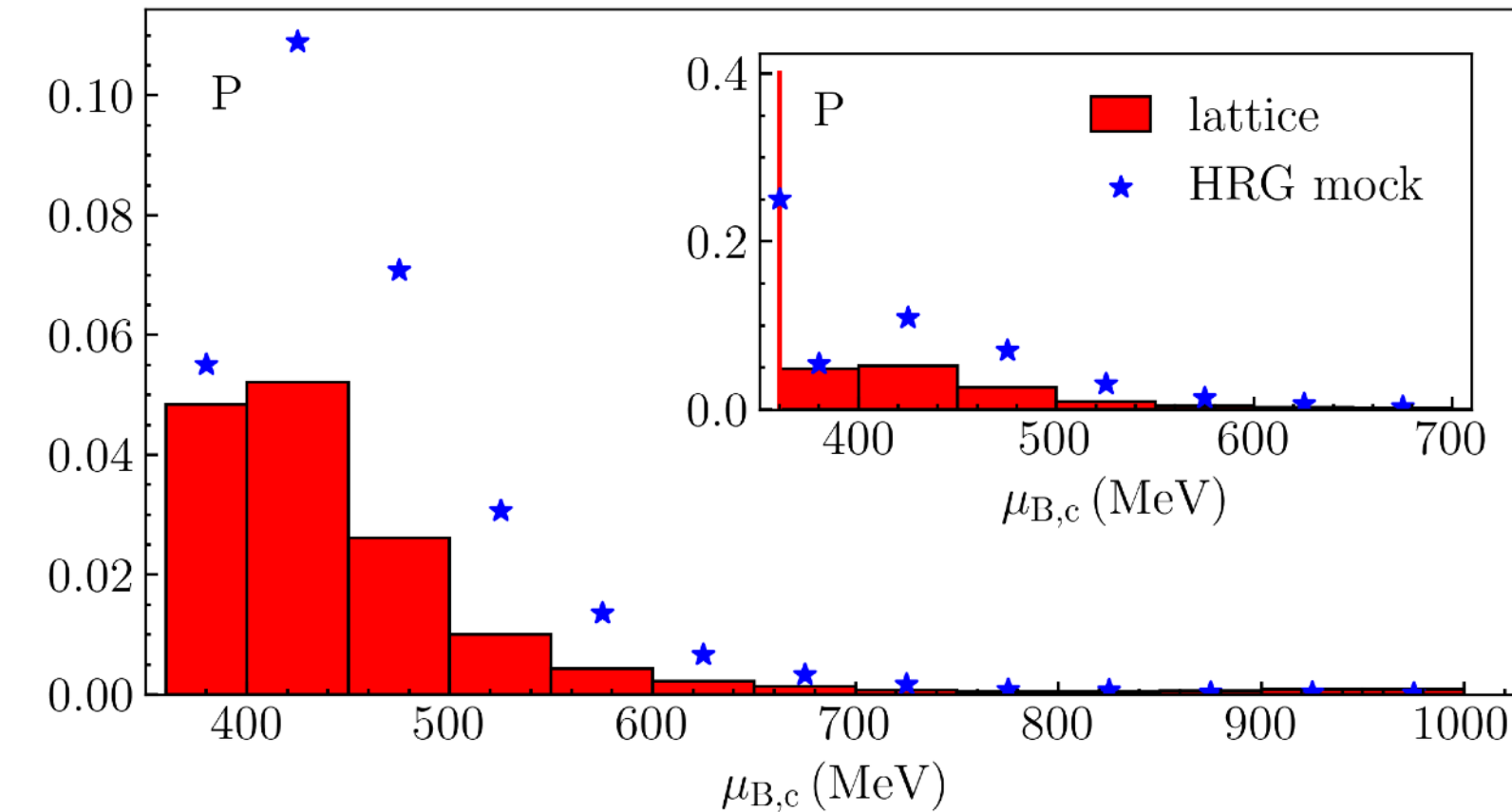
📌 Nonzero false-positive $P_{\text{HRG}} > P_{\text{LAT}}$!

Region $\mathcal{R}$ (MeV)	$P(\mathcal{R})$	
	lattice	HRG mock
387–502 (Predicted Region I)	9.1%	20.4%
420–750 (Experimental Search Region I)	7.3%	19.4%
420–632 (Experimental Search Region II)	7.1%	18.7%
540.0–664.2 (Predicted Region II)	0.86%	2.6%

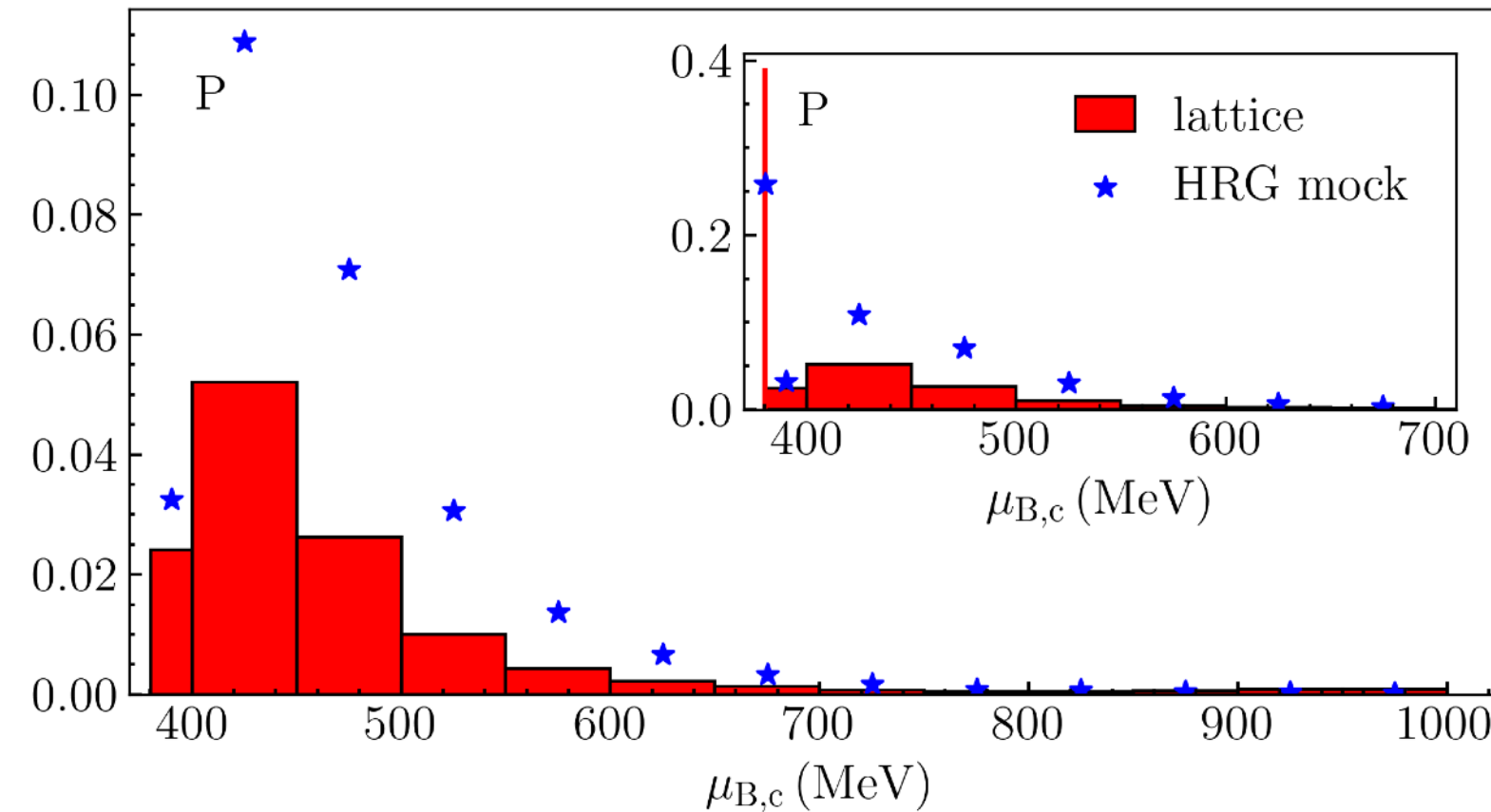
📌 Accumulation of large samples at
lower bound

