Finite-volume analysis and universal scaling near chiral phase transition in (2+1)-flavor QCD

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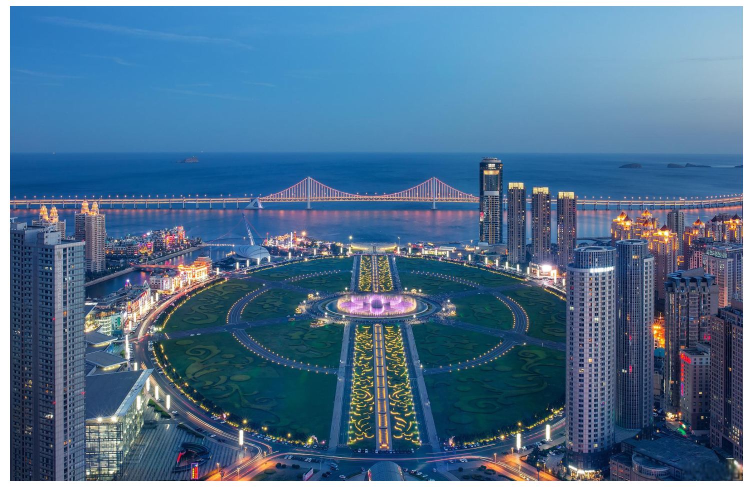
The QCD phase diagram : From theory to experimental signatures

8-11 October , Dalian workshop 2025



THANKS to my collaborators

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- Christian Schmidt
- Jishnu Goswami

Bielefeld University, Germany

- Mugdha Sarkar (NTU , Taiwan)
- Sipaz Sharma (TuM Munich, Germany)

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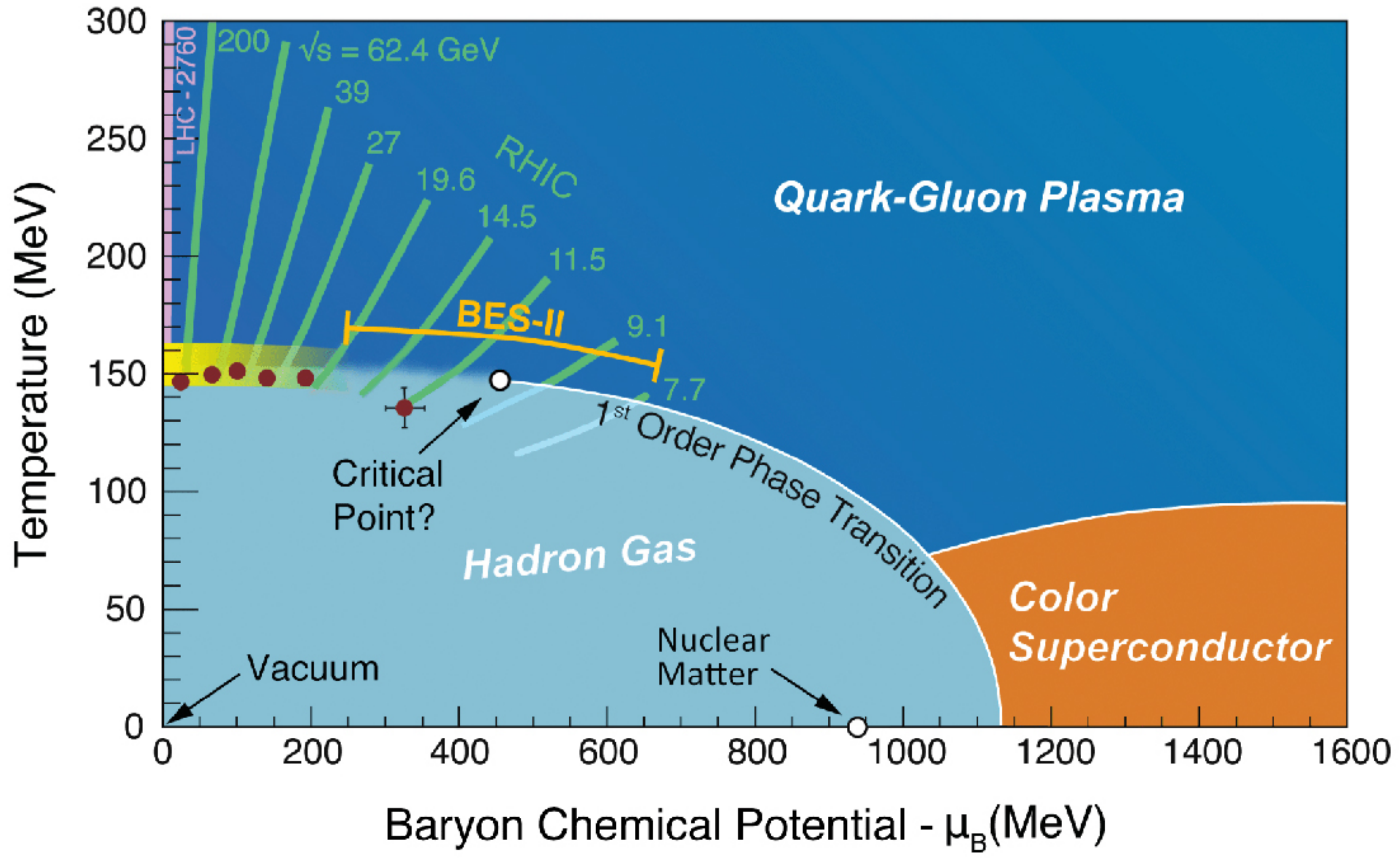
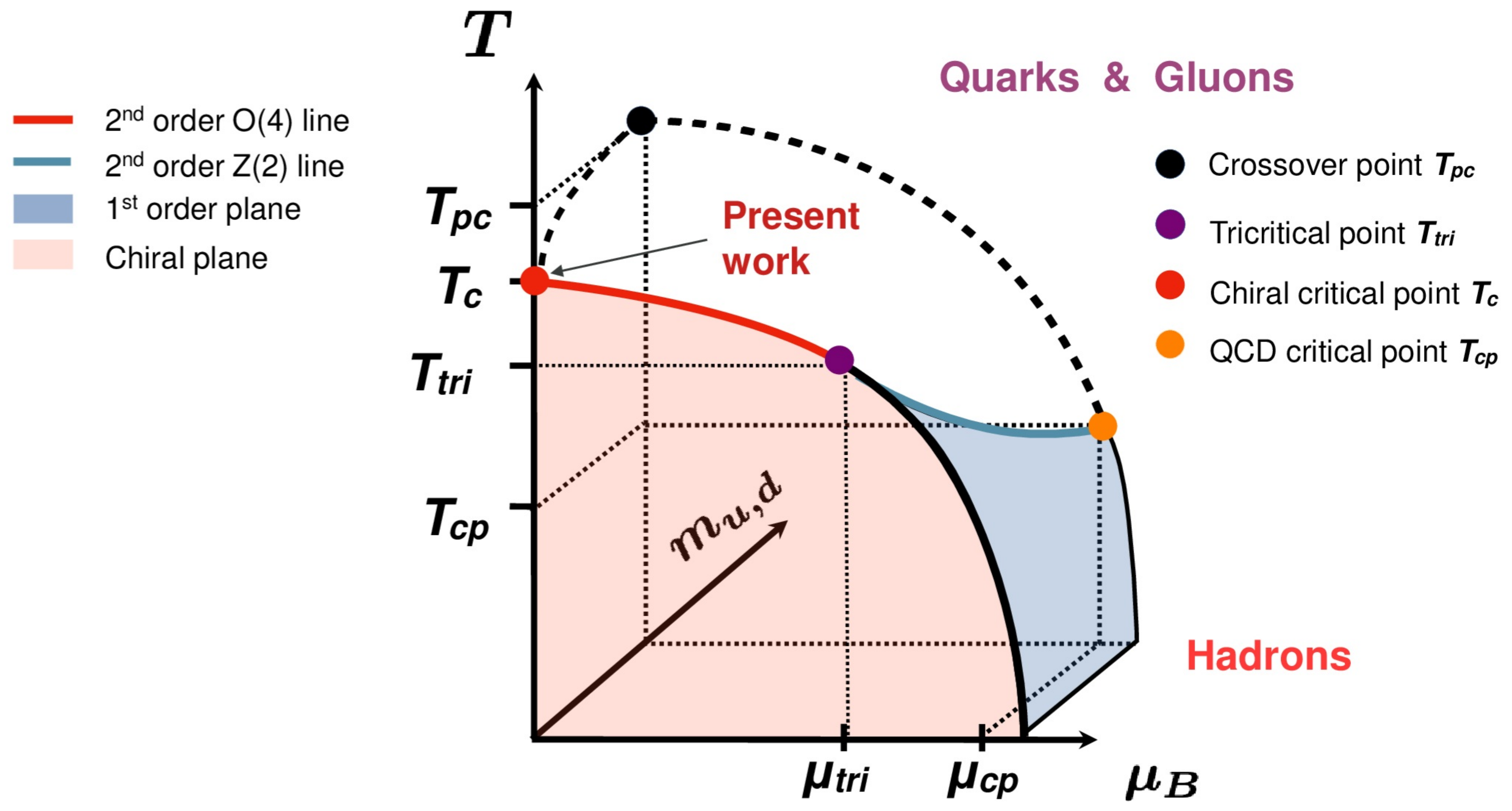
Motivation and Introduction



Credit : CERN courier

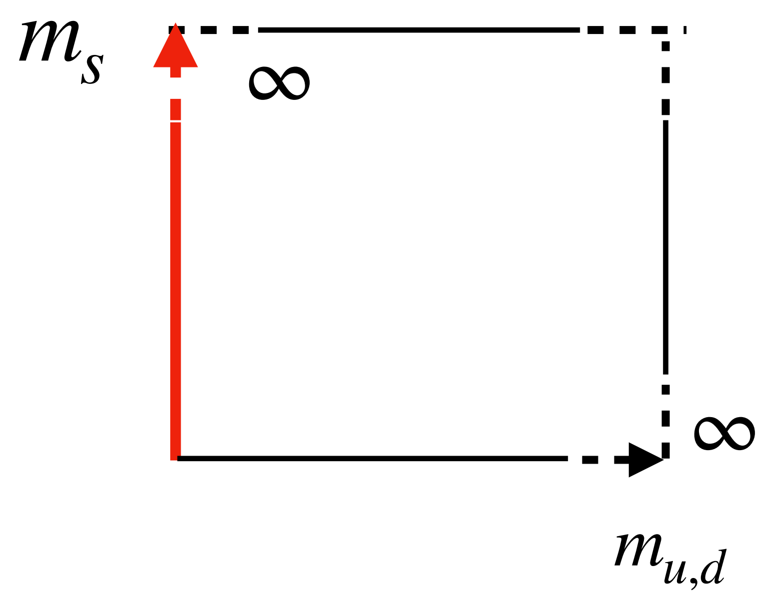


Numerous interesting talks in this workshop

- For **narrowing** onto the **QCD critical point**, one needs knowledge of **chiral phase transition** (cPT) .
- It is also important for constructing early universe QCD, explain heavy-ion experiments etc. ······
- This provides an **upper bound** on the QCD critical point temperature T_{cp} .

