

Phase diagram at intermediate densities: constraints from lattice QCD

Szabolcs Borsányi

Wuppertal-Budapest collaboration.

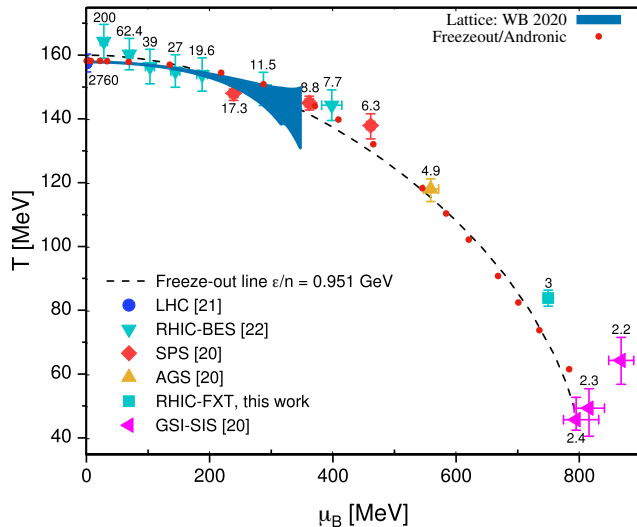
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The QCD Phase Diagram: From Theory to Experimental Signatures
Dalian, 8-11 Oct 2025



QCD phase diagram



What can Lattice QCD add to this phase diagram?

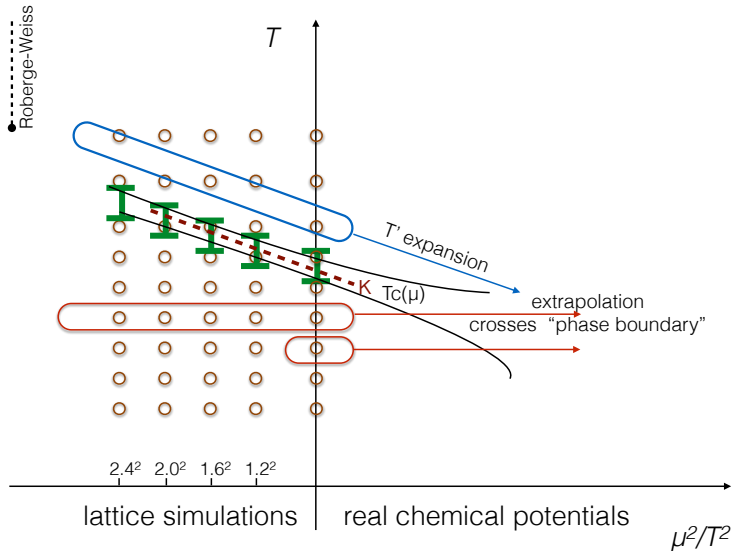
Phase diagram compiled by

[Vovchenko et al 2408.06473]

Lattice result on the chiral transition line:

[Wuppertal-Budapest PRL 125 (2020) 052001]

Extrapolation schemes on the phase diagram



- *Taylor coefficients* of the pressure

$$\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \frac{1}{2!} \frac{\mu_B^2}{T^2} \chi_2^B(T) + \frac{1}{4!} \frac{\mu_B^4}{T^4} \chi_4^B(T) + \frac{1}{6!} \frac{\mu_B^6}{T^6} \chi_6^B(T) + \dots$$

- These Taylor coefficients are equal to the Grand Canonical fluctuations

$$\chi_2^B(T) = \langle B^2 \rangle - \langle B \rangle^2 = \frac{1}{VT} \frac{\partial^2 \log Z(V, T, \mu_B, \mu_Q, \mu_S)}{\partial \mu_B^2}$$

- Higher fluctuations are the Taylor coefficients of lower fluctuations

$$\chi_2^B(\mu_B) = \chi_2^B(\mu_B = 0) + \frac{1}{2!} \frac{\mu_B^2}{T^2} \chi_4^B(\mu_B = 0) + \frac{1}{4!} \frac{\mu_B^4}{T^4} \chi_6^B(\mu_B = 0) + \dots$$

- Taylor coefficients can be used to reveal analytic structure of the thermodynamic potential
 - Repulsive interactions beyond ideal HRG
 - Searching the critical end point
- Hints for chiral O(4) universality

How to calculate the χ coefficients?

$$\chi_{200}^{uds} = +\langle B_u \rangle + \langle A_u A_u \rangle$$

$$\begin{aligned}\chi_{400}^{uds} = & +\langle D_u \rangle + 3\langle B_u B_u \rangle - 3\langle B_u \rangle \langle B_u \rangle + 4\langle A_u C_u \rangle \\ & +\langle A_u A_u A_u A_u \rangle - 3\langle A_u A_u \rangle \langle A_u A_u \rangle + 6\langle A_u A_u B_u \rangle - 6\langle B_u \rangle \langle A_u A_u \rangle\end{aligned}$$

