Large density QCD phase structure from strangeness fluctuations on the lattice

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The QCD Phase Diagram: From Theory to Experimental Signatures

with
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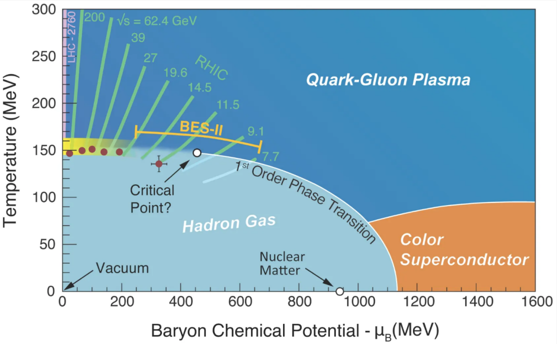
The phase diagram of QCD

What do we know about QCD thermodynamics at finite T, μ_B?

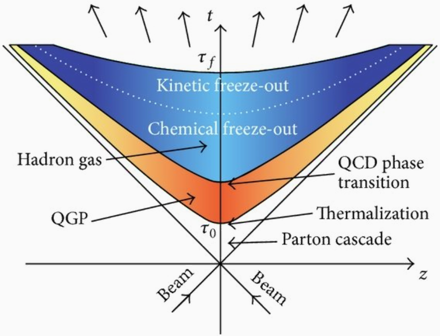
From a combination of approaches (experiment, models, first principle calculations, ...), *we are pretty sure* of some things, and *suspect* others.

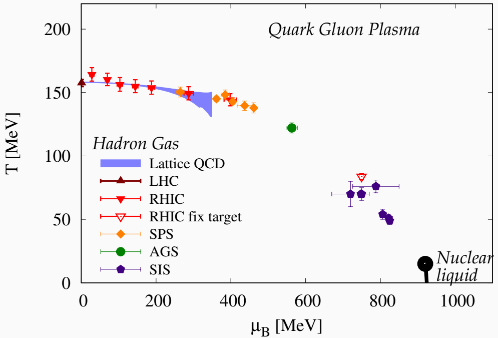
- Hadron phase at low $T \,\&\, \mu$, QGP at high $T \,||\, \mu$
- Crossover at zero density at $T \simeq 160\,\mathrm{MeV}$
- **Heavy-ion collisions** probe high T, varying density with energy scans
- Ordinary nuclear matter at $T \simeq 0$ and $\mu_B \simeq 922\,\mathrm{MeV}$
- **Critical point?** Exotic phases?



Experiment: heavy ion collisions

Measuring the number of final-state hadrons we have an "experimental" sketch of the phase diagram → **chemical freeze-out**





⇒ First-principle theoretical methods don't yet reach as far: **what's beyond?**

Does/where does the QCD transition detach from FO line?

Finite density QCD: the sign/complex action problem

Euclidean path integrals are calculated with MC methods using importance sampling and the Boltzmann weights $\det M[U]\, e^{-S_G[U]}$

$$\begin{aligned} Z(V,T,\mu) &= \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_F(U,\psi,\bar{\psi})-S_G(U)} \\ &= \int \mathcal{D}U\; \det M(U) e^{-S_G(U)} \end{aligned}$$

When a chemical potential is introduced, a problem appears:

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

in general the determinant is complex and cannot serve as a statistical weight.

However, that is **not the case if**:

- there is particle-antiparticle-symmetry ($\mu = 0$):
- the chemical potential is *purely imaginary* ($\mu^2 < 0$)

$$[\det M(\mu)]^* = \det M(-\mu^*) = \det M(\mu) \in \mathbb{R}$$

The sign/complex action problem

Because finite-μ_B physics is of great interest, alternatives have been widely explored:

- **Taylor expansion** around $\mu_B = 0$
  One calculates derivatives of the QCD pressure at $\mu_B = 0$, then construct the expansion:

$$\frac{p(T,\mu_B)}{T^4} = \sum_n c_{2n}(T) \left(\frac{\mu_B}{T}\right)^{2n} , \qquad c_n(T) = \frac{1}{n!} \left.\frac{\partial^n (p/T^4)}{\partial \mu_B^n}\right|_{\mu_B=0} = \frac{1}{n!}\chi_n(T)$$

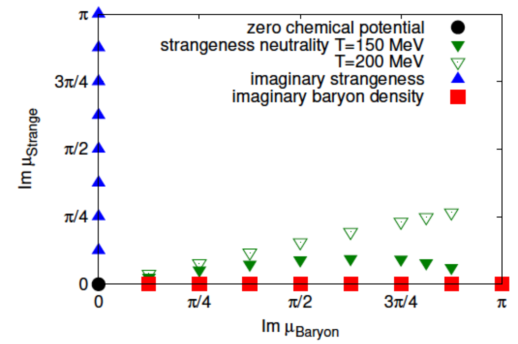
- **Reweighting**
  Because the Boltzmann weight contains $\det M(\mu_B)$, which is complex, move it to the observable, e.g.:

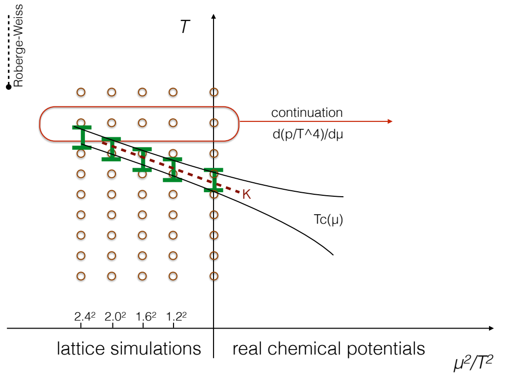
$$\det M(\mu_B) = \frac{\det M(\mu_B)}{\det M(\mu_B = 0)} \det M(\mu_B = 0)$$

- **Imaginary chemical potential**
  Simulate at imaginary μ_B, then (somehow) analytically continue to real μ_B

Simulations at imaginary chemical potential

Since the QCD transition is analytic, we can analytically continue from $\mu_B^2 < 0$ to $\mu_B^2 > 0$



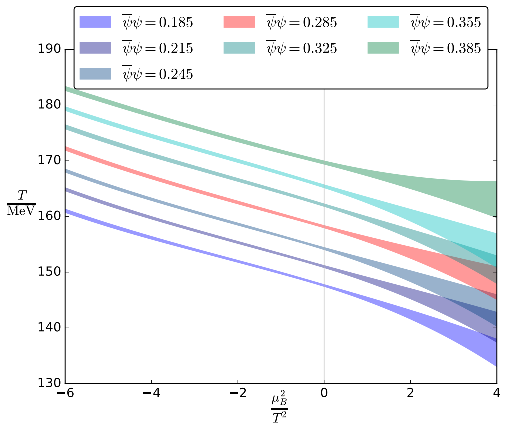


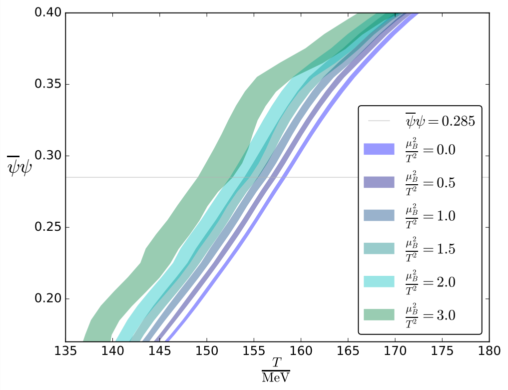
To mimic experiment, one can impose **strangeness neutrality**, i.e. fix the chemical potentials μ_B, μ_Q, μ_S associated to B, Q, S such that:

$$\langle n_S \rangle = 0 \qquad \text{(sometimes also } \langle n_Q \rangle = 0.4 \langle n_B \rangle \text{)}$$

Finite density QCD from imaginary chemical potentials

Already in 2020 we extrapolated contours of constant $\langle\bar\psi\psi\rangle$ (left) to real μ_B (right)





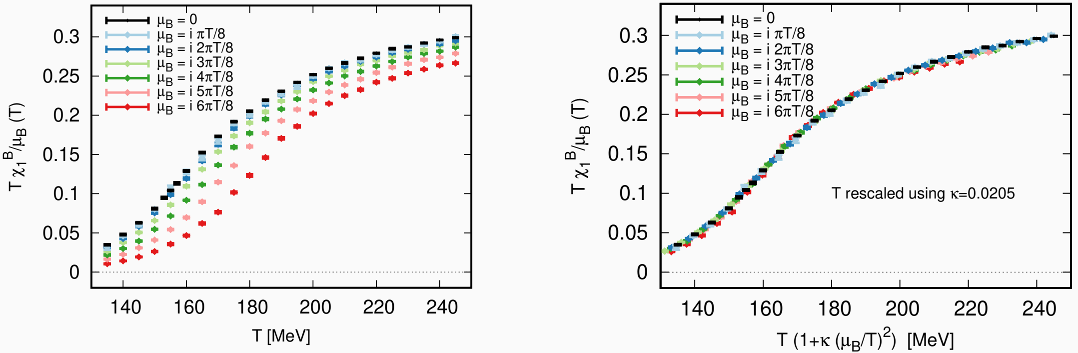
Quite precise extrapolated $\langle\bar\psi\psi\rangle$ at finite μ_B for $\mu_B < 3T$

Borsányi, PP, *at el.*, PRL 125 (2020) 052001

T' expansion

We *observed* that $\chi_1^B(T,\hat{\mu}_B)$ at (imaginary) $\hat{\mu}_B = \mu_B/T$ appears to be differing from $\chi_2^B(T,0)$ mostly by a redefinition of T: **Borsányi, P.P. *et al.*, PRL 126 (2021) 232001**

$$\frac{\chi_1^B(T,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)\ , \quad T' = T\left(1+\kappa\,\hat{\mu}_B^2\right)$$



Borsányi, PP *et al.*, PRL 126 (2021) 232001

$\Rightarrow$ Likely a consequence of chiral O(4) scaling w/ $T' = T(1+\kappa\,\hat{\mu}_B^2)$ as **scaling variable**

T' expansion

- In general, the expansion of an observable F can be made systematic as:

$$\boxed{\frac{F(T,\,\hat\mu_B)}{\overline{F}(\,\hat\mu_B)} = \frac{F(T'_F,0)}{\overline{F}(0)} \qquad\qquad T'_F = T(1+\lambda_2^F(T)\,\hat\mu_B^2+\lambda_4^F(T)\mu_B^4+\mathcal{O}(\,\hat\mu_B^6))}$$

  where the overline indicates the Stefan-Boltzmann limit, and the λ_n^F depend on the observable F. By construction $\lambda_2^F\to 0$ at large T.