So, something brief about cPT

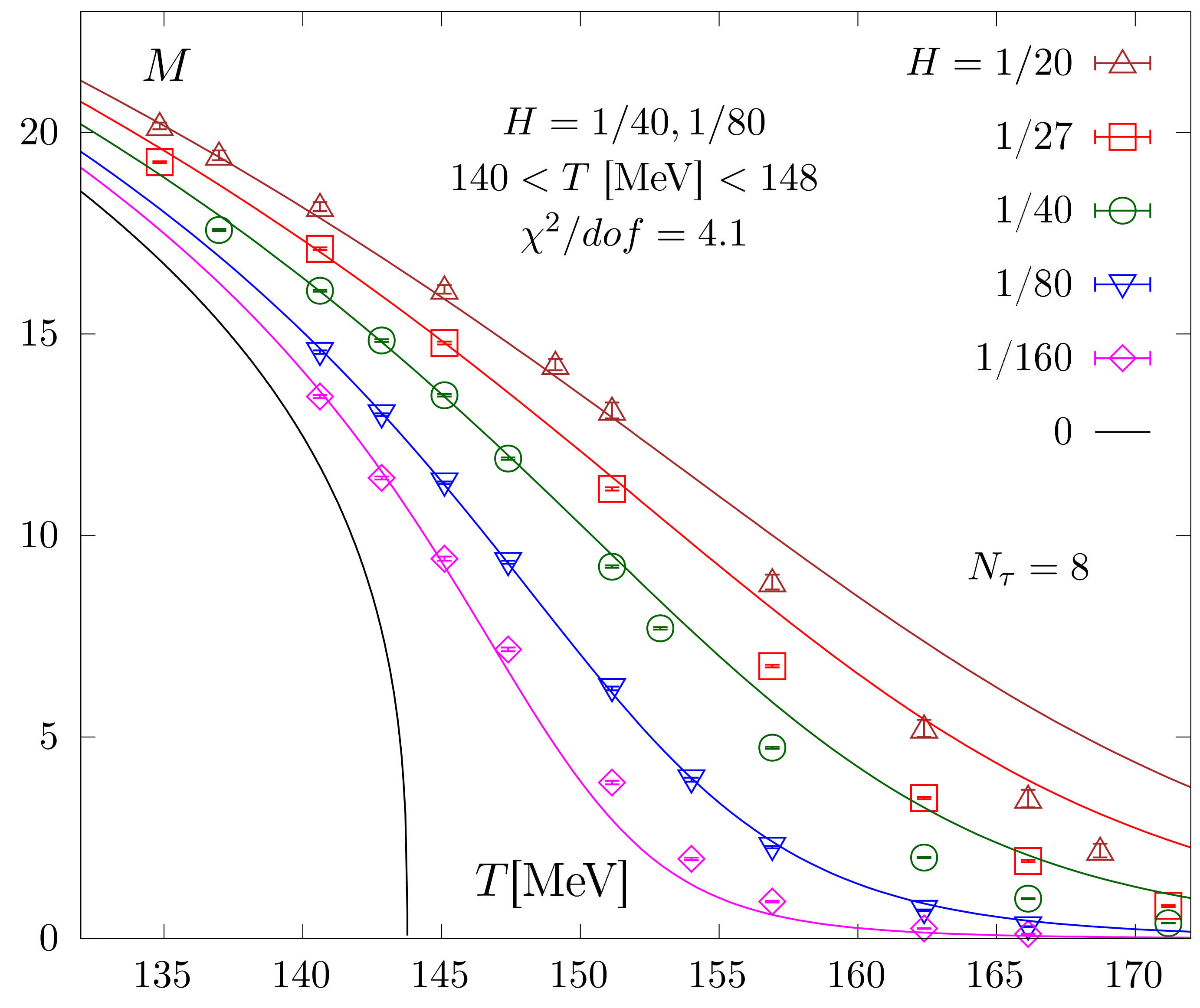
Chiral Phase Transition (cPT) : Brief

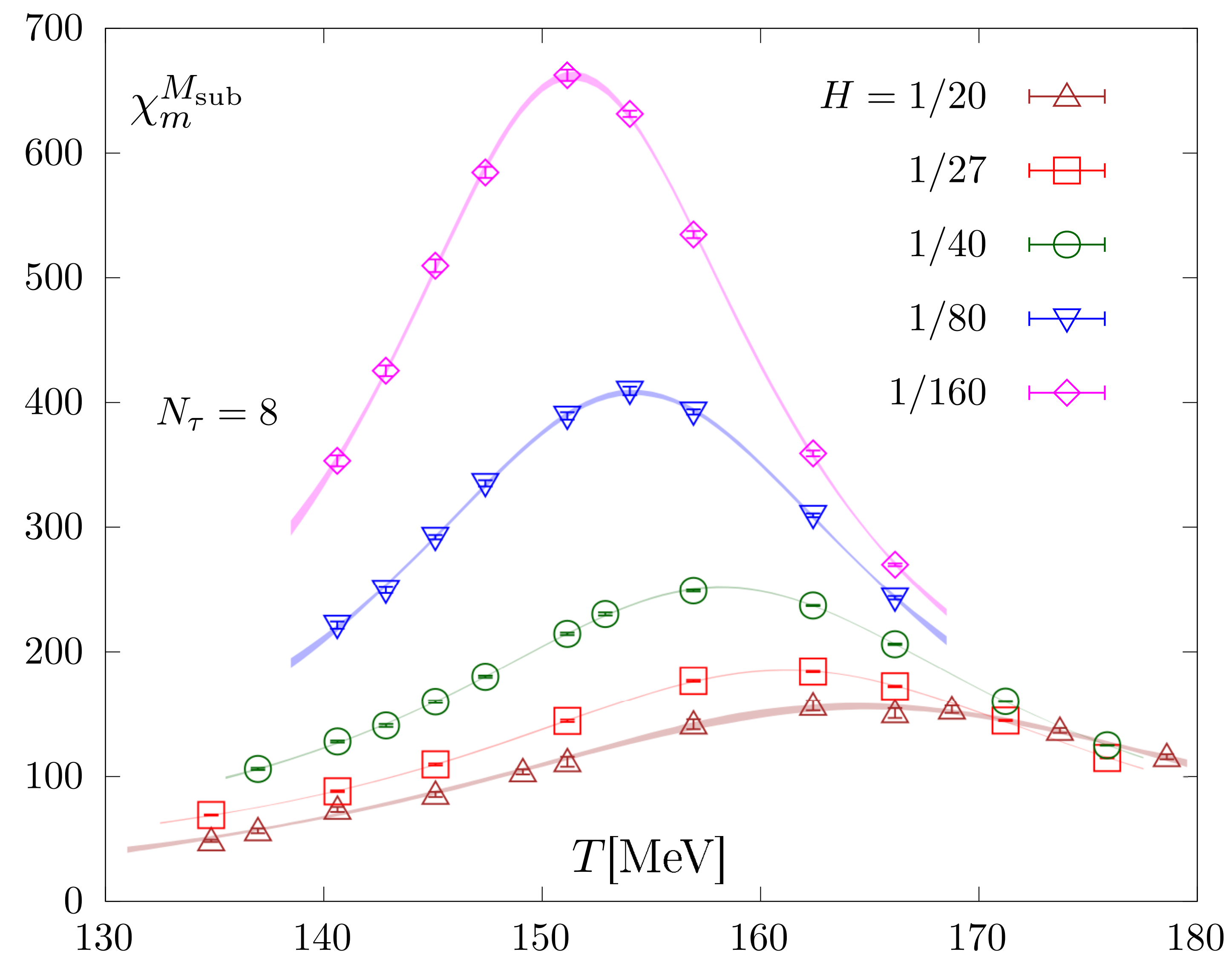


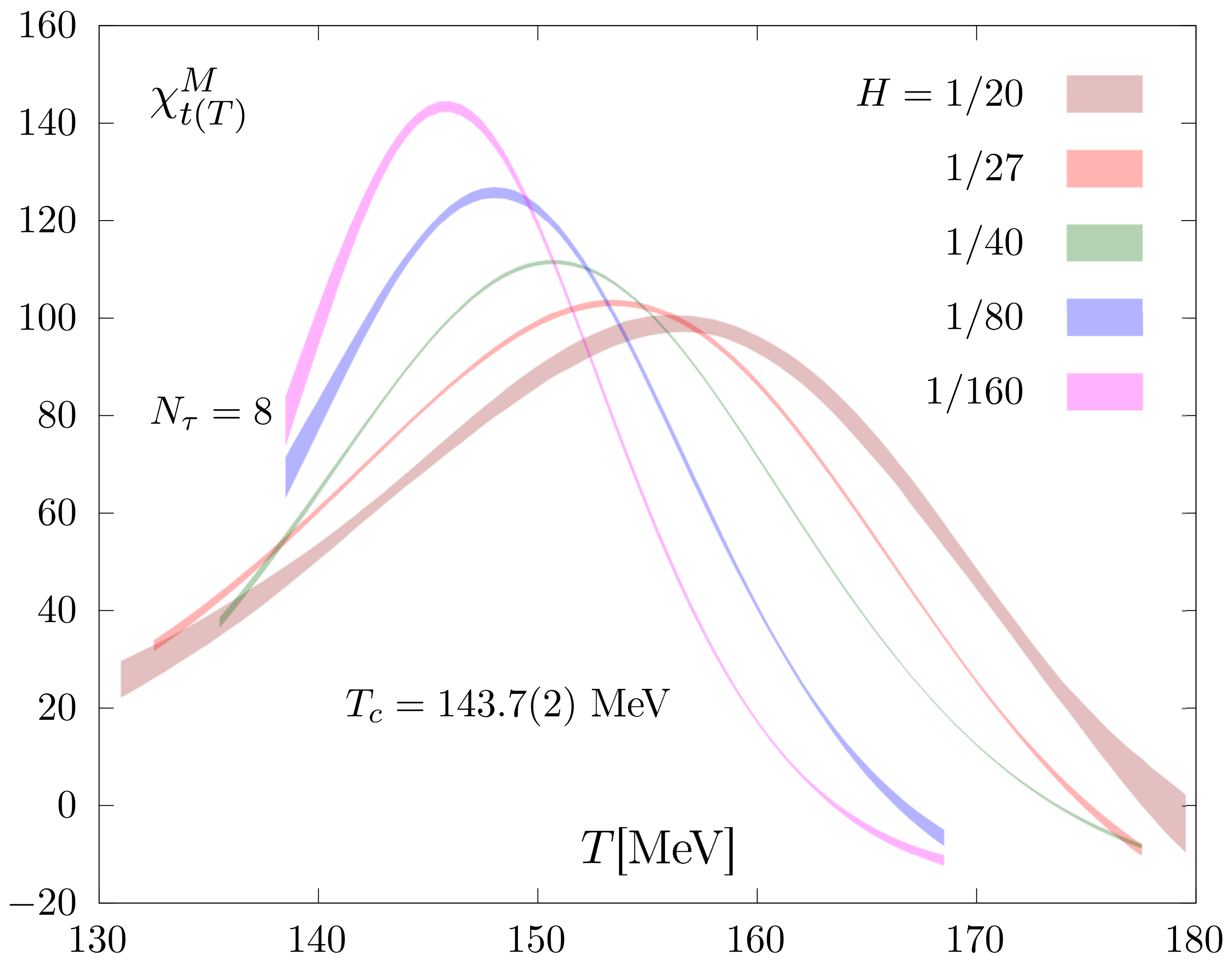
The “phase transition” between the high T - quark-gluon plasma (QGP) and the low T - hadronic phase

- In principle, this happens in the exact chiral limit $\left(m_u = m_d = m_{u,d} = 0\right)$
- This is a **second order** phase transition, with spontaneous symmetry breaking of the chiral symmetry group $SU(2)_L \times SU(2)_R \to SU(2)_V$
- The **order parameter** $M_{u,d} \sim \partial_{m_{u,d}} f$ remains **continuous, but non-differentiable** across $T = T_c$, for which there are **divergences in the chiral susceptibility** $\chi_{u,d} \sim \partial^2_{m_{u,d}} f$
- The universality class (UC) of this phase transition determines the **fate of axial $U(1)_A$ symmetry** (restored / broken) **at** $T = T_c$.

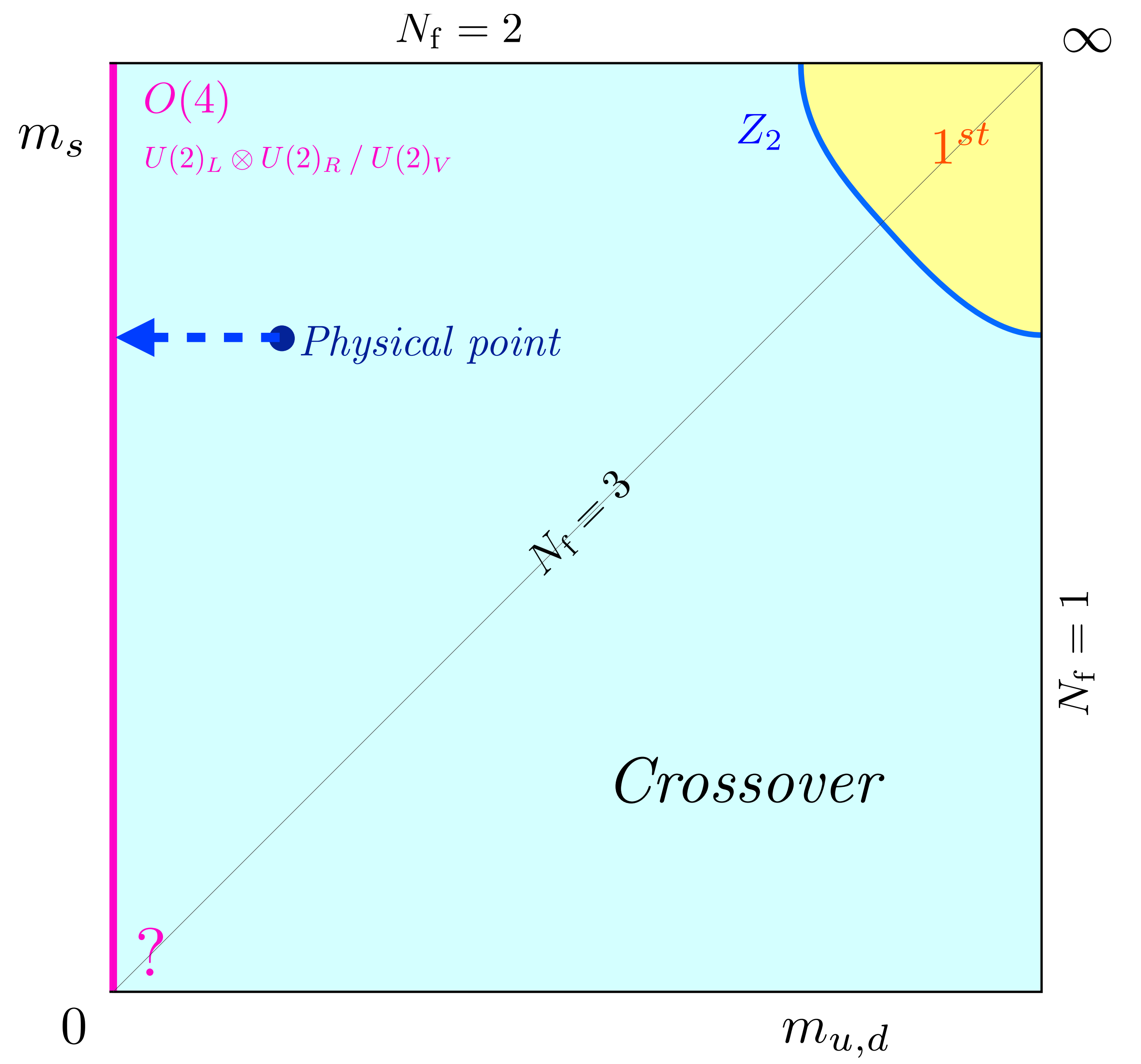
The present state-of-the-art status ?







Ding et. al. , PRD 109 (2024) 114516



Cuteri et. al. , JHEP 11 (2021) 141

Ding, Cuteri

Lattice QCD (LQCD) $\xrightarrow[\text{behaviour}]{M\,,\,\chi}$ 2^{nd} **order nature** of cPT (increasingly)