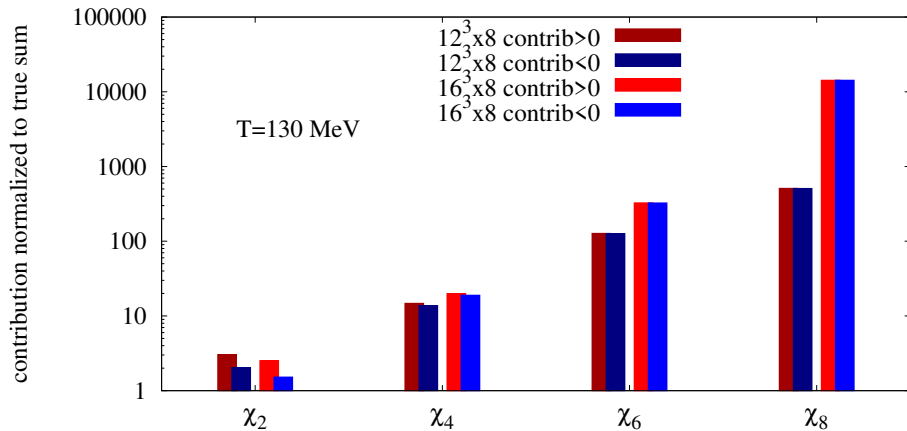
$$\begin{aligned}\chi_{600}^{uds} = & +\langle F_u \rangle + 10\langle C_u C_u \rangle + 15\langle B_u D_u \rangle + 15\langle B_u B_u B_u \rangle + 6\langle A_u E_u \rangle + 60\langle A_u B_u C_u \rangle + 15\langle A_u A_u D_u \rangle \\ & +45\langle A_u A_u B_u B_u \rangle + 20\langle A_u A_u A_u C_u \rangle + 15\langle A_u A_u A_u A_u B_u \rangle + \langle A_u A_u A_u A_u A_u A_u \rangle - 15\langle D_u \rangle \langle B_u \rangle \\ & -15\langle D_u \rangle \langle A_u A_u \rangle - 45\langle B_u \rangle \langle B_u B_u \rangle - 60\langle B_u \rangle \langle A_u C_u \rangle - 90\langle B_u \rangle \langle A_u A_u B_u \rangle \\ & -15\langle B_u \rangle \langle A_u A_u A_u A_u \rangle - 45\langle B_u B_u \rangle \langle A_u A_u \rangle - 60\langle A_u C_u \rangle \langle A_u A_u \rangle - 90\langle A_u A_u \rangle \langle A_u A_u B_u \rangle \\ & -15\langle A_u A_u \rangle \langle A_u A_u A_u A_u \rangle + 30\langle B_u \rangle \langle B_u \rangle \langle B_u \rangle + 90\langle B_u \rangle \langle B_u \rangle \langle A_u A_u \rangle \\ & +90\langle B_u \rangle \langle A_u A_u \rangle \langle A_u A_u \rangle + 30\langle A_u A_u \rangle \langle A_u A_u \rangle \langle A_u A_u \rangle\end{aligned}$$

$$\chi_{800}^{uds} = 79 \text{ terms } \dots$$

$A, B, C, \dots$ are defined as $[\det M(\mu_u)]^{1/4} = [\det M(0)]^{1/4} \exp \left(1 + A_u \mu_u + \frac{B_u}{2!} \mu_u^2 + \frac{C_u}{3!} \mu_u^3 + \dots \right)$

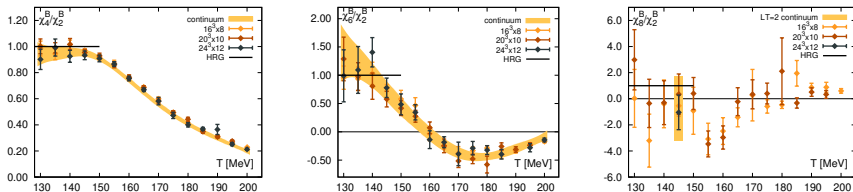
Data analysis uses computer generated code.

Sign problem in the Taylor coefficients

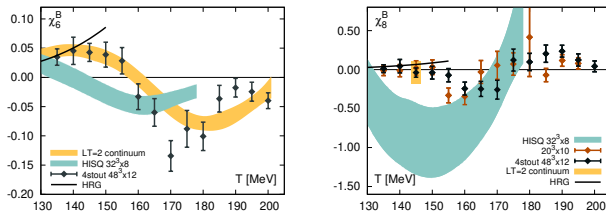


High order coefficients in a $LT = 2$ box

4Hex continuum result [Phys.Rev.D 110 (2024) 1, L011501]



Comparison with literature



4stout data (imaginary μ_B , $48^3 \times 12$ lattice) : [Wuppertal-Budapest, 1805.04445]

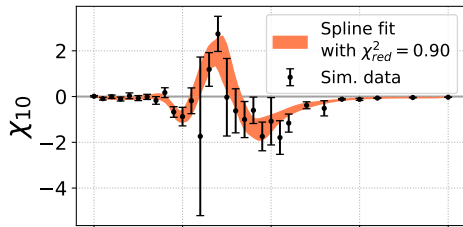
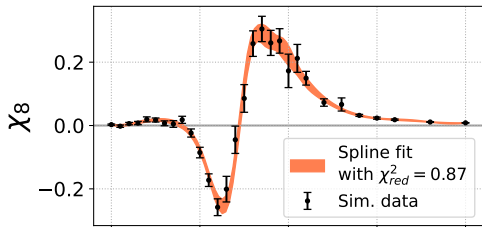
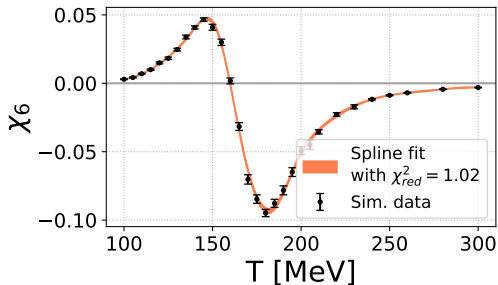
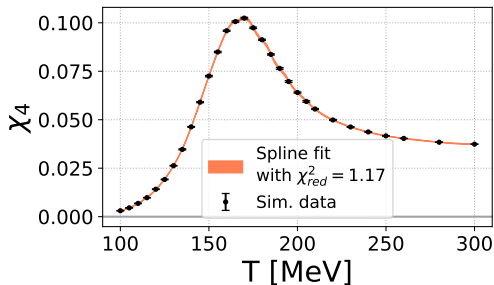
HISQ data: ($\mu_B = 0$, $32^3 \times 8$ lattice, inexact charge conservation) [BNL-Bielefeld, 2202.09184,2212.09043]

$16^3 \times 8$ lattice, 4HEX action, full μ -dependence is recorded.