- This is a convenient re-organization of the Taylor expansion, expanding instead in

$$\Delta T_F = T - T'_F = T\left(\lambda_2^F(T)\,\hat\mu_B^2+\lambda_4^F(T)\,\hat\mu_B^4+\mathcal{O}(\,\hat\mu_B^6)\right)$$

- The coefficients are in fact related to the χ_n^B. E.g. if $F=\chi_1^B/\,\hat\mu_B$:

$$\lambda_2^{BB}(T) = \frac{1}{6T\chi_2^{B\prime}(T)}\left(\chi_4^B(T)-\frac{\overline{\chi}_4^B(0)}{\overline{\chi}_2^B(0)}\chi_2^B(T)\right)$$
$$\lambda_4^{BB}(T) = \frac{1}{360\chi_2^{B\prime}(T)^3}\left(5\text{ terms w/ up to }\chi_6^{\rm B},\chi_2^{\rm B\prime\prime}(\rm T)\right)$$

T' expansion for the equation of state

The $\lambda_n^{BB}(T)$ can be determined via zero and/or imaginary μ_B simulations

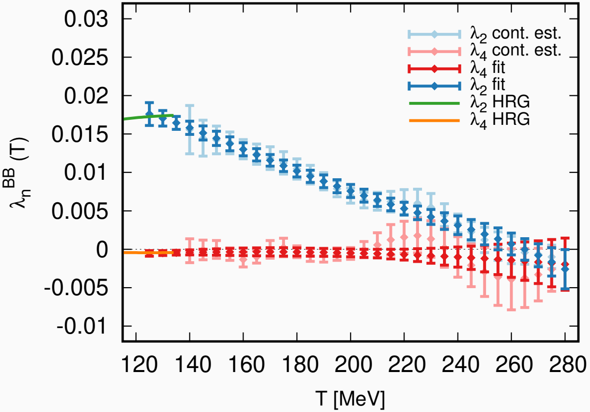
- Then get back the thermodynamics:

$$\frac{\chi_1^B(T,\hat{\mu}_B)}{\overline{\chi}_1^B(\hat{\mu}_B)} = \frac{\chi_2^B(T',0)}{\overline{\chi}_2^B(0)}$$

with $T' = T(1 + \lambda_2^{BB}(T)\,\hat{\mu}_B^2 + \lambda_4^{BB}(T)\,\hat{\mu}_B^4)$.

- From the baryon density one finds the pressure:

$$\frac{p(T,\hat{\mu}_B)}{T^4} = \frac{p(T,0)}{T^4} + \int_0^{\hat{\mu}_B} \mathrm{d}\hat{\mu}'_B \, \frac{\chi_1^B(T,\hat{\mu}'_B)}{T^3}$$

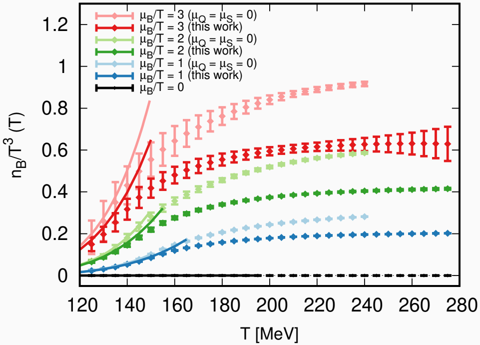


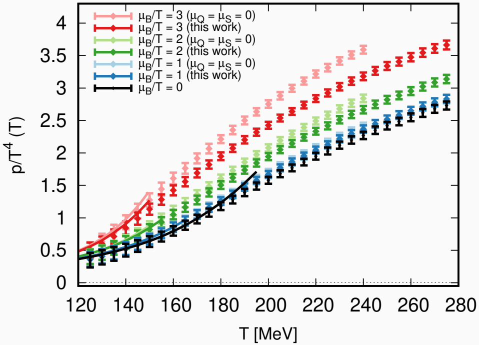
Borsanyi, PP *et al.*, PRD 105 (2022) 114504

$\Rightarrow$ From the pressure, all other quantities are easily obtained

Strangeness neutral or not

- The difference between the two cases is driven by different chemical potentials
- The quality of the results is comparable

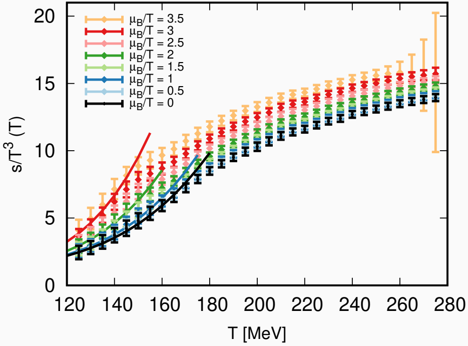


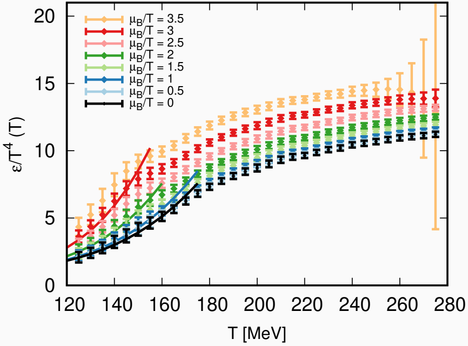


Difference in the pressure is less visible, because dominated by $\mu_B = 0$ contribution

T' expansion: thermodynamics

- Reach out to $\hat{\mu}_B \simeq 3.5$ with reasonable uncertainties and agreement with hadron resonance gas (HRG) model
- No unphysical non-monotonic behavior





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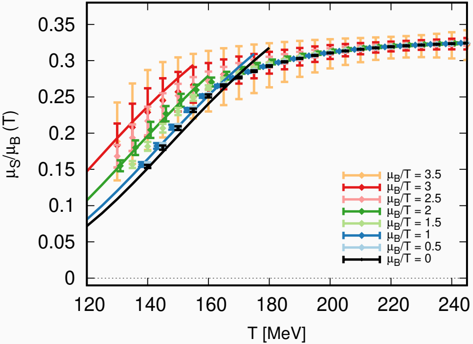
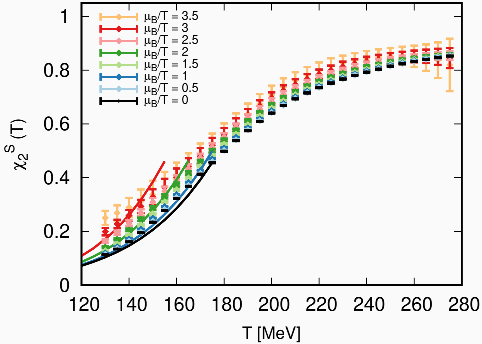
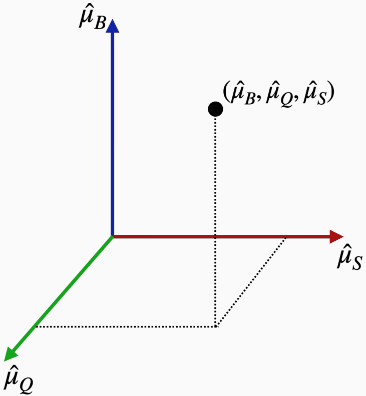




The T' expansion in 4D for the equation of state

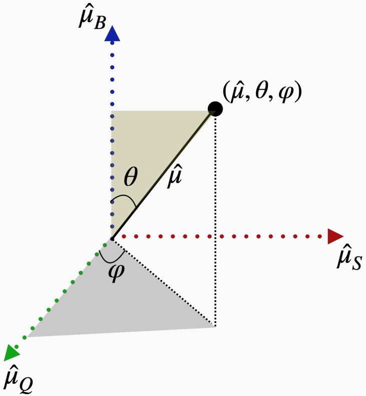
Generalize the 1D T' expansion to an arbitrary direction in 3D space of $\vec{\mu}$
Abuali, PP *et al.*, PRD (2025)



Cartesian Coordinates

$$\hat{\mu}_B = \hat{\mu}\cos(\theta)$$
$$\hat{\mu}_Q = \hat{\mu}\sin(\theta)\cos(\varphi)$$
$$\hat{\mu}_S = \hat{\mu}\sin(\theta)\sin(\varphi)$$

$$\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}$$
$$\theta = \arccos\left(\hat{\mu}_B/\hat{\mu}\right)$$
$$\varphi = \mathrm{sgn}(\hat{\mu}_S)\arccos\left(\hat{\mu}_Q/\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}\right)$$



Spherical Coordinates

Work in spherical coordinates and apply the procedure along direction defined by θ, φ

The T' expansion in 4D for the equation of state

Define the generalized susceptibilities in the arbitrary direction θ, φ:

$$X_n^{\theta,\varphi}(T) = \frac{\partial^n p/T^4}{\partial\,\hat{\mu}^n}\Bigg|_{\hat{\mu}=0}^{\theta,\varphi}$$

simply linear combinations of χ_{ijk}^{BQS} with $i+j+k=n$. Now expand in the direction θ, φ thanks to the $X_n^{\theta,\varphi}$:

$$X_1^{\theta,\varphi}(T,\hat{\mu}) = \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T'^{\,\theta,\varphi}(T,\hat{\mu}),0)$$

and:

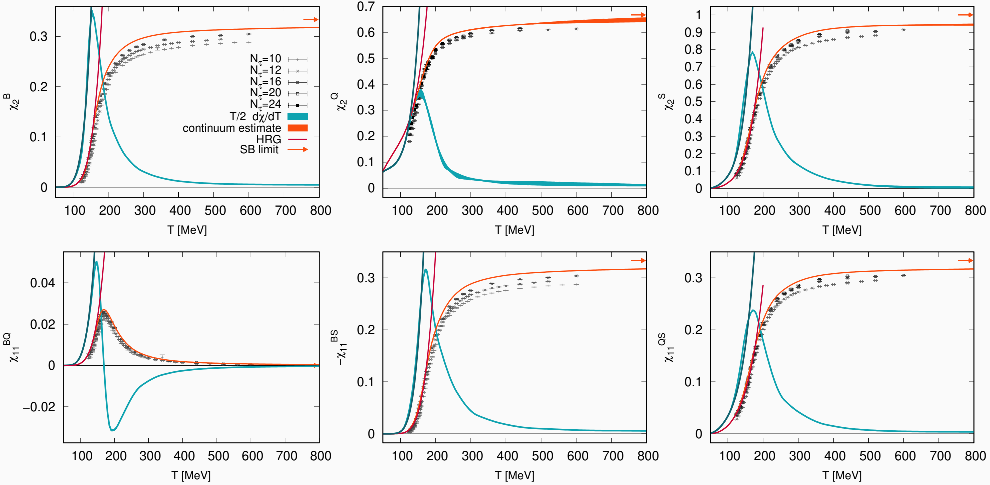
$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T(X_2^{\theta,\varphi}(T))'}\left(X_4^{\theta,\varphi}(T) - \frac{\overline{X_4}^{\theta,\varphi}(0)}{\overline{X_2}^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T)\right)$$

Just like before, along the direction θ, φ, the pressure will be:

$$\hat{p}(T,\,\hat{\mu}) = \hat{p}(T,\,\hat{\mu}=0) + \int_0^{\hat{\mu}} d\hat{\mu}' X_1(T,\,\hat{\mu}')$$

Input: order $\mathcal{O}(\hat{\mu}_B^2)$ (now downloadable)

We "only" need all $\chi_{ijk}^{BQS}(T)$ as input: merge **HRG + lattice continuum estimate + SB limit**



Abuali, PP *et al.*, PRD 112 (2025) 5, 054502

Input: order $\mathcal{O}(\hat{\mu}_B^4)$ (only 6 of 15 shown, also downloadable)

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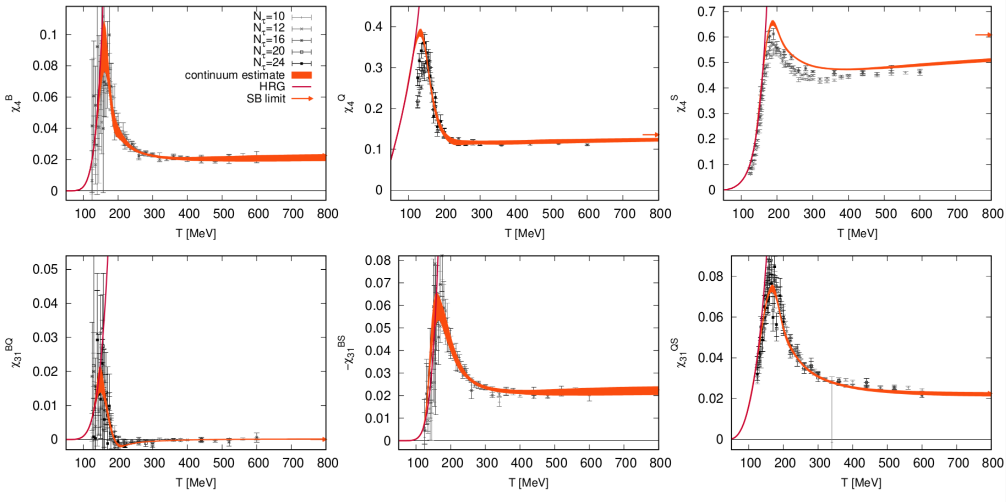


Abuali, PP *et al.*, PRD 112 (2025) 5, 054502

Limit of applicability

We truncate the expansion at LO($=\lambda_2=\chi_4$, because χ_6 unavailable from the lattice).
Hence, we can't check convergence by comparing against NLO.

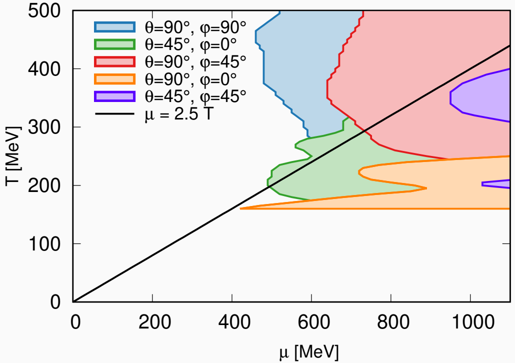
However, since:

$$X_1(T,\hat{\mu}) = \frac{\overline{X}_1(\hat{\mu})}{\overline{X}_2(0)} X_2(T'(T,\hat{\mu}),0)$$

and we know that:

- In the crossover region, X_1 should be increasing with T
- X_2 is monotonic with T

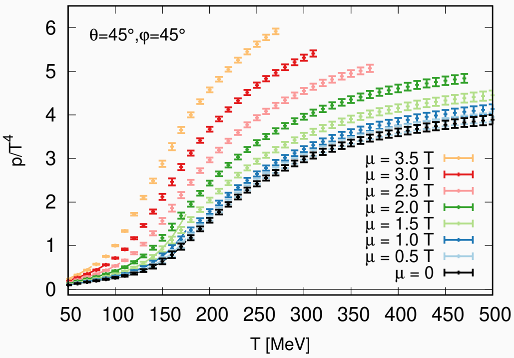
Hence, the construction fails if $dT'/dT < 0$

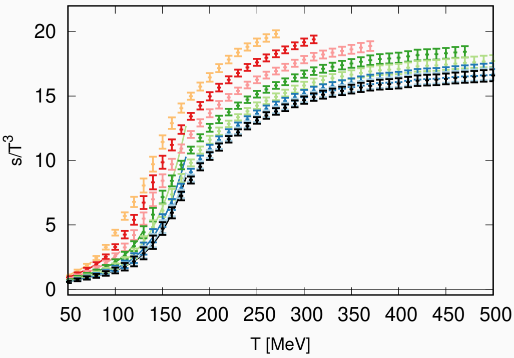


→ We can map where our expansion breaks down for some directions.

Thermodynamics in 4D

Example: $\theta = \varphi = \pi/4$ (mixing B, Q, S)





No pathological behavior and more extended coverage than Taylor

Thermodynamics in 4D

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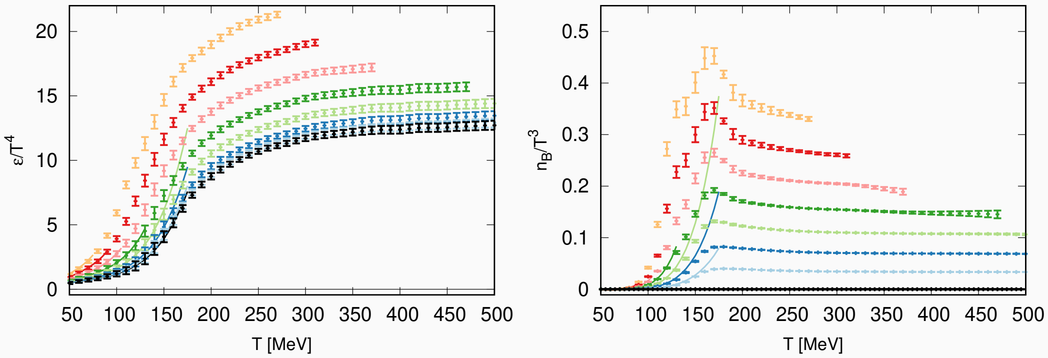

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Thermodynamics in 4D

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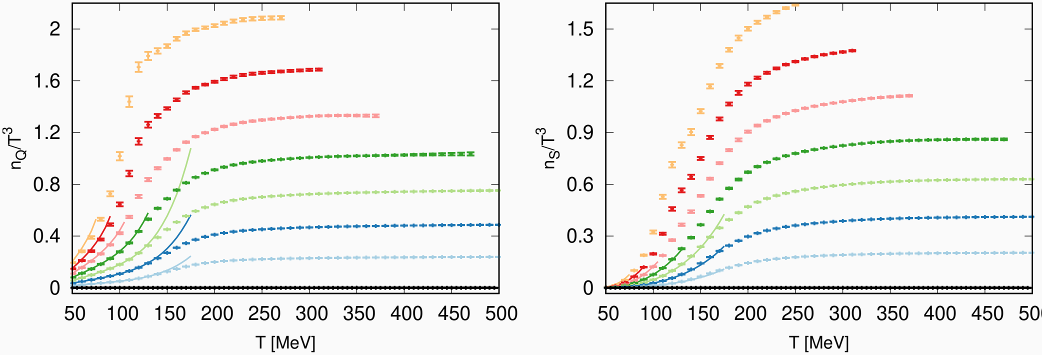
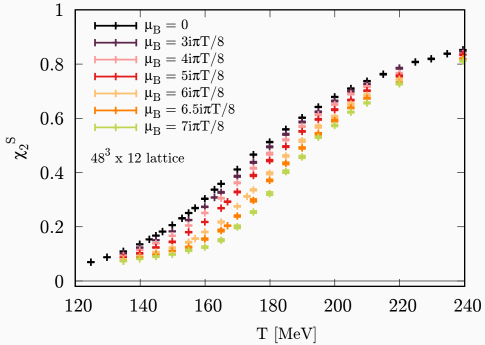


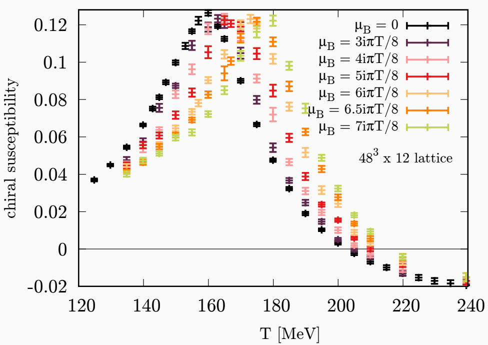
No pathological behavior and more extended coverage than Taylor

Strangeness susceptibility: proxy of T_c?