Dependence on the lower bound & # of bootstrap samples

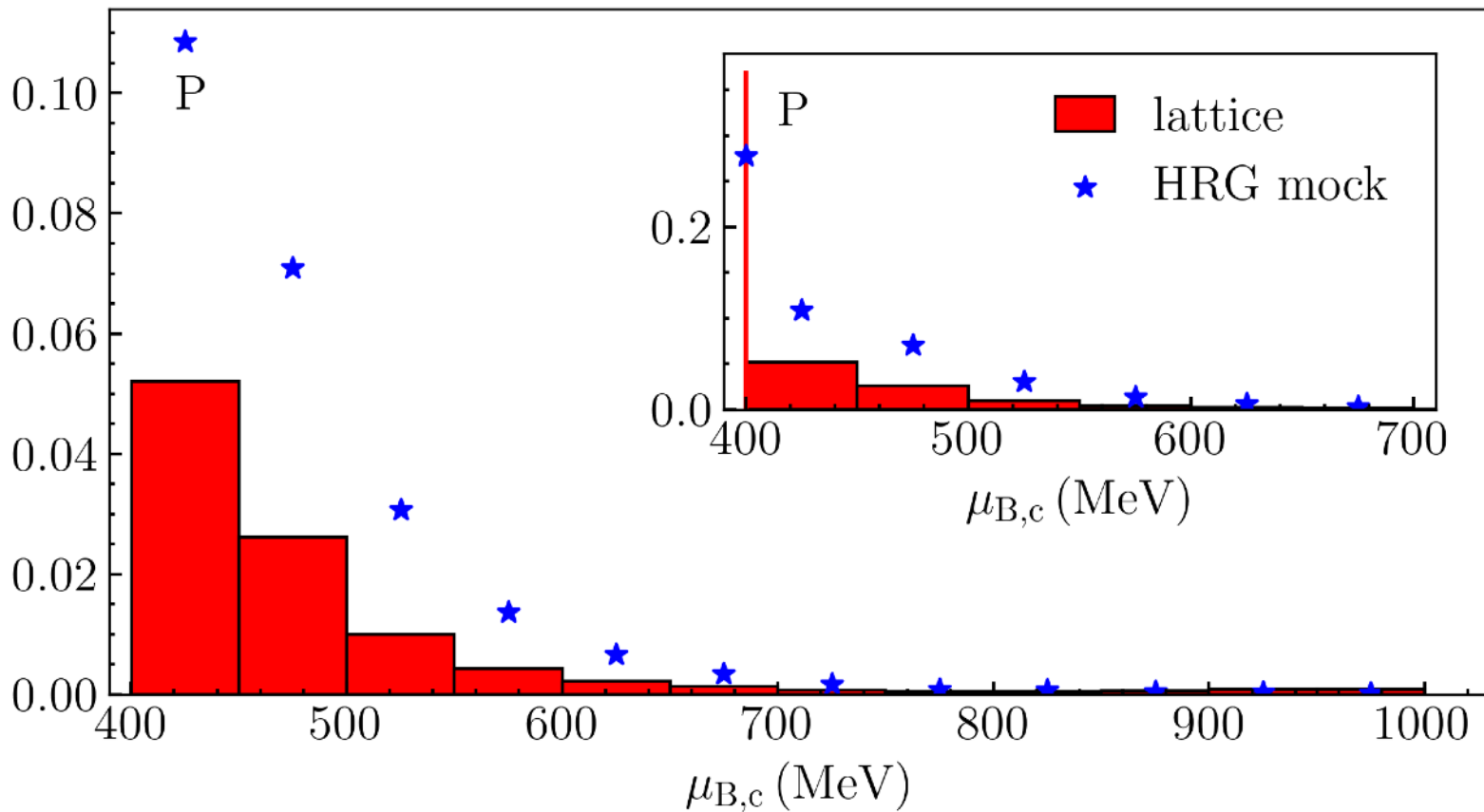
Lower bound 360 MeV



Lower bound 380 MeV

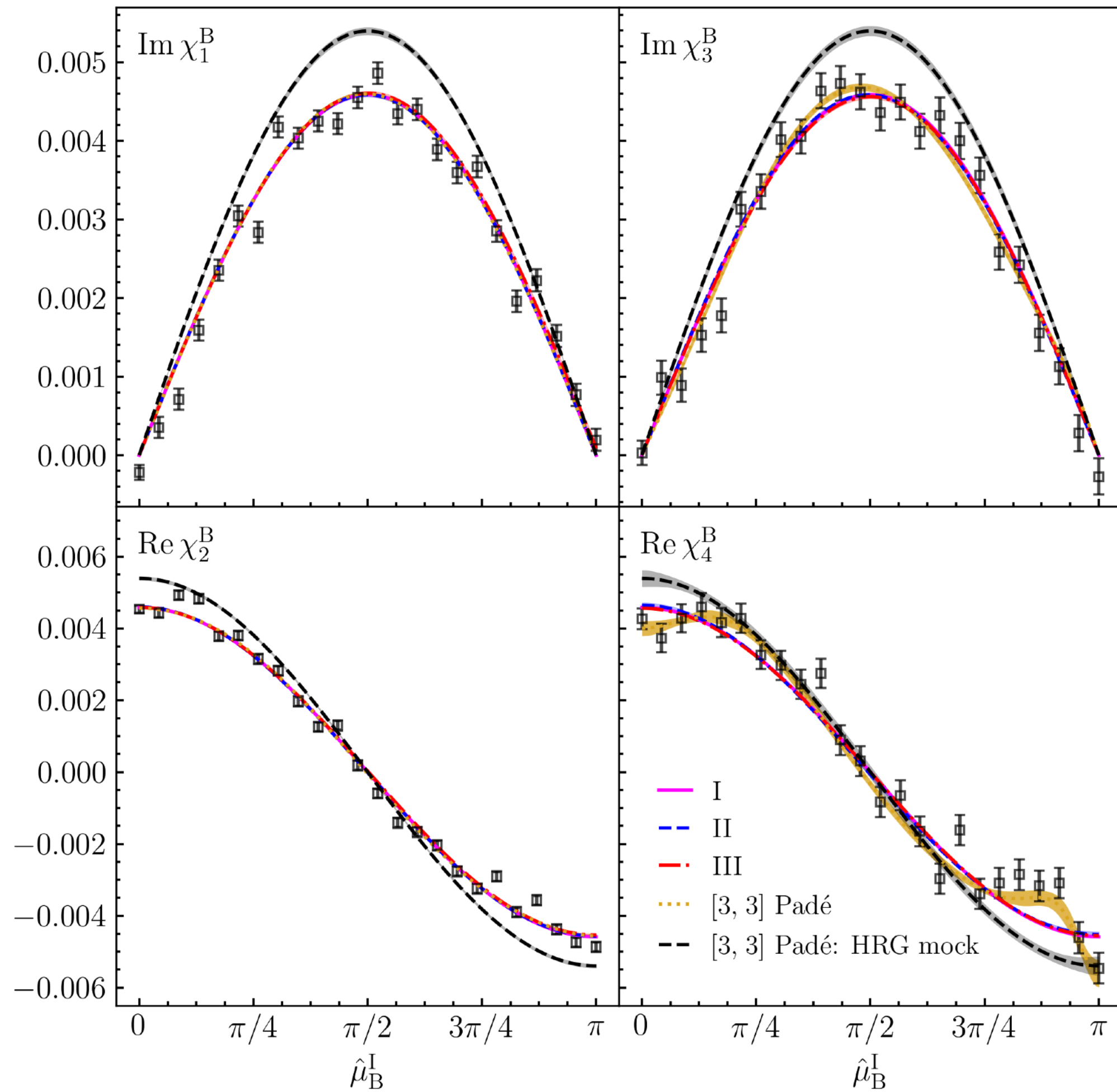


Lower bound 400 MeV



Region $\mathcal{R}$ (MeV)	$P(\mathcal{R})$											
	100K Samples				200K Samples				300K Samples			
	350	360	380	400	350	360	380	400	350	360	380	400
387–502	0.091	0.091	0.091	-	0.092	0.091	0.091	-	0.092	0.092	0.092	-
420–750	0.074	0.074	0.074	0.074	0.073	0.073	0.073	0.073	0.073	0.073	0.073	0.073
420–632	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071
540.0–664.2	0.0086	0.0086	0.0087	0.0087	0.0087	0.0087	0.0086	0.0086	0.0085	0.0085	0.0084	0.0085

Dependence on the reduced/full data sets



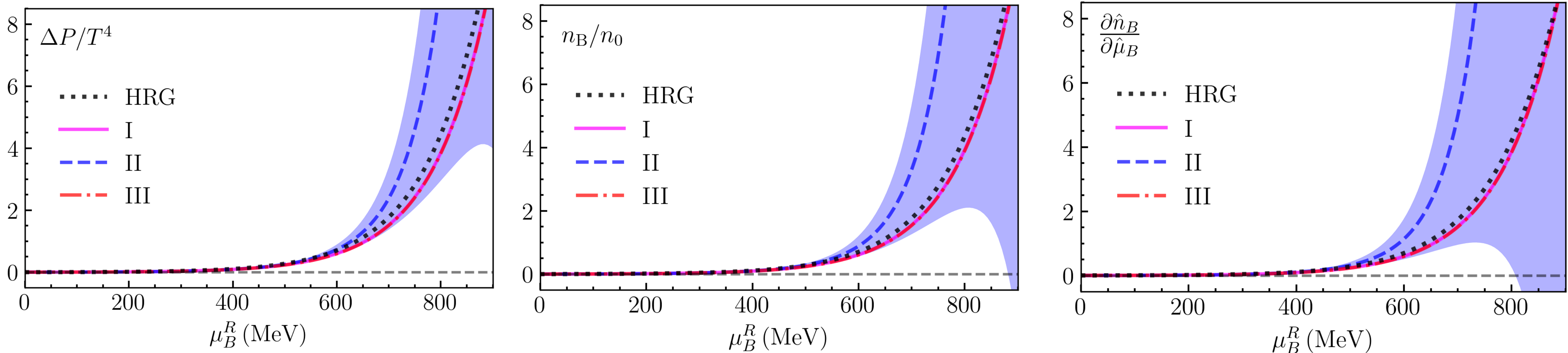
Lattice

Region $\mathcal{R}$ (MeV)	$P(\mathcal{R})$							
	All 24 points				Random 12 points			
	350	360	380	400	350	360	380	400
387–502	0.092	0.092	0.092	-	0.124	0.124	0.124	-
420–750	0.073	0.073	0.073	0.073	0.133	0.133	0.132	0.133
420–632	0.071	0.071	0.071	0.071	0.126	0.126	0.126	0.126
540.0–664.2	0.0085	0.0085	0.0085	0.0085	0.023	0.023	0.022	0.023