- However , **finite-volume effects (FVE's)** are **yet to be controlled**, specially **close to the chiral limit** . Thus,
- Demanding a **detailed finite-volume analysis (FVA)** here, and re-estimate the **cPT temperature** T_c and the **UC** .

The focus of this work

Some quick basic theory first ...

Theory

Close to $\left(T = T_c\ ,\ m_l = m_{u,d} = m_u = m_d = 0\right)$ for finite volumes, one can write :

$$M_\ell\,(T,H,L) = h_0^{-1/\delta} H^{1/\delta} f_G\,(z,z_L) + M_{\ell,\,reg}$$

$$\chi_\ell\,(T,H,L) = h_0^{-1/\delta} f_\chi\,(z,z_L)\, H^{1/\delta-1} + \chi_{\ell,\,reg}$$

Non-analytic singular and analytic regular terms

Scaling functions

Critical exponent δ

Scaling variables z, z_L : on $N_\sigma^3 \cdot N_\tau$ lattice

Lattice volume V :

$V \sim L^3\ ,\ L = N_\sigma / N_\tau$

$$\left.\begin{aligned} z &= z_0 \left(\frac{T - T_c}{T_c}\right) H^{-1/\beta\delta} \\ z_L &= z_{L,0}\ \frac{N_\tau}{N_\sigma}\, H^{-\nu/\beta\delta} \end{aligned}\right\}$$

$H = m_\ell / m_s$

Non-universal constants $h_0\ ,\ z_0\ ,\ z_{L,0}$

Heng-Tong's talk

Due to additive divergences in M_ℓ , ...