The strangeness susceptibility was one of the observables we applied the T' to.

As $\mathrm{Im}(\hat{\mu}_B)$ varies, the curve moves as a whole

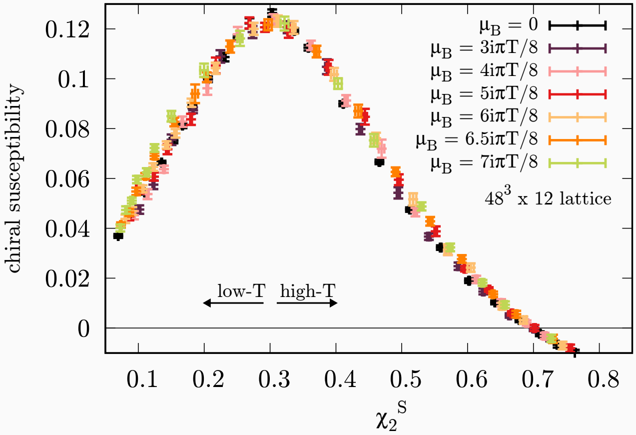




At the same time, the peak of the chiral susceptibility (i.e. the chiral transition) shifts

Strangeness susceptibility: proxy of T_c?

In fact, plotting χ vs χ_2^S we get a curve that looks independent of $\mathrm{Im}(\hat{\mu}_B)$, but only in the case of **strangeness neutrality**



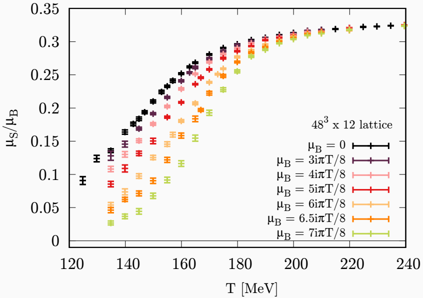
$\Rightarrow$ use strangeness susceptibility χ_2^S as proxy to locate QCD transition

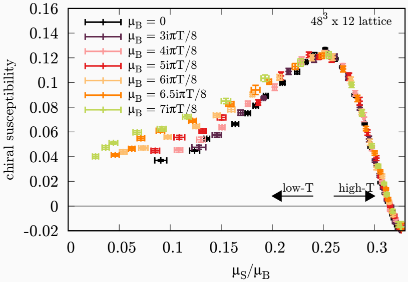
Strangeness susceptibility: proxy of T_c?

Why does it work?

- As in the T' expansion, clearly the physics at $\mathrm{Im}\mu_B$ looks self-similar
- We are likely in some sort of scaling region $\rightarrow$ O(4)?
- Moreover, the SB limit of χ_2^S is independent of μ_B **in the strangeness neutral case**

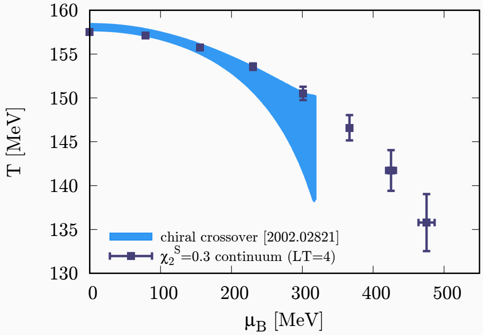
A similar scenario appears in μ_S/μ_B, but less cleanly:

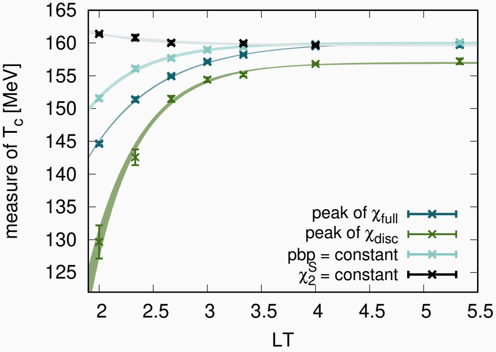




Strangeness susceptibility: proxy of T_c?

The condition $\chi_2^S = 0.3$ returns a line in agreement with chiral transition





And shows a remarkably weak dependence on the volume

It does look like a very good proxy for charting the QCD transition. Now, what about criticality?

Net-hadron distributions and criticality

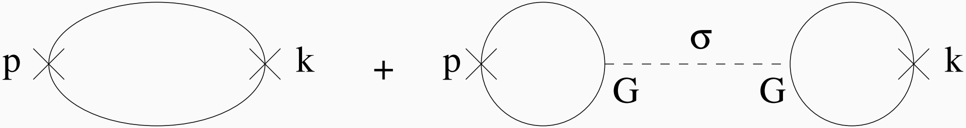
- Hadron fluctuation observables diverge at the critical point as powers of the correlation length, e.g.:

$$\chi_2^B \sim \xi^{\gamma/\nu} + \text{noncritical}$$

- Critical behavior stems from the coupling with the σ field, which becomes critical $m_\sigma \sim 1/\xi \to \infty$ at the critical point
- Nucleons couple to the σ field via a Yukawa term:

$$\mathcal{L}_{\sigma\bar{N}N} = G\sigma\bar{N}N$$

- E.g. second order *net-proton* fluctuations get a contribution from this interaction:



(left) is non-critical, **(right) is critical**

The zero-momentum propagator of the σ field drives the divergence for $m_\sigma \to \infty$.

Rajagopal, Shuryak, Stephanov, PRD 60 (1999) 114028; Stephanov, PRD 65 (2002) 096008, PRL 91 (2003) 102003

Net-hadron distributions and criticality

One can write second order correlations as integrals over momentum modes:

$$\left\langle \Delta N^i \Delta N^j \right\rangle = V \int_p \int_k \left\langle \delta n_p^i \delta n_k^j \right\rangle$$

where δn_p^i is the *net-number* fluctuation of species i in momentum mode p.

The critical contribution (σ field exchange) reads:

$$V \left\langle \delta n_p \delta n_k \right\rangle = \frac{G^2}{m_\sigma^2 T} \frac{4m_p^2}{E_p E_k} \left[n_p^+ \left(1 - n_p^+\right) - n_p^- \left(1 - n_p^-\right) \right] \\ \times \left[n_k^+ \left(1 - n_k^+\right) - n_k^- \left(1 - n_k^-\right) \right]$$

where:

- $n_p^{\pm} = \left[\exp((E_p \mp \mu_B)/T) + 1\right]^{-1}$ are the Dirac distributions for $p, \bar{p}$;
- m_p is the proton mass, $E_p = \sqrt{p^2 + m_p^2}$;
- $n_p^{\pm}(1 - n_p^{\pm})$ accounts for the Pauli principle.

Rajagopal, Shuryak, Stephanov, PRD 60 (1999) 114028; Stephanov, PRD 65 (2002) 096008, PRL 91 (2003) 102003

What about strangeness and criticality?

We want to **estimate the critical contribution to the strangeness susceptibility**.

Applying the same idea, replacing nucleons with *consituent quarks* of flavour u, d, s, the correlations between net-numbers for flavours i, j will be:

$$\left\langle \Delta N^i \Delta N^j \right\rangle = V \int_p \int_k \left\langle \delta n_p^i \delta n_k^j \right\rangle$$

with, as before:

$$V \left\langle \delta n_p^i \delta n_k^j \right\rangle = \frac{G^2}{m_\sigma^2 T} \frac{4 m_i m_j}{E_p^i E_k^i} \left[n_p^{i,+} \left(1 - n_p^{i,+}\right) - n_p^{i,-} \left(1 - n_p^{i,-}\right) \right] \times \left[n_k^{j,+} \left(1 - n_k^{j,+}\right) - n_k^{j,-} \left(1 - n_k^{j,-}\right) \right]$$

where now $m_u = m_d = 340$ MeV, $m_s = 500$ MeV. This factorizes:

$$V \left\langle \delta n_p^i \delta n_k^j \right\rangle = \frac{G^2}{m_\sigma^2 T} F_p^i F_k^j \qquad \text{where} \quad F_p^i = \frac{2 m_i}{E_p^i} \left[n_p^{i,+} \left(1 - n_p^{i,+}\right) - n_p^{i,-} \left(1 - n_p^{i,-}\right) \right]$$

Remind that this is always only the critical contribution

Strangeness and criticality

Then the flavour-flavour correlation reads:

$$\left\langle \Delta N^i \Delta N^j \right\rangle = \frac{G^2}{m_\sigma^2 T} \int_p F_p^i \int_p F_p^j$$

How large is the critical contribution to χ_2^S? Take the ratio:

$$R(T, \mu_B) = \int_p F_p^u \Big/ \int_p F_p^s$$

Combining u, d, s into conserved charges B, Q, S, we get (with degenerate u,d, quarks):

$$\frac{\chi_2^S}{\chi_2^B}(T, \mu_B) = \frac{4}{9} R(T, \mu_B)^2 + \frac{4}{9} R(T, \mu_B) + \frac{1}{9} + \mathrm{noncritical}(\mathrm{T}, \mu_\mathrm{B})$$

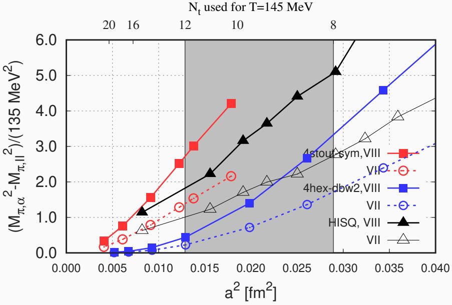
- $\mu_S = 0 \longrightarrow \chi_2^S \simeq (0.3 - 0.4)\chi_2^B$
- $n_S = 0 \longrightarrow \chi_2^S \simeq 0.01\chi_2^B$

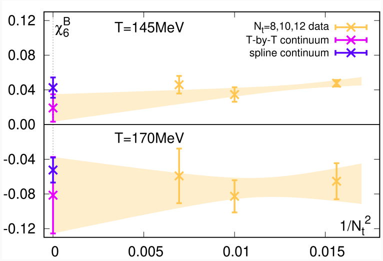
With strangeness neutrality, χ_2^S seems oblivious to criticality!