HRG

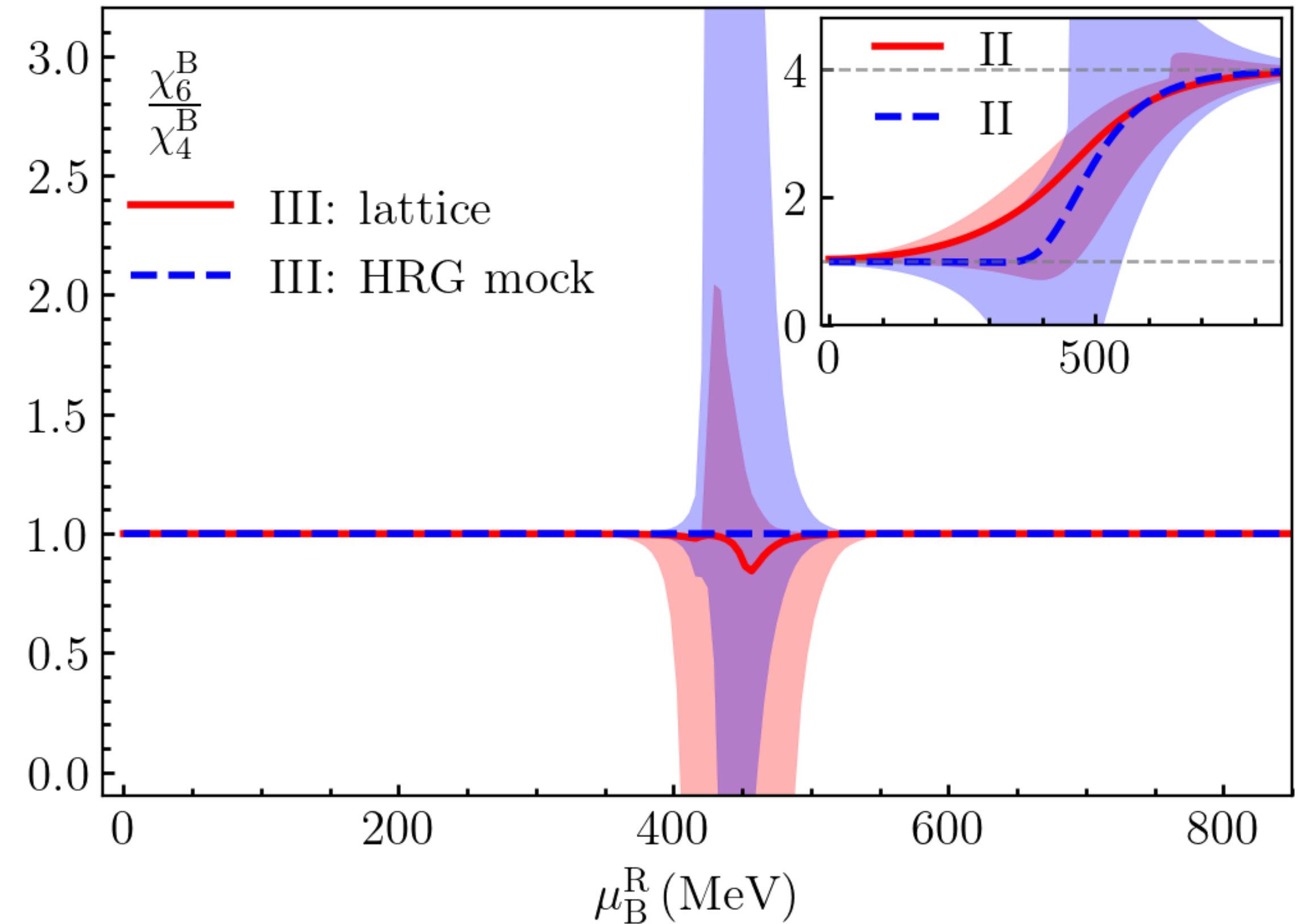
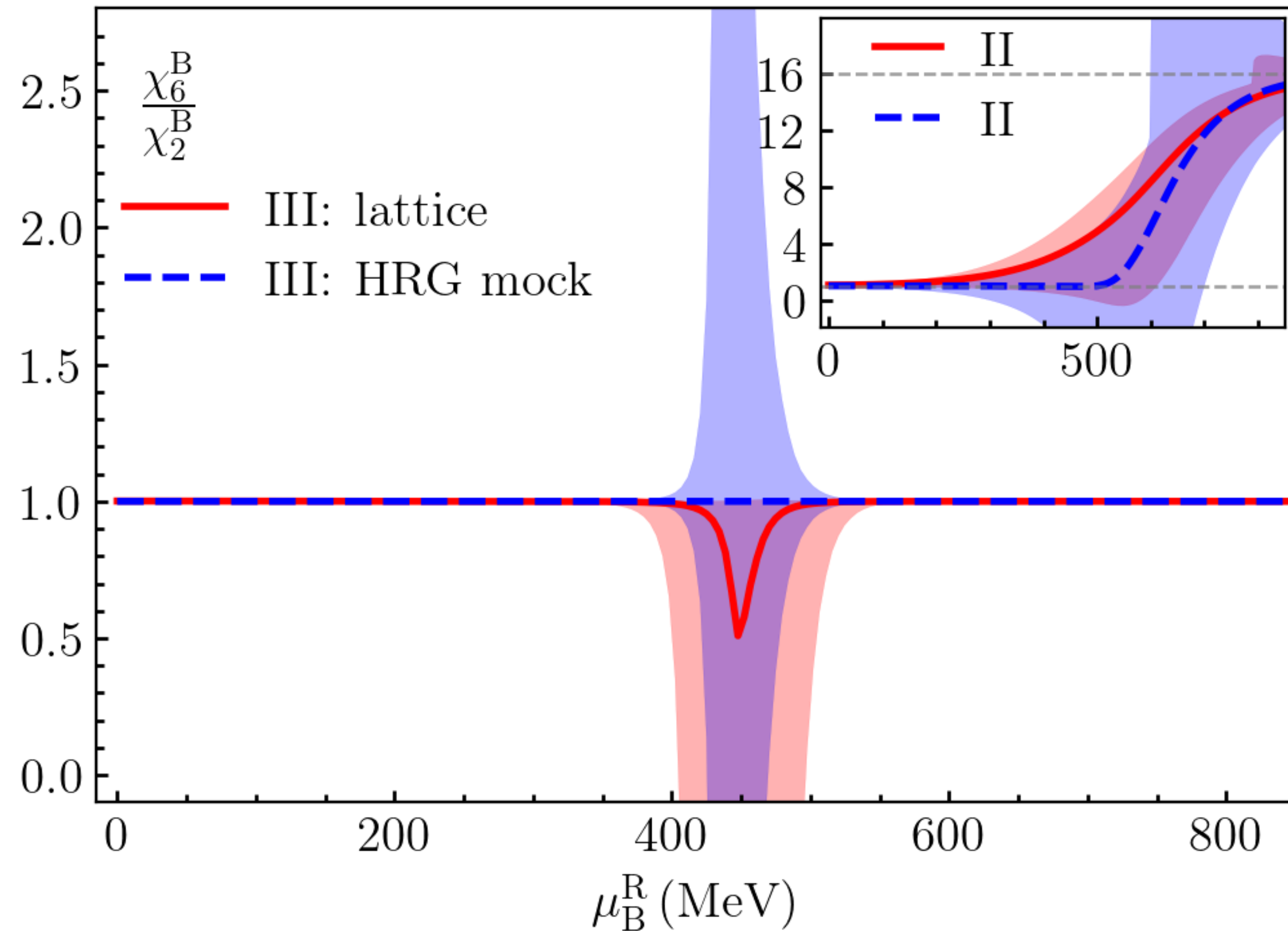
Region $\mathcal{R}$ (MeV)	$P(\mathcal{R})$							
	All 24 points				Random 12 points			
	350	360	380	400	350	360	380	400
387–502	0.204	0.204	0.204	-	0.263	0.225	0.197	-
420–750	0.194	0.194	0.194	0.194	0.301	0.218	0.175	0.184
420–632	0.187	0.187	0.187	0.187	0.293	0.211	0.168	0.177
540.0–664.2	0.0258	0.0259	0.0258	0.0258	0.0327	0.0266	0.0233	0.0247

Analytic continuation



- 📌 Baryon number density $\hat{n}_B$ and $\partial \hat{n}_B / \partial \hat{\mu}_B$ shall be positive, which restricts $\mu_B^R \lesssim 800 \text{ MeV}$
- 📌 Results from ansatz I (non-interacting) and III (critical) are very much similar
- 📌 Results from ansatz II (interacting) differ from others at $\mu_B^R \gtrsim 500 \text{ MeV}$

Manifestation in the high order susceptibilities



Although about 7% bootstrap samples allowed to have $\mu_{B,c} \in (420, 750)$ MeV
 No significant sign change, deviation from HRG etc are observed