Improved order parameter and scaling function

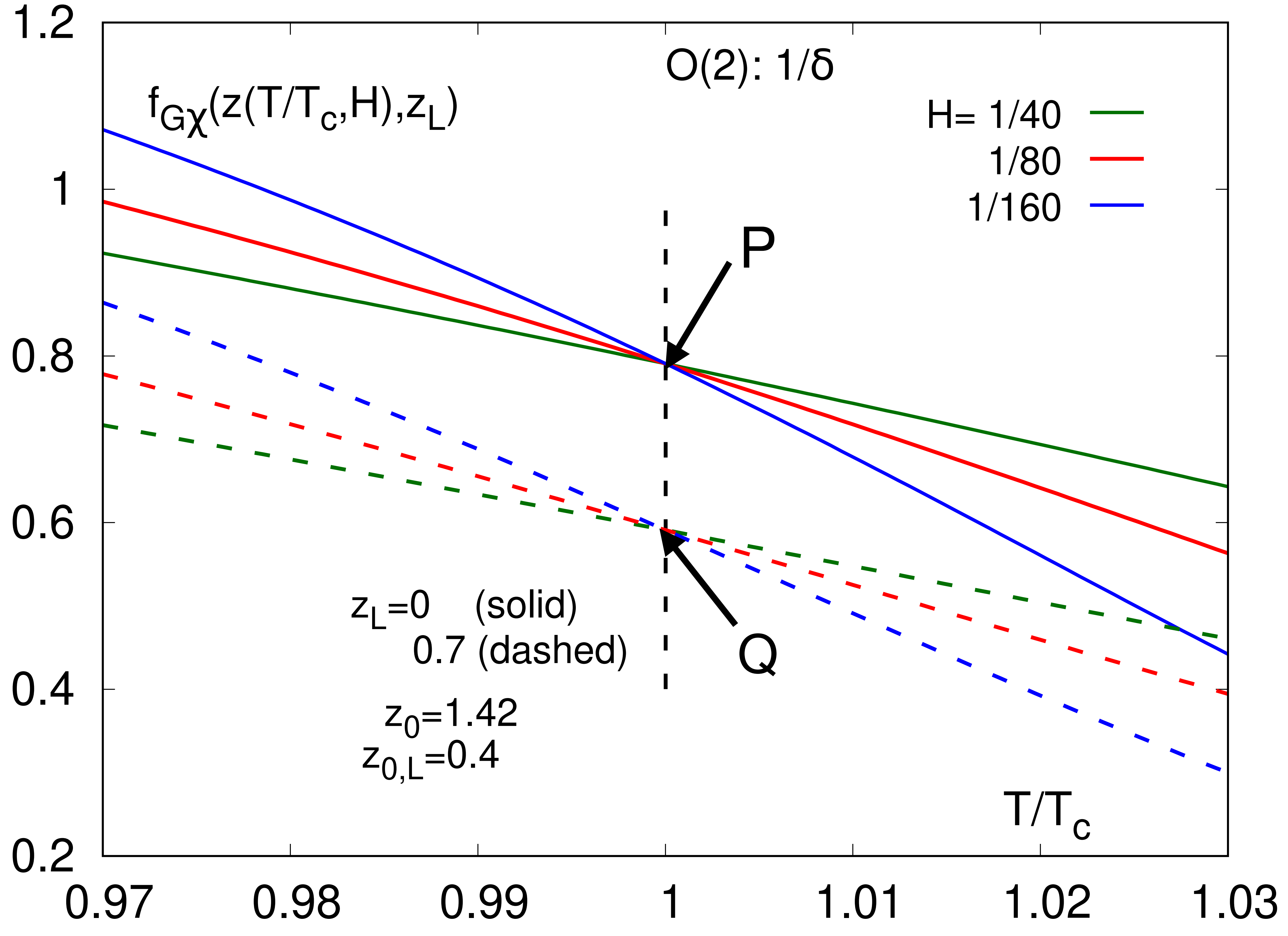
We use an $\longrightarrow$ "Improved" order parameter

$$M = M_\ell - H\chi_\ell = h_0^{-1/\delta} H^{1/\delta} f_{G\chi}(z, z_L) + M_{reg}$$

"Improved" scaling function
$f_{G\chi} = f_G - f_\chi$

- This has **no** additive divergences .
- Therefore, it is **well-defined** in the **continuum and chiral limits** .
- This also has **reduced regular** (reg) contribution in H , $\mathcal{O}\left(H^3\right) < \mathcal{O}(H)$
- **Augmenting singular behaviour** $\rightarrow$ making scaling analysis more **convenient** (relatively **straightforward** extraction of T_c and exponent δ)

Let's explore this new scaling function



$$f_{G\chi} = \left(1 - \frac{1}{\delta}\right) f_G + \frac{z}{\beta\delta} f_G^{(z)} + \frac{z_L \nu}{\beta\delta} f_G^{(z_L)}$$

A unique intersection point for all H lines at $T = T_c$

$V \to \infty$ $\quad z_L = 0$ $\quad$ P

$z_L \neq 0$ $\quad$ Q $\quad V < \infty$

Obtain δ estimate <u>directly</u>

Not exact δ, finite-volume $\equiv \mathcal{O}(z_L)$ corrections

$$f_{G\chi}(z = 0, z_L = 0) = (1 - 1/\delta)$$

Here, we used δ of $O(2)$

Encountered in Lattice calculations

So …

- **On lattice**, it is **not possible** to **extract δ directly** $\leftarrow$ **finite-volume corrections**
- Hence, a careful finite-volume analysis is very much required .
- To enable **reliable infinite $V-$ extrapolation** , and thus, determine T_c and δ
- Following a parameter-free approach directly from LQCD calculations (**no prior assumption of the universality class required** before conducting LQCD estimates)

A quick snippet into this parameter-free method

Mitra et. al. , PoS Lattice2024 (2025) 187

Parameter-free method

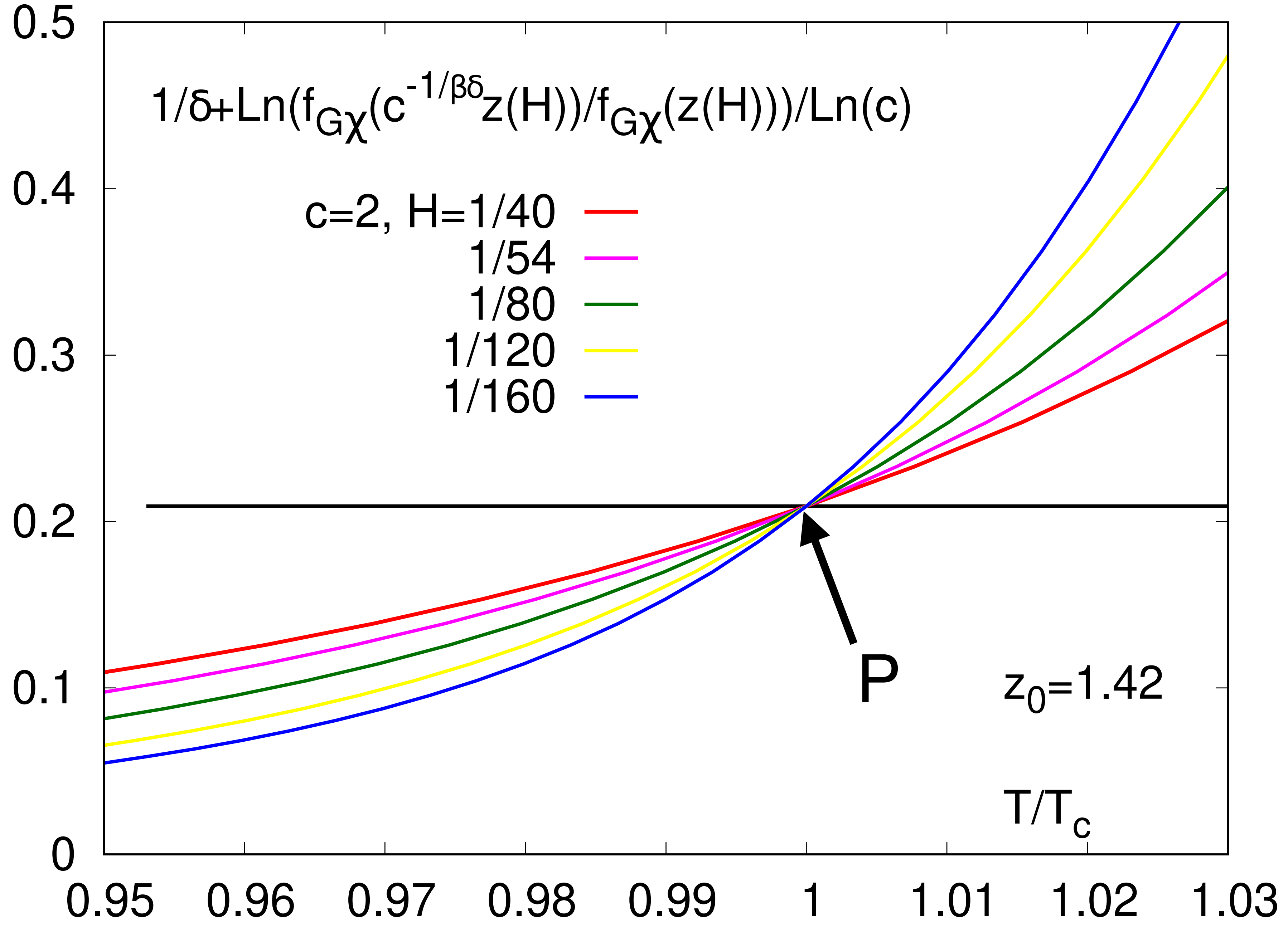
LATTICE 2024 talk

In $V \to \infty$, construct

$$B(T, H_1, H_2) = \frac{\ln\left[\frac{M(T,H_1)}{M(T,H_2)}\right]}{\ln\left[\frac{H_1}{H_2}\right]} = \frac{1}{\delta} + \ln\left[\frac{f_{G\chi}(z_1(T,H_1))}{f_{G\chi}(z_2(T,H_2))}\right]$$

Fully from lattice

In chiral limit ($H \to 0$)



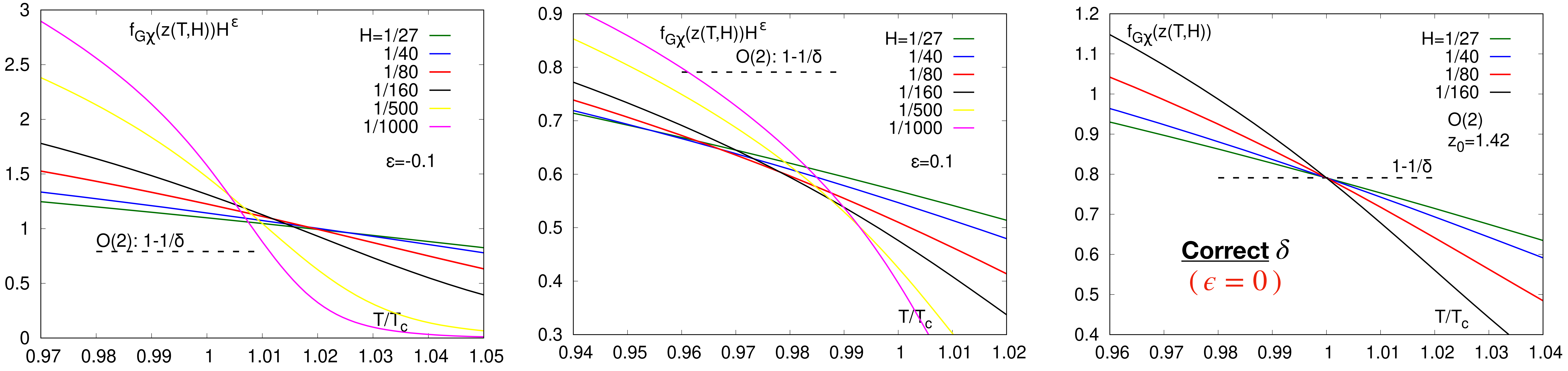
- $B(T_c, H_1, H_2) = 1/\delta$ with $H_2 = cH$, $H_1 = H$.
- We see the unique intersection point **P** $\left(T_c, \frac{1}{\delta}\right)$ in B vs T plane

A way to extract T_c and δ from **direct LQCD calculations**

Feature of uniqueness important

As we will see now …

Uniqueness as indicator



We compute here

<u>Wrong</u> δ $(\,\epsilon \neq 0\,)$

$$\widetilde{M} = H^{-1/\delta} M = \quad h_0^{-1/\delta} \; H^{\epsilon} \; f_{G\chi}(z, z_L) \; + M_{reg\,,H}$$

calculated on lattice

Exploit this feature of uniqueness of intersection in LQCD calculations

We use δ of $O(2)$ in our LQCD estimates (educated ansatz)

But before that, ...