Large density QCD transition: the analysis

Proxy for T_c w/ almost no critical behavior $\rightarrow$ perfect scenario to chart T_c at large density

- Single $16^3 \times 8$ lattice with huge statistics $\sim \mathcal{O}(10^6)$ for each $T = 110 - 300$ MeV
- $N_f = 2 + 1$ flavours of physical quark masses
- Our recent 4HEX action has smaller discretization effects (i.e., $N_\tau = 8$ is not that coarse)



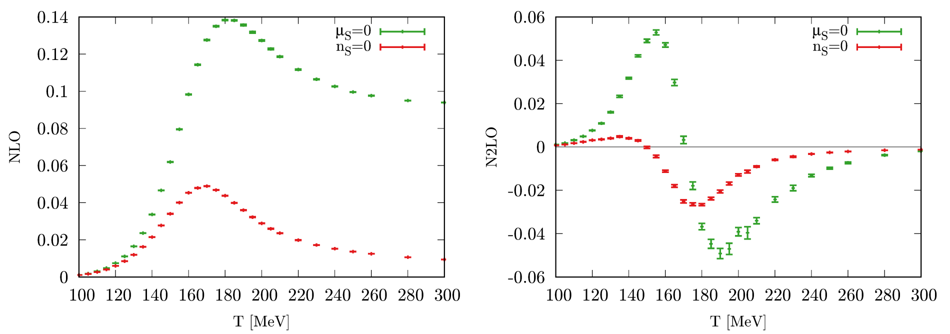


Borsanyi, PP *et al.*, PRD 110 (2024) L011501

Strangeness susceptibility: Taylor expansion

To get χ_2^S over the QCD phase diagram – up to large baryon chemical potential – we use a Taylor expansion:

$$\chi_2^S(T,\mu_B) = \sum_{n\in\text{even}} \frac{1}{n!} c_n(T) \left(\frac{\mu_B}{T}\right)^n \qquad c_n(T) = \left.\frac{\partial^n \chi_2^S}{\partial \mu_B^n}\right|_{n_S=0\,\text{OR}\,\mu_S=0} .$$



with (red) or without (green) strangeness neutrality.

Strangeness susceptibility: Taylor expansion

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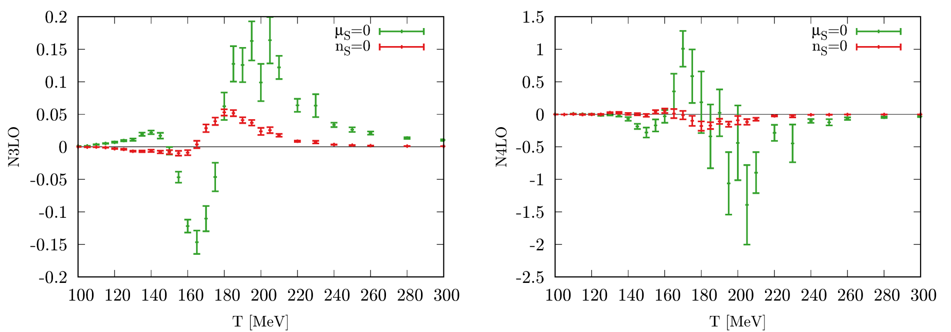
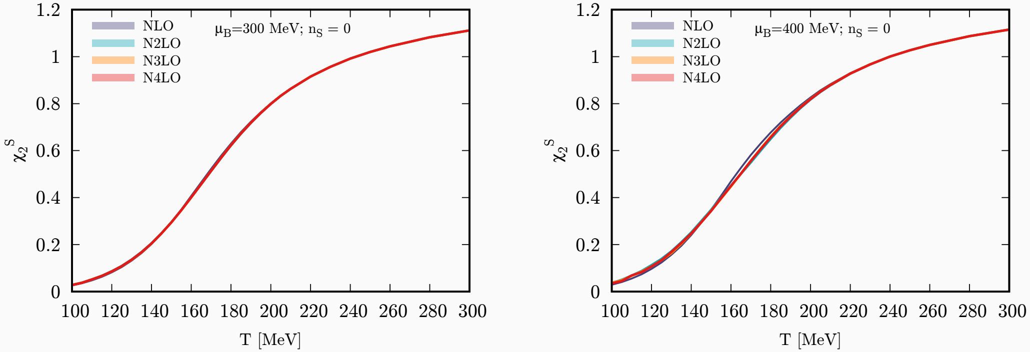


with (red) or without (green) strangeness neutrality.

Strangeness susceptibility: Taylor expansion

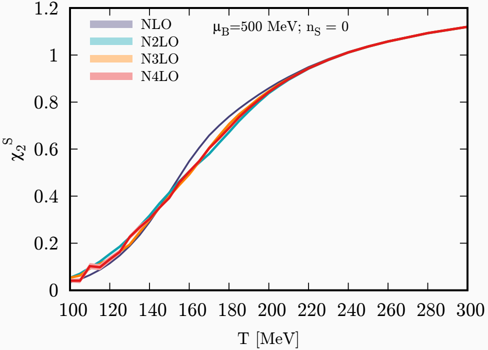
With strangeness neutrality, we see remarkably good convergence of the Taylor series

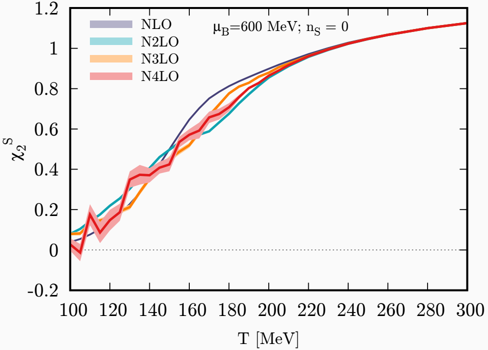


Virtually no difference at all orders up to $\mu_B = 400$ MeV

Strangeness susceptibility: Taylor expansion

In the case of strangeness neutrality, we see remarkably good convergence of the Taylor series:

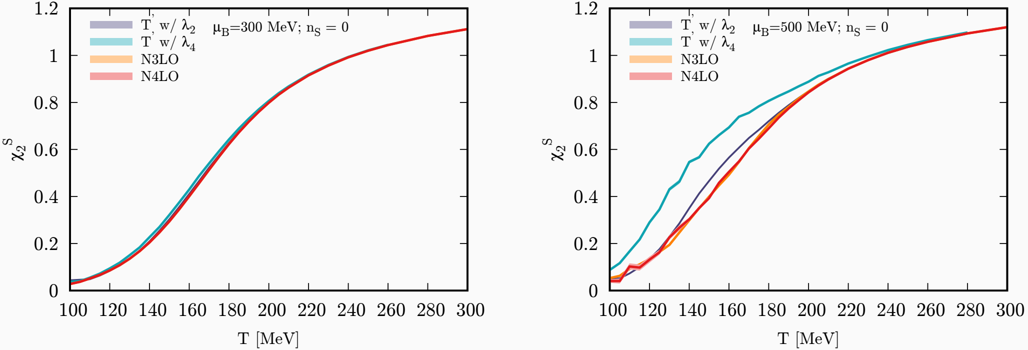




Only above $\mu_B = 500$ MeV does N4LO differ from N3LO

Strangeness susceptibility: Taylor vs T' expansion

Why not the T' expansion?

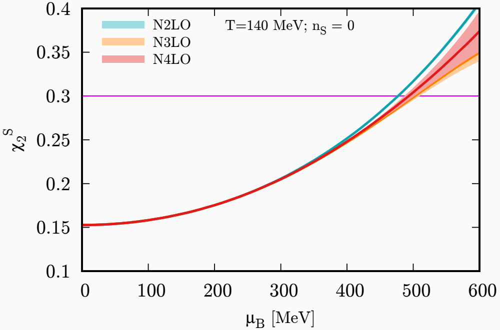


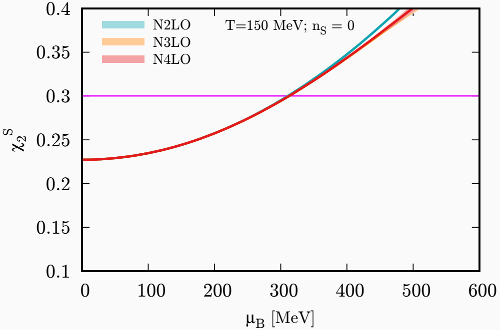
On a single lattice it does not converge well (volume-dependent SB limit).