No signal in Exp

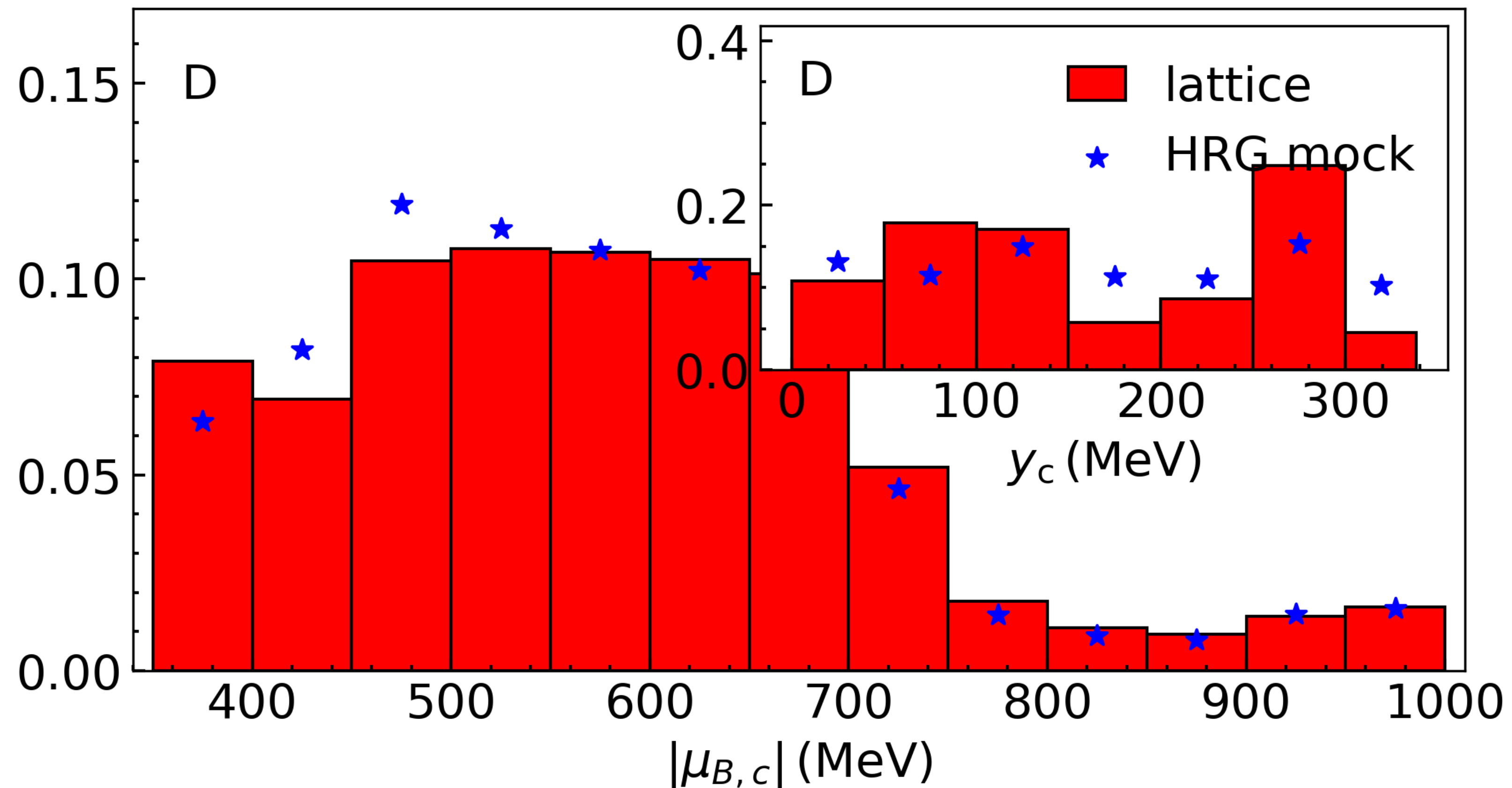
Criticality in the complex μ_B plane

$$f(\hat{\mu}_B) = A(T) \cosh(\hat{\mu}_B) + C (\hat{\mu}_B - \hat{\mu}_{B,c})^{\alpha_{LY}} + C (-\hat{\mu}_B - \hat{\mu}_{B,c})^{\alpha_{LY}} \\ + C^* (\hat{\mu}_B - \hat{\mu}_{B,c}^*)^{\alpha_{LY}} + C^* (-\hat{\mu}_B - \hat{\mu}_{B,c}^*)^{\alpha_{LY}}$$

$$C, \hat{\mu}_{B,c} \in \mathbb{C}$$

$$\alpha_{LY} = 1.085$$

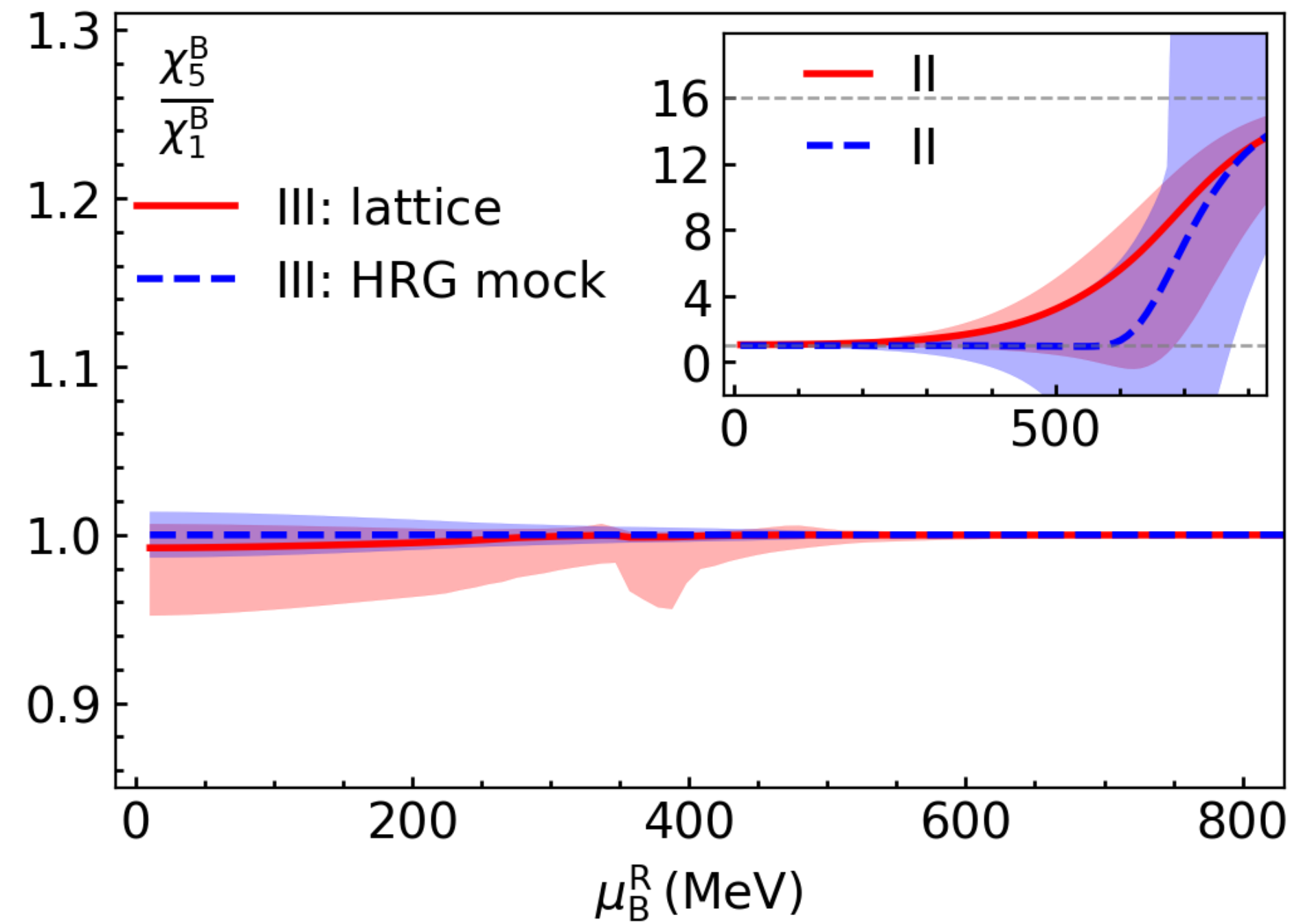
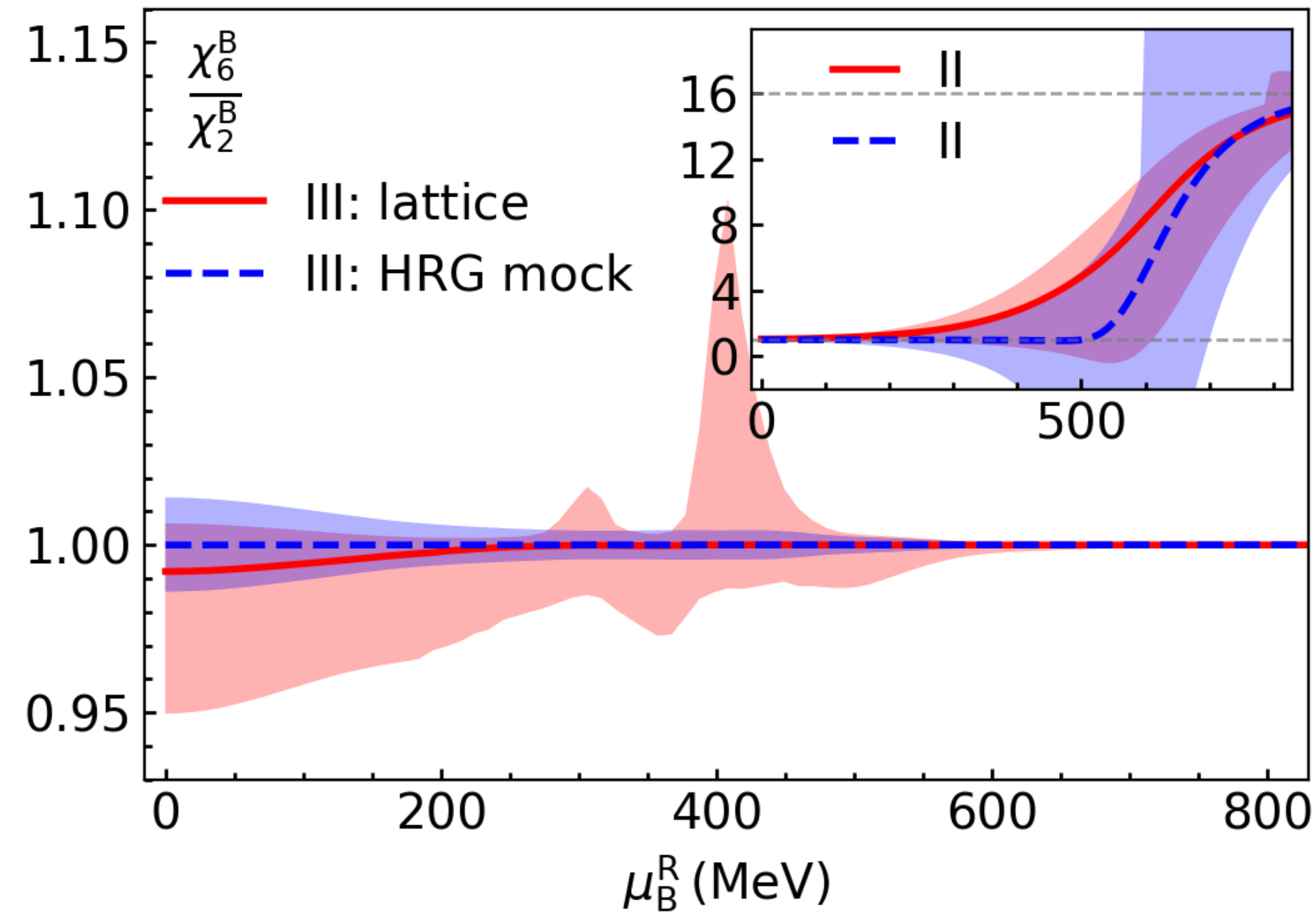
Gliozzi and Rago,
JHEP 10 (2014) 042