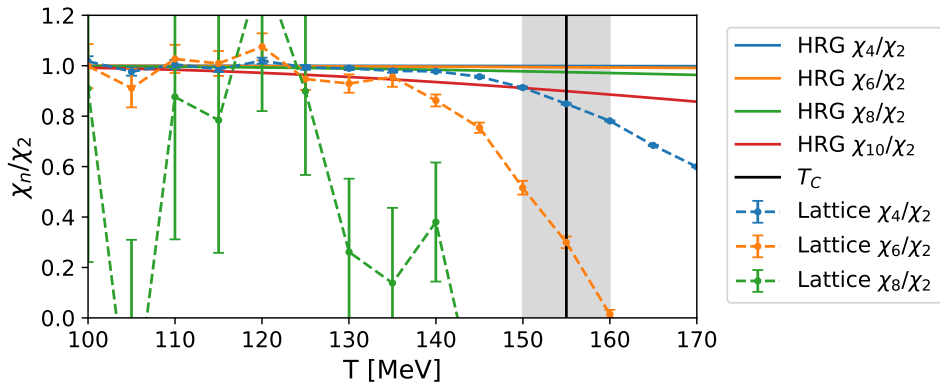
T [MeV]	# configurations at $\mu_B = i\pi(j/8)$								
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
100	0.9M	-	-	-	-	-	-	-	-
105	0.9M	-	-	-	-	-	-	-	-
110	0.8M	-	-	-	-	-	-	-	-
115	1.1M	-	-	-	-	-	-	-	-
120	1.1M	-	-	-	-	-	-	-	-
125	1.7M	-	-	-	-	-	-	-	-
130	2.1M	-	96k	-	96k	-	96k	-	96k
135	1.5M	111k	105k	104k	120k	103k	104k	103k	103k
140	4.0M	100k	105k	105k	359k	104k	99k	103k	114k
145	2.6M	107k	106k	105k	122k	104k	103k	102k	115k
150	1.7M	114k	114k	112k	117k	110k	98k	96k	109k
155	1.6M	109k	109k	107k	126k	104k	103k	101k	116k
160	2.2M	112k	111k	109k	128k	104k	96k	101k	109k
165	1.8M	104k	102k	90k	121k	96k	93k	87k	105k
170	1.2M				-				

Status of $16^3 \times 8$ simulations

Baryon cumulant ratios



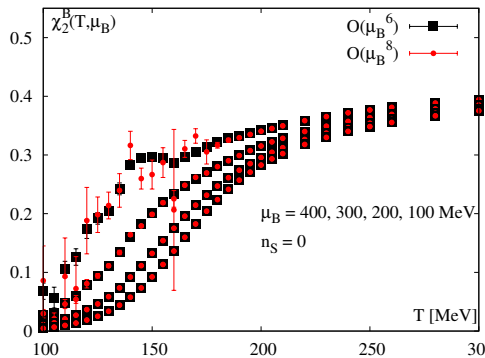
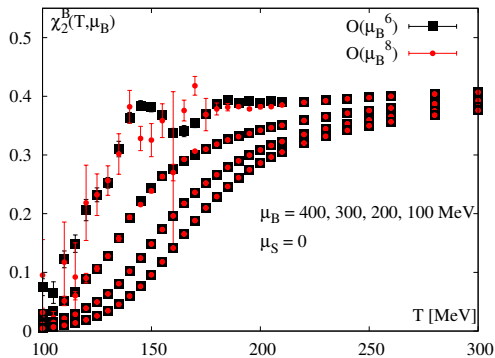
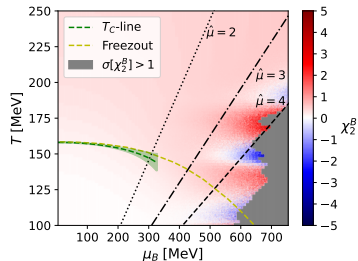
Do we see any differences to the HRG model?



We see a clear signal from baryon interactions for $T \leq 130$ MeV.

Baryon 2nd cumulant $\chi_2^B(\mu_B)$ to $\mathcal{O}(\mu_B^8)$ order:

[Wuppertal-Budapest 2507.13254]



Baryon fluctuations in two scenarios

Stephanov [0809.3450,1104.1627] :

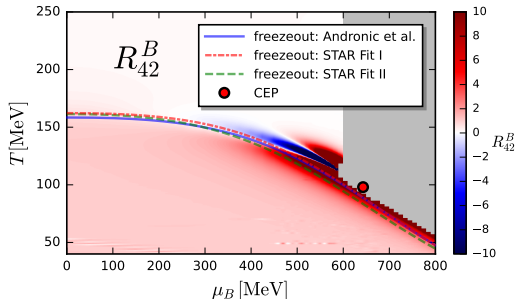
Non-monotonic high order fluctuations near a critical end point.

Detailed predictions for χ_4/χ_2 are given in various setups.