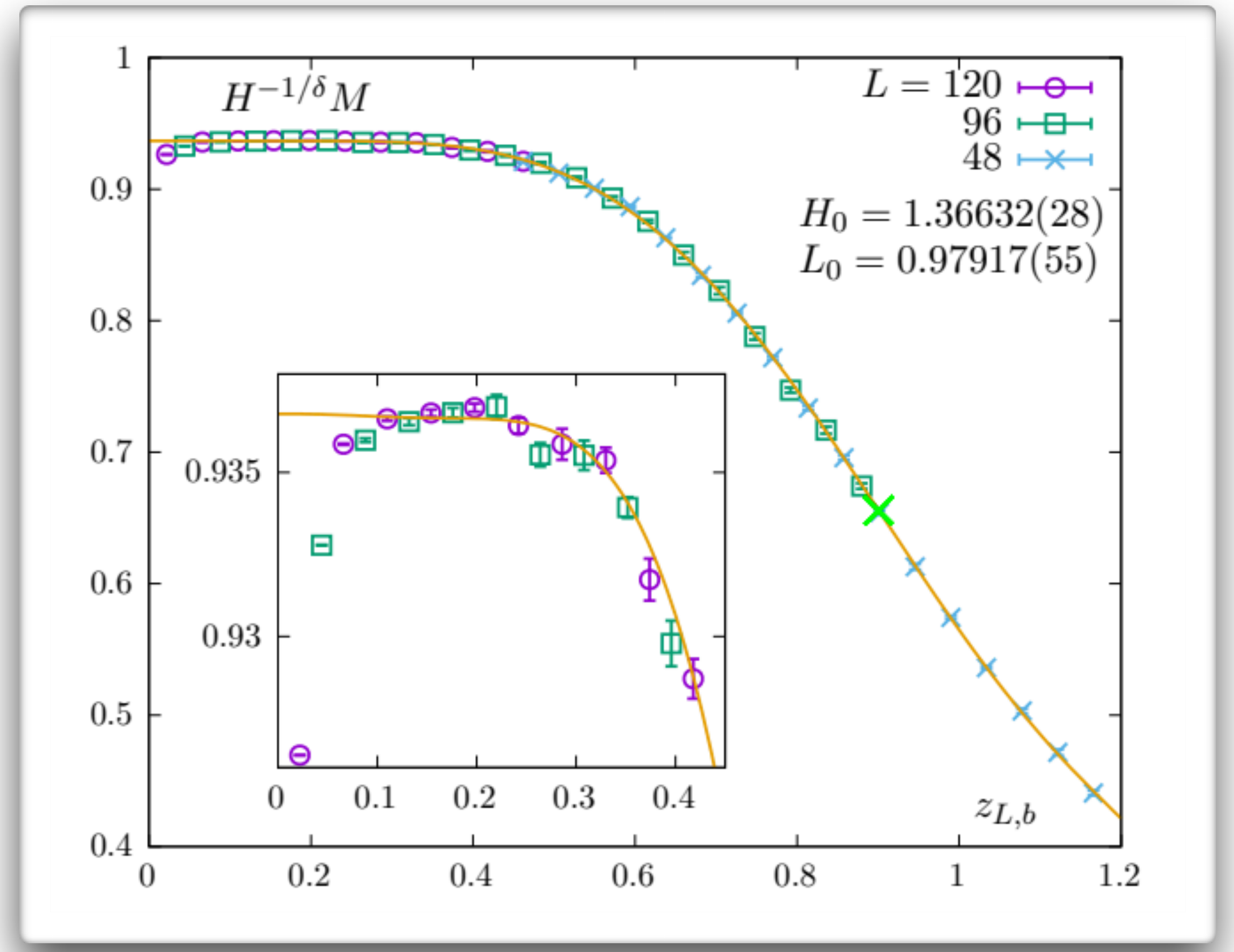
Volume dependence in scaling function

Taylor expansion of $f_{G\chi}$ around $z_L = 0$ Karsch et. al. , PRD 108, 014505

$z_{L,b} = L^{-1} H^{-\nu/\beta\delta}$

$$\boxed{f_{G\chi}(z, z_L) = f_{G\chi}(z,0) + \sum_{n=0}^{n_u} \sum_{m=3}^{m_u} \left(1 - \frac{1}{\delta} + \frac{n + m\nu}{\beta\delta}\right) a_{nm}\, z^n\, z_L^m}$$