Also, matching the order of Taylor expansion would require multiple (4) T-derivatives

Contours of χ_2^S in the phase diagram

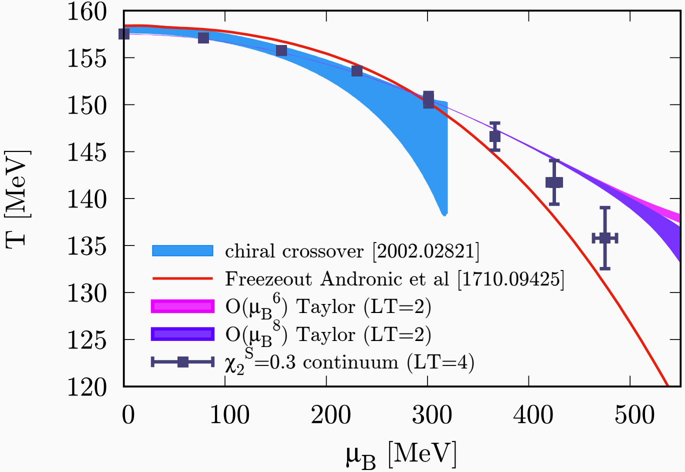
Then, we search for fixed values of $\chi_2^S = \text{const}$ while varying μ_B:





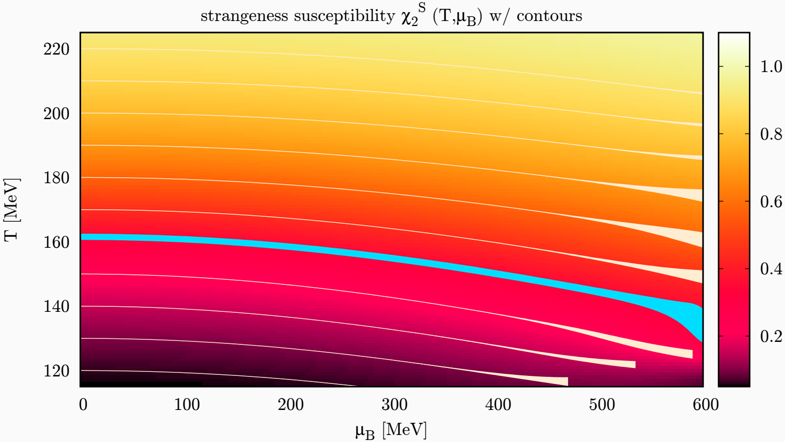
T_c from χ_2^S in the phase diagram

The $\chi_2^S = 0.3$ condition in our analysis is in agreement with continuum result(s) (only small tension)



The resulting estimate for T_c sits some 15 MeV higher that FO line at $\mu_B = 500$ MeV

Contours of χ_2^S in the phase diagram

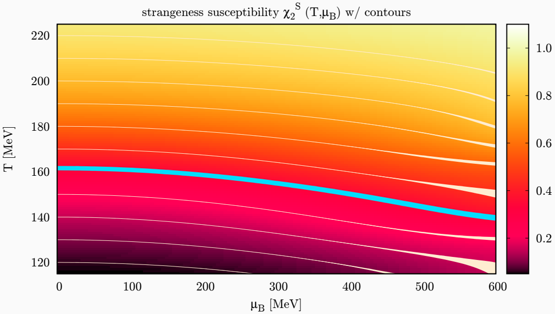


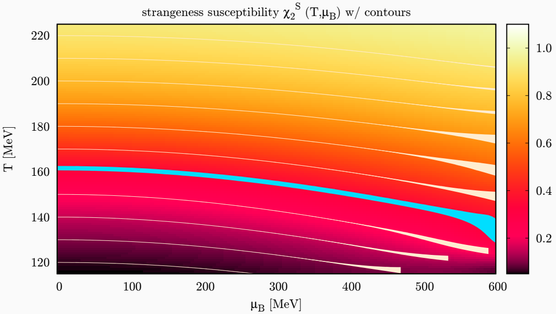
Different contours are virtually parallel up to $\mu_B = 500$ MeV, and errors are tiny

For illustration: blue curve defined by $\chi_2^S =$ const at the $\mu_B = 0$ transition temperature obtained from peak of S_Q ($T_c \simeq 161 \pm 1$ MeV on the $16^3 \times 8$ lattice).

Contours of χ_2^S in the phase diagram

N3LO vs N4LO

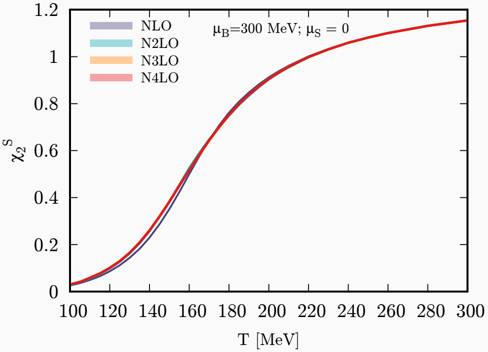


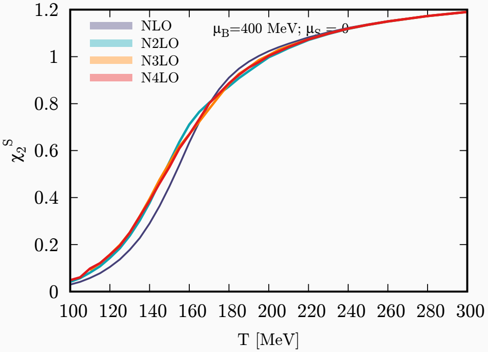


very good agreement as expected

Strangeness susceptibility with $\mu_S = 0$

Without strangeness neutrality, unsurprisingly convergence is poorer:

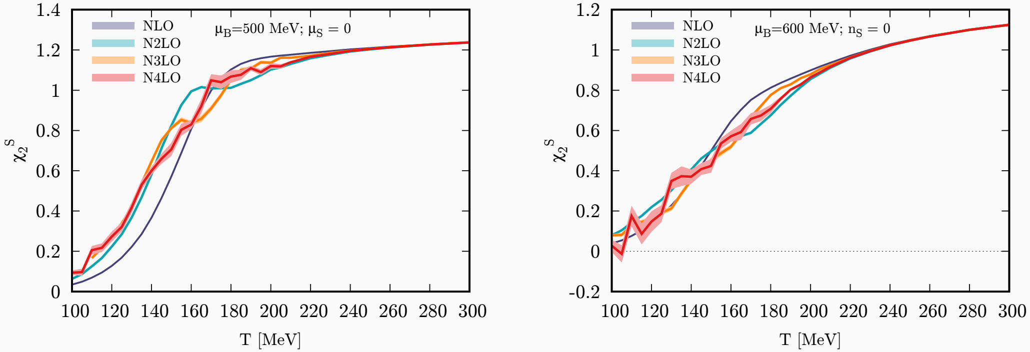




Differences appear already at $\mu_B = 400$ MeV

Strangeness susceptibility with $\mu_S = 0$

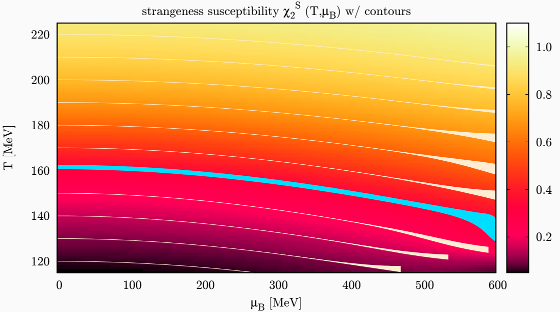
Without strangeness neutrality, unsurprisingly convergence is poorer:

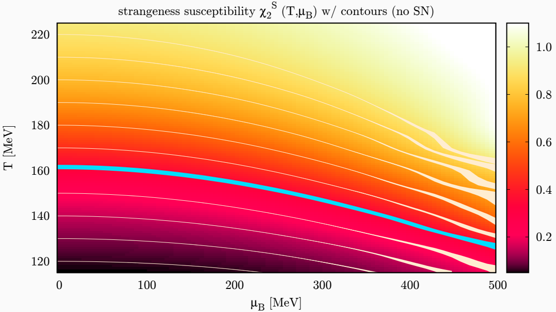


Above $\mu_B = 500$ MeV convergence is lost

Contours of χ_2^S in the phase diagram with $\mu_S = 0$

$n_S = 0$ (N4LO) vs $\mu_S = 0$ (N4LO)



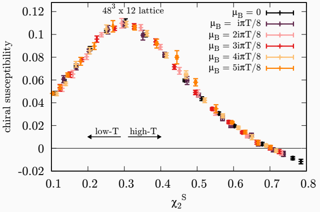


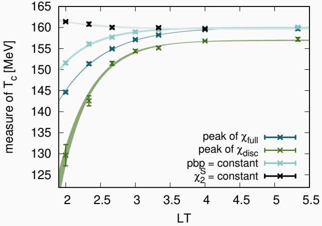
This also shows up in the heatmap, even though we can reach very large μ_B also for $\mu_S = 0$

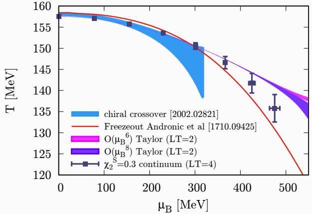
We saw that when $\mu_S = 0$, then χ_2^S is **not** insensitive to criticality. Contours almost parallel up to $\mu_B \simeq 450$ MeV, in agreement with continuum exclusion range

Summary

- We wish to chart large density QCD: e.g. **where is the chiral transition**?
- Imaginary μ_B simulations show that $\chi_2^S = 0.3$ virtually always at the chiral transition, **in the case of strangeness neutrality**. Robust vs volume and vs continuum limit
- Extreme statistics **Taylor reaches** $\mu_B \simeq 550$ **MeV** with N3LO, N4LO in agreement
- This proxy of T_c **at** $\mu_B \simeq 500$ **MeV sits** ~ 15 **MeV above freeze-out**