FRG-Lattice assisted LEFT

Fu et al. [2308.15508]

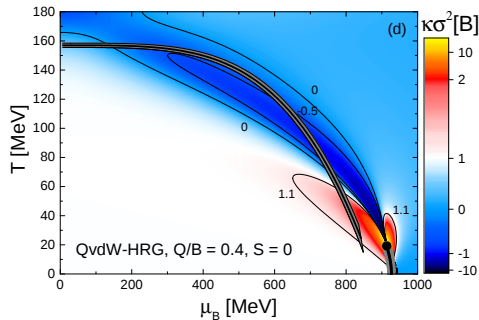
CEP = (98 MeV, 643 MeV)



QVdW gas

Vovchenko et al [1906.01954]

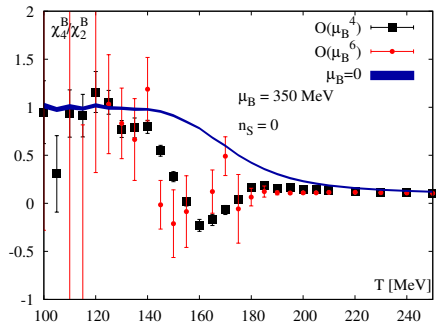
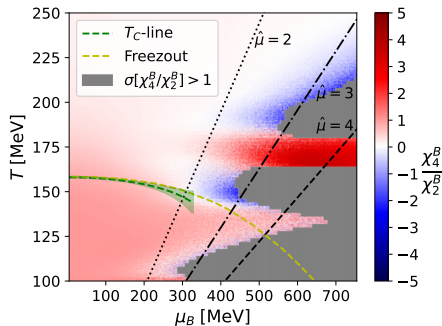
Liquid-Gas CEP = (19.7 MeV, 922 MeV)



Searching for criticality in baryon fluctuations

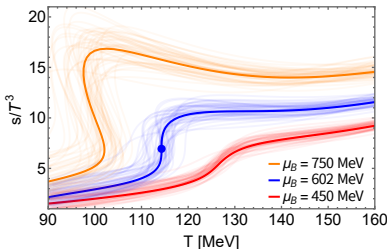
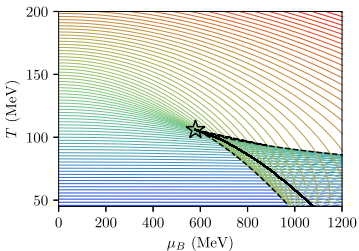
Using the baryon Taylor coefficients we express fluctuation ratios, e.g.

$$\frac{\chi_4^B(\mu_B)}{\chi_2^B(\mu_B)} = \frac{\chi_4^B + \frac{\hat{\mu}_B^2}{2!} \chi_6^B + \frac{\hat{\mu}_B^4}{4!} \chi_8^B + \frac{\hat{\mu}_B^6}{6!} \chi_{10}^B}{\chi_2^B + \frac{\hat{\mu}_B^2}{2!} \chi_4^B + \frac{\hat{\mu}_B^4}{4!} \chi_6^B + \frac{\hat{\mu}_B^6}{6!} \chi_8^B + \frac{\hat{\mu}_B^8}{8!} \chi_{10}^B}$$



The Houston critical endpoint

Idea: look at contours of constant entropy/ T^3 , find spinodal regions.



[Black-Hole-Engineering model Hippert et al[2309.00579]]

[QCD: Shah et al [2410.16206]]

Leading order: Entropy contours are exact parabolas: $T' = A + B\mu_B^2$

Optimistic assumption on error propagation.

Critical endpoint estimate: $T_c = 114.3 \pm 6.9$ MeV, $\mu_{B,c} = 602.1 \pm 62.1$ MeV

Is this a first principles result on a CEP?

No, an expansion is defined, where

each order can be computed from first principles,

this expansion does not automatically break down near the CEP.

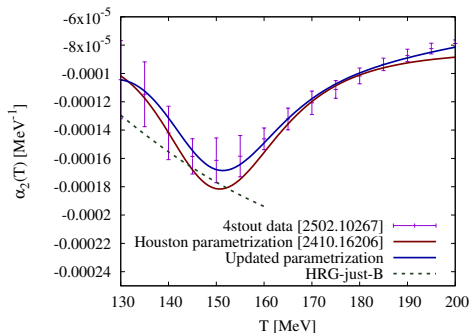
New implicit T' -expansion for the entropy [Houston]

The Houston group introduced a systematic scheme to expand the entropy [2410.16206]

$$s[T'(\mu_B; T), \mu_B] = s[T, \mu_B = 0]$$

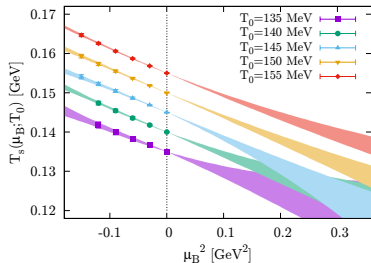
$$T'(\mu_B; T) = T + \alpha_2(T) \frac{1}{2} \mu_B^2 + \dots$$

$$\mu_{B,c} = \sqrt{-2/\alpha_2'(T_*)}, \quad \text{where } \alpha_2''(T_*) = 0$$