Infinite volume scaling function

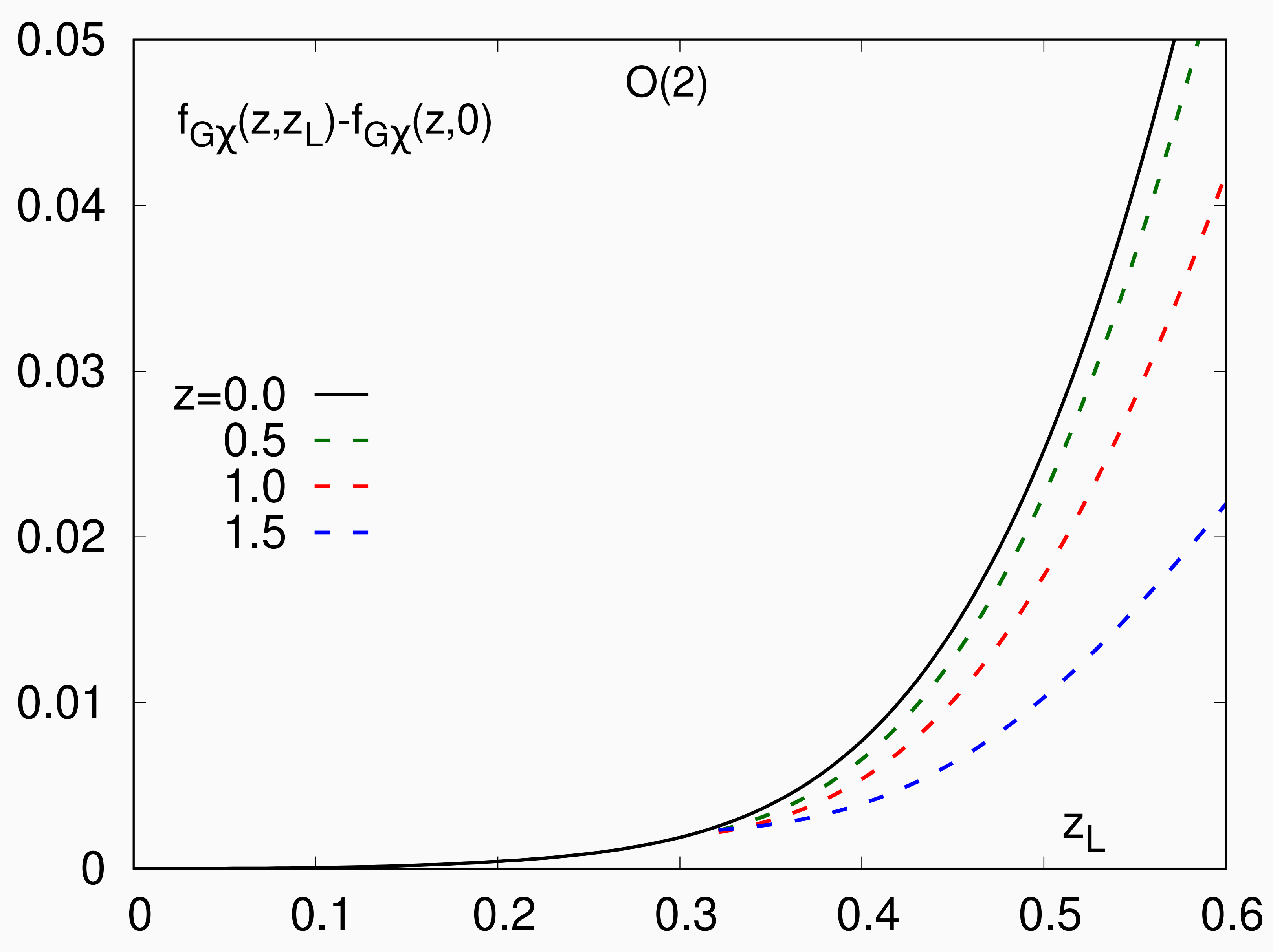


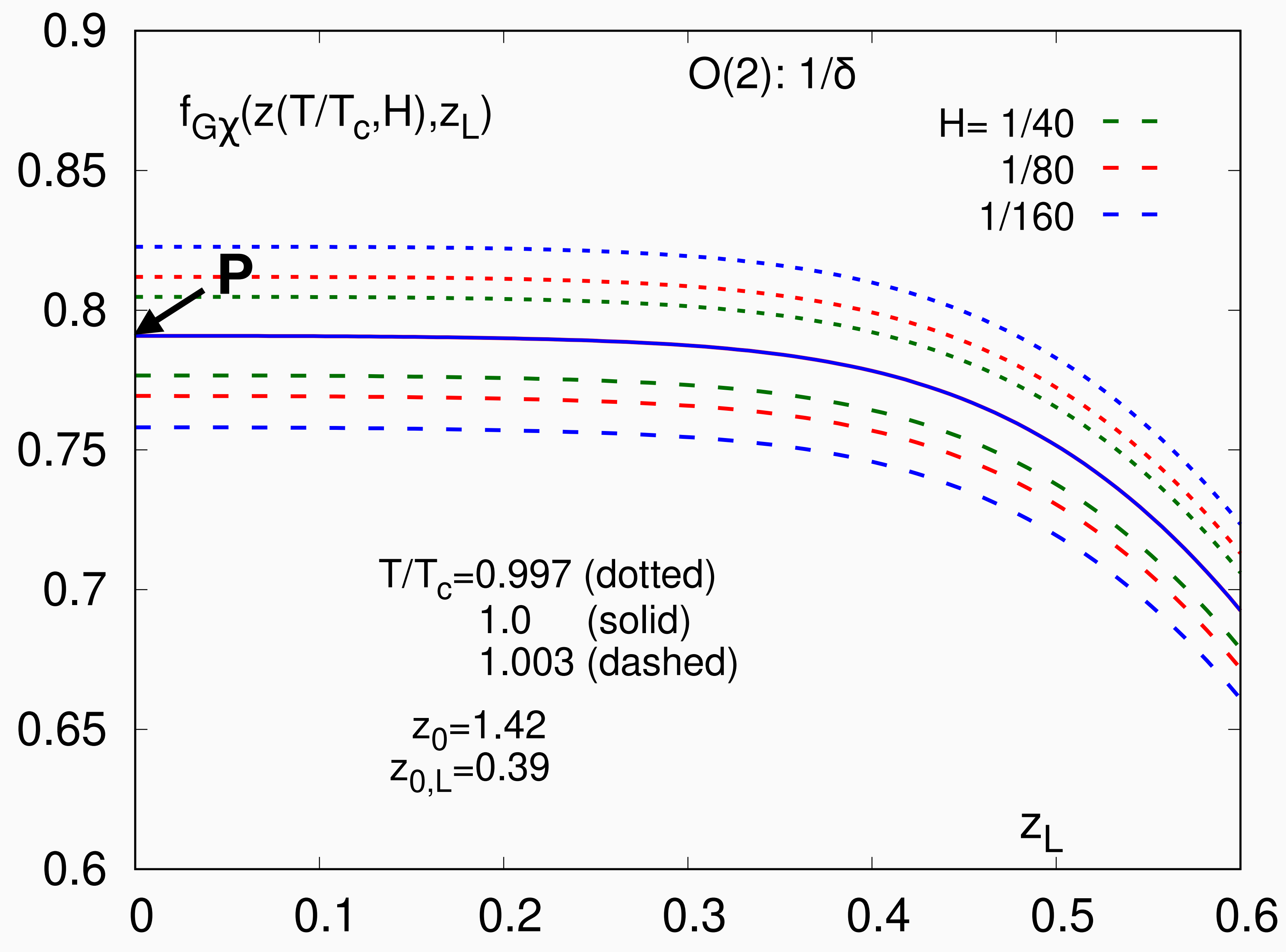
Karsch et. al. , PRD 108, 014505

- $m_l = 3$, as Leading order correction $\sim \mathcal{O}(1/V)$
- $n_u = 5$, $m_u = 8$ $\leftarrow$ best fit to Monte-Carlo data on different lattices
- $\beta \sim 0.35$, $\delta \sim 4.78$, $\nu = \beta(\delta + 1)/3$ $\in$ 3-d $O(2)$ universality class
  - $f_{G\chi}(0,0) = (1 - 1/\delta)$, at $T = T_c$, $V^{-1} = 0$

Some FVE observations …

Finite-volume improved scaling function





- FVE's are **more severe** (**sharp rise**) as one goes **towards** T_c i.e. **lower** z (**greater sensitivity**)
- **Well-controlled** finite-volume effects (FVE's) for $z_L \leq 0.4$, for $T \to T_c$ Karsch et. al. , PRD 108, 014505
- **One unique line** for $T = T_c\,(\,z = 0\,)$, whereas **no such uniqueness** for $T \neq T_c$.
- This translates to **unique intersection point in** T **plane** for $T = T_c$.
- $H-$ dependence for $T \neq T_c$: **ordering reversal** for $z < 0\,(\,T < T_c\,) \iff z > 0\,(\,T > T_c\,)$

Note : No LQCD here, these are model-driven results,

What does LQCD say … ?

Going towards the lattice : Plan forward

- Use δ of $O(2)$ in the pre-factor $H^{-1/\delta}$
- Use the finite-volume dependent form of the scaling function $f_{G\chi}$ (shown before)
- Thus, compare the lattice data with these benchmarks, and
- Try to improve the finite-volume analysis and its systematics .

Lattice setup …

Lattice setup

- We use highly improved staggered quarks / fermions (HISQ).
- All our simulations are done presently on $N_\tau = 8$ lattices.
- We use δ of 3-d $O(2)$ universality class , for finite $V \sim L^3$ calculations here
- We use $T_c = 145.1$ MeV

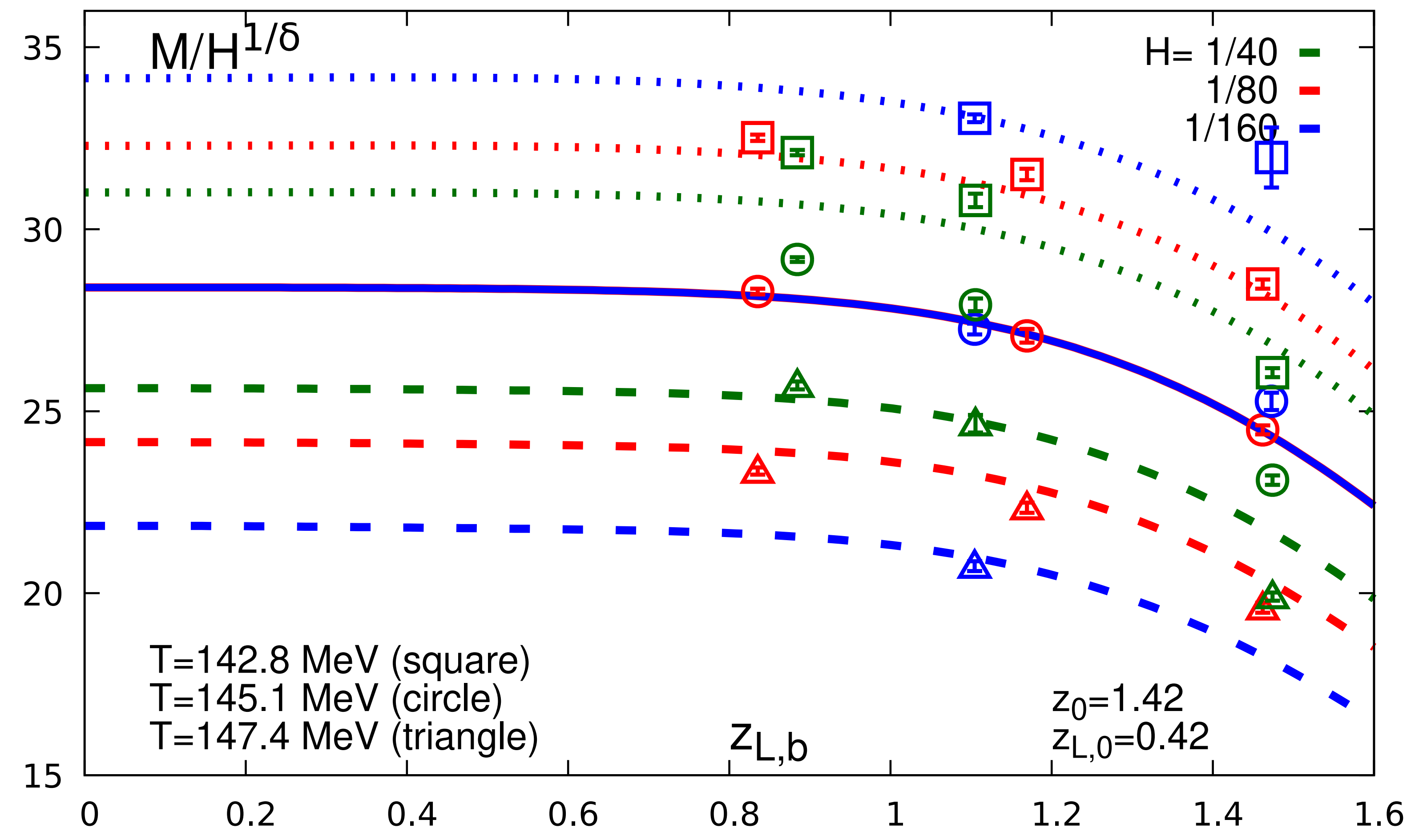
Staggered fermions retain only $O(2)$ part of the full chiral symmetry

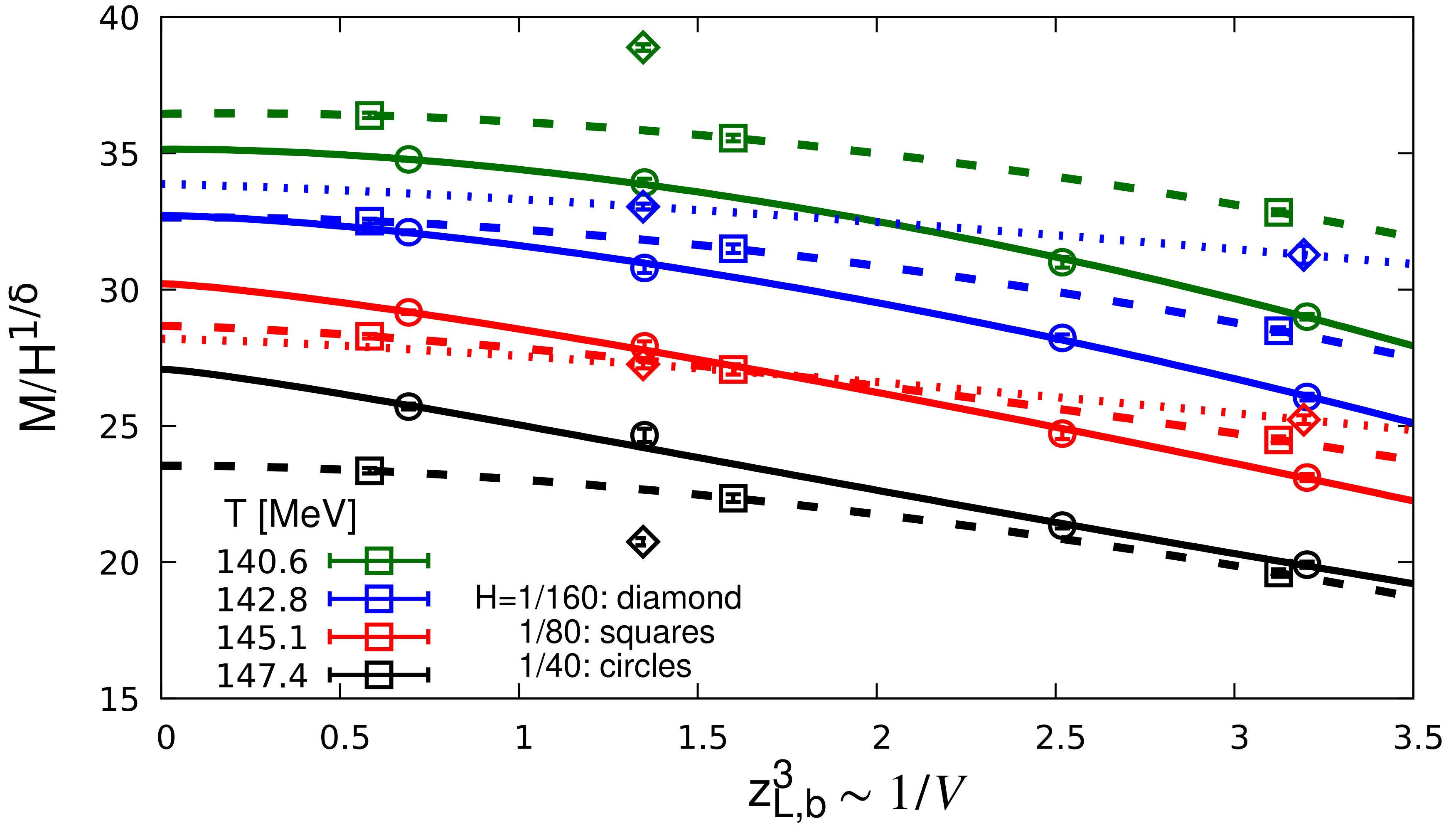
The parameters T, H, L are tuned as follows :