Summary

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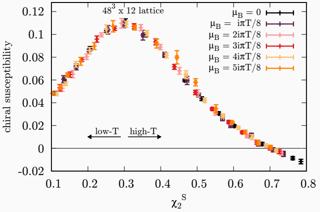
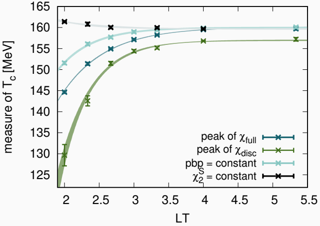
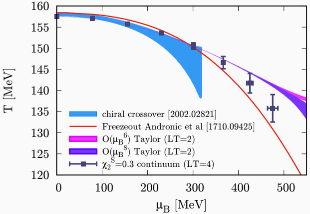


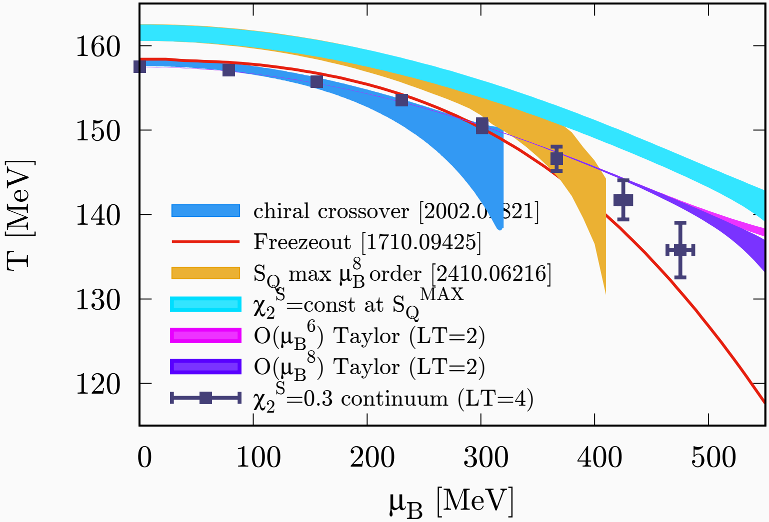




THANK YOU!

BACKUP

QCD transition

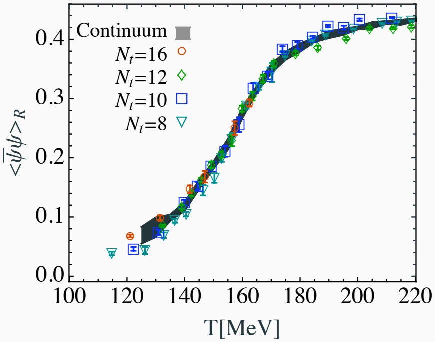


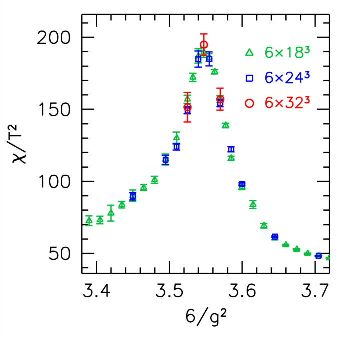
QCD transition: chiral symmetry breaking

Chiral symmetry: exact for $m_q \to 0$. **Chiral condensate** $\langle\bar{\psi}\psi\rangle$ (order parameter) and **chiral susceptibility**:

$$\langle\bar{\psi}\psi\rangle = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial m} \qquad\qquad \chi = \frac{T}{V}\frac{\partial^2 \log \mathcal{Z}}{\partial m^2}$$

Symmetric phase $\langle\bar{\psi}\psi\rangle = 0$ at high-T, and SSB $\langle\bar{\psi}\psi\rangle \neq 0$ at low-T





Borsanyi *et al.*, JHEP 1009:073 (2010); Aoki *et al.*, Nature 443, 675–678 (2006)

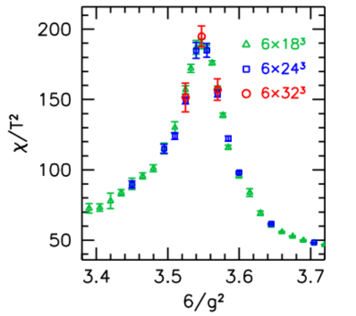
No volume scaling in susceptibility → **QCD transition is a crossover**

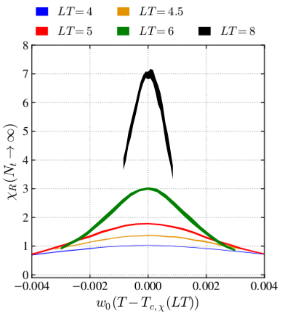
The QCD transition: crossover vs. first order

On the lattice we study the volume scaling of certain quantities to determine the order of the transition

Left: physical masses

Right: infinite masses (pure gauge)



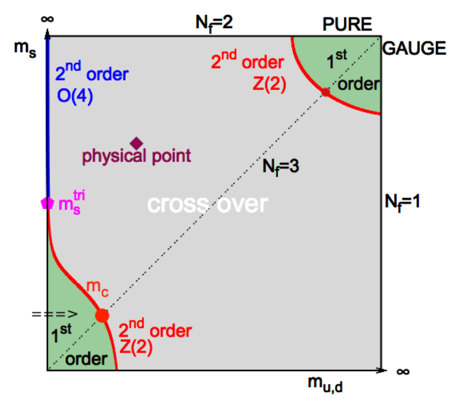


- For a crossover (left), **the peak height is independent of the volume**
- For a first order transition, **it scales linearly with the volume**

Aoki *et al.* Nature 443 (2006), Borsányi, PP *et al.*, PRD 105 (2022)

The QCD transition: Columbia plot

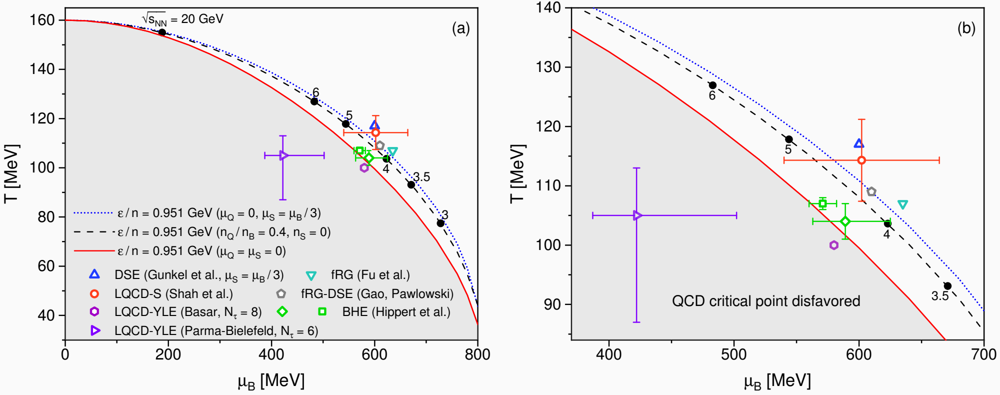
As a function of the light (u,d) and strange quark masses, the order of the transition changes



- At the physical point $m_s/m_{ud} \simeq 27$, the transition is a smooth crossover!
- In the heavy-quark limit (pure gauge), the transition is first order

The QCD critical point

From the theory side, many different models predicting a critical point

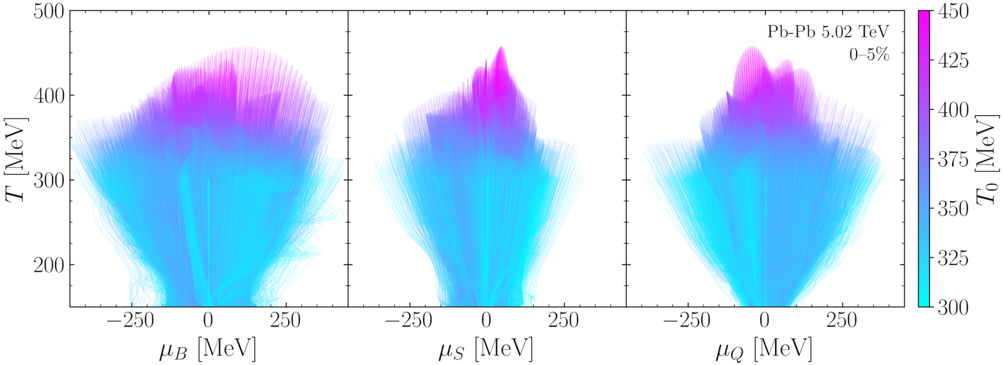


Vovchenko, PRC 111 (2025) 054903

→ and recent estimates seem to “converge”

B,Q,S: a 4D equation of state

Full trajectories of hydro cells until hadronization. We can have a picture of density fluctuations in all B, Q, S, as a function of T



Plumberg *et al.*, PRC 111 (2025) 044905

Full 4D equation of state needed even at LHC, where $\langle \mu_i \rangle = 0$.