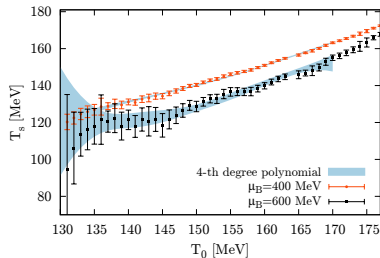


Steps that lead to an exclusion range

1. Calculate entropy at imaginary μ_B
2. Extrapolate constant s contours to $\mu_B > 0$

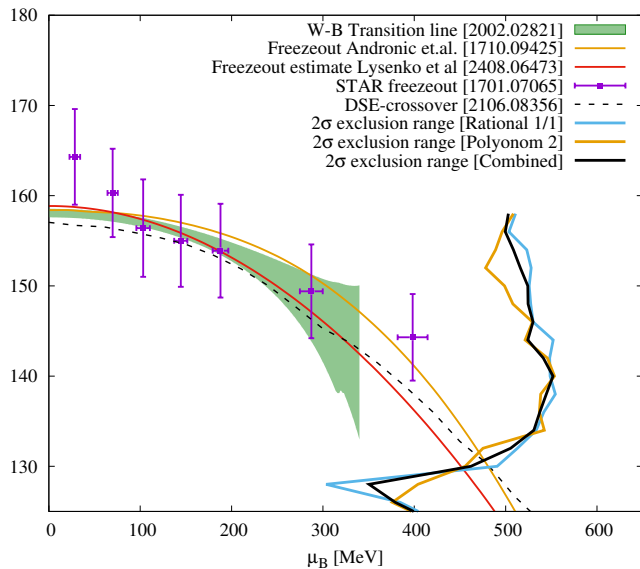


3. Pick a μ_B : $s(T_s, \mu_B) = s(T_0, 0)$
4. $\mu_B \geq \mu_{B,c}$ means multivalued s , that is $T_s(T_1) = T_s(T_2)$.



*For how big μ_B can we show that $T_s(T_0)$ is strictly monotonic?
That will be the exclusion range.*

Result for an exclusion range



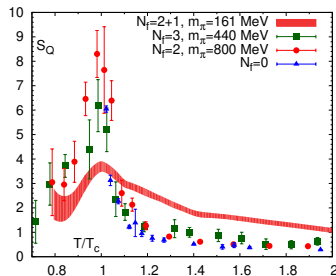
T_c from the Polyakov loop

$$P = \frac{1}{V} \left\langle \text{Tr} \prod_i U_i \right\rangle$$

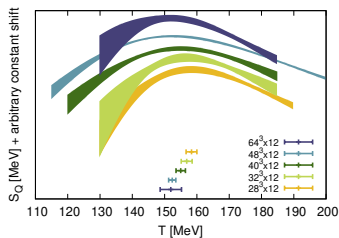
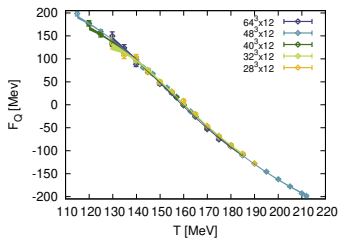
$$F_Q = -T \log P$$

$$S_Q = - \frac{\partial F_Q(T, \mu_B)}{\partial T}$$

T_c : *peak of S_Q in T*



[Left: TUM QCD, Bazavov et al 1603.06637]



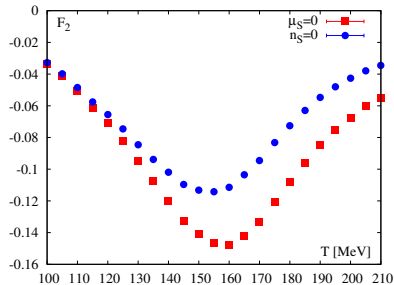
[Right: Wuppertal-Budapest 2405.12320]

The Taylor coefficients for $F_Q(T)$

$$F = -\frac{T}{2} \log |\langle P \rangle|^2 = T \sum_{n=0,2,\dots} \frac{F_n}{n!} \left(\frac{\mu_B}{T} \right)^n$$

$$F_0 = -\frac{1}{2} \log(|P|^2)$$

$$F_2 = -\frac{1}{2|P|^2} \frac{\partial^2}{(\partial \mu_B/T)^2} |P|^2$$



[This result: Wuppertal-Budapest 2410.06216, first computed: D'Elia et al 1907.09461]

Expansion of the Polyakov loop in imaginary μ

$$P = P_R + P_I$$

Magenta color: real and C-even.

Blue color: real and C-odd. Imagine ∂_j as imaginary- μ derivative!

A straightforward application of these rules yields up to second order

$$\begin{aligned}\partial_j \langle P_R \rangle|_{\mu \equiv 0} &= 0, \\ \partial_j \langle P_I \rangle|_{\mu \equiv 0} &= \langle A_j P_I \rangle, \\ \partial_j \partial_k \langle P_R \rangle|_{\mu \equiv 0} &= \delta_{jk} [\langle B_j P_R \rangle - \langle B_j \rangle \langle P_R \rangle] \\ &\quad \langle A_j A_k P_R \rangle - \langle A_j A_k \rangle \langle P_R \rangle, \\ \partial_j \partial_k \langle P_I \rangle|_{\mu \equiv 0} &= 0.\end{aligned}$$

These can be combined into the following relation (*all terms are real, even*):

$$\begin{aligned}\frac{1}{2} \partial_j \partial_k |\langle P \rangle|^2 &= +\delta_{jk} \left[\langle P_R \rangle \langle B_j P_R \rangle - \langle P_R \rangle^2 \langle B_j \rangle \right] \\ &\quad + \langle P_R \rangle \langle A_j A_k P_R \rangle - \langle A_j P_I \rangle \langle A_j P_I \rangle \\ &\quad - \langle A_j A_k \rangle \langle P_R \rangle^2,\end{aligned}$$