Kai-Fan Ye, work in progress

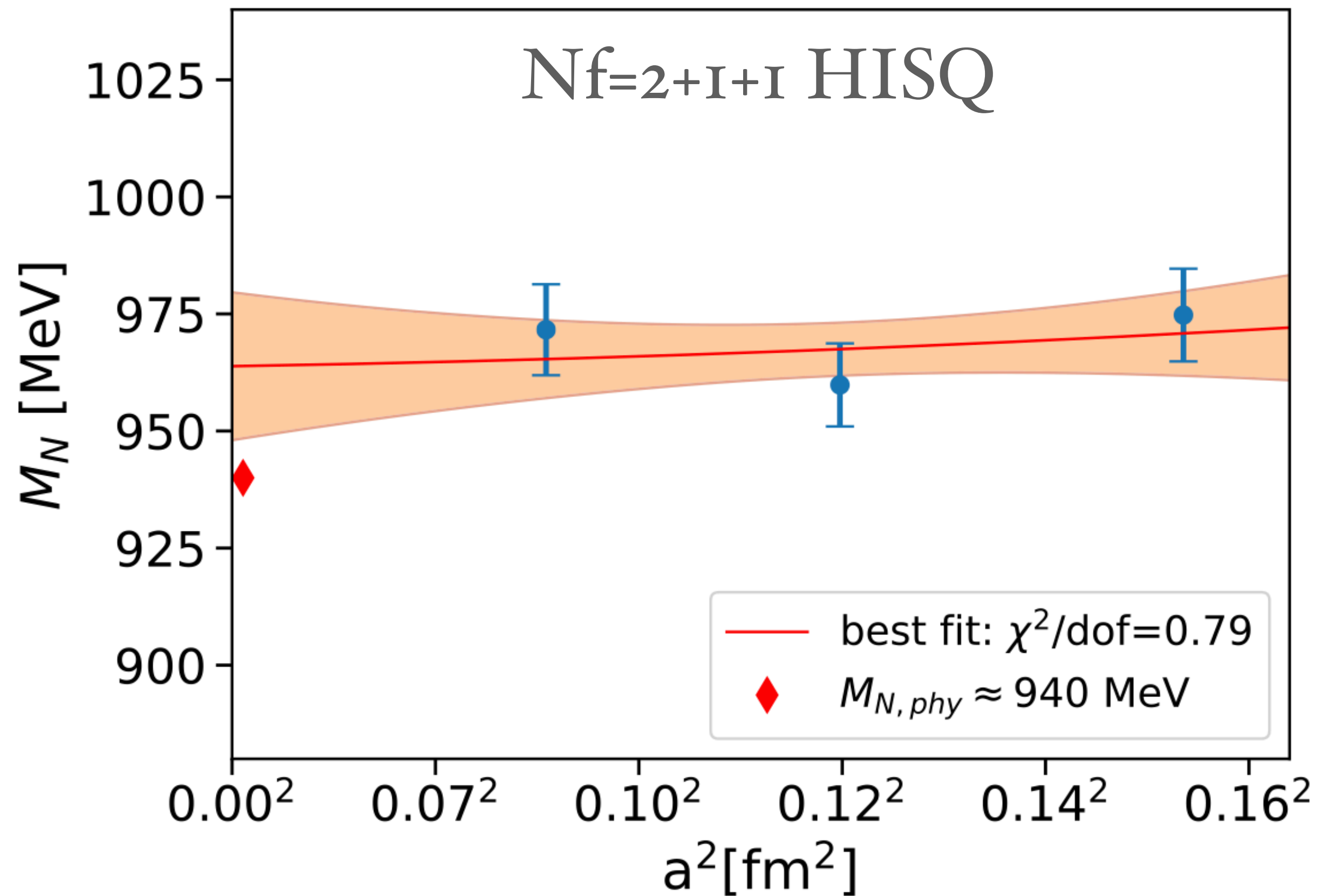
Ratios of susceptibilities at real μ_B

$$f(\hat{\mu}_B) = A(T) \cosh(\hat{\mu}_B) + C (\hat{\mu}_B - \hat{\mu}_{B,c})^{\alpha_{LY}} + C (-\hat{\mu}_B - \hat{\mu}_{B,c})^{\alpha_{LY}} \\ + C^* (\hat{\mu}_B - \hat{\mu}_{B,c}^*)^{\alpha_{LY}} + C^* (-\hat{\mu}_B - \hat{\mu}_{B,c}^*)^{\alpha_{LY}}$$



Kai-Fan Ye, work in progress

Lattice cutoff effects



- $N_t=10$ at $T=108$ MeV $\Rightarrow a \approx 0.18$ fm

- proton mass is inflated by about 4-5%, ~ 980 MeV at $a \approx 0.18$ fm, as estimated from the difference in A fit parameter

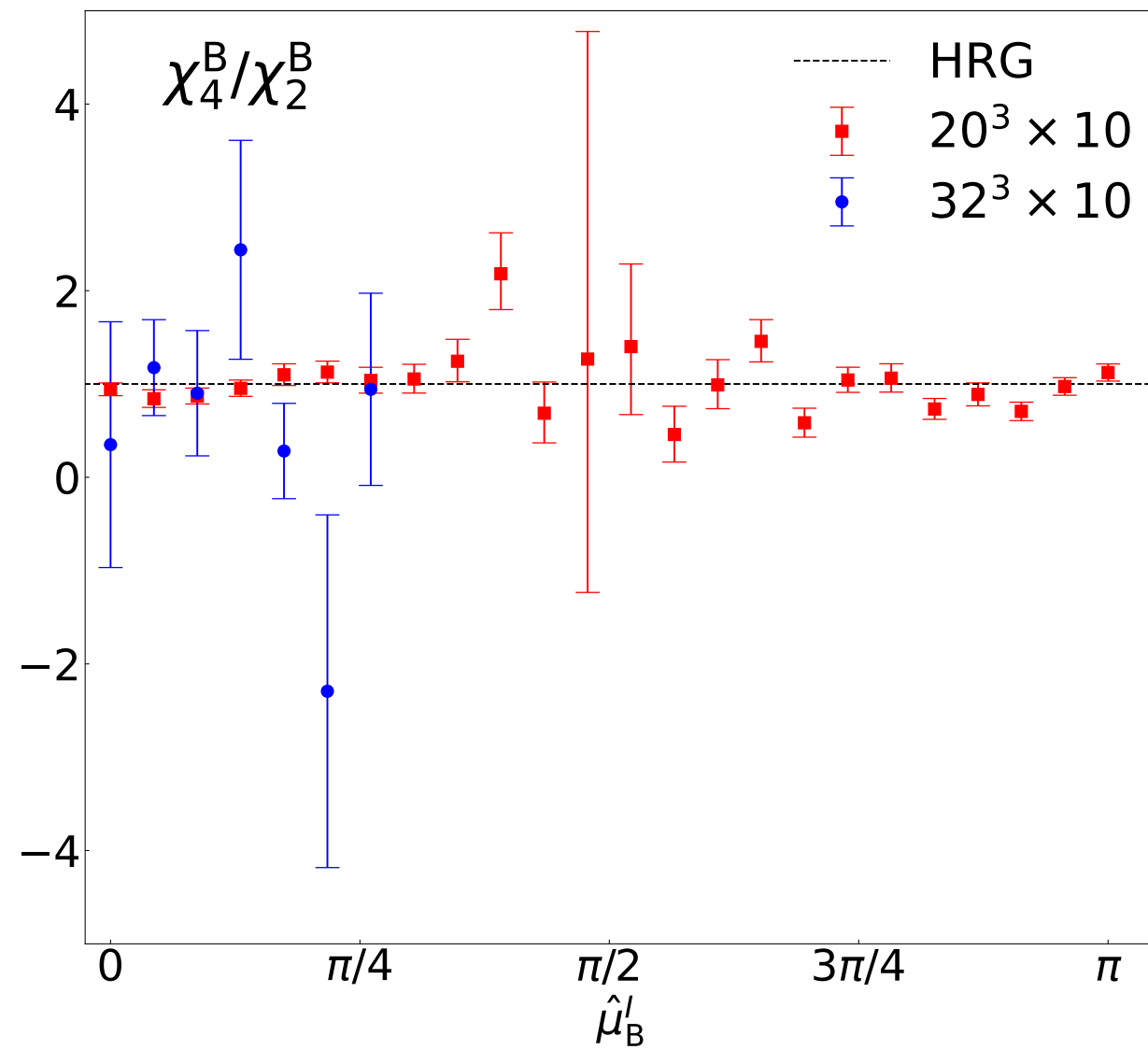
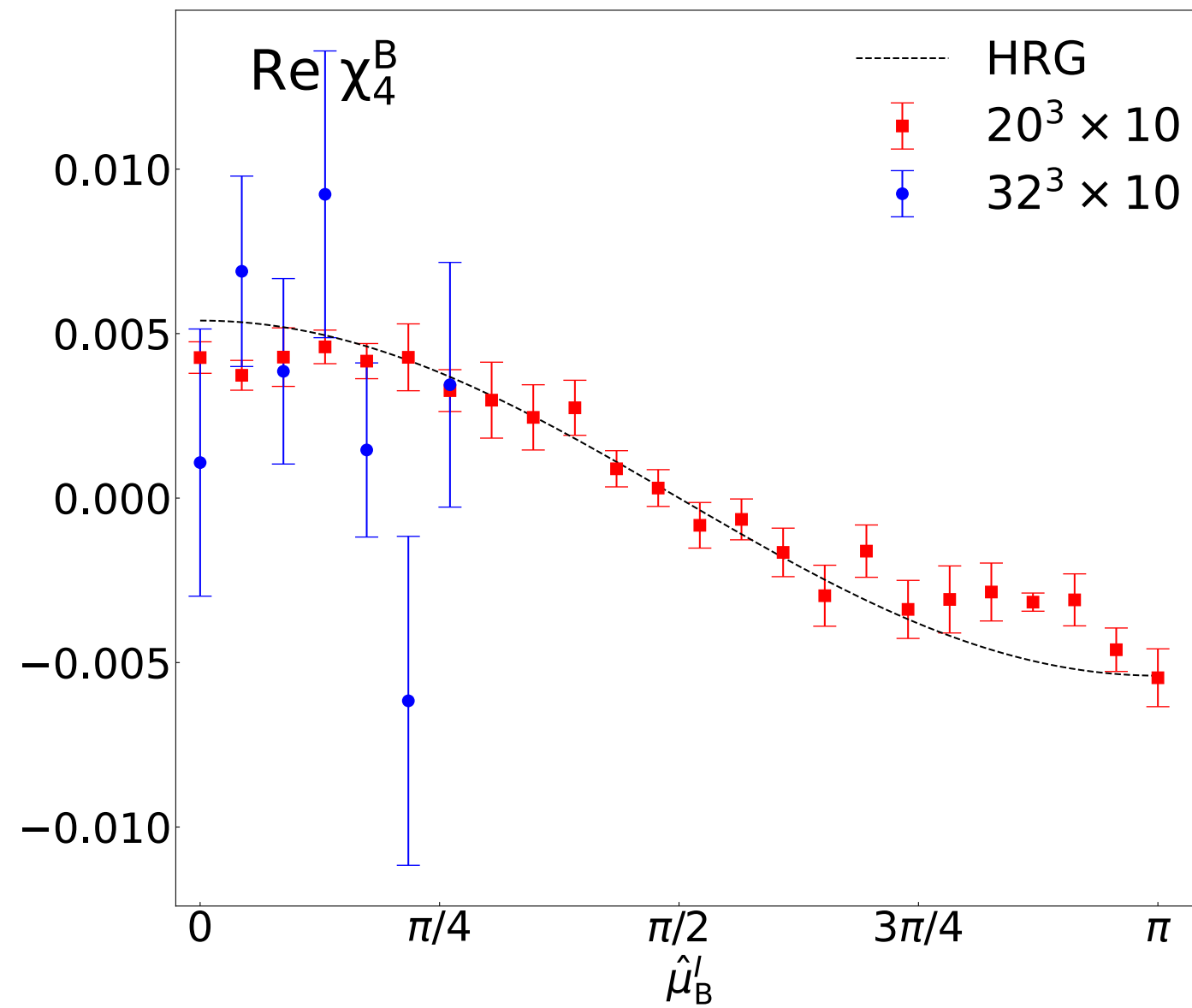
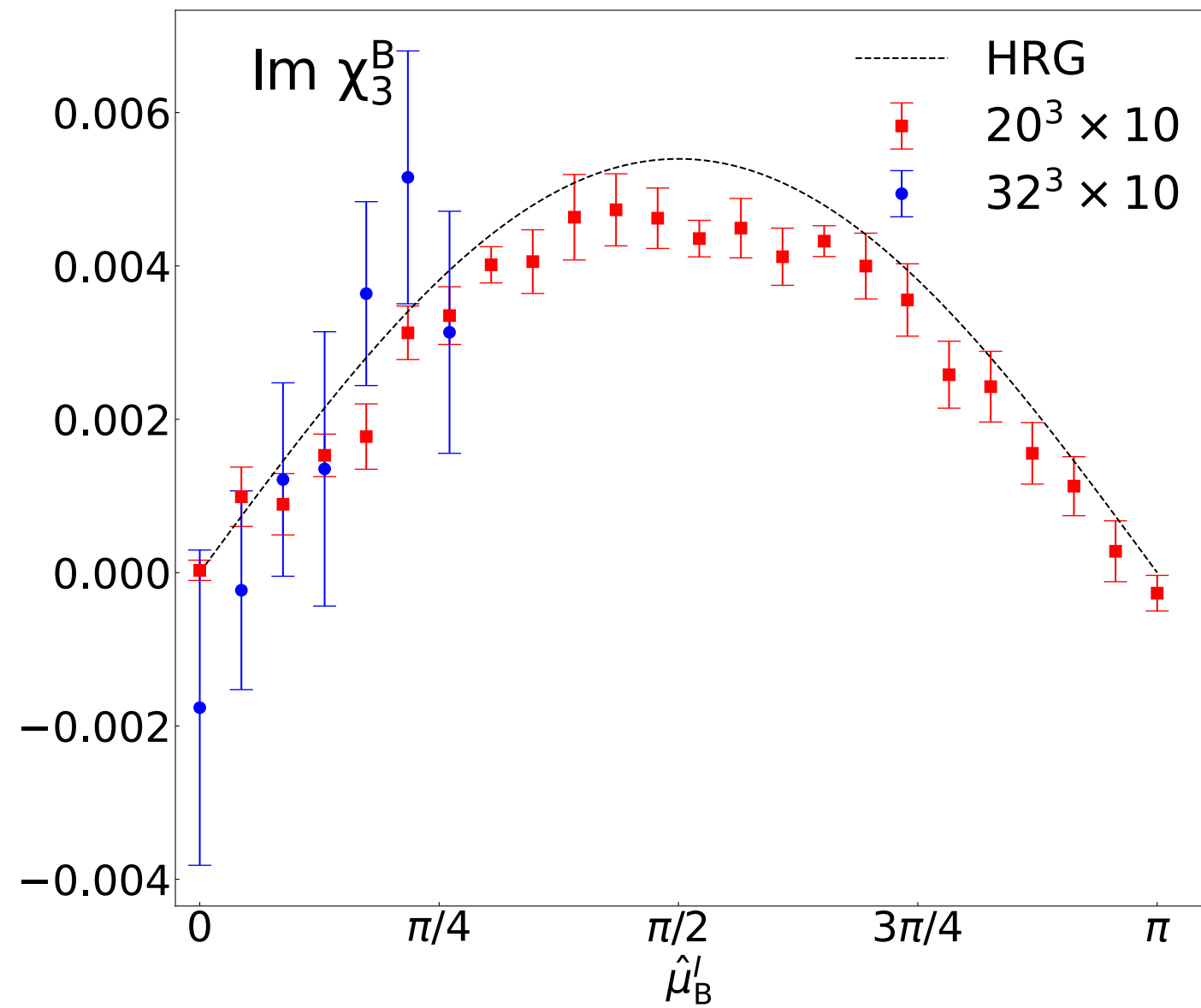
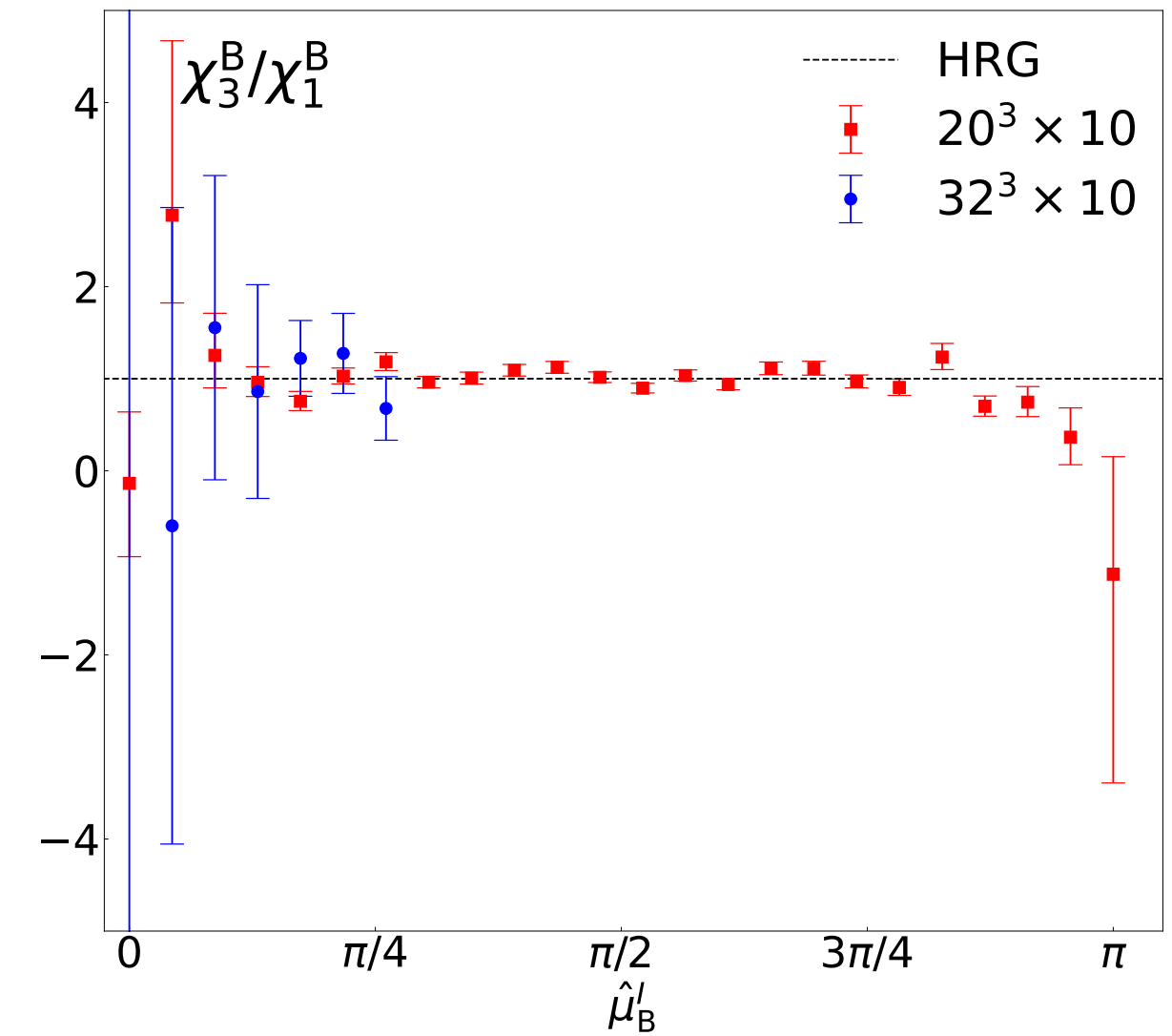
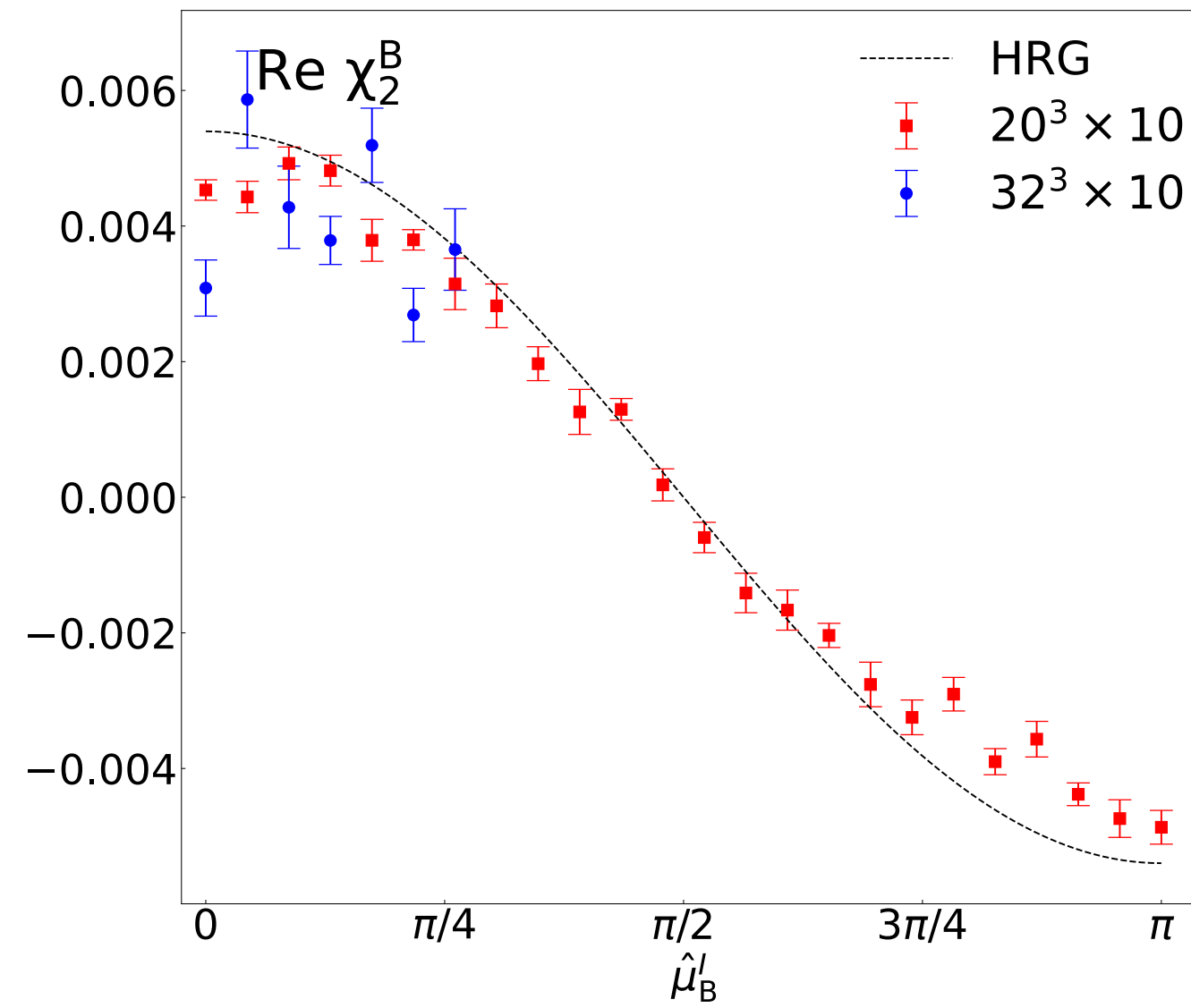
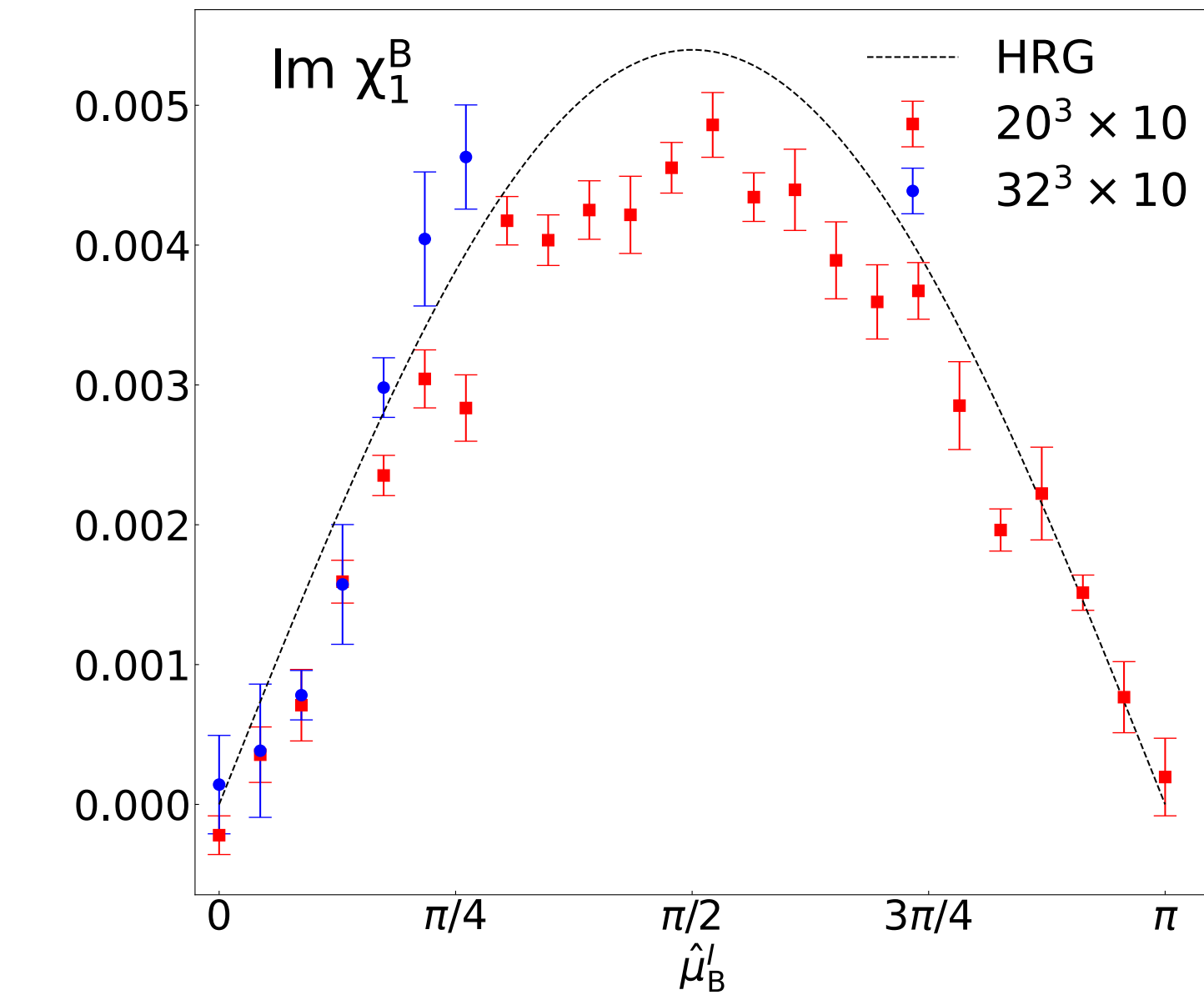
$$A_{\text{LAT}} = 4.584(35)$$

$$A_{\text{HRG}} = 5.397(42)$$

- Consistent with direct computation in Nf=2+1+1 QCD

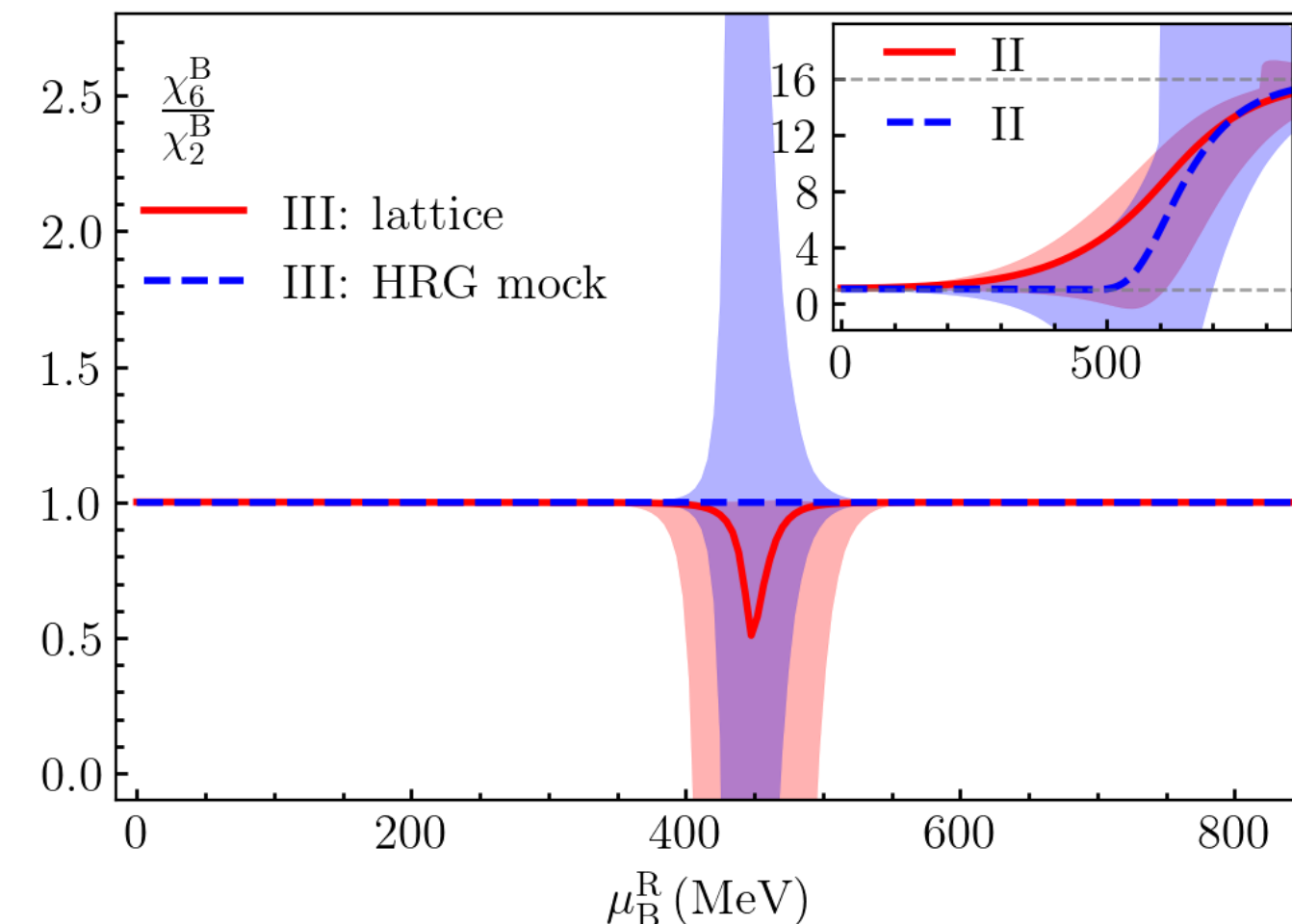
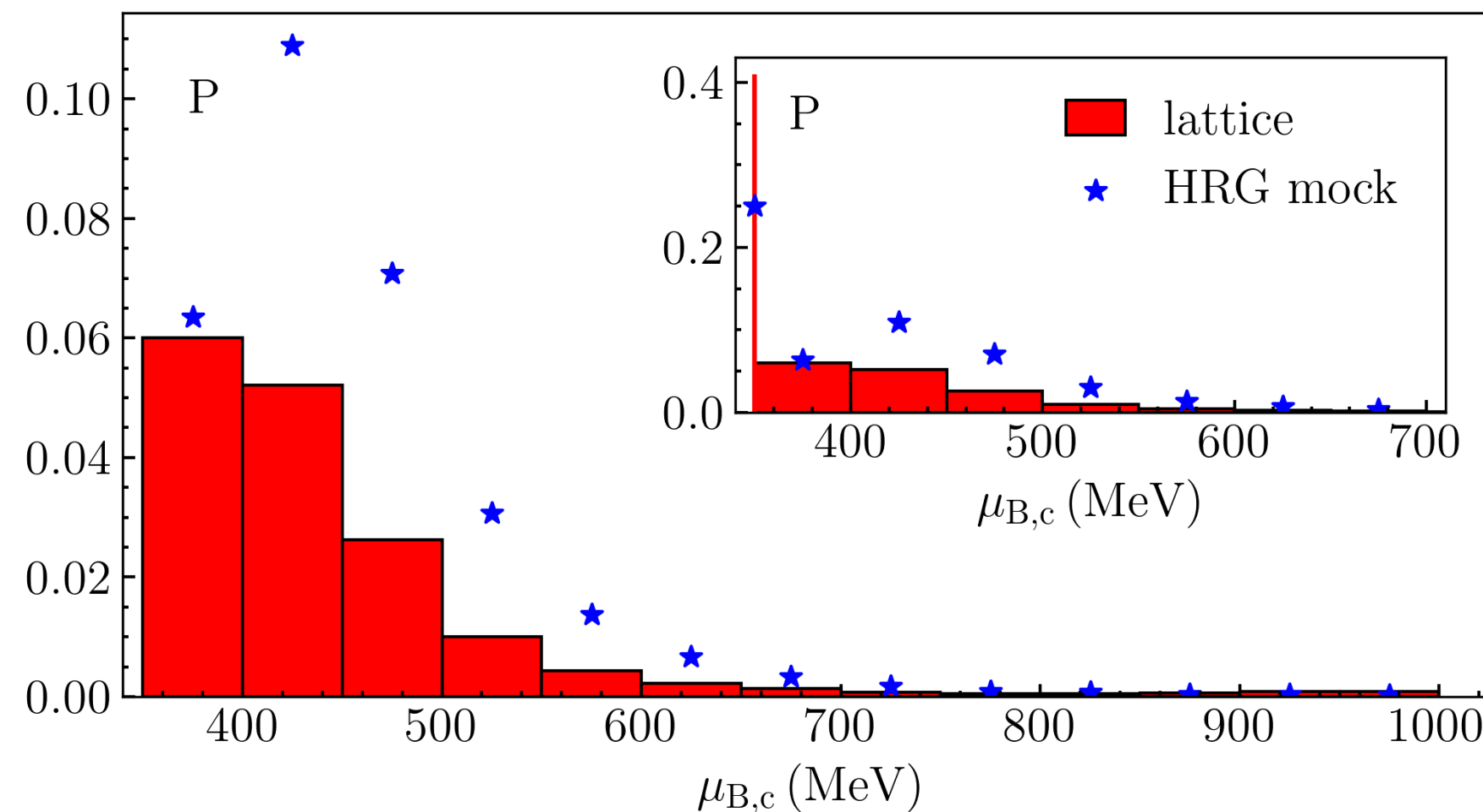
Yin Lin et al., *Phys.Rev.D* 103 (2021) 3, 034501

Volume dependences



Summary

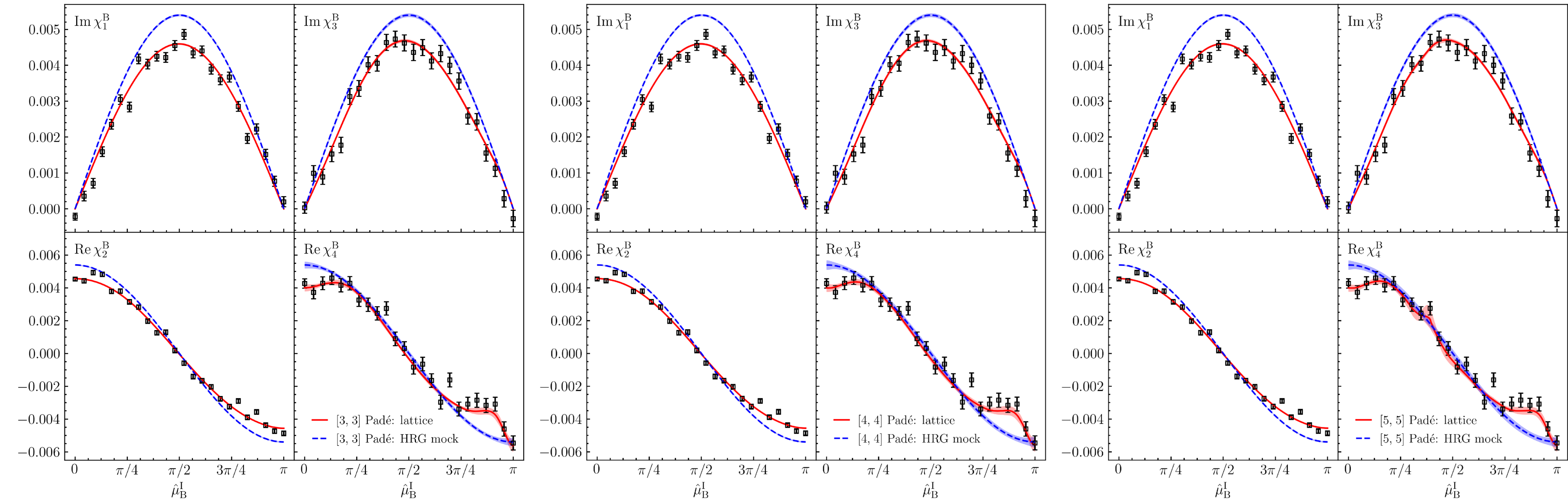
- Recent predictions on $T_c^{\text{CEP}} \approx 83 - 120 \text{ MeV} < T_c^0$, extrapolated based on the lattice simulations at $T \gtrsim 120 \text{ MeV}$
- Our LQCD simulations at $\text{Im}(\mu_B)$ @ $T=108 \text{ MeV}$: data compatible with smooth, analytic thermodynamics, obtained using fits with direct Padé approximants, physical motivated non-interacting, interacting and critical ansatz
- Only $\sim 7\%$ bootstrap samples gives for a CEP in $\mu_{B,c}^{\text{CEP}} \in 420\text{-}750\text{ MeV}$ at $T=108 \text{ MeV}$, however, it does manifest in the highest baryon susceptibilities that EXP can measure with reasonable precisions



Backup

Multi-point Padé fits

$$P(x) \equiv R_n^m(x) = \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^n b_j x^j}, x = \cosh(\mu_B) - 1$$



[3,3]

[4,4]

[5,5]