Existing results based on Taylor expansion **Monnai:2019hkn**, **Noronha-Hostler:2019ayj**, **Monnai:2024pvy**

Fluctuations of conserved charges

- **Theory**: grand canonical fluctuations are derivatives of the free energy:

$$Z(V,T,\mu_B,\mu_Q,\mu_S) = \sum_{B,Q,S} e^{B\mu_B} e^{Q\mu_Q} e^{S\mu_S} Z_C(V,T,B,Q,S)$$

wrt the associated chemical potentials:

$$\chi_{ijk}^{BQS}(T,\mu_B,\mu_Q,\mu_S) = \frac{1}{VT^3} \frac{\partial^{i+j+k} \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)^i \partial\left(\mu_Q/T\right)^j \partial\left(\mu_S/T\right)^k}$$

- **Experiment**: moments/cumulants $\langle(\Delta N)^n\rangle_{\text{events}}$ of net-particle distributions:

$$\langle B\rangle = \frac{1}{VT^3} \frac{\partial \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)} = \chi_1^B$$

$$\left\langle B^2\right\rangle - \langle B\rangle^2 = \frac{1}{VT^3} \frac{\partial^2 \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)^2} = \chi_2^B$$

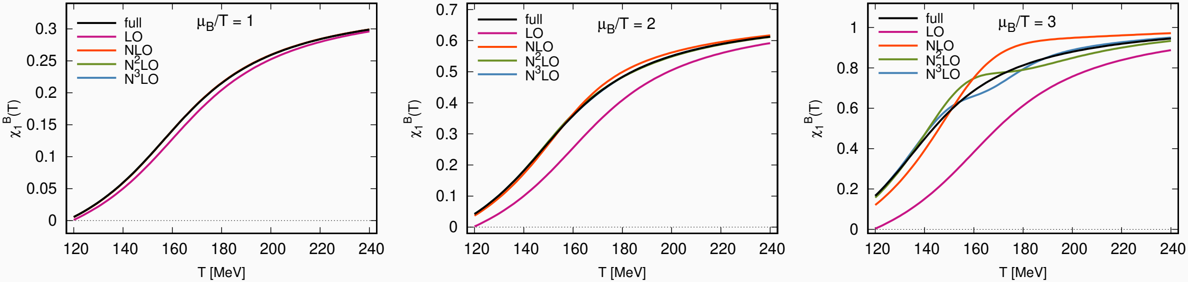
$$\langle BS\rangle - \langle B\rangle\langle S\rangle = \frac{1}{VT^3} \frac{\partial^2 \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)\partial\left(\mu_S/T\right)} = \chi_{11}^{BS}$$

Taylor expanding a (shifting) sigmoid

Assume we have a sigmoid function $f(T)$ which shifts with $\hat{\mu}$, with a simple T-independent shifting parameter κ. How does Taylor cope with it?

$$f(T,\,\hat{\mu}) = f(T',0)\ , \qquad\qquad T' = T(1+\kappa\,\hat{\mu}^2)\ ,$$

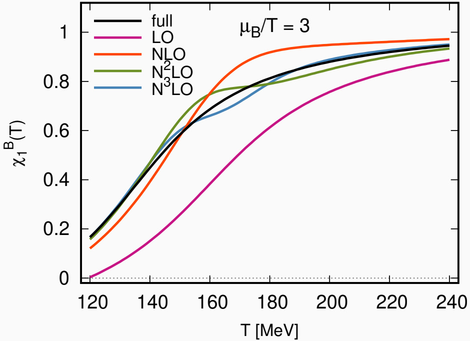
We fitted $f(T,0) = a + b\arctan(c(T-d))$ to $\chi_2^B(T,0)$ data for a 48×12 lattice

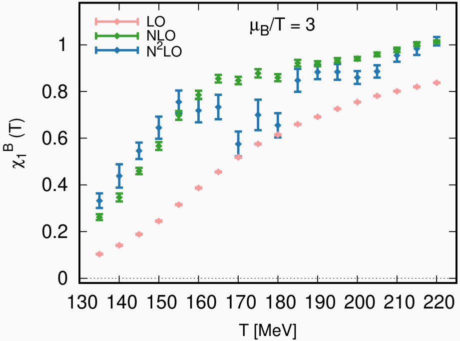


Borsányi, PP *et al.* PRL 126 (2021) 232001

Taylor expanding a (shifting) sigmoid

- The Taylor expansion seems to have problems reproducing the original function (left)
- Quite suggestive comparison with actual Taylor-expanded lattice data (right)





- Problems at T slightly larger than T_{pc} $\Rightarrow$ influence from structure in χ_6^B and χ_8^B

Borsányi, PP *et al.* PRL 126 (2021) 232001

Determine κ_n

I. Directly determine $\kappa_2(T)$ at $\hat{\mu}_B = 0$ from the previous relation

II. From imaginary $\hat{\mu}_B$ simulations we calculate:

$$\frac{T' - T}{T\,\hat{\mu}_B^2} = \kappa_2(T) + \kappa_4(T)\,\hat{\mu}_B^2 + \mathcal{O}(\,\hat{\mu}_B^4) = \Pi(T)$$

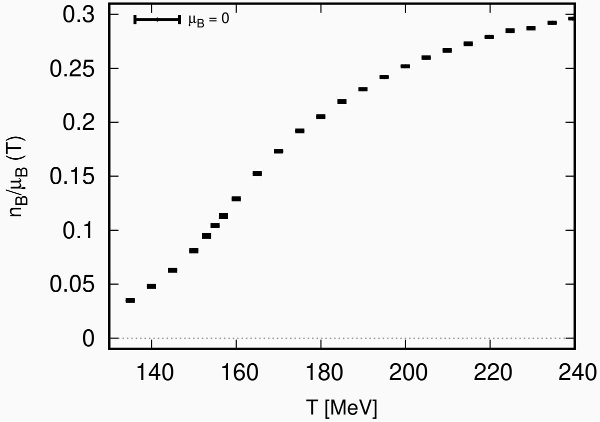
III. Calculate $\Pi(T, N_\tau, \hat{\mu}_B^2)$ for $\hat{\mu}_B = in\pi/8$ and $N_\tau = 10, 12, 16$

IV. Perform a combined fit of the $\hat{\mu}_B^2$ and $1/N_\tau^2$ dependence of $\Pi(T)$ at each temperature, yielding a continuum estimate for the coefficients

$\Rightarrow$ The $\mathcal{O}(1)$ and $\mathcal{O}(\,\hat{\mu}_B^2)$ coefficients of the fit are $\kappa_2(T)$ and $\kappa_4(T)$

Determine κ_n

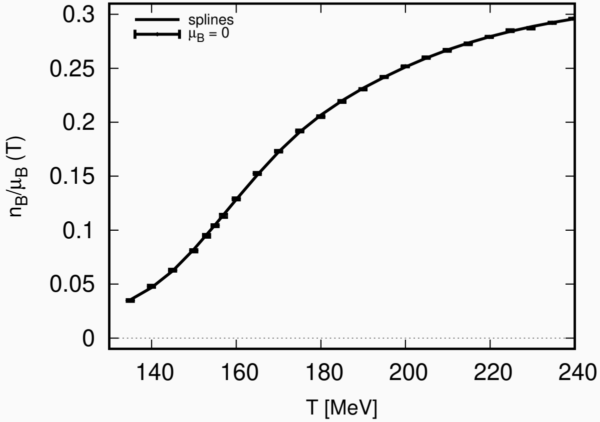
The procedure, visualized:



Spline fit both at $\mu_B = 0$ and $\mu_B \neq 0$, then determine $T - T'$ (horizontal segments)

Determine κ_n

The procedure, visualized:



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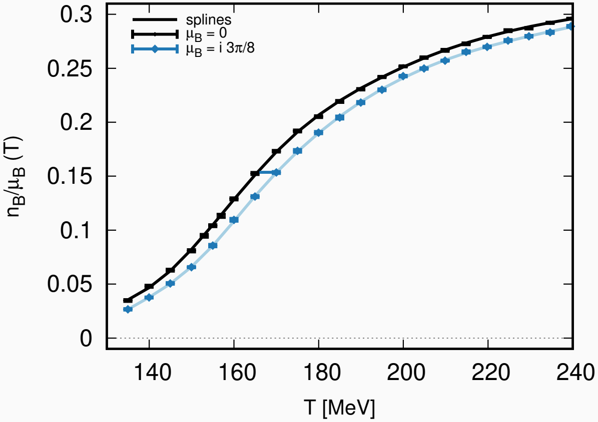

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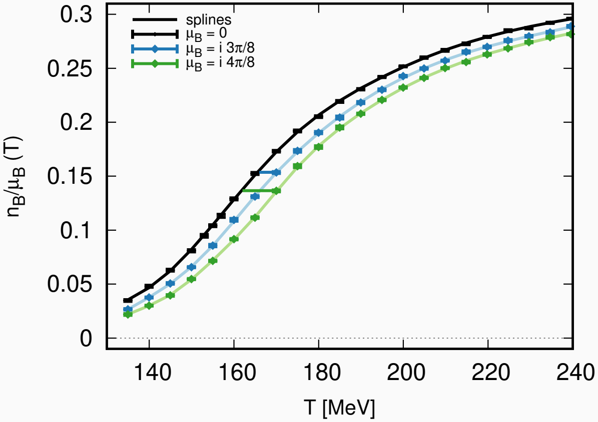

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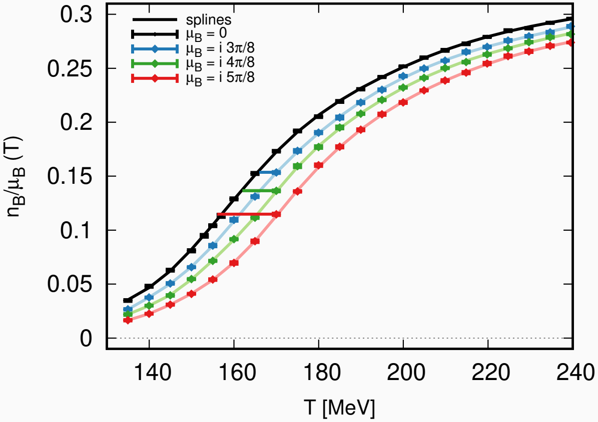

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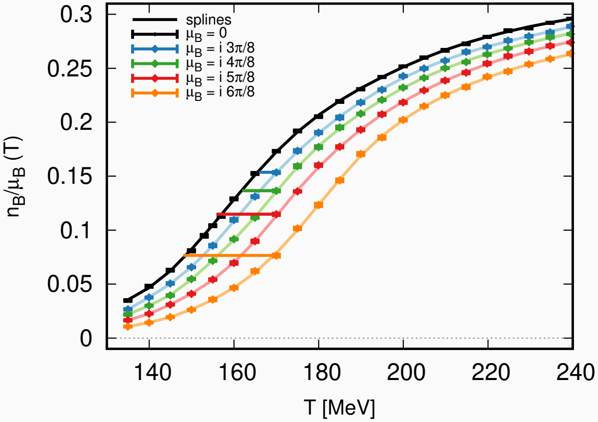

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