Expansion of the Polyakov loop in imaginary μ

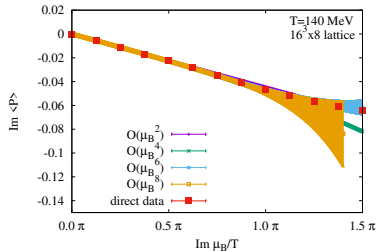
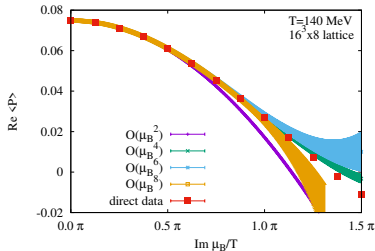
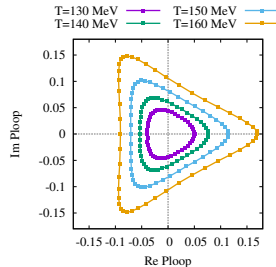
$$P = P_R + P_I$$

$$\partial_j \langle P_R \rangle|_{\mu \equiv 0} = 0,$$

$$\partial_j \langle P_I \rangle|_{\mu \equiv 0} = \langle A_j P_I \rangle,$$

$$\partial_j \partial_k \langle P_R \rangle|_{\mu \equiv 0} = \delta_{jk} [\langle B_j P_R \rangle - \langle B_j \rangle \langle P_R \rangle] \\ \langle A_j A_k P_R \rangle - \langle A_j A_k \rangle \langle P_R \rangle,$$

$$\partial_j \partial_k \langle P_I \rangle|_{\mu \equiv 0} = 0.$$



Varying the imaginary μ_B , P_R stays real, P_I stays imaginary.

Expansion of the Polyakov loop in real μ

$$P = P_R + P_I$$

Magenta color: C-even (real or imaginary).

Blue color: C-odd (real or imaginary)

$$\begin{aligned}\partial_j \langle P_R \rangle|_{\mu \equiv 0} &= 0, \\ \partial_j \langle P_I \rangle|_{\mu \equiv 0} &= \langle A_j P_I \rangle, \\ \partial_j \partial_k \langle P_R \rangle|_{\mu \equiv 0} &= \delta_{jk} [\langle B_j P_R \rangle - \langle B_j \rangle \langle P_R \rangle] \\ &\quad \langle A_j A_k P_R \rangle - \langle A_j A_k \rangle \langle P_R \rangle, \\ \partial_j \partial_k \langle P_I \rangle|_{\mu \equiv 0} &= 0.\end{aligned}$$

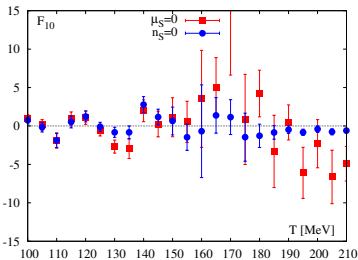
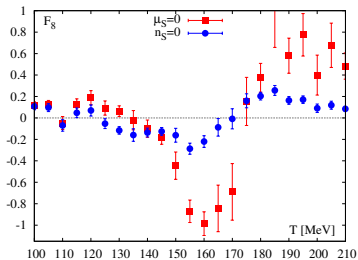
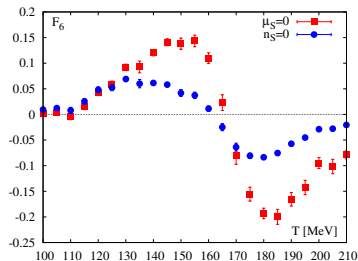
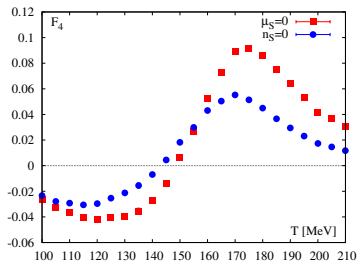
If ∂_j is a **real chemical potential** derivative:

$$\langle P_I(\mu_B) \rangle = \underbrace{\langle A_j P_I(\mu_B = 0) \rangle}_{\text{real}} \mu_B + \dots,$$

$\langle P_I \rangle$ was meant to be the imaginary part, but in fact,
it has a **real expectation value**, while being C-odd.

The Taylor coefficients for $F_Q(T)$

Higher orders:



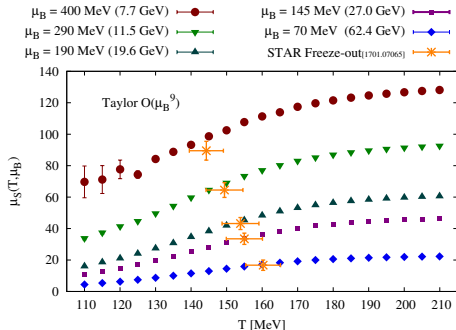
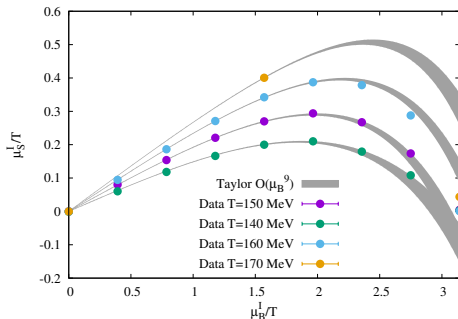
Strangeness neutrality in a crosscheck

Besides light baryons hyperons are also generated with $\mu_B > 0$: $\Rightarrow \langle S \rangle < 0$

In experiment (at chemical freeze-out) $\langle S \rangle = 0$.

We achieve this by adding $\mu_S > 0$, this is T and μ_B dependent.