- We vary T close to T_c , following $T \in \left(0.99\, T_c\, ,\, 1.01\, T_c\right)$
- With physical m_s , m_ℓ is varied with $1/240 \leq H \leq 1/27$
- Different spatial volumes **on** $N_\tau = 8$: $L = N_\sigma / N_\tau \in (3\, , 10)$

$$|z| < 0.5\, ,$$
$$0.4 < z_L < 1$$

Lattice Results





Better agreement between Lattice data $\Longleftrightarrow$ $O(2)$ benchmarks :

- For **lower** $z_{L,b}$ $\Longleftrightarrow$ closer to infinite V limit $\left(z_{L,b} = 0 \right)$
- For **lower** H values $\Longleftrightarrow$ approaching chiral limit $(H \to 0)$

Bare scaling variable

$$z_{L,b} = z_L / z_{L,0} = \frac{N_\tau}{N_\sigma} H^{-\nu/\beta\delta}$$

Ongoing $\longrightarrow$ $H = 1/160$, on $78^3 \cdot 8$ lattices for $T = 142.8\,,\, 145.1\,,\, 147.4$ MeV

A third blue point for three T

$z_{L,b} \sim 0.8\ ,\ z^3_{L,b} \sim 0.5$

Closer

Conclusions

- We work with an **improved order parameter** M having **well-defined continuum and chiral limits**, **reduced regular contributions** and **no additive divergences**.

- **Uniqueness is an important signature** to identify correct δ (essence of a unique point)

- We find **good agreement** between the expected $O(2)$ singular behaviour and the Lattice QCD results . (An **increasingly convincing indication** about the $O(2)$ **nature of cPT universality class**) .

- This agreement **improves** as expected for **smaller** H (chiral limit), and **smaller** $z_{L,b}$ (infinite-V limit)

- **More well-controlled** finite-volume effects for $z_{L,b} < 1 \Rightarrow$ need for **higher** N_σ **, for smaller** H (demanding more computational expansivity)

Still a lot going on, and a lot to do

Outlook

- Come up with the results of the new $H = 1/240$ data point
- Quantify regular contributions and analyse their volume dependence .
- Continue with our finite-volume work $\Rightarrow$ infinite V extrapolation $\Rightarrow$ parameter-independent way of extraction of δ from direct LQCD calculations **Mitra et. al. , PoS Lattice2024 (2025) 187**
- Add more $(T\ ,\ m_\ell)$ data points to attain higher precision, with the goal to distinguish between δ of $O(2)$ and that of $O(4)$ (in continuum) .
- Approach continuum limit and re-estimate volume corrected, continuum estimate of T_c (how different is it ??) .
  - **Ding et. al. , PRL123 (2019) 6, 062002**

THANK YOU

for your

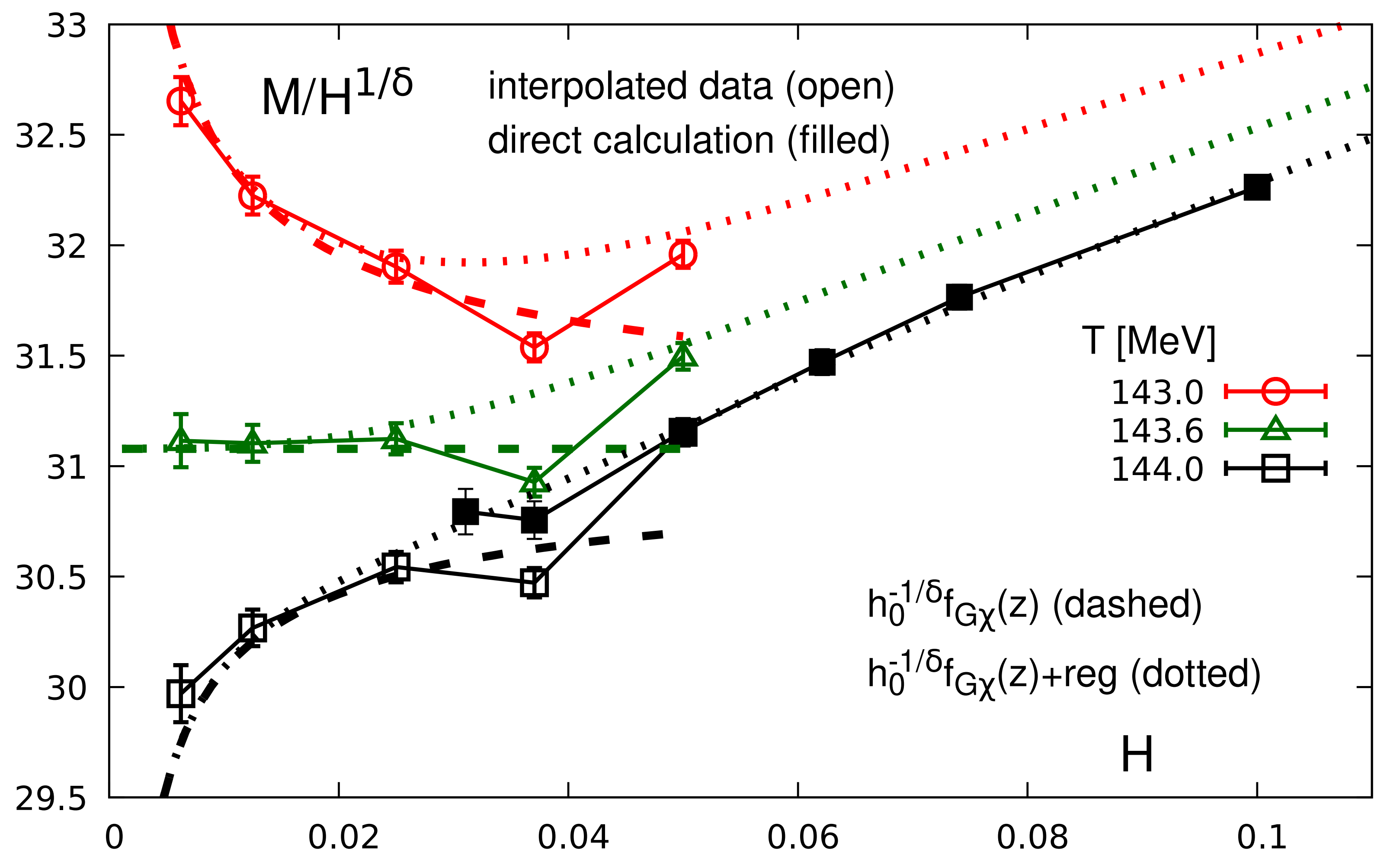
PATIENCE and

ATTENTION

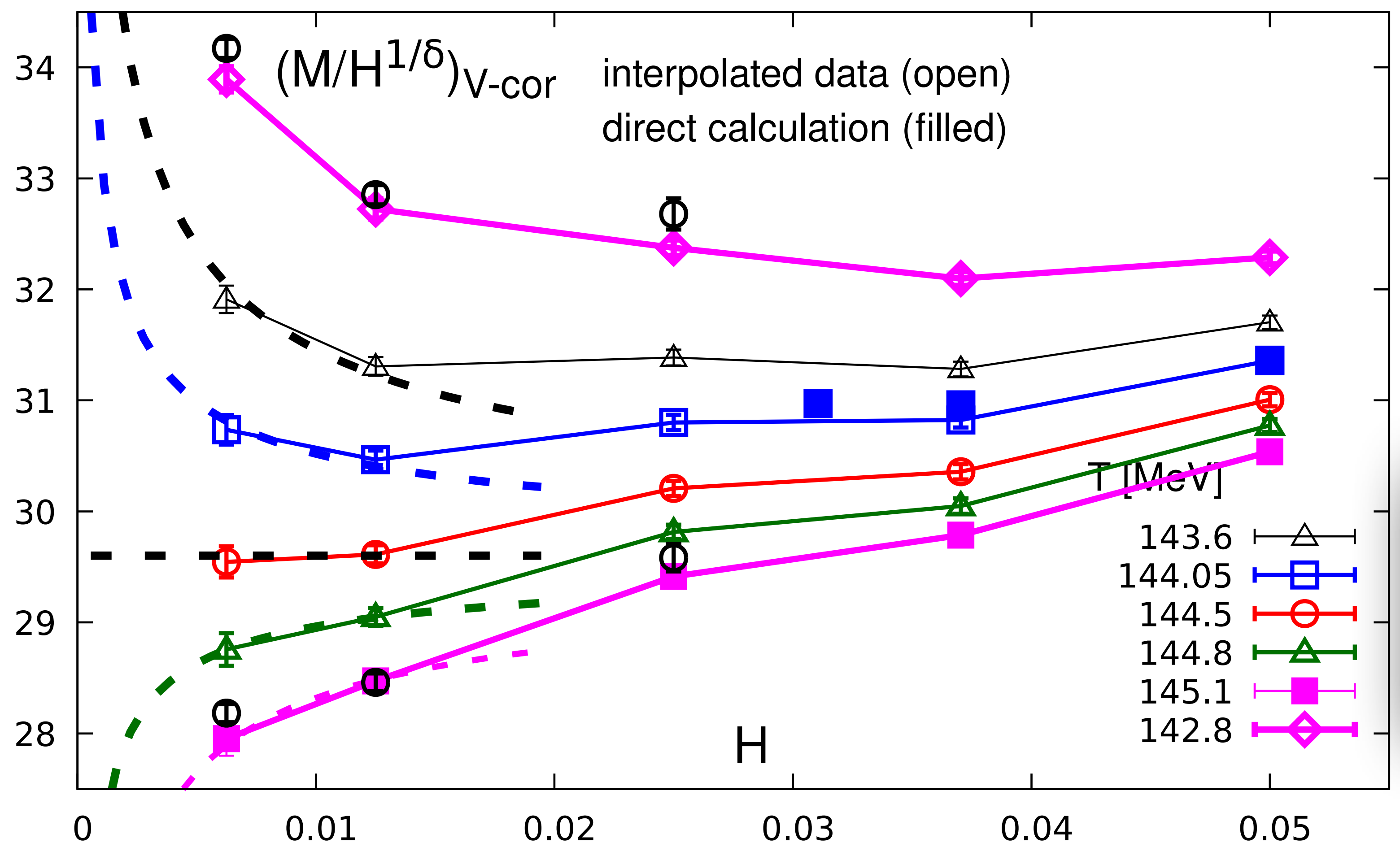
Credit: Organisers

Backup slides

Lattice QCD results



with $T_c = 143.6$ MeV



with controlled FVE's

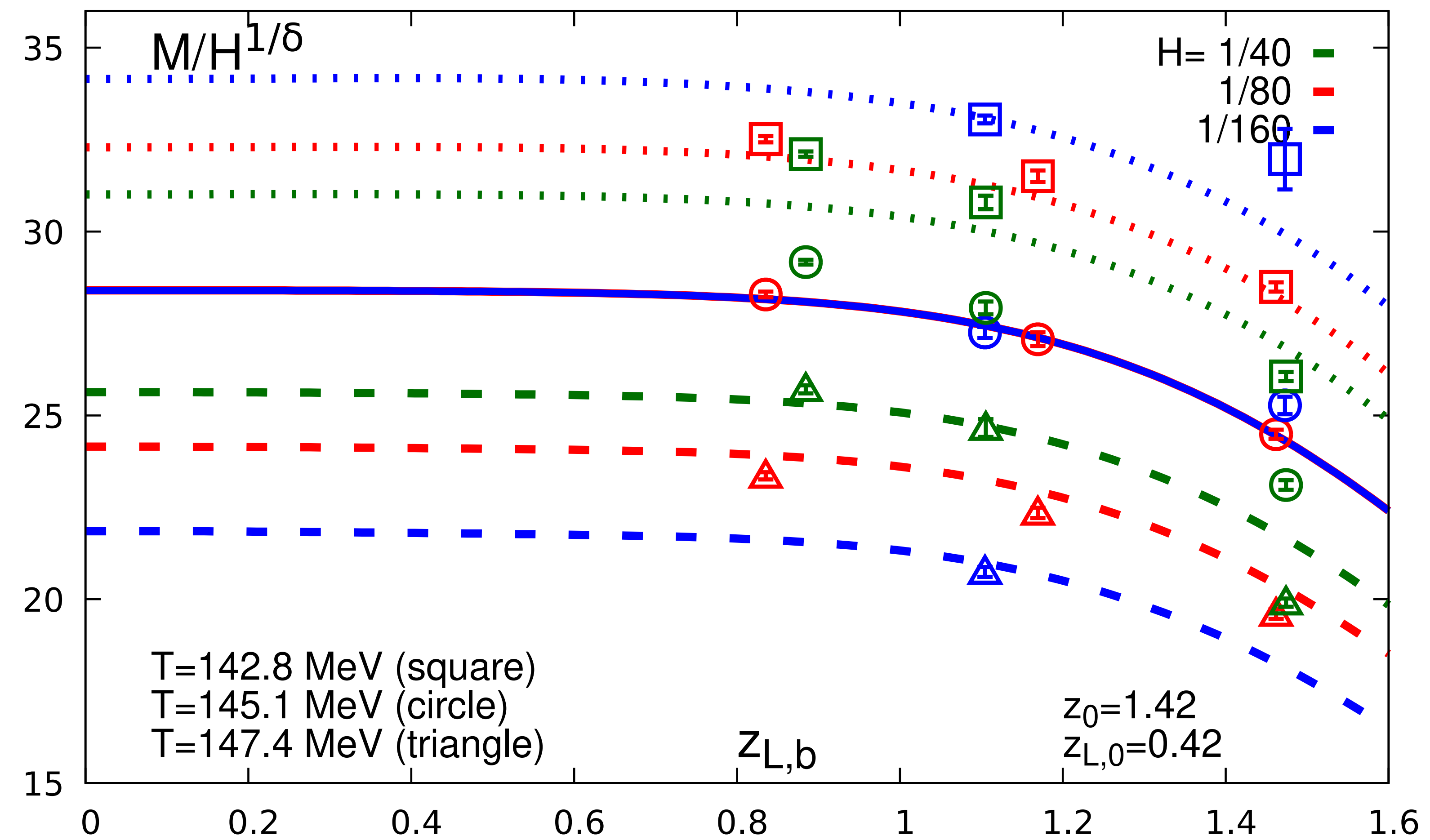
LQCD results for 142.8 , 144.05 , 145.1 MeV

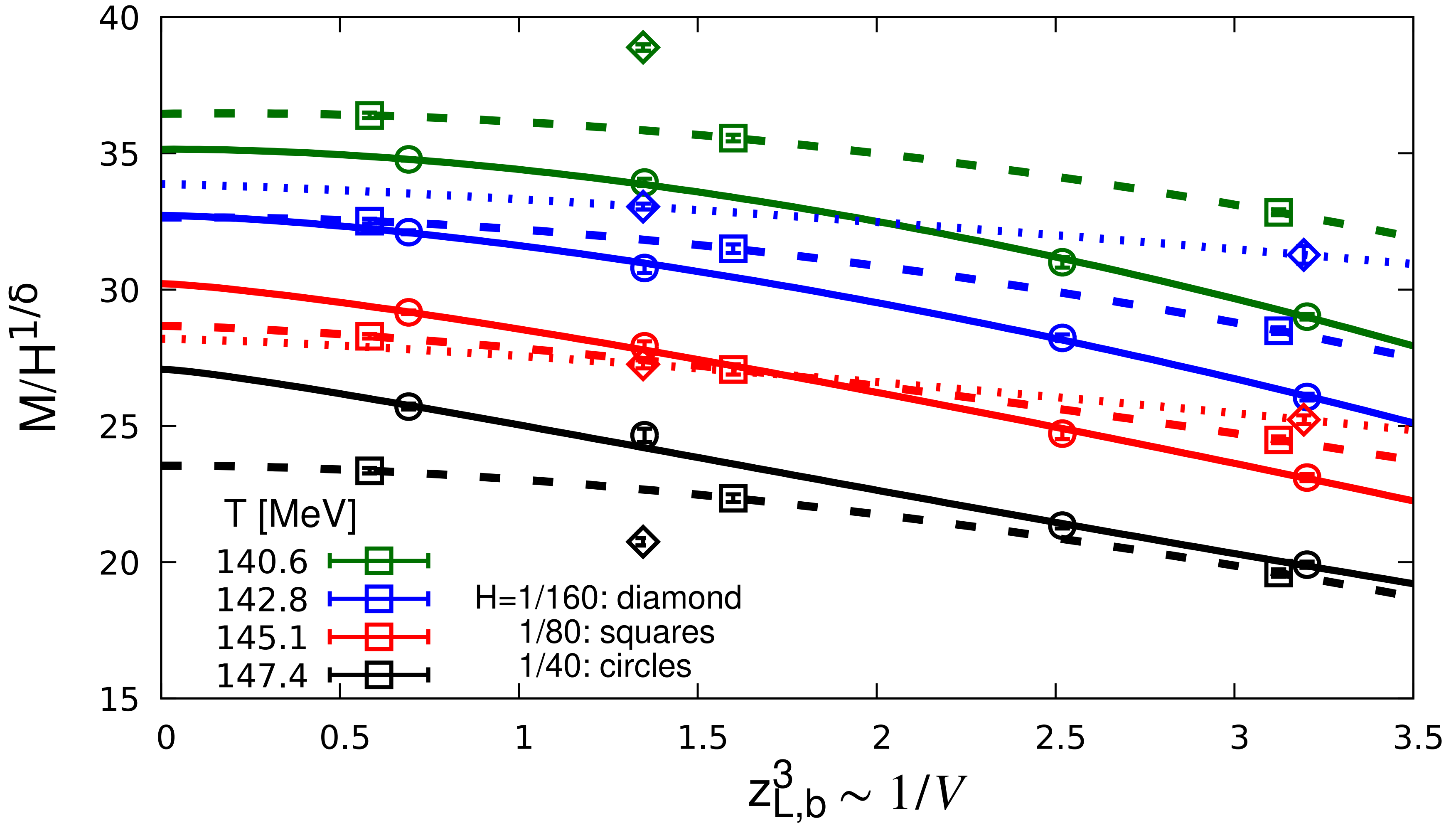
- Good agreement with universal (singular) behaviour apparent for sufficiently low $H < 1/40 = 0.025$ (expected in chiral limit)
- Also, visible deviations from this for "heavier" $m_\ell \equiv$ higher H
- Regular contributions still to be quantified

- Similar story of agreement and deviations, offering universal manifestations in small H.
- More control on FVE's on lowest $H = 1/160$ data point , with two lattice data values computed

Causing new estimate of $T_c = 144.5 > 143.6$ MeV

More Work on higher N_σ in Progress ...





$$f(x) = a + b\,x^{4/3} + c\,x^{5/3} + \mathcal{O}\left(z_{L,b}^{6}\right)$$

$$x = z_{L,b}^{3}$$