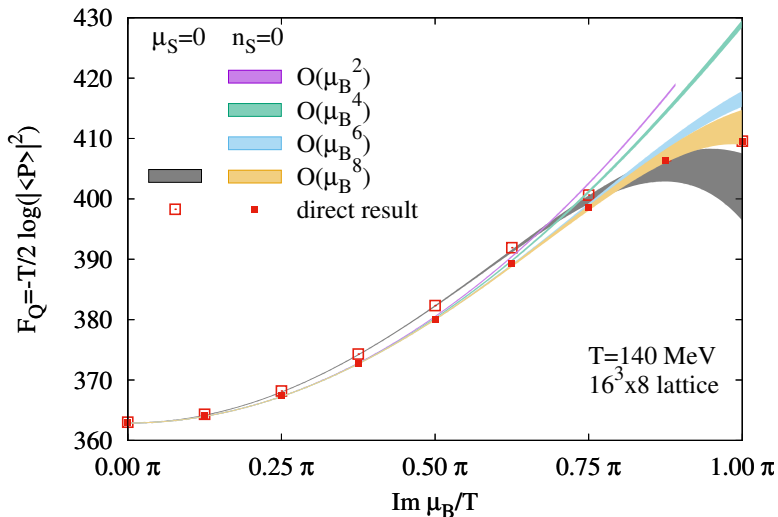
$$\mu_S^*(\mu_B, T) = s_1(T)\mu_B + s_3(T)\mu_B^3 + s_5(T)\mu_B^5 + s_7(T)\mu_B^7 + s_9(T)\mu_B^9 + \dots$$



Wuppertal-Budapest preliminary data, $16^3 \times 8$ lattice.

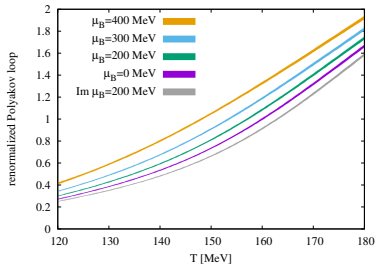
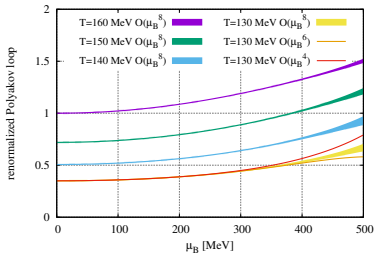
Testing the expansion at imaginary μ_B

Testing the strangeness neutral extrapolation:



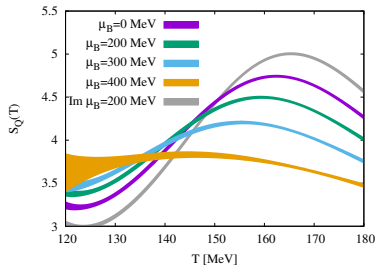
The extrapolated static quark entropy

The renormalized Polyakov loop:



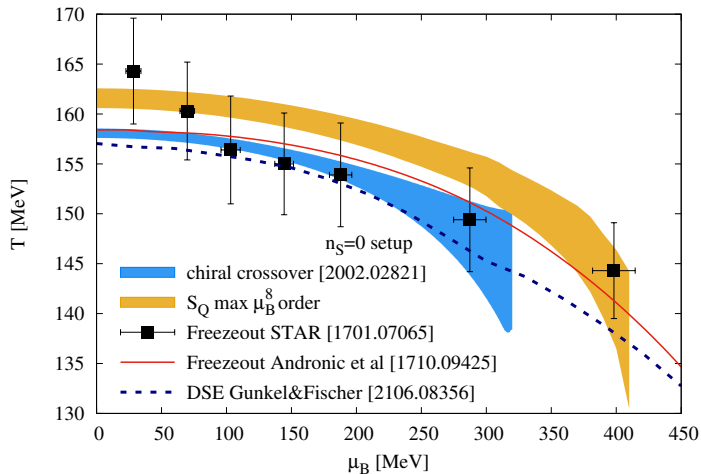
$$F_Q = -\frac{T}{2} \log(|P|^2)$$

$$S_Q = -\frac{\partial F_Q(T)}{\partial T}$$



S_Q itself can be extrapolated ($16^3 \times 8$ lattices).

The temperature of the S_Q -peak is calculated for each μ_B .

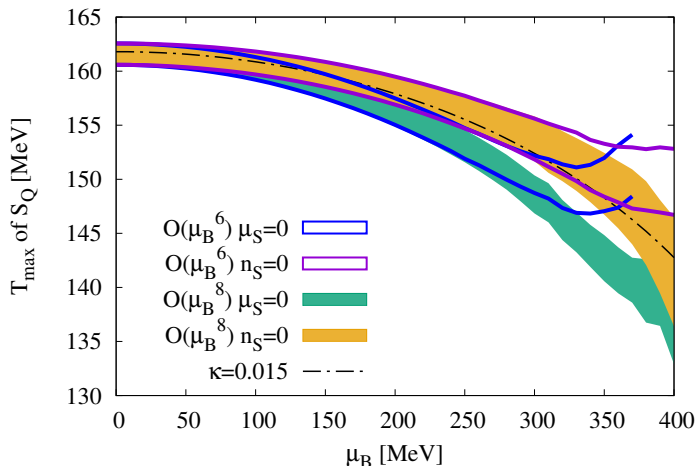


T_c extrapolated to eighth order

F_Q is extrapolated ($16^3 \times 8$ lattices).

Strangeness neutral ($n_S = 0$) and normal ($\mu_S = 0$) extrapolations differ.

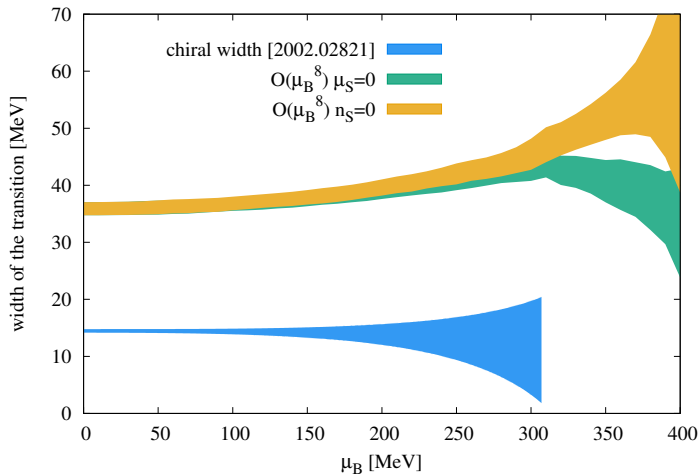
The temperature of the S_Q -peak is calculated for each μ_B for both cases.



Width of the transition extrapolated to eight order

S_Q itself can be extrapolated ($16^3 \times 8$ lattices).

The **width** of the S_Q -peak is calculated for each μ_B .



It is still very expensive to access high density physics from the lattice.

- Continuum extrapolated lattices in large volume:

- Updated equation of state
- Exclusion range for the critical point

[2502.10267]

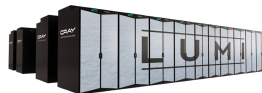
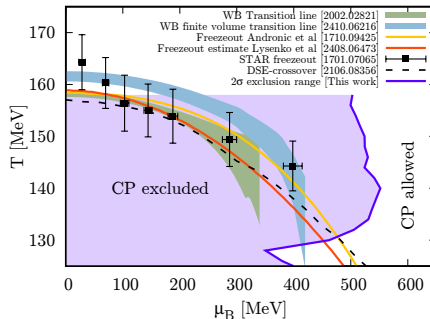
- Extreme statistics on $16^3 \times 8$ lattices

($\frac{1}{2}$ years in Jülich
+1 year on LUMI)

one can attempt to extrapolate to so far unattainable parts of the phase diagram.

[2410.06216, 2507.13254]

(today only on coarse lattices)

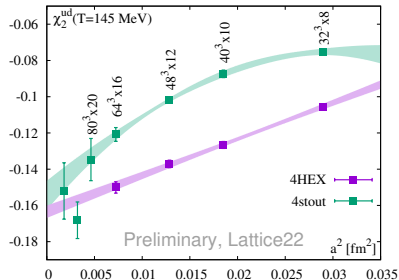
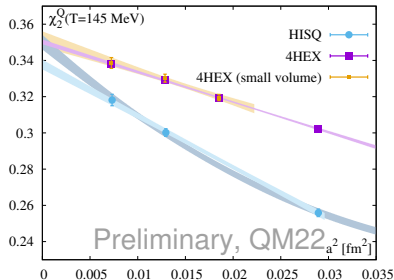


Testing continuum limits with the 4HEX action

4HEX staggered action with strongly reduced taste breaking.

Let's look at those fluctuations e.g. charge, that is sensitive to it.

Continuum extrapolation $T = 145$ MeV with large volume up to $N_t = 16$

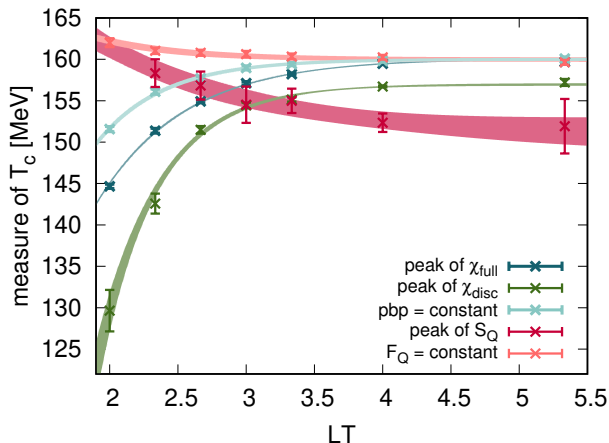


4STOUT: Wuppertal-Budapest (2013–2023) [Wuppertal-Budapest [1507.04627]]

HISQ: BNL-Bielefeld (2011–...) [HotQCD [2107.10011]]

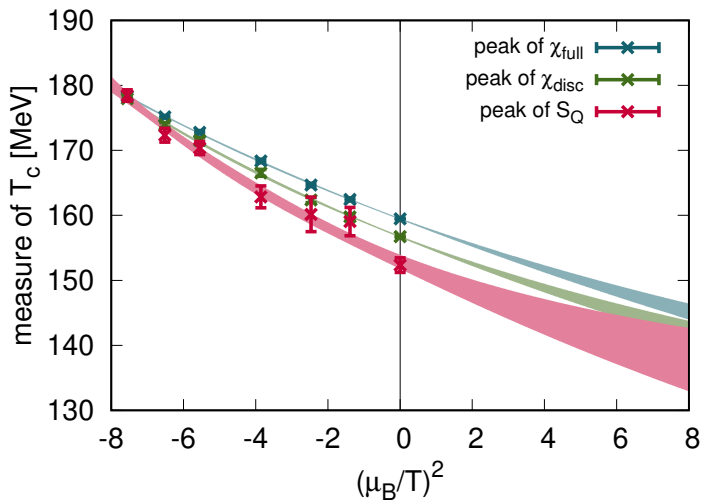
4HEX: Wuppertal-Budapest (2022–...) [Wuppertal-Budapest QM2022]

How is physics distorted in a smaller volume?



Chiral observables suffer from the reduced volume, but heavy quark observables (e.g. Polyakov loop) are much less affected. [Wuppertal-Budapest 2405.1232]

We repeat the T_c calculation at many imaginary chemical potentials.



This plot: $48^3 \times 12$ lattice, 4stout

Testing the expansion at imaginary μ_B

Two temperatures in the $\mu_S = 0$:

