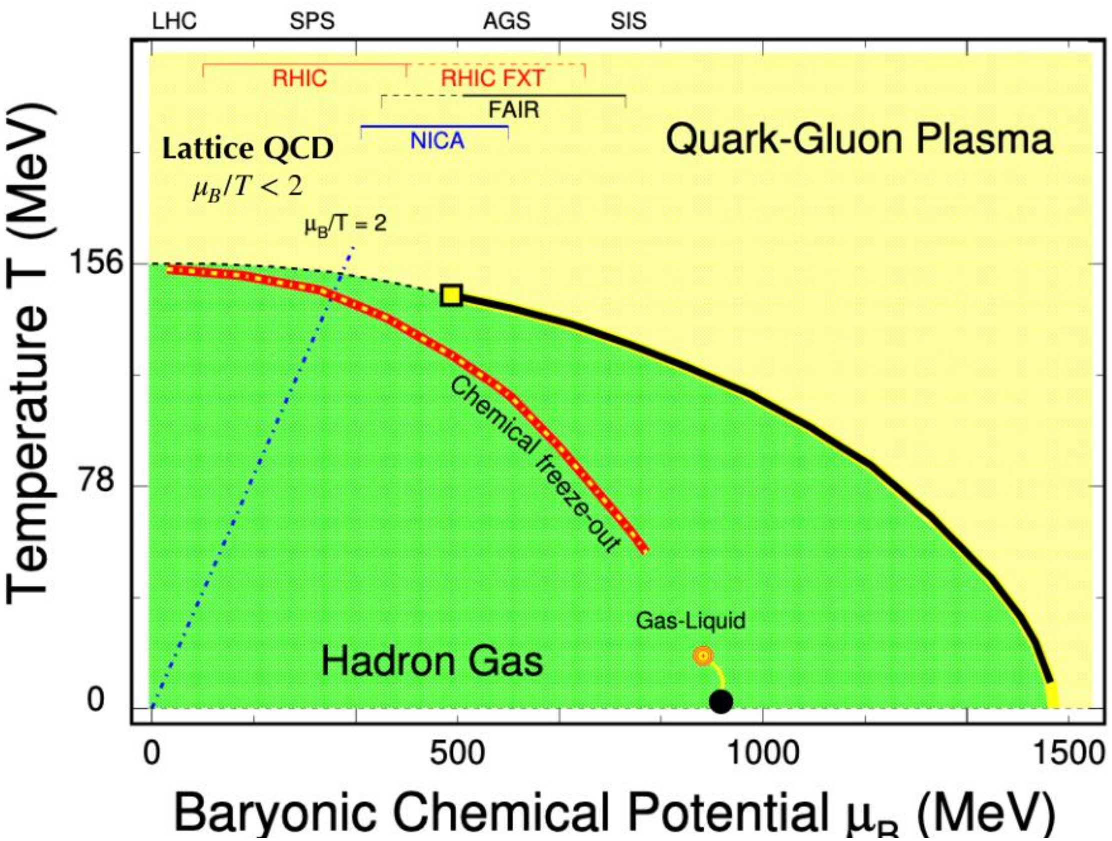
Stochastic Charge Transport in Relativistic Hydrodynamics

Baochi Fu

with Nicolas Borghini and Sören Schlichting

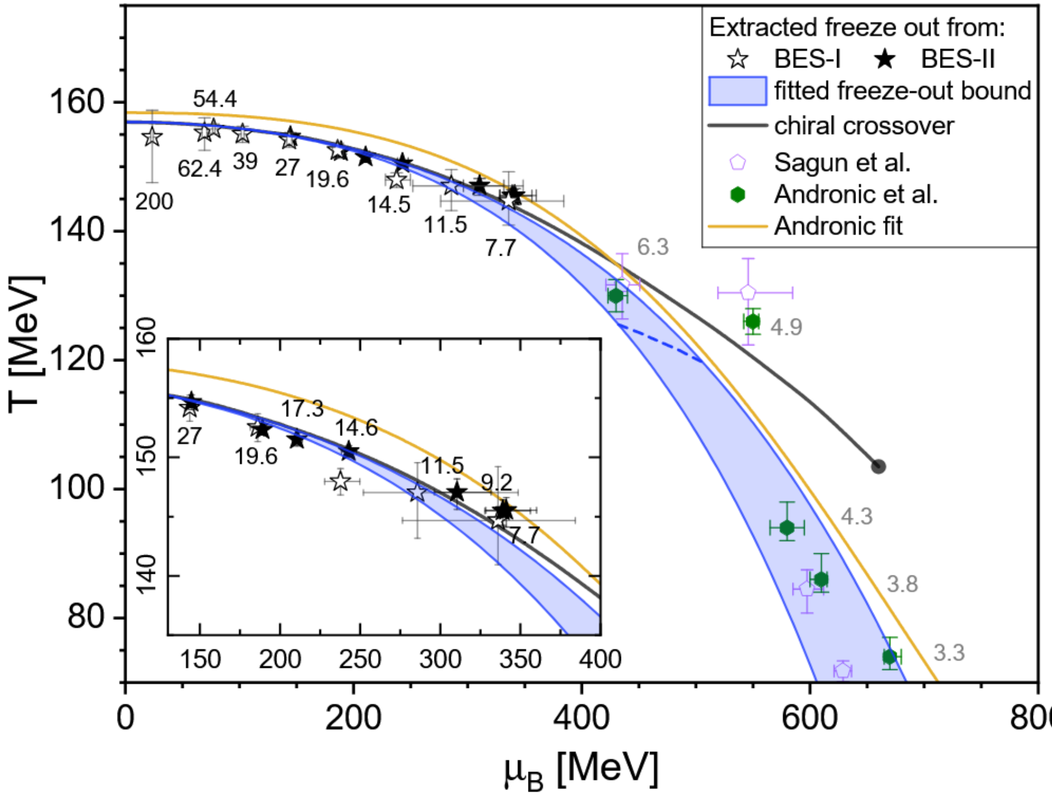
The QCD Phase Diagram: From Theory to Experimental Signatures,
8-11, Oct. 2025, Dalian

Probing the QCD Phase Diagram Through Fluctuations



X. Luo and N. Xu, Nucl.Sci.Tech. 28 (2017) 8, 112

Talks by Y. Lu and F. Gao

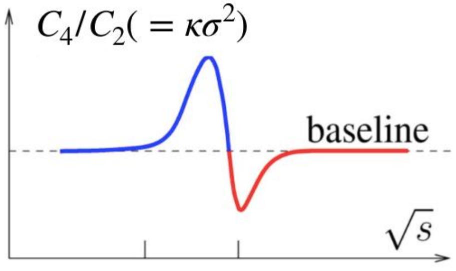


- HIC provide unique opportunities to probe QCD phase diagram
- Important observable: cumulants (fluctuation) ←→ susceptibilities
- How to compare equilibrium calculations and off-eq. signals?

$$C_2 = \langle (\Delta N)^2 \rangle$$

$$C_4 = \langle (\Delta N)^4 \rangle$$

$$\frac{C_4}{C_2} = \frac{\chi_4}{\chi_2}$$



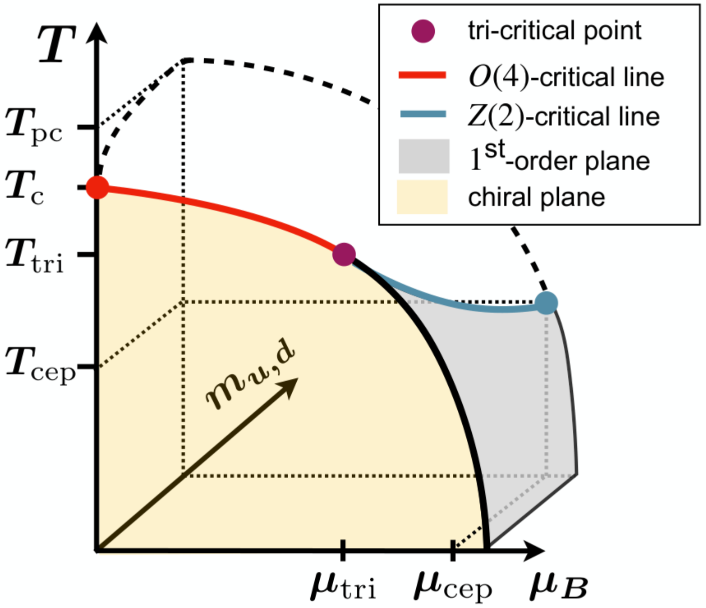
M. A. Stephanov, PRL 107 (2011) 052301

Assumption: Thermodynamic equilibrium

Importance of Dynamical Modeling

Pics@Xin An

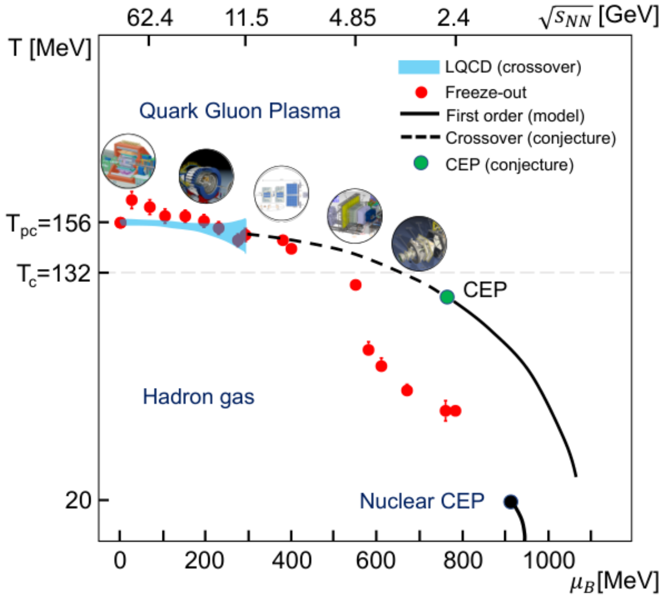
Theory



H.-T. Ding, et al.,
PRD 109 (2024) 114516

- Long lived and spatially infinite
- Uniform, particle heat bath
- And static

Experiments



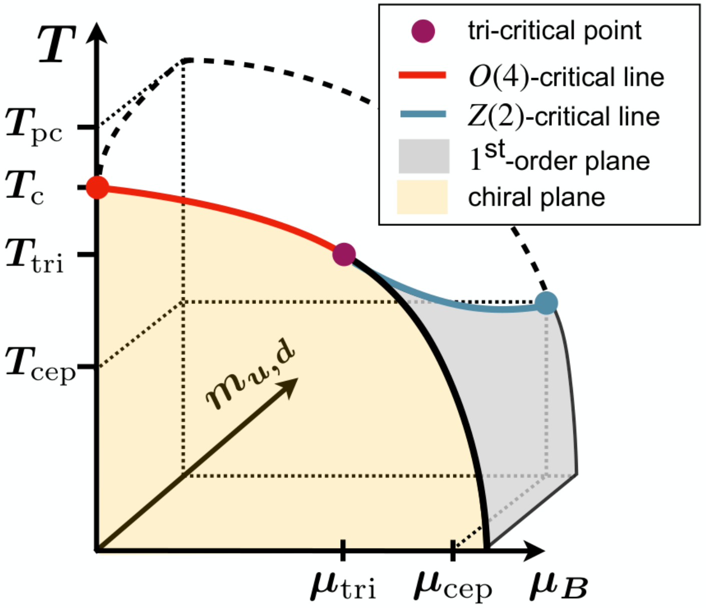
F. Gross, et al.,
EPJC 83 (2023) 1125

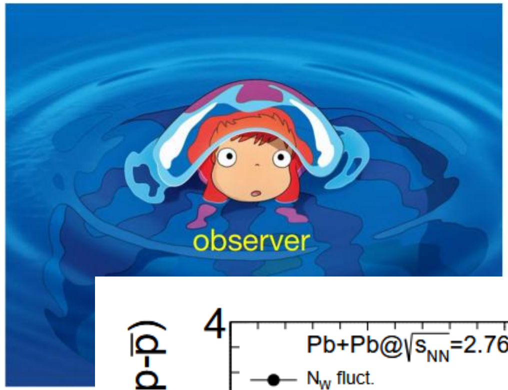
- Short lived and spatially small (~10 fm)
- Inhomogeneous
- And highly dynamical!

Importance of Dynamical Modeling

Pics@Xin An

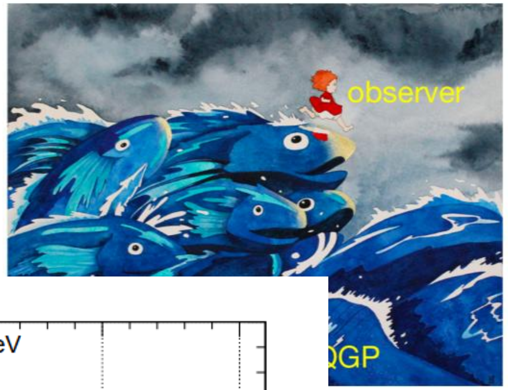
Theory



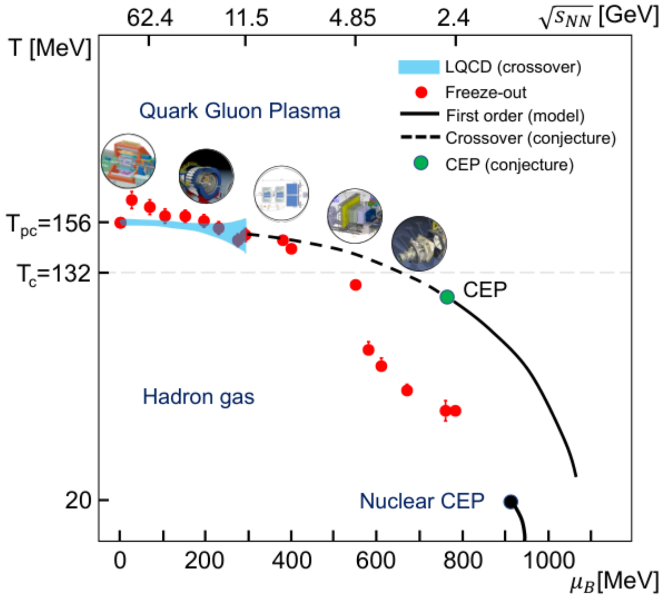


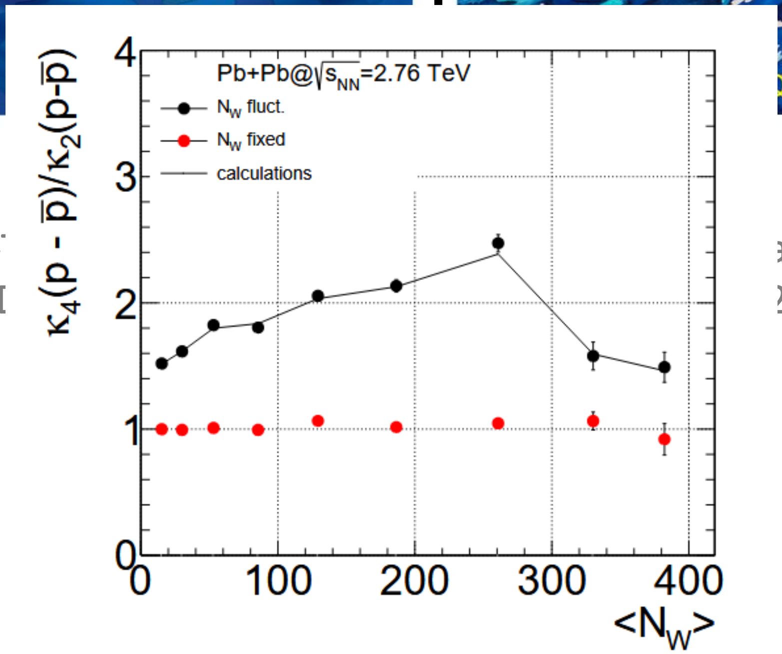
H.-
PRI

Experiments



al.,
23) 1125





P. B.-Munzinger, A. Rustamov, J. Stachel, NPA 960 (2017) 114

- Long lived and spatially infinite
- Uniform, particle heat bath
- And static

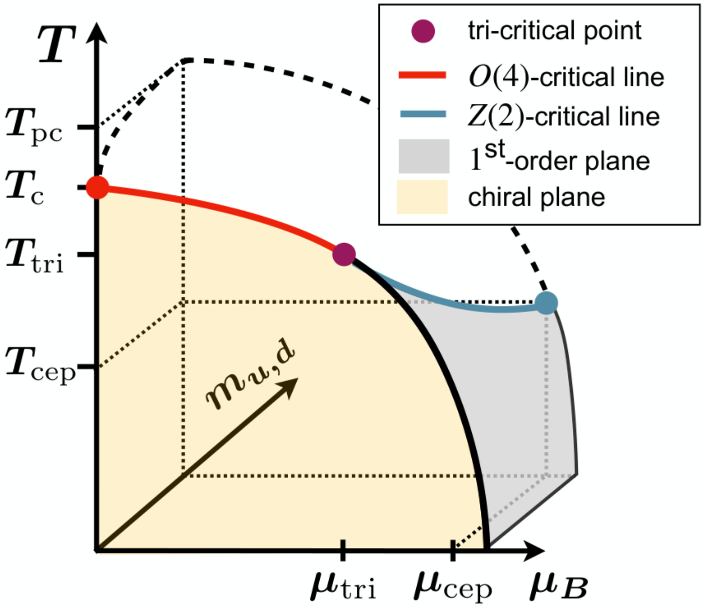
- Short lived and spatially small (~10 fm)
- Inhomogeneous
- And highly dynamical!

Finite volume effect: S. Mitra's talk (Thus.)

Importance of Dynamical Modeling

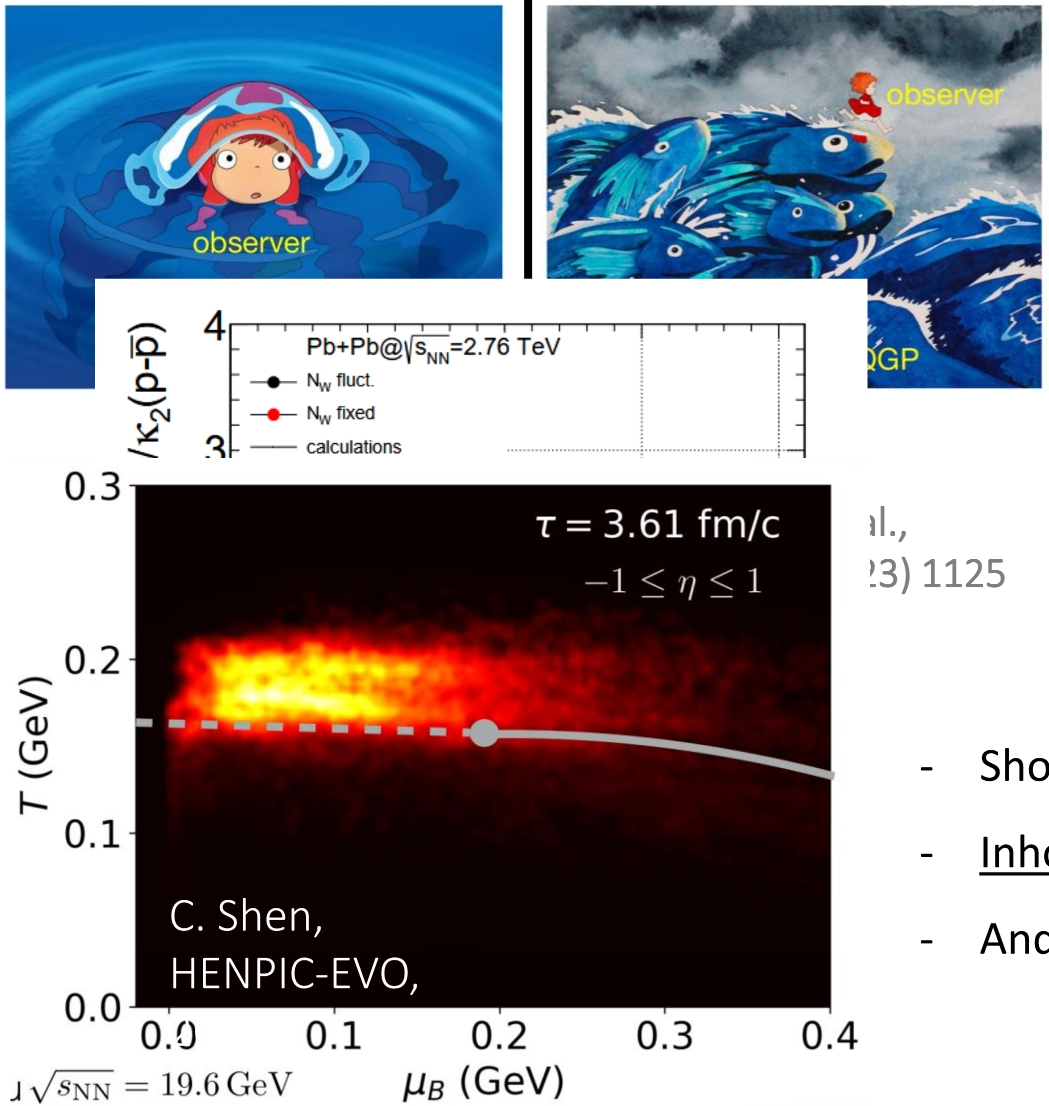
Pics@Xin An

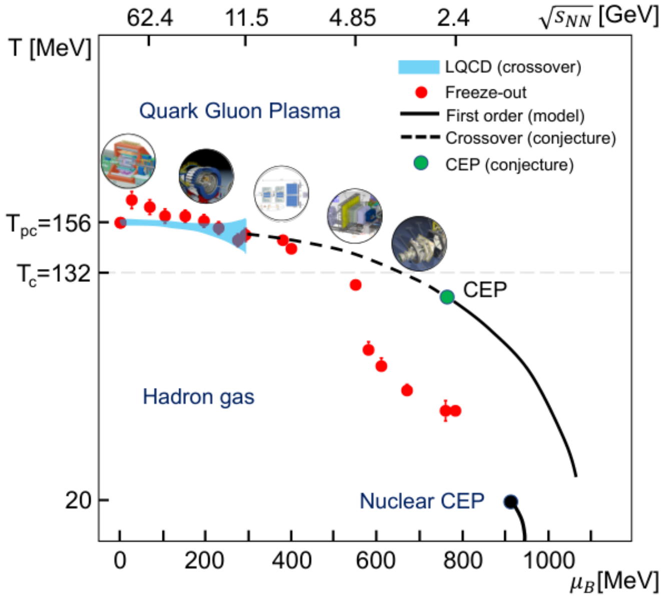
Theory



- Long lived and spatially infinite
- <u>Uniform, particle heat bath</u>
- And static

Experiments



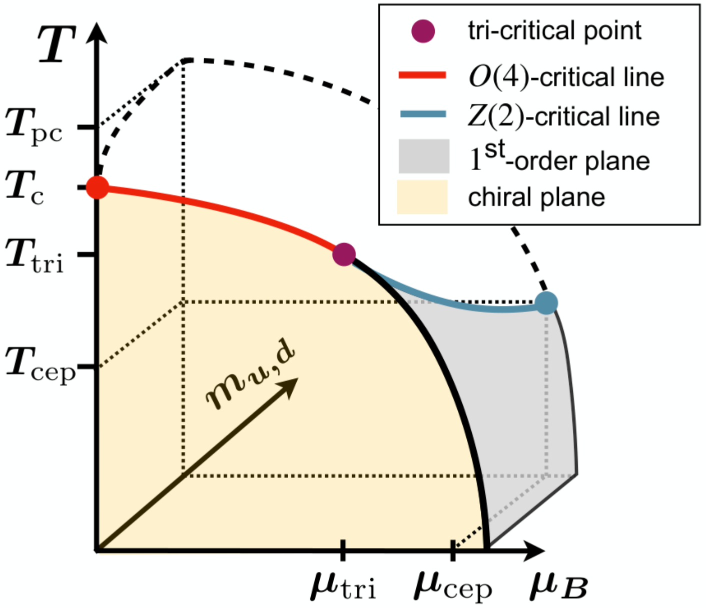


- Short lived and spatially small (~10 fm)
- <u>Inhomogeneous</u>
- And highly dynamical!

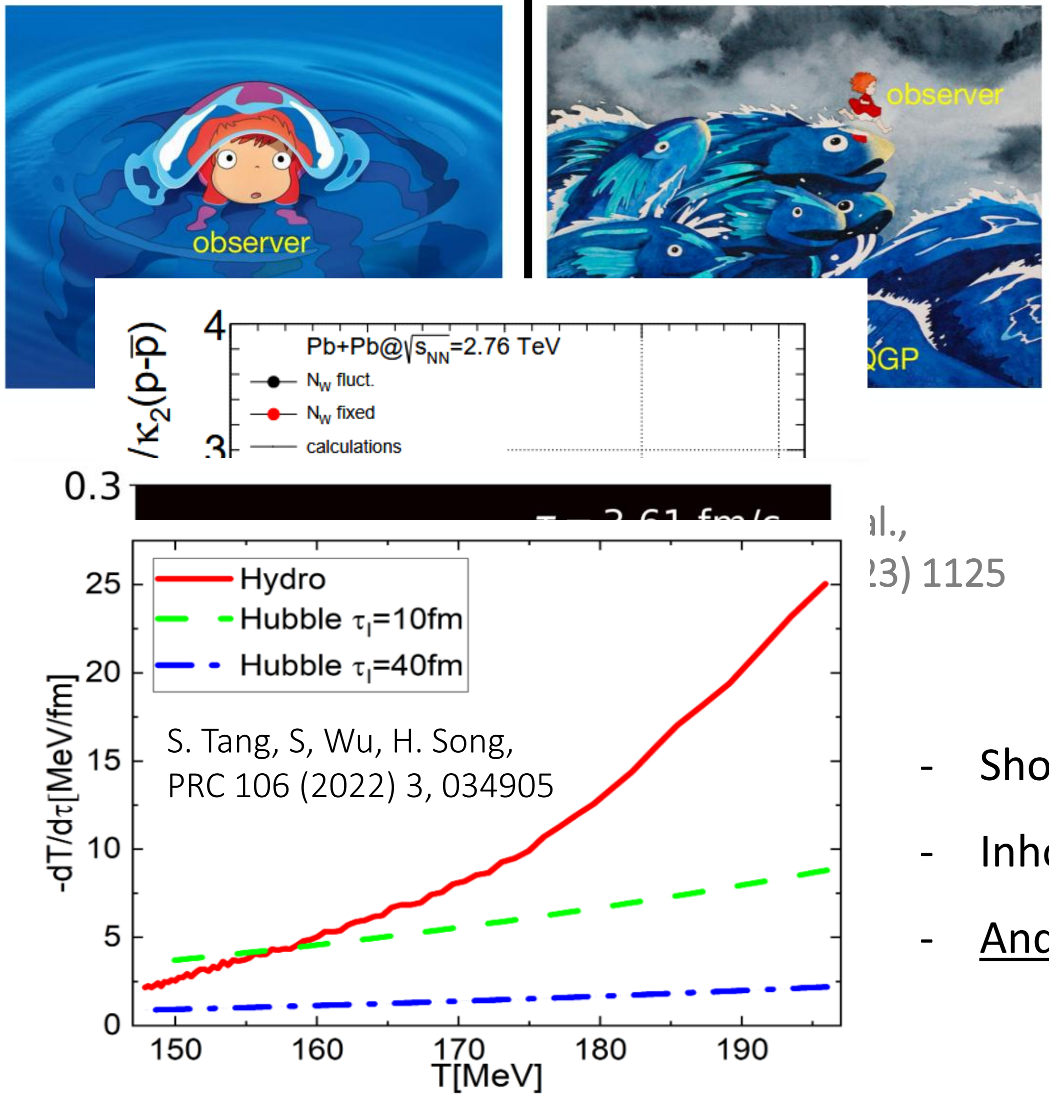
Importance of Dynamical Modeling

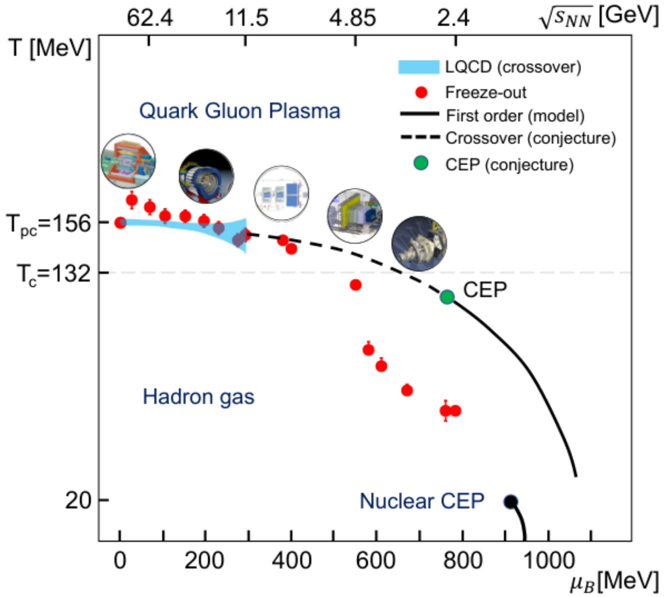
Pics@Xin An

Theory



Experiments





- Long lived and spatially infinite
- Uniform, particle heat bath
- <u>And static</u>

- Short lived and spatially small (~10 fm)
- Inhomogeneous
- <u>And highly dynamical!</u>

Dynamical Approaches for Hydro Fluctuation

Hydro-kinetics (Deterministic)

- **Hydrodynamic + correlation modes**
- One sample, millions of equations (3+3+1)
- Application: Hydro +

Model B: S. Wu's talk (Fri.)
Model H: J. Chao's talk (Sat.)
Inhomogeneous: S. Shi's talk (Sat.)

Recent Developments:

(No-Gaussian modes) X. An, et al., PRL 127 (2021) 072301
(Path integral) N. Mullins, et al., PRL 134 (2025) 23, 232302
(Effective action) J. Chao and T. Shaefer, JHEP 06 (2023) 057
(Hydro + with CEP) K. Rajagopal, et al., PRC 102 (2020) 094025
(With freeze-out) M. Pradeep, et al., PRD 106 (2022) 036017

Fluctuating Hydro (Stochastic)

- **Hydrodynamic + stochastic noise**

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{\text{viscous}} \boxed{+ S^{\mu\nu}_{\text{noise}}},$$
$$\partial_\mu J^{\mu} = 0, \quad J^{\mu} = J^{\mu}_{\text{ideal}} + J^{\mu}_{\text{viscous}} \boxed{+ I^{\mu}_{\text{noise}}}.$$

- Fluctuation-Dissipation Relation (FDR)

$$\langle I^\mu(x_1) I^\nu(x_2)\rangle = 2T\sigma\Delta^{\mu\nu}\delta^{(4)}(x_1 - x_2)$$

E. Wang and U. Heinz, PRD 66 (2002) 025008
J. Kapusta, B. Muller, M. Stephanov, PRC 85 (2012) 054906

- One equation, millions of samples
- **Advantages: Observables, IC models, hard probes, etc.**

(Metropolis Ideal) T. Shaefer, et al., PRD 106 (2022) 014006
(Metropolis Visc) J. Bhambure, et al., PRC 111 (2025) 064909
(Density frame) J. Bhambure, et al., PRC 111 (2025) 064910
(Vs. hydro-kinetic) A. De, et al., PRC 106 (2022) 054903

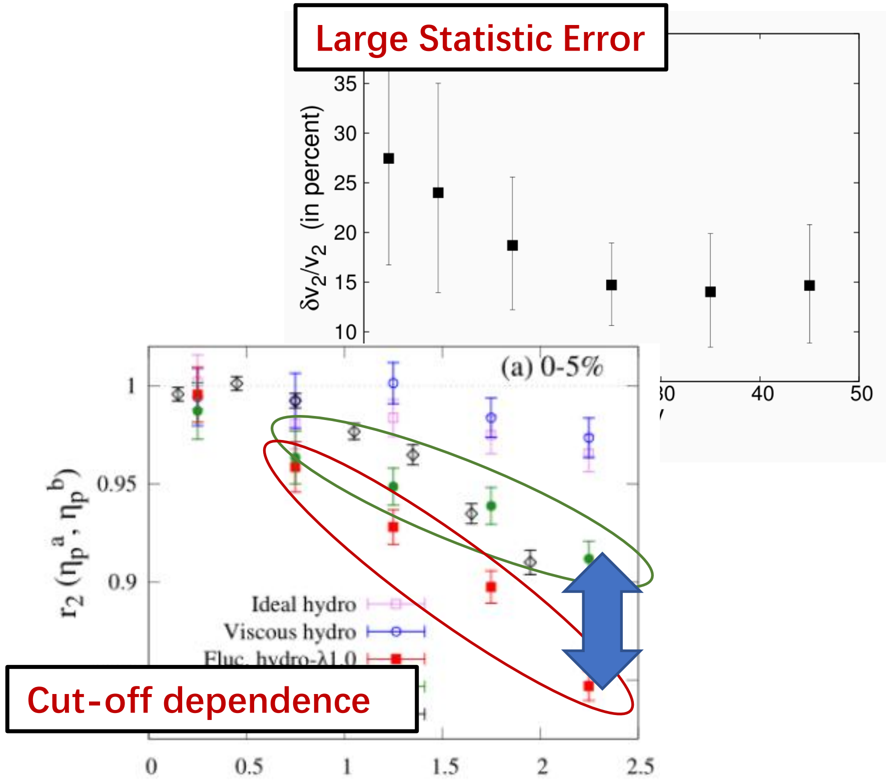
Progress and Challenges of Stochastic Hydrodynamics

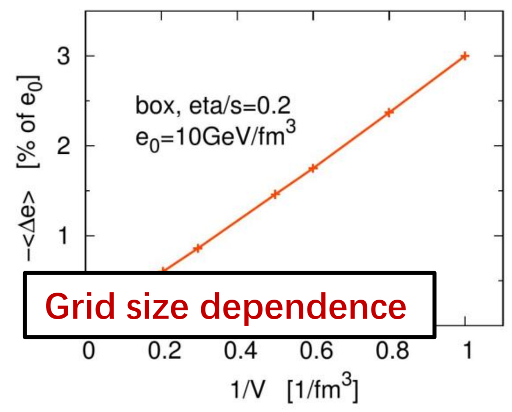
- Stochastic hydro models have been developed and applied in
  - Anisotropic flow, UCC flow puzzle (Murase, et al.)
  - Anisotropic flow at LHC (Singh, Shen, et al.)
  - Critical signatures (Nahrgang, et al.)

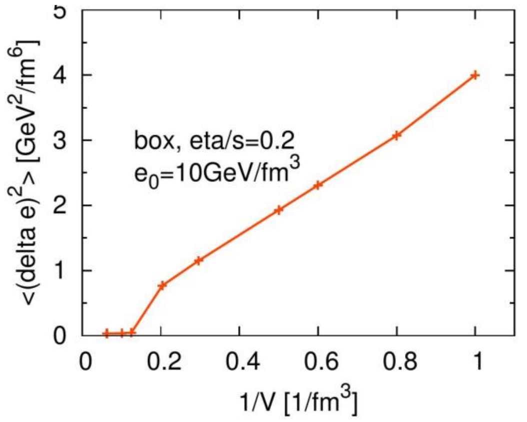
- Several issues arise from the discretization of the Dirac-Delta function

$$\langle \xi^{\mu}(x)\xi^{\nu}(x')\rangle = 2\kappa_B \Delta^{\mu\nu}\delta^{(4)}(x-x') \sim 1/(\Delta t \Delta x^3)$$

- Instabilities and challenges for PDE solver
- Sensitive on cut-off and grid spacing
- Requires large statistics







A. De, C. Shen, J. Kapusta, PRC 106 (2022) 5, 054903
A. Sakai, K. Murase, T. Hirano, PRC 102 (2020) 6, 064903
M. Singh, et al., Nucl.Phys.A 982 (2019) 319

(In this talk) Linearized Stochastic Charge Transport

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\mathrm{ideal}} + T^{\mu\nu}_{\mathrm{viscous}} \cancel{+ S^{\mu\nu}_{\mathrm{noise}}}\,,$$
$$\partial_\mu J^{\mu} = 0, \quad J^{\mu} = J^{\mu}_{\mathrm{ideal}} + J^{\mu}_{\mathrm{viscous}} + I^{\mu}_{\mathrm{noise}}\,.$$

- Neglect $T^{\mu\nu}$ fluctuations and focus on n_B fluctuation
- Separately calculate the smooth and neutral background + n_B fluctuations (at high energy)

Background ($T^{\mu\nu}$) + Stochastic n_B transport (N^μ)

- Application in 3+1-d MUSIC
- Stable and cut-off independent solution
- Save ~98% computation time
- Towards mixed BQS-charge transport

Hydrodynamics
+
Linearized Baryon Transport

Hydrodynamics

Conservation Eqs.

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = eu^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

$$\boxed{\partial_\mu N_i^\mu = 0}$$

$$N_i^\mu = n_i u^\mu + q_i^\mu$$

+ EoS

+ Transport equations

$$\Delta^{\mu\nu} D q_\nu = -\frac{1}{\tau_q}(q^\mu - q_{NS}^\mu)$$

Hydrodynamics

Conservation Eqs.

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = e u^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

$$\boxed{\partial_\mu N_i^\mu = 0}$$

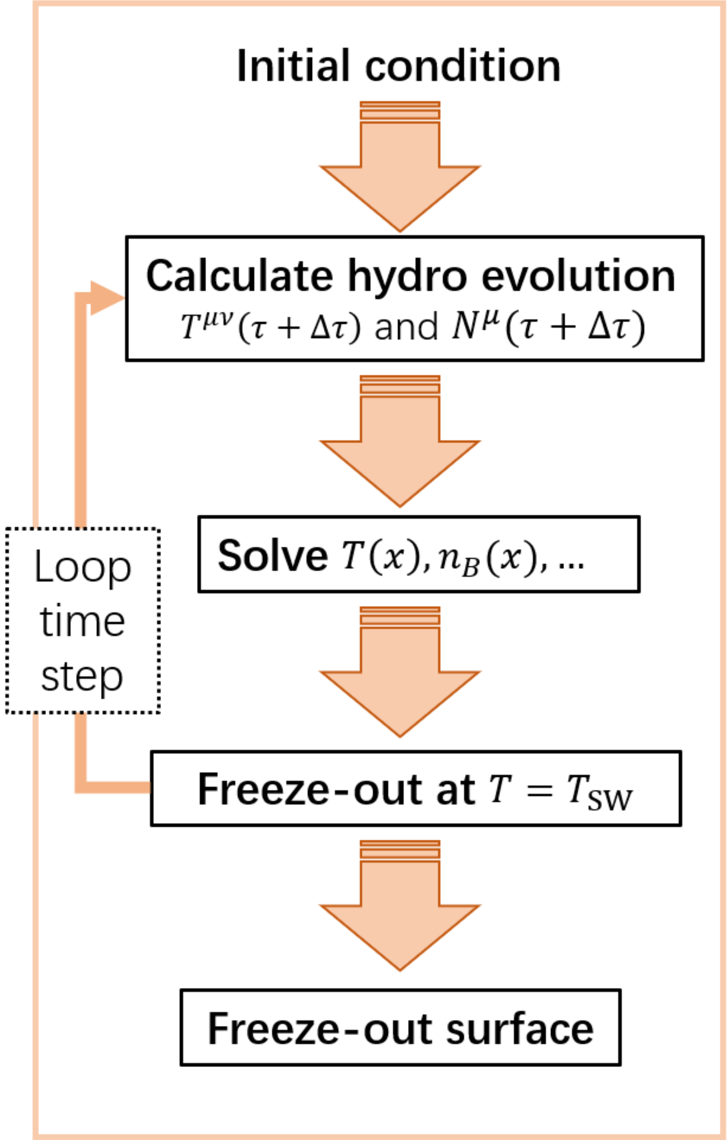
$$N_i^\mu = n_i u^\mu + q_i^\mu$$

\+ EoS

\+ Transport equations

$$\Delta^{\mu\nu} D q_\nu = -\frac{1}{\tau_q}\left(q^\mu - q_{NS}^\mu\right)$$

Ordinary MUSIC



1. **Initial Condition:**
   - provide $T^{\mu\nu}(\tau_0, \boldsymbol{x})$ and $N^\mu(\tau_0, \boldsymbol{x})$
   - lots of IC models on market (Trento, etc.)
   - (in later comparison) 1+1-d boost invariant IC

$$e(\tau_0, \boldsymbol{x}) = \text{const.}\,,\; u^\mu(\tau_0, \boldsymbol{x}) = (1, \boldsymbol{0})$$

2. **Hydro evolution**
   - MUSIC: a 3+1-d viscous hydrodynamic code
   - input EoS and transport coefficients (η/s, κ_B, …)

3. **Freeze-out**
   - mapping the field info onto constant T hyper-surface

Hydrodynamics + Baryon fluctuation

Background

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = e u^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

Baryon Fluc. + Trans.

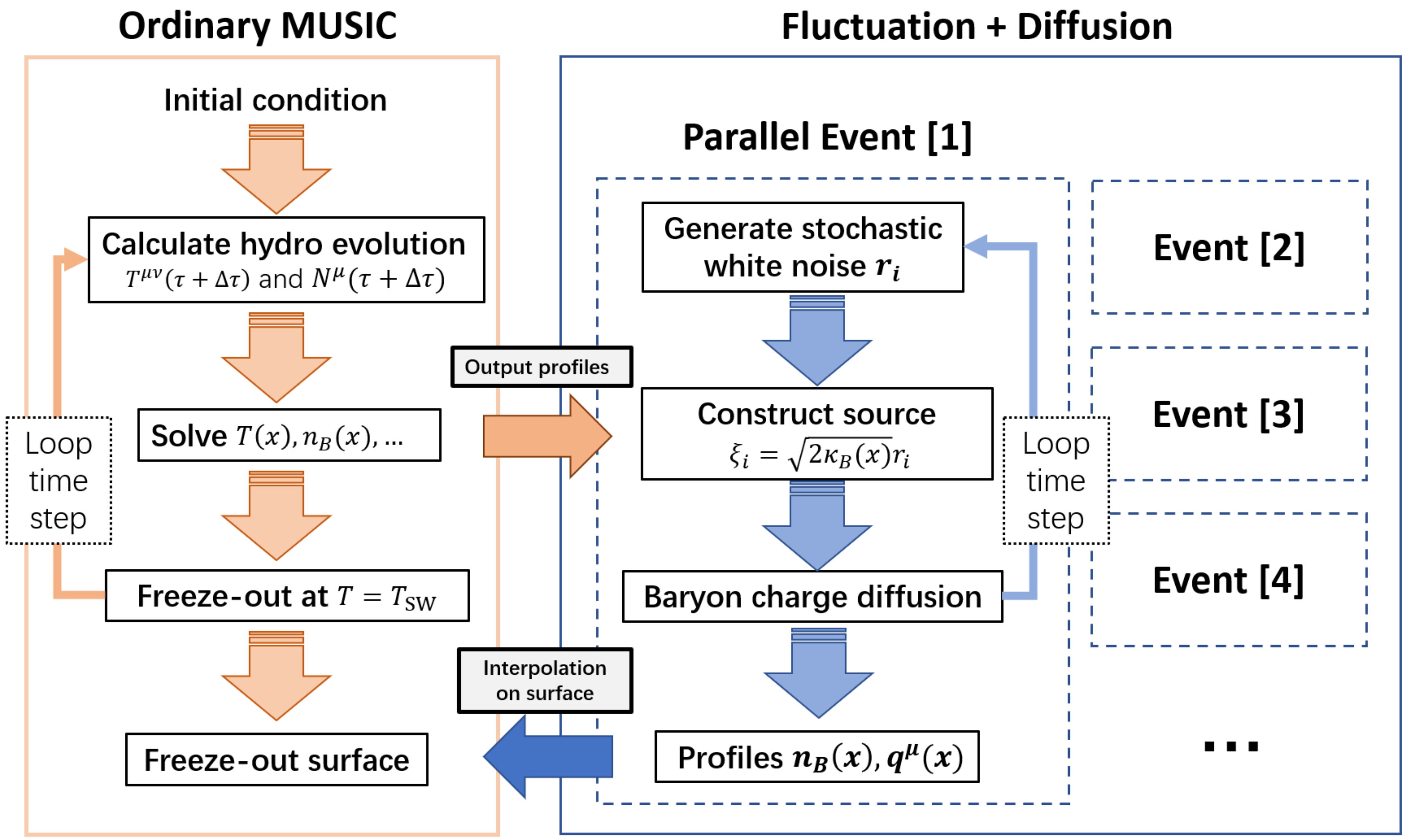
$$\boxed{\partial_\mu N_i^\mu = 0}$$

$$N_i^\mu = n_i u^\mu + q_i^\mu$$

Diffusion with noise

$$\Delta^{\mu\nu} D q_\nu = -\frac{1}{\tau_q}\left(q^\mu - q_{NS}^\mu - \xi^\mu\right)$$

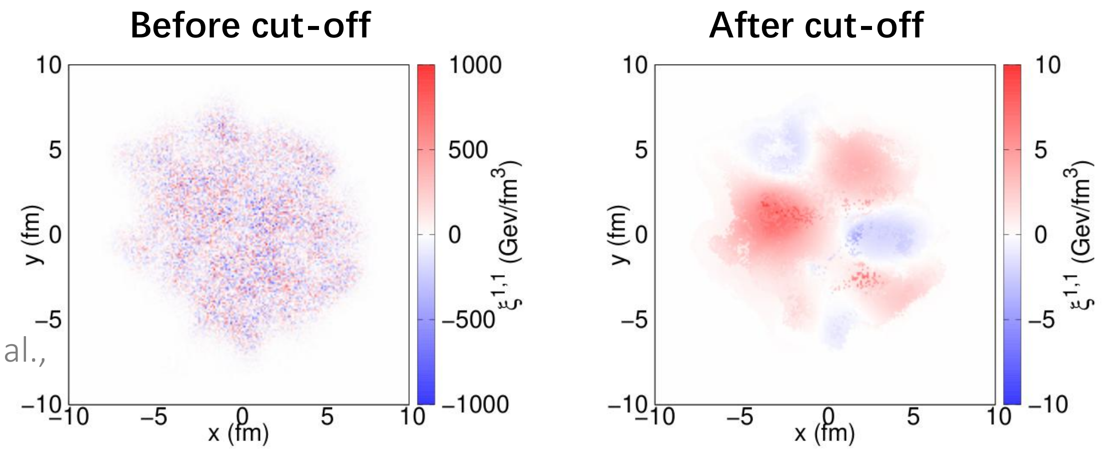
$$\langle \xi^\mu(x)\xi^\nu(x')\rangle = 2\kappa_B \Delta^{\mu\nu}\delta^{(4)}(x-x')$$



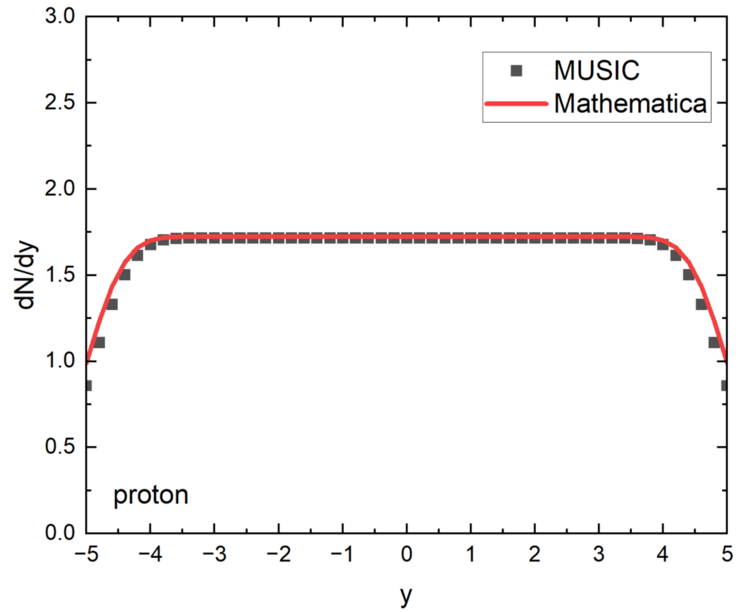
N. Borghini, BF, S. Schlichting, Phys. Rev. D 111 (2025) 11, 116019

Observables

- Zero baryon density in background, finite baryon distribution induced by fluctuations $\langle \xi^\mu(x)\xi^\nu(x')\rangle = 2\kappa_B \Delta^{\mu\nu}\delta^{(4)}(x-x') \sim 1/\Delta x$



M. Singh, C. Shen, et al., NPA 982 (2019) 319



A. De, C. Plumberg, J. Kapusta
Phys. Rev. C 102 (2020) 2, 024905

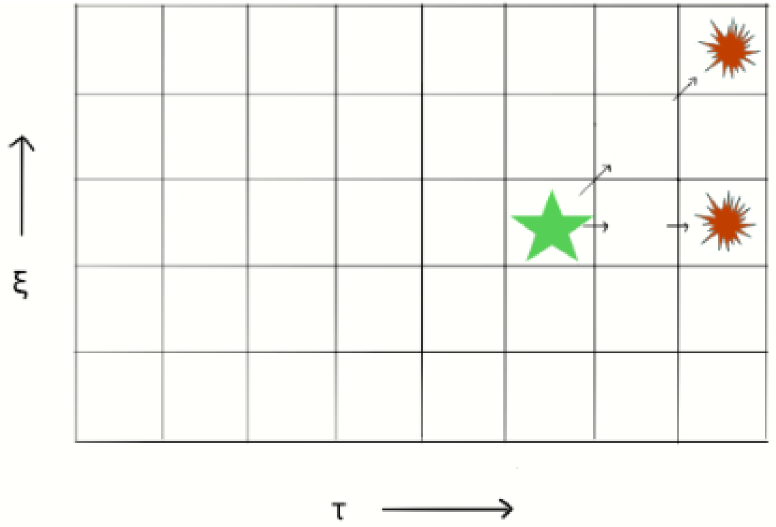
- one-point function vanish

$$\langle \xi^\mu(x)\rangle = 0$$

- Possible observable – 2-point correlation

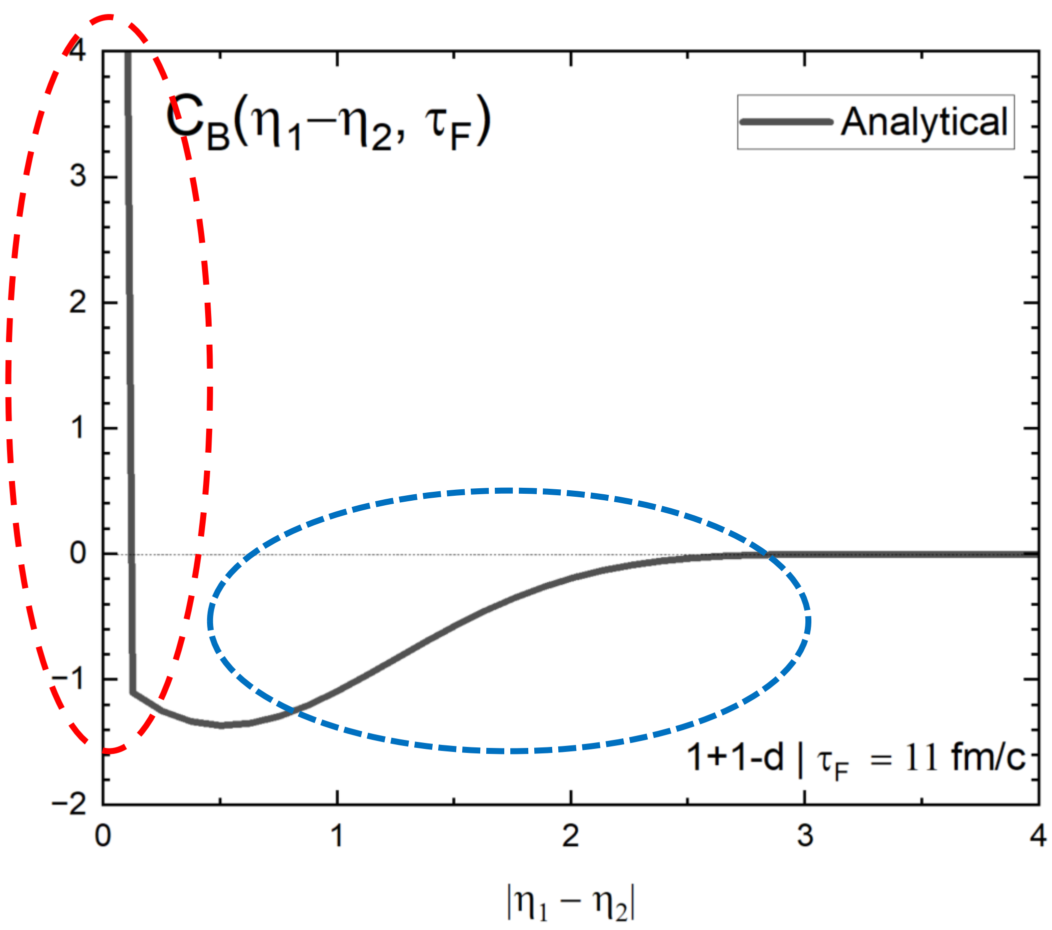
$$C_B(\eta_1-\eta_2,\tau_F) \equiv \langle n_B(\eta_1)\, n_B(\eta_2)\rangle \tau_F^2$$

$$\tilde{C}_B(k_\eta,\tau_F) \equiv \frac{\tau_F^2}{V}\langle \tilde{n}_B(k_\eta,\tau_F)\, \tilde{n}_B(-k_\eta,\tau_F)\rangle$$

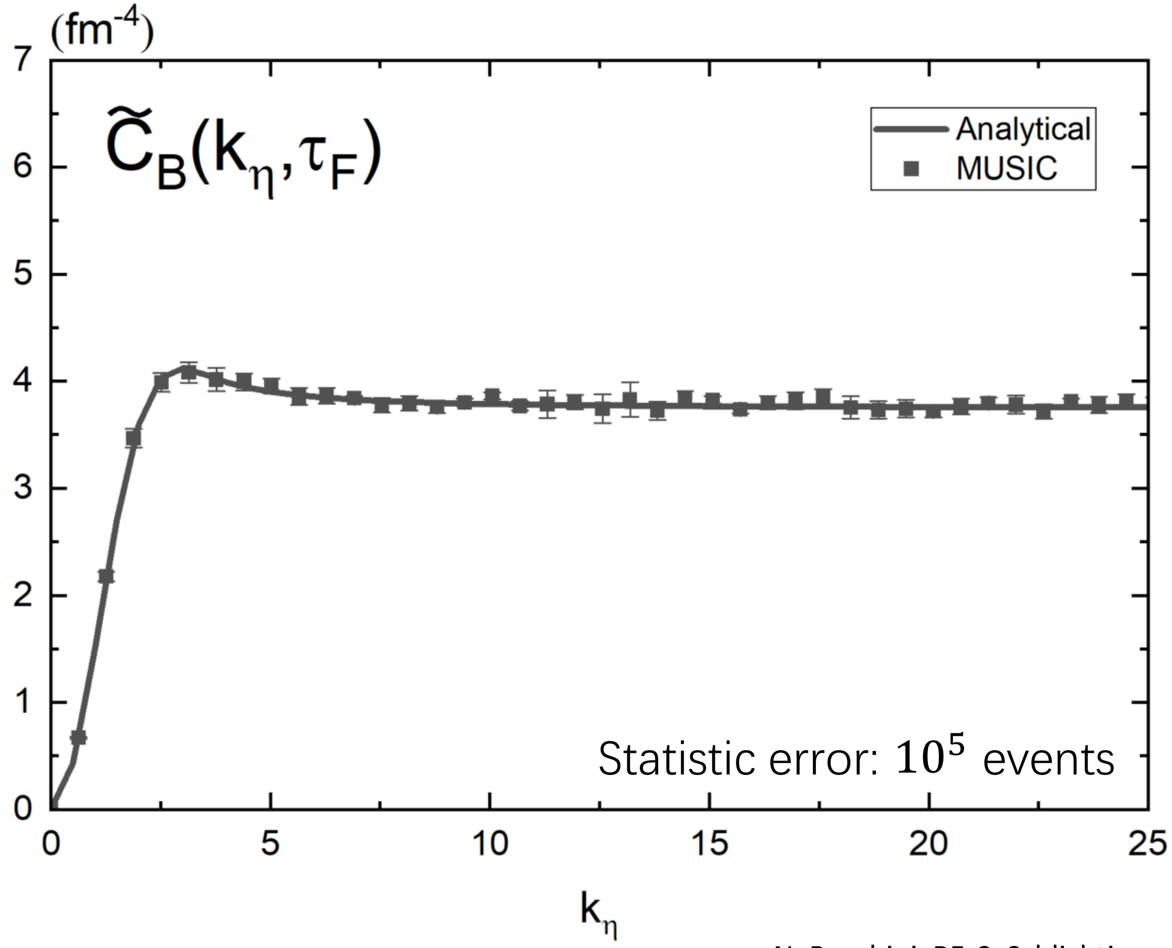


Observable: 2-point correlation

- 2PC in coordinate space (analytical solution)



- 2PC in Fourier space



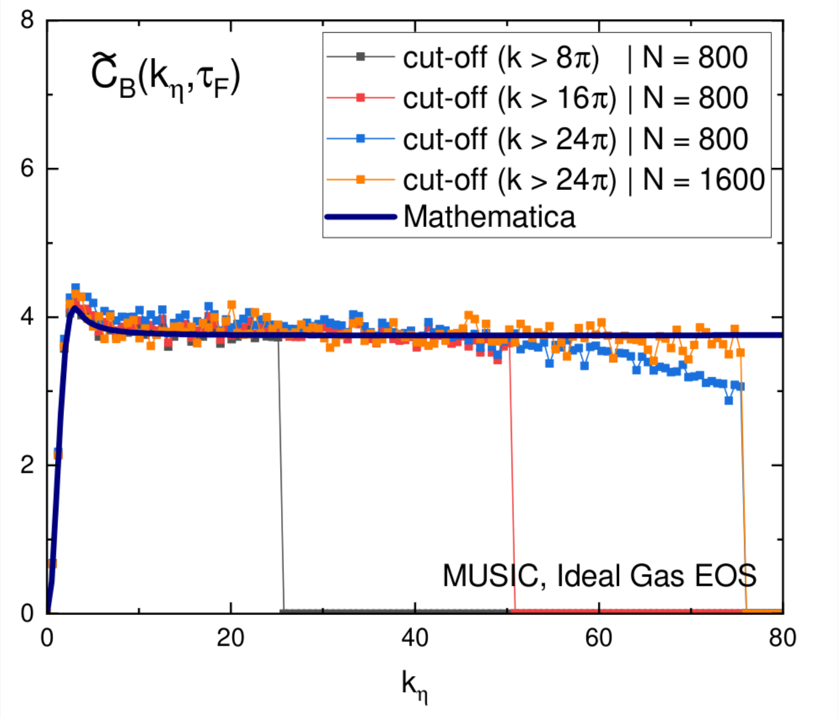
N. Borghini, BF, S. Schlichting
Phys. Rev. D 111 (2025) 11, 116019

Short range correlation: **(FDR)** $\langle \xi^\mu(x)\xi^\nu(x')\rangle = 2\kappa_B \Delta^{\mu\nu}\delta^{(4)}(x-x')$

Long range correlation: negative due to conservation

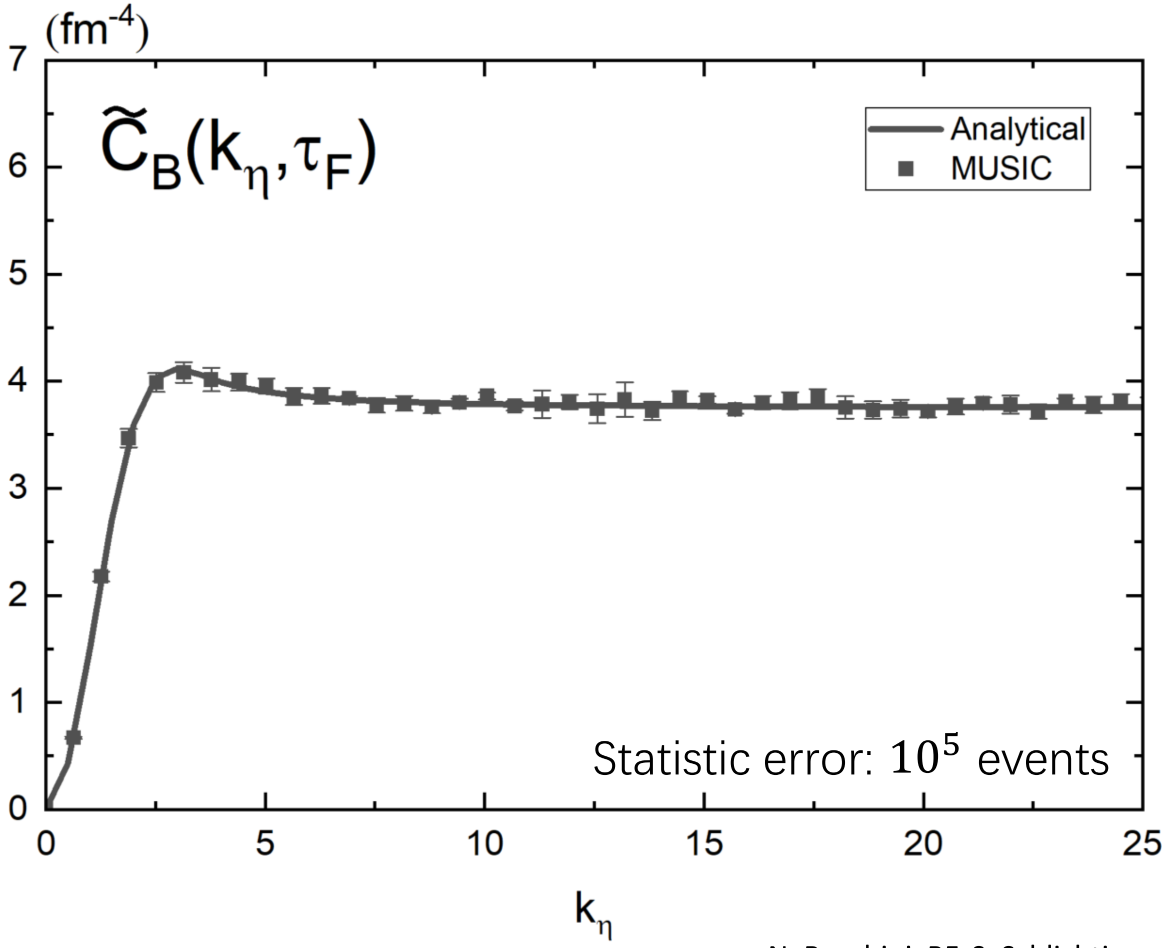
Observable: 2-point correlation

- Stable (K-T algorithm)
- and cut-off / grid size insensitive solution



- Saves ~98% computation time comparing with traditional fluctuating hydro.

- 2PC in Fourier space

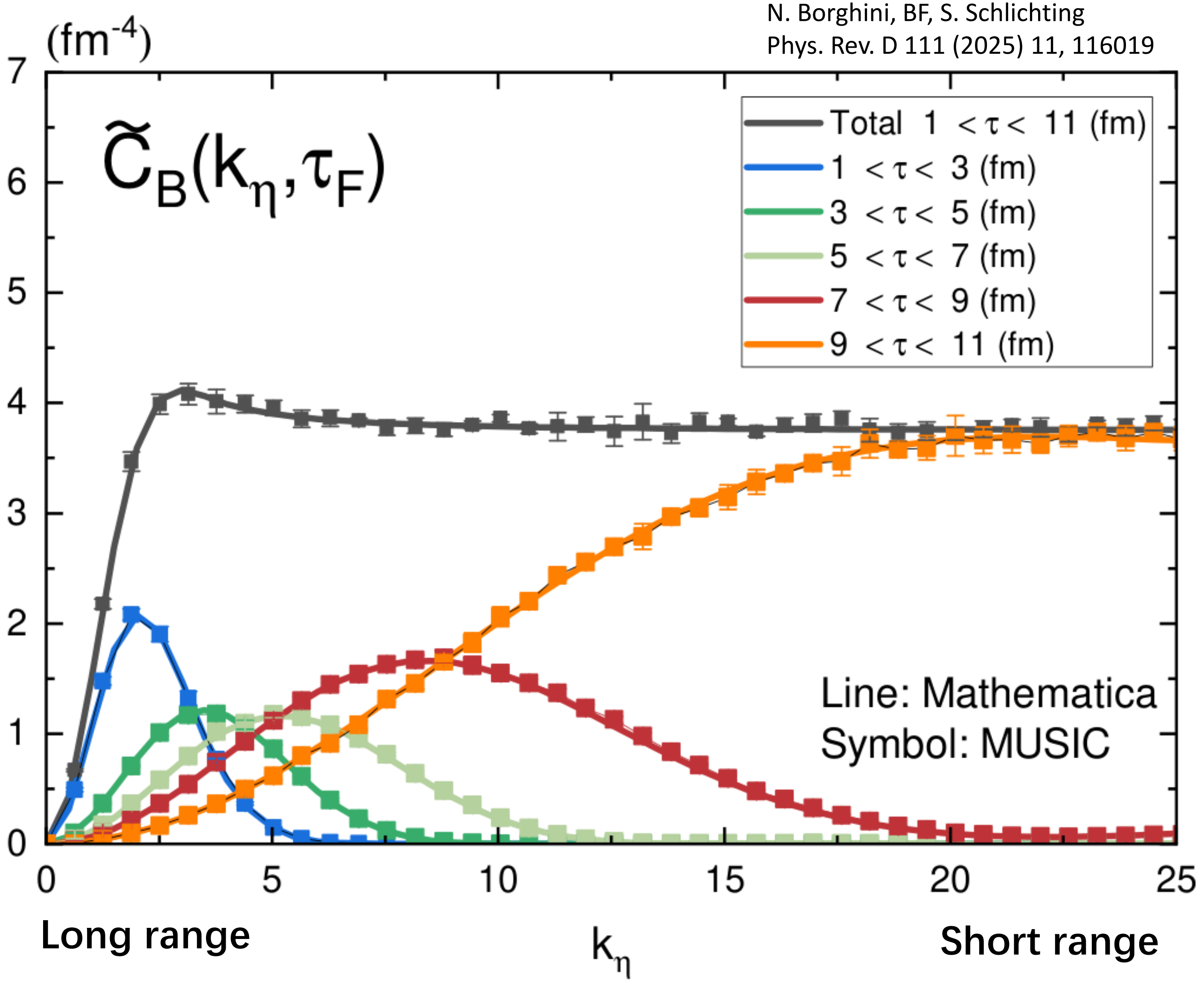


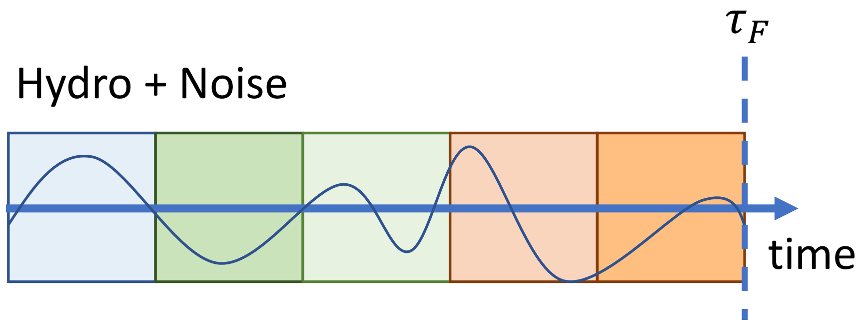
$^{4)}(x - x')$

N. Borghini, BF, S. Schlichting
Phys. Rev. D 111 (2025) 11, 116019

2PC: Time dependence

N. Borghini, BF, S. Schlichting
Phys. Rev. D 111 (2025) 11, 116019



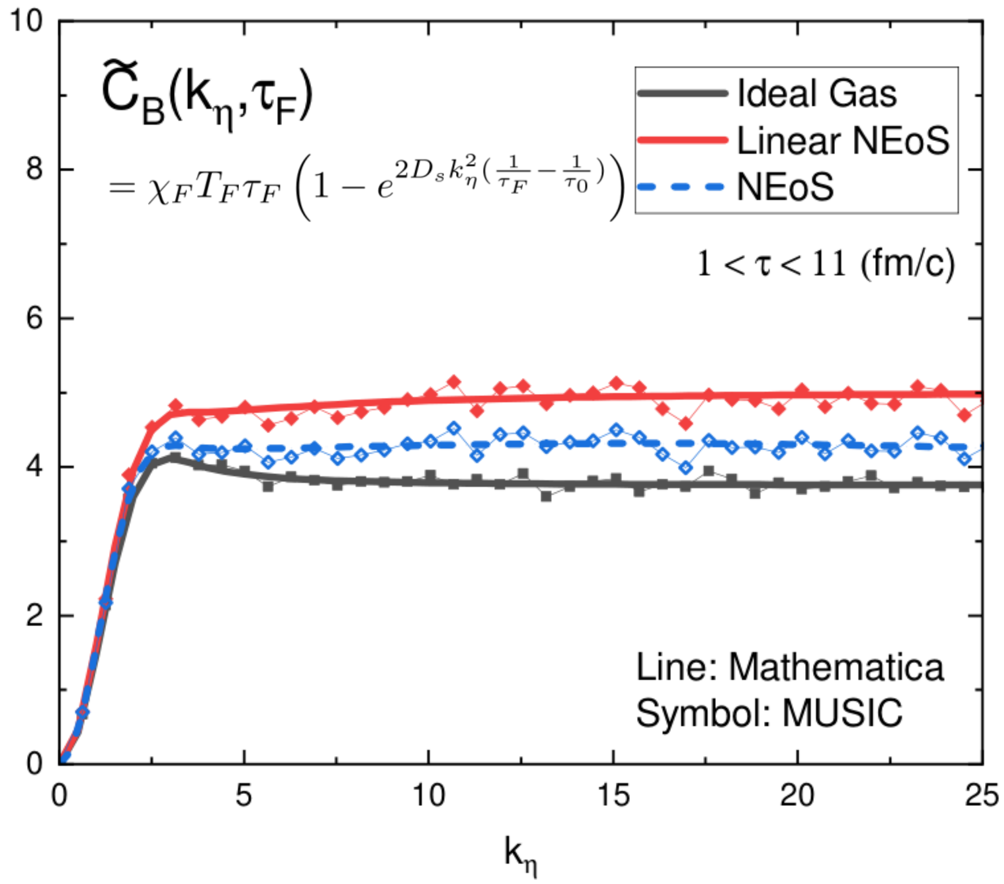


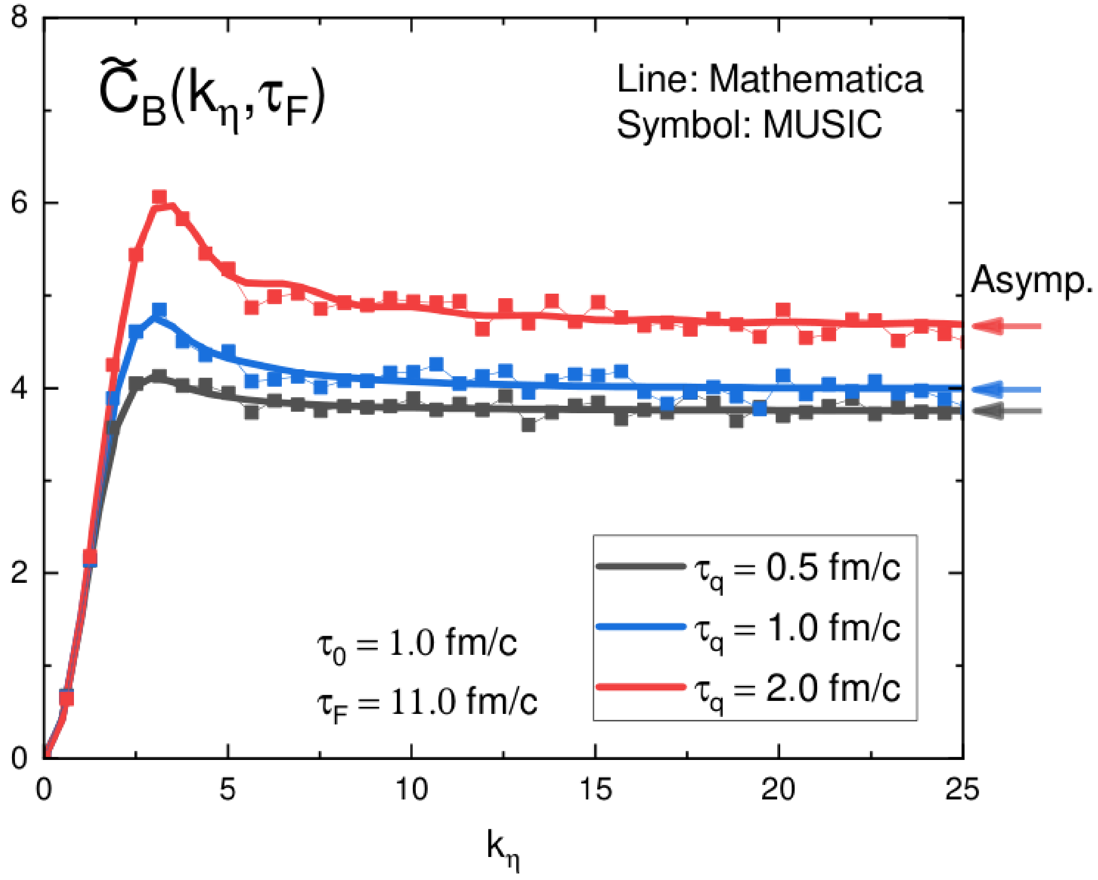
- Obvious time ordering
- Long range correlation – early noise (diffusion)
- Short range correlation – late noise (equilibrium)
- Explore the fluctuations near freeze-out with large momentums

2PC dependence on EoS / Transport Coeff.

- High momentum 2PC sensitive to EoS through susceptibilities
- Similarly, 2PC also sensitive to τ_q, consistent with analytical solutions at large-k limit

N. Borghini, BF, S. Schlichting, Phys. Rev. D 111 (2025) 11, 116019

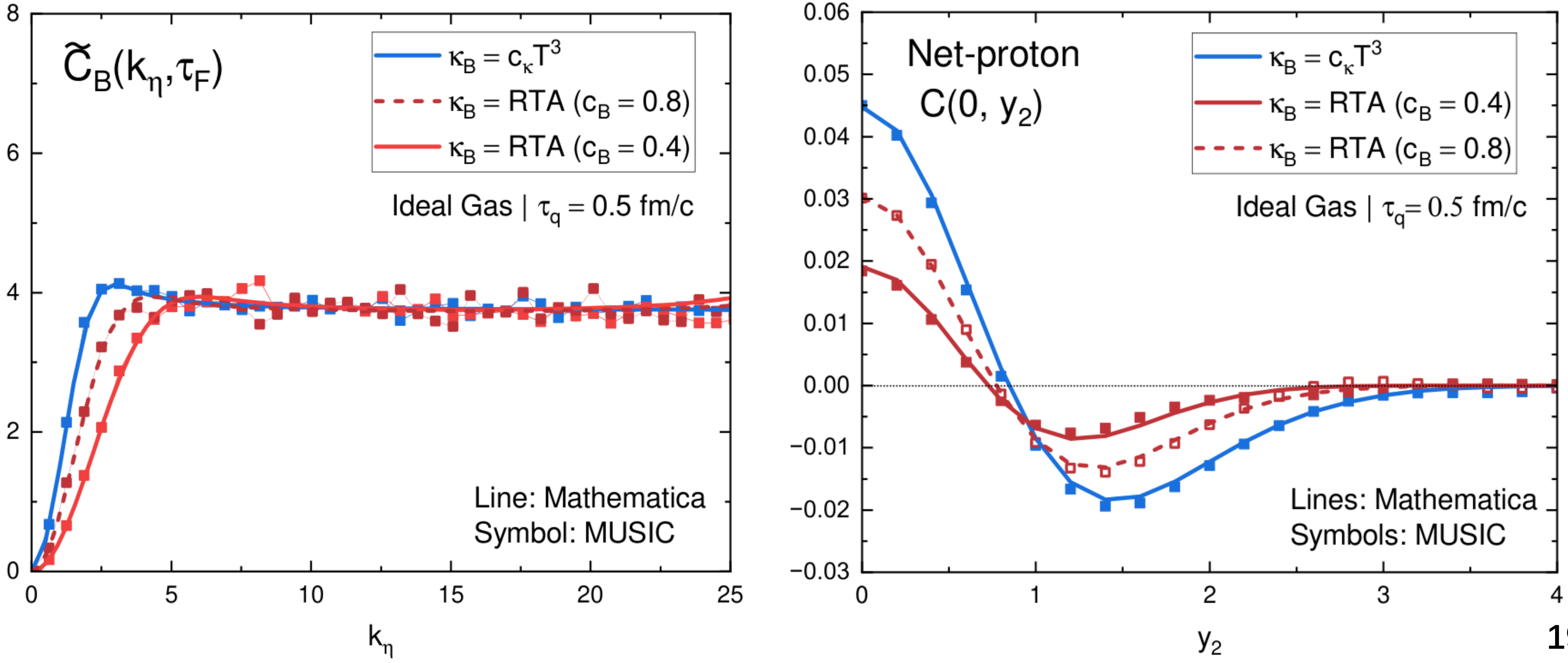




2PC dependence on EoS / Transport Coeff.

- Low momentum 2PC sensitive to κ_B that characterize the diffusion at large space scale.
- Such κ_B dependence survives after Cooper-Frye freeze-out and consistent with analytical solution.

N. Borghini, BF, S. Schlichting, Phys. Rev. D 111 (2025) 11, 116019



Towards mixed B,Q,S fluctuations …

Challenges of mixed-charge fluctuation

- 4D-Equation of State

$$(e, n_B, n_Q, n_S) \Rightarrow (T, \mu_B, \mu_Q, \mu_S)$$

- Proper input of transport coefficients

$$\boxed{\tau_{ij}}\Delta^{\mu\nu} D q_{j,\nu} + q_i^{\mu} = \boxed{\kappa_{ij}\nabla^{\mu}(\mu_j/T)} + \xi_i^{\mu} \qquad i,j = B,Q,S$$

- Cross terms in diffusion equation and FDR

$$\langle \xi_i^{\mu}(x)\xi_j^{\alpha}(x')\rangle = -2T\kappa_{ij}\Delta^{\mu\alpha}\delta^{(4)}(x-x') - \boxed{T\sum_{k=1}^{n}(\tau_{ik}\kappa_{kj} - \tau_{jk}\kappa_{ki})\Delta^{\mu\alpha}\mathrm{D}\delta^{(4)}(x-x')}$$

K. Murase, Annals Phys. 411 (2019) 167969

No ordinary hydrodynamic model includes mixed-charge diffusion so far.

Challenges of mixed-charge fluctuation

- 4D-Equation of State

$$(e, n_B, n_Q, n_S) \Rightarrow (T, \mu_B, \mu_Q, \mu_S)$$

- Proper input of transport coefficients

$$\boxed{\tau_{ij}}\Delta^{\mu\nu} D q_{j,\nu} + q_i^{\mu} = \boxed{\kappa_{ij}\nabla^{\mu}(\mu_j/T)} + \xi_i^{\mu} \qquad i,j = B, Q,$$

- Cross terms in diffusion equation and FDR

$$\langle \xi_i^{\mu}(x)\xi_j^{\alpha}(x')\rangle = -2T\kappa_{ij}\Delta^{\mu\alpha}\delta^{(4)}(x-x') - \boxed{T\sum_{k=1}^{n}(\tau_{ik}\kappa_{kj} - \tau_{jk}\kappa_{ki})\Delta^{\mu\alpha}\mathrm{D}\delta^{(4)}(x-x')}$$

K. Murase, Annals Phys. 411 (2019) 167969

In this work

- Linearized EoS $\mu_i = \chi_{ij}^{-1} n_j$
- κ_{ij} and τ_{ij} from kinetic model with RTA
  - τ_{ij} is a diagonal matrix
- Generate stochastic noise by Onsager relation

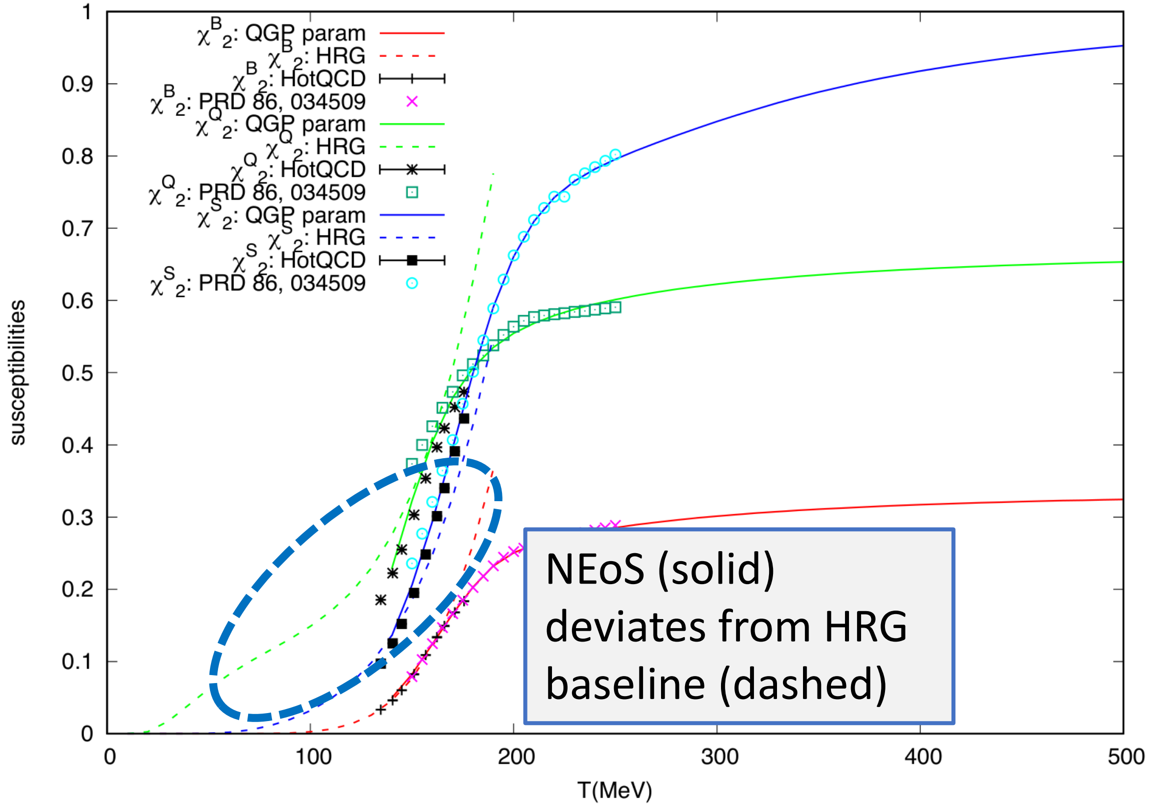
No ordinary hydrodynamic model includes mixed-charge diffusion so far.

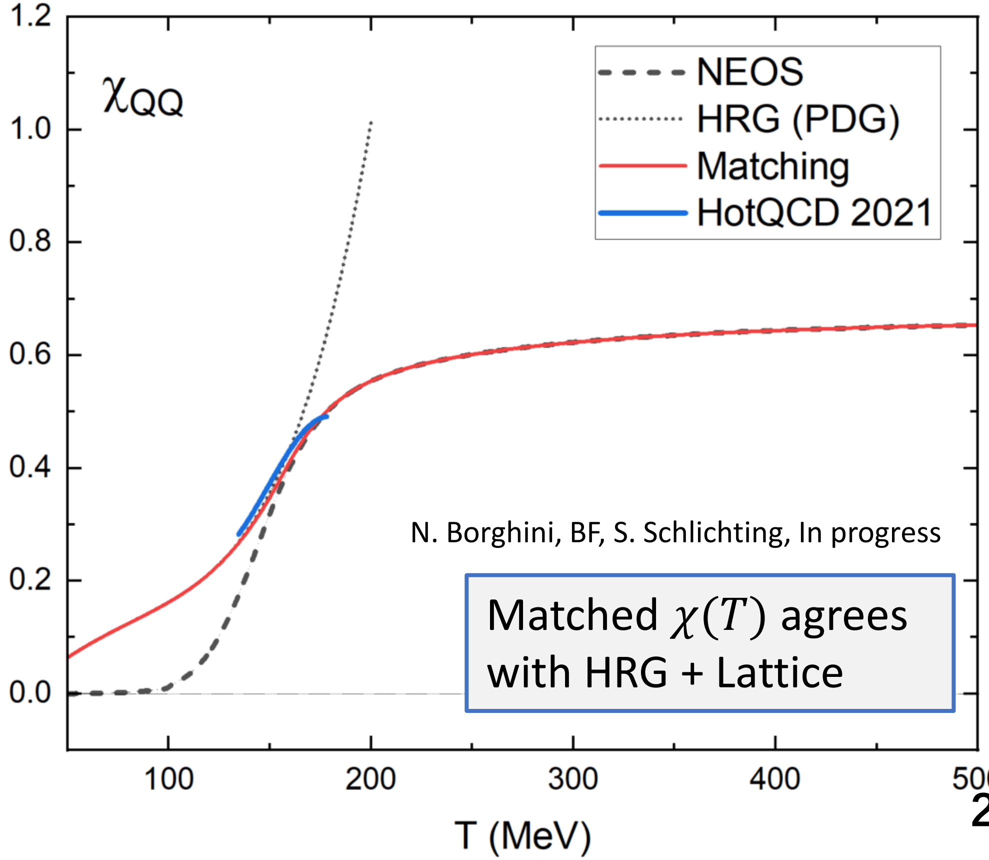
Linearized EoS

- Background: $T(e, n_i = 0)$ and $P(e, n_i = 0)$ from NEOS (HRG + Lattice)
- Linearized fluctuating chemical potentials $\mu_i = \chi_{ij}^{-1} n_j$
- Matching $\chi(T)$ by HRG (low T) + NEoS (high T)

A. Monnai, B. Schenke, C. Shen,
Phys.Rev.C 100 (2019) 2, 024907

Hot QCD,
Phys.Rev.D 104 (2021) 7, 074512

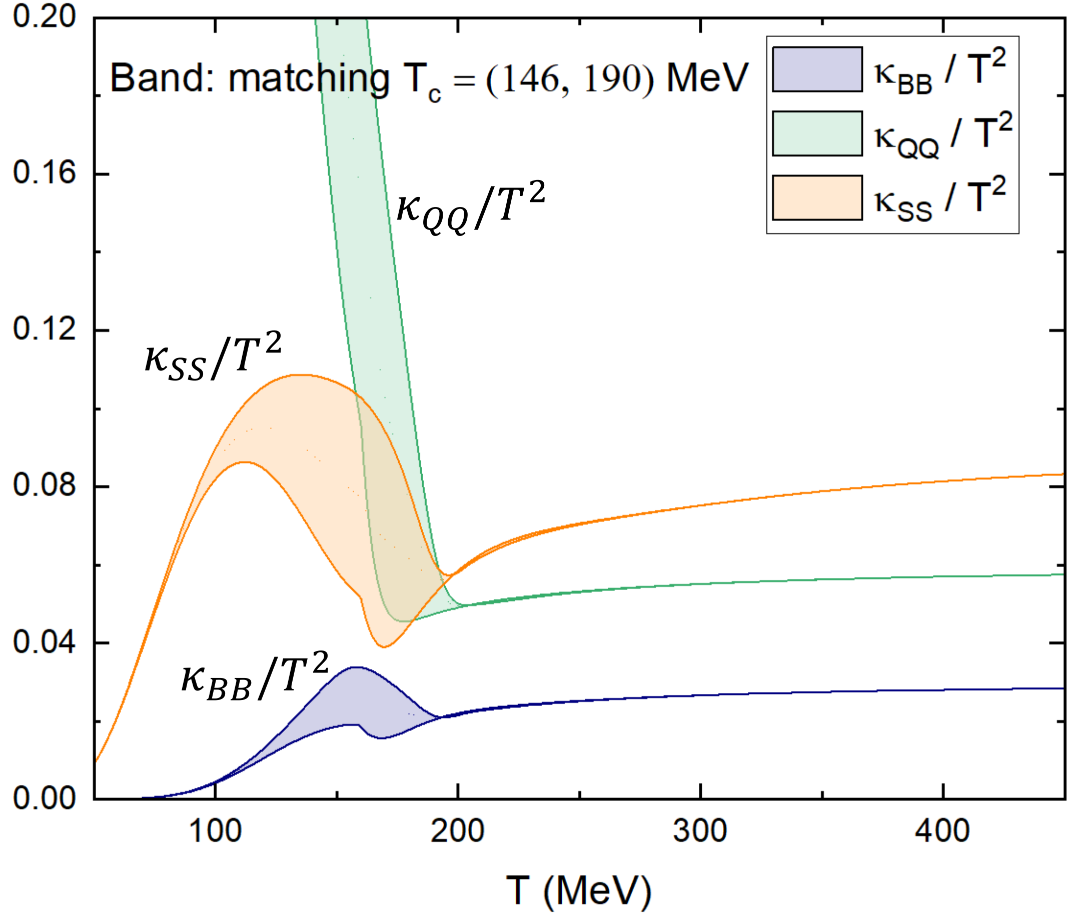


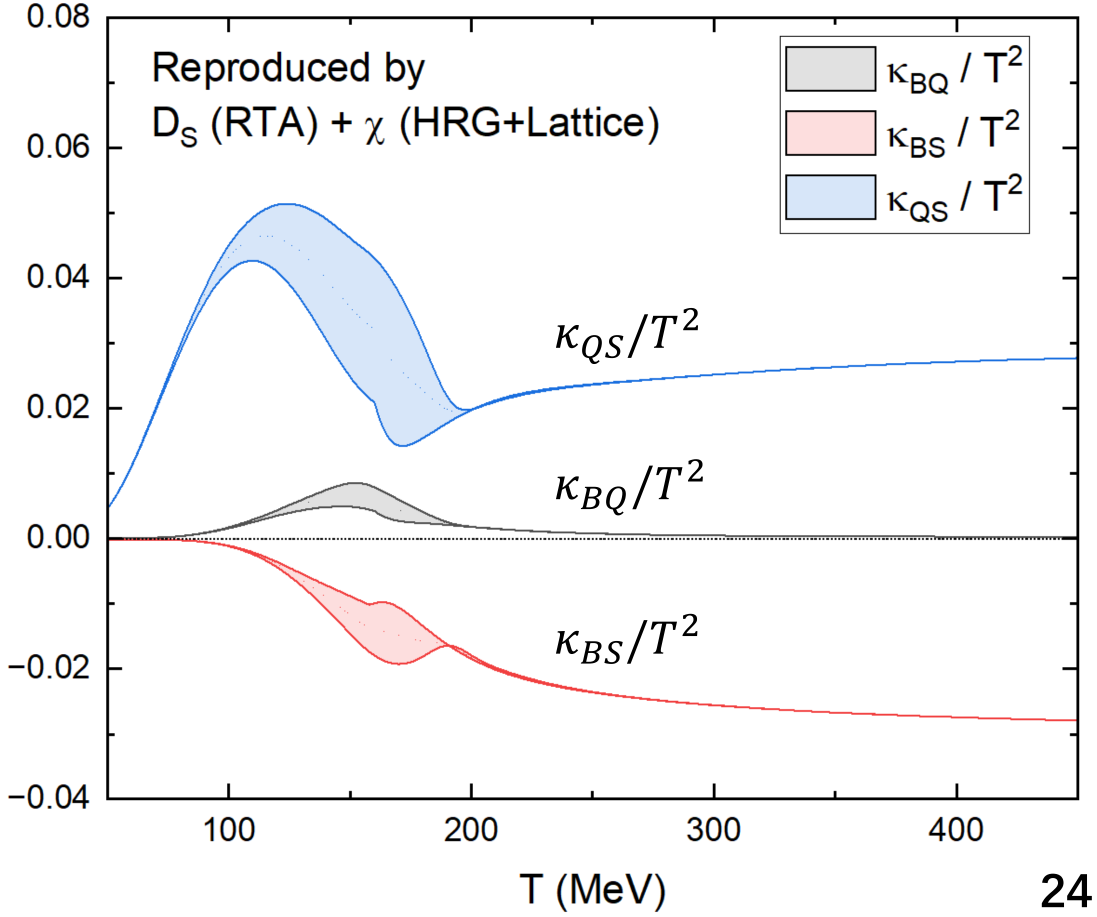


Transport Coefficients

- Equivalently, we can reproduce κ_{ij} from D_{ij} and χ_{ij}
- Model uncertainties (bands) by varying matching temperature in (146, 190) (MeV)

N. Borghini, BF, S. Schlichting, In progress

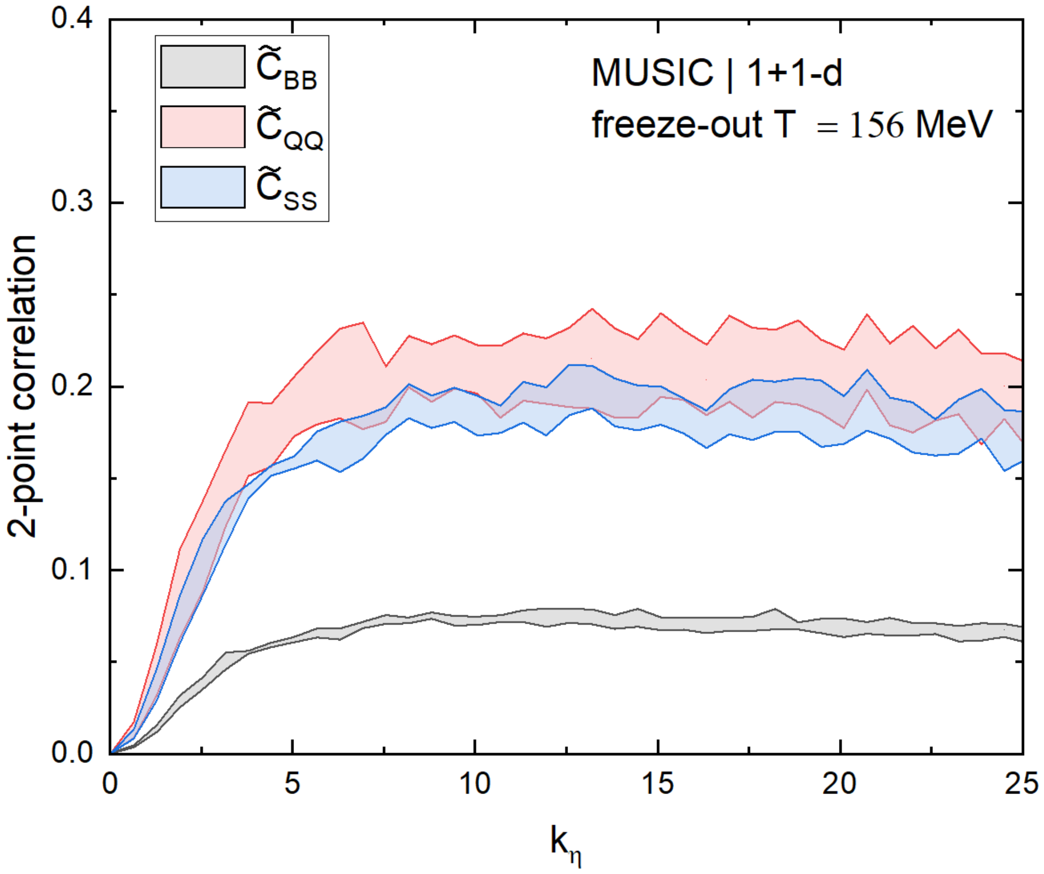


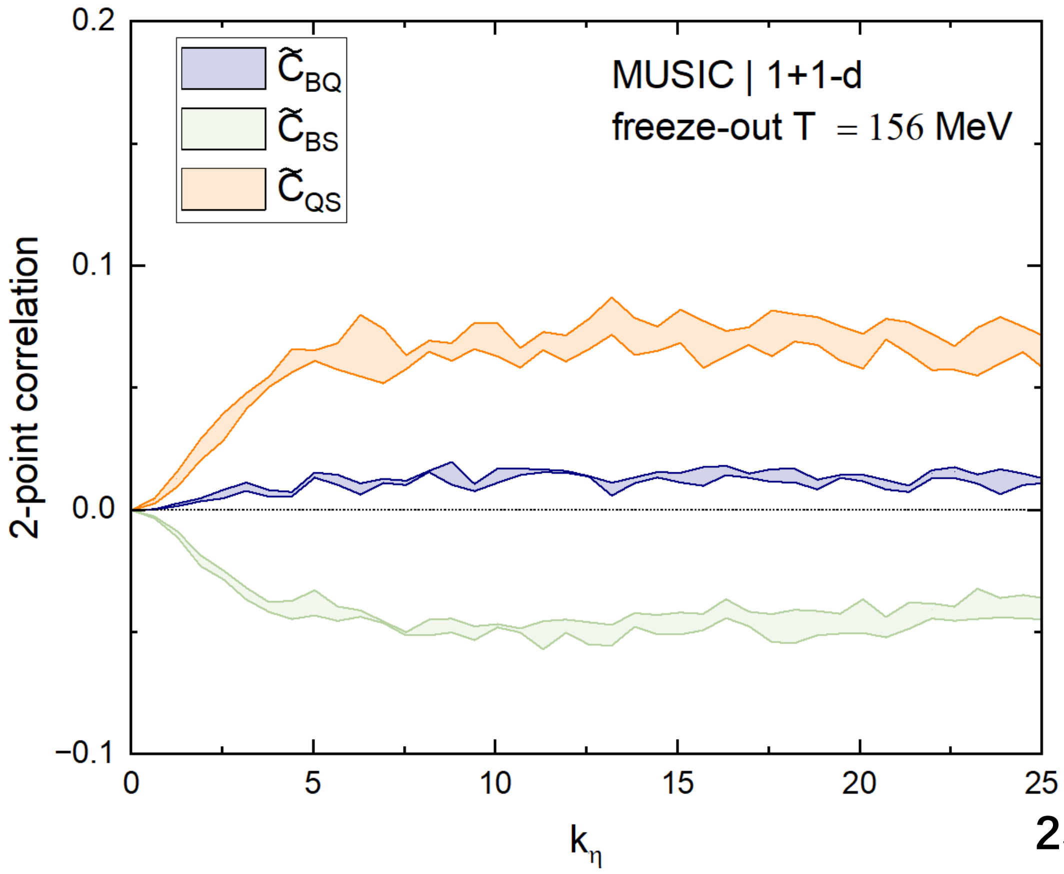


2PC with multiple charges

- Similar and significant QQ and SS correlations
- Sizeable off-diagonal correlations: mixed charge transport
- Model uncertainties in quantitative level

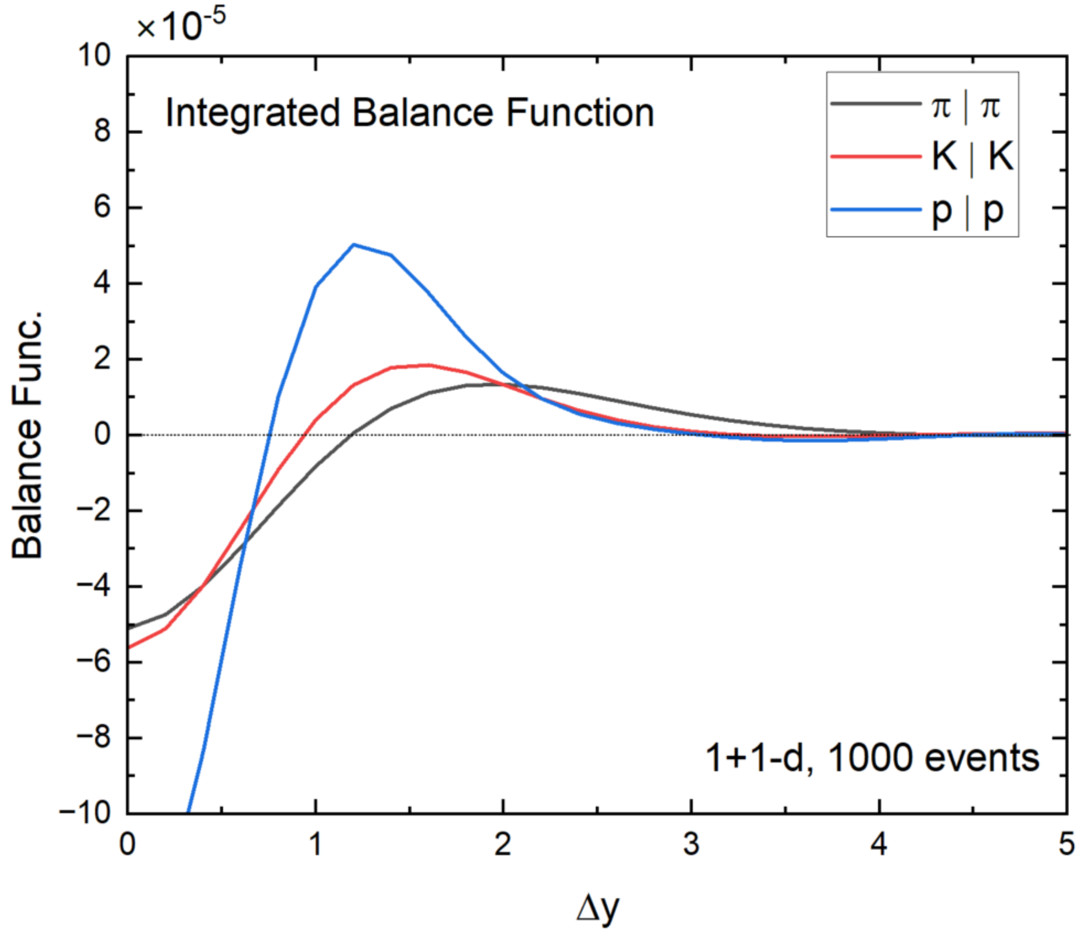
N. Borghini, BF, S. Schlichting, In progress



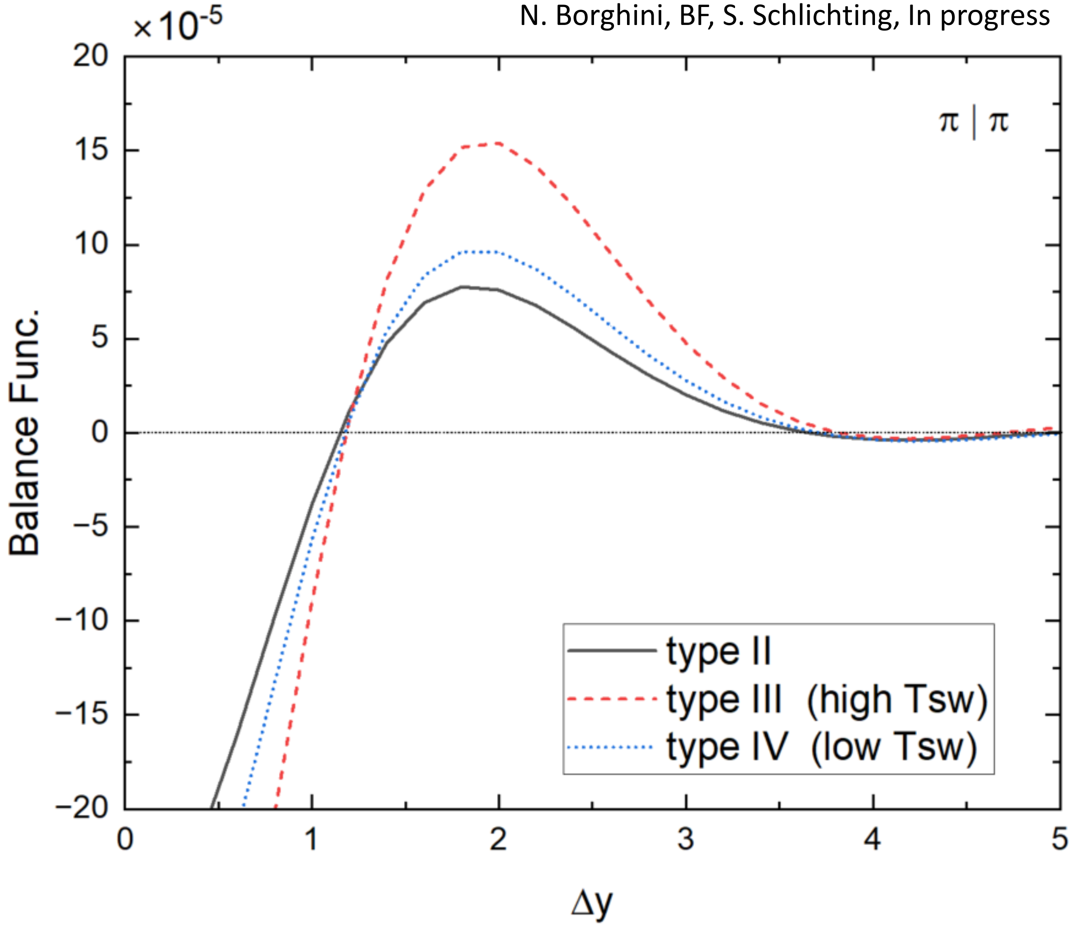


Balance Function

$$B_{h|h'}(p|p') = \frac{1}{2N_{h'}(p')}\langle[N_{\bar{h}}(p) - N_h(p)]N_{h'}(p')\rangle + \frac{1}{2N_{\bar{h}'}(p')}\langle[N_h(p) - N_{\bar{h}}(p)]N_{\bar{h}'}(p')\rangle$$



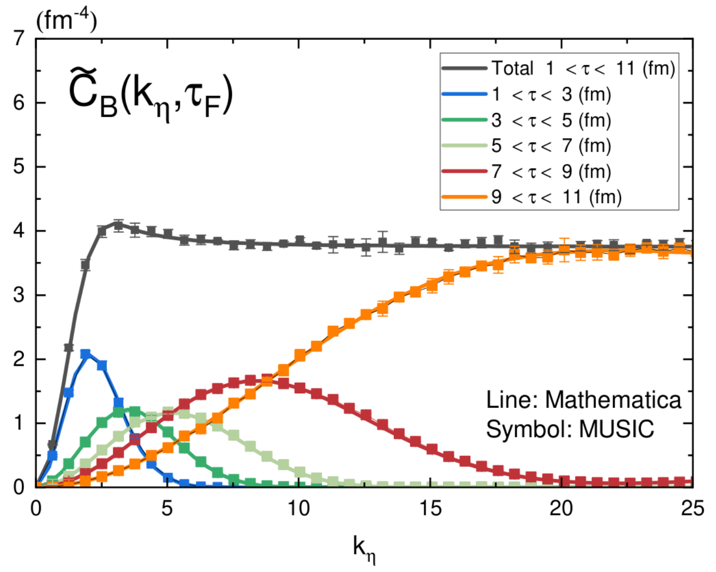
N. Borghini, BF, S. Schlichting, In progress

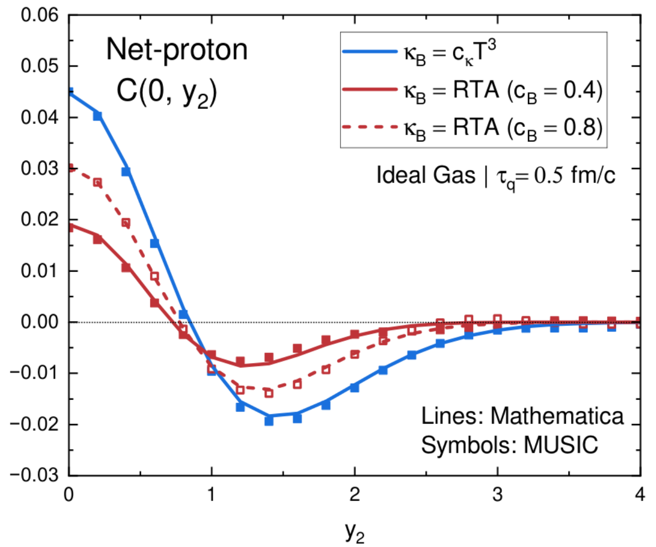


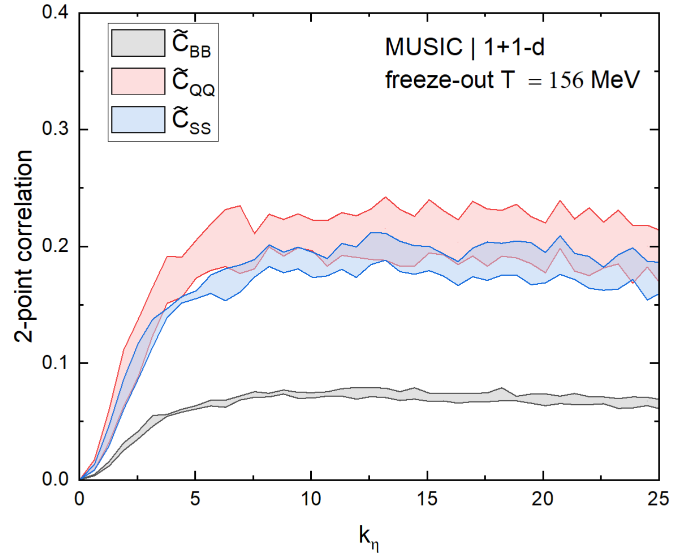
Summary & Outlook

- Linearized stochastic charge transport model: stable, efficient and cut-off independent
- 2PC: time ordering, sensitive on κ_B
- Significant mixed-charge transport signals

- Include CE/MCE freeze-out and hadronic afterburner to finalize this hybrid-hydrodynamic model
- Investigate observables in realistic 3+1-d heavy-ion system







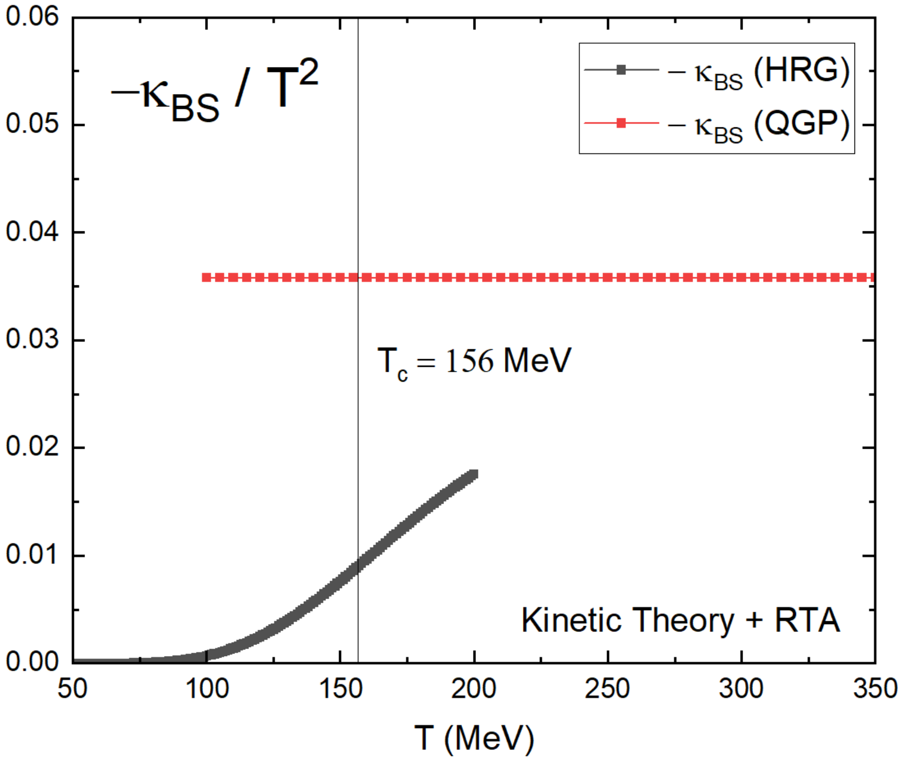
Transport Coefficients

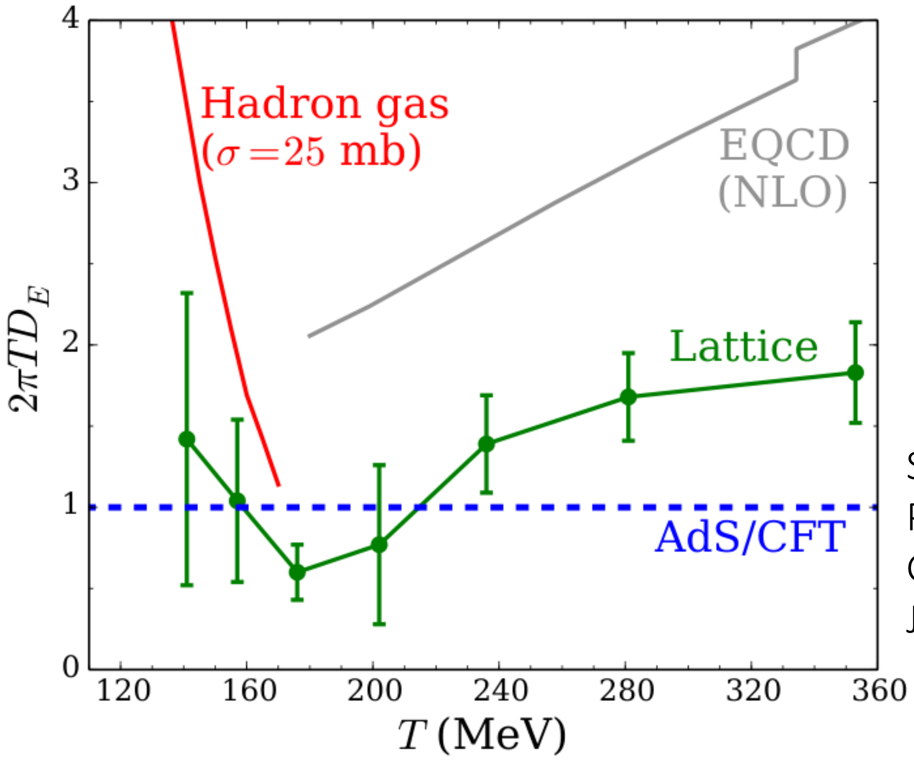
- Transport coefficients (κ_{ij}, τ_q) calculated by kinetic model with RTA
- Upgrades on the QGP phase assumption ($m_q = 0$)

$$D_{ij} \equiv \kappa_{ik}\, \chi_{kj}^{-1}\, T^{-1}$$

- Using D_{ij} instead of κ_{ij} as charge diffusion parameter

J. Fotakis, et al.,
Phys.Rev.D 101 (2020) 7, 076007





S. Pratt, C. Plumberg
Phys.Rev.C 102 (2020) 4, 044909
G. Aarts, et al.,
JHEP 1502, 186 (2015)

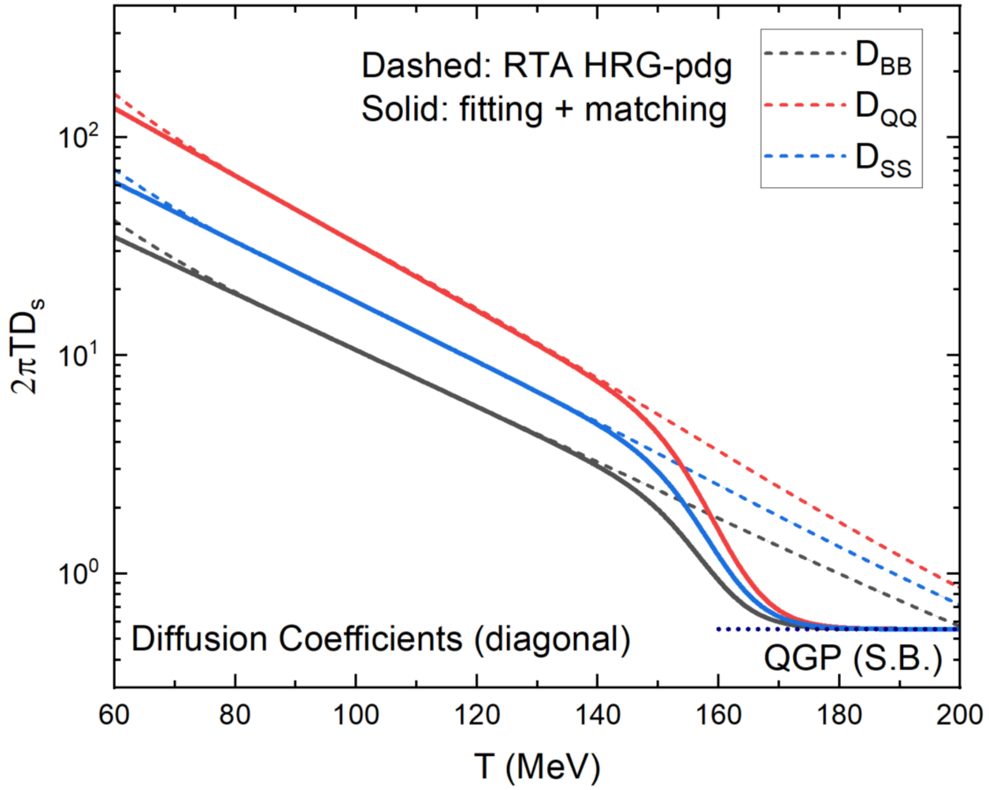
Transport Coefficients

- Transport coefficients (κ_{ij}, τ_q) calculated by kinetic model with RTA
- Upgrades on the QGP phase assumption $(m_q = 0)$

$$D_{ij} \equiv \kappa_{ik}\, \chi_{kj}^{-1}\, T^{-1}$$

- Using D_{ij} instead of κ_{ij} as charge diffusion parameter

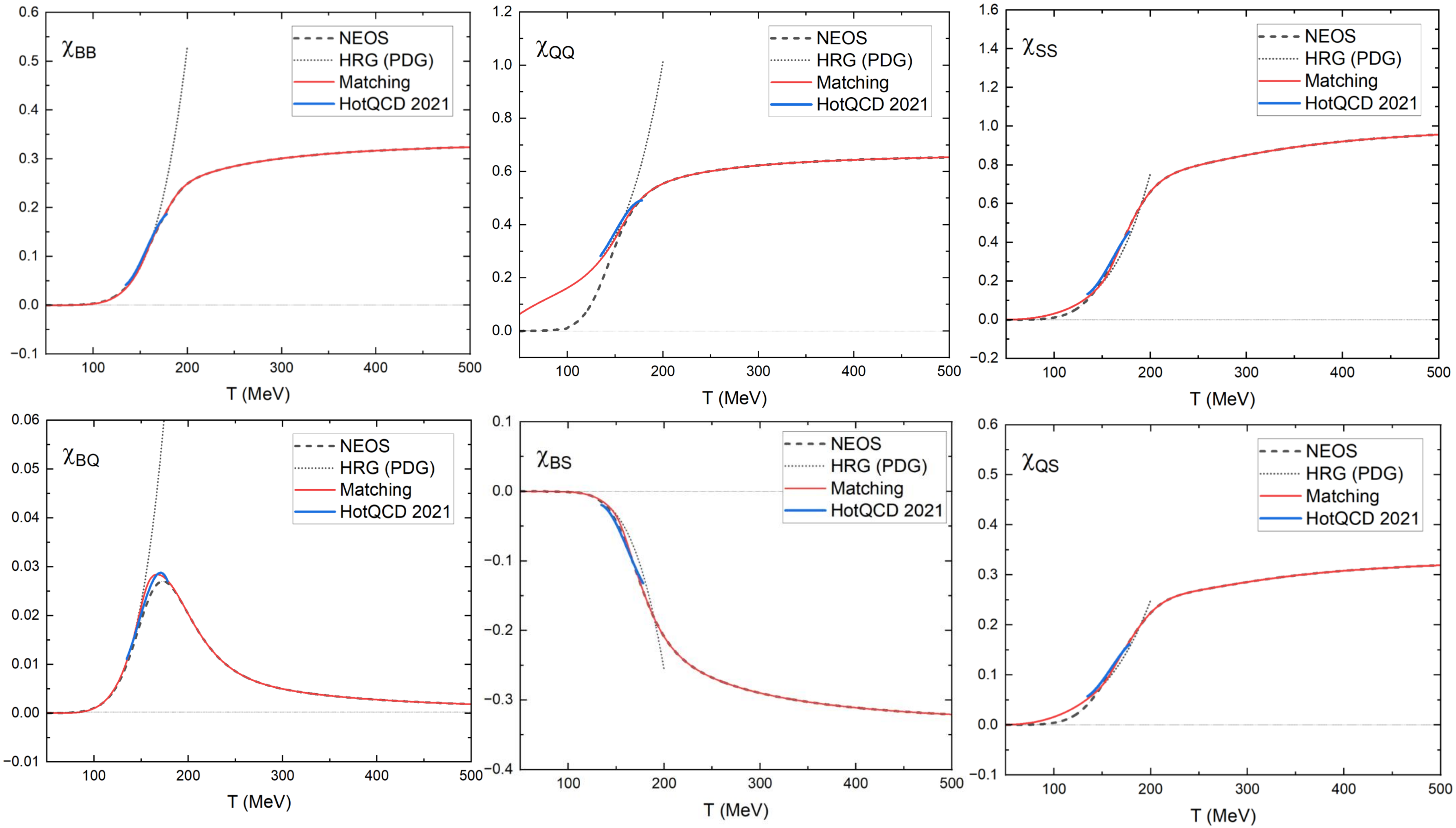
J. Fotakis, et al.,
Phys.Rev.D 101 (2020) 7, 076007



N. Borghini, BF, S. Schlichting, In progress

Linearized EoS

N. Borghini, BF, S. Schlichting, In progress



Hydrodynamics

Conservation Eqs.

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = eu^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

$$\boxed{\partial_\mu N_i^\mu = 0}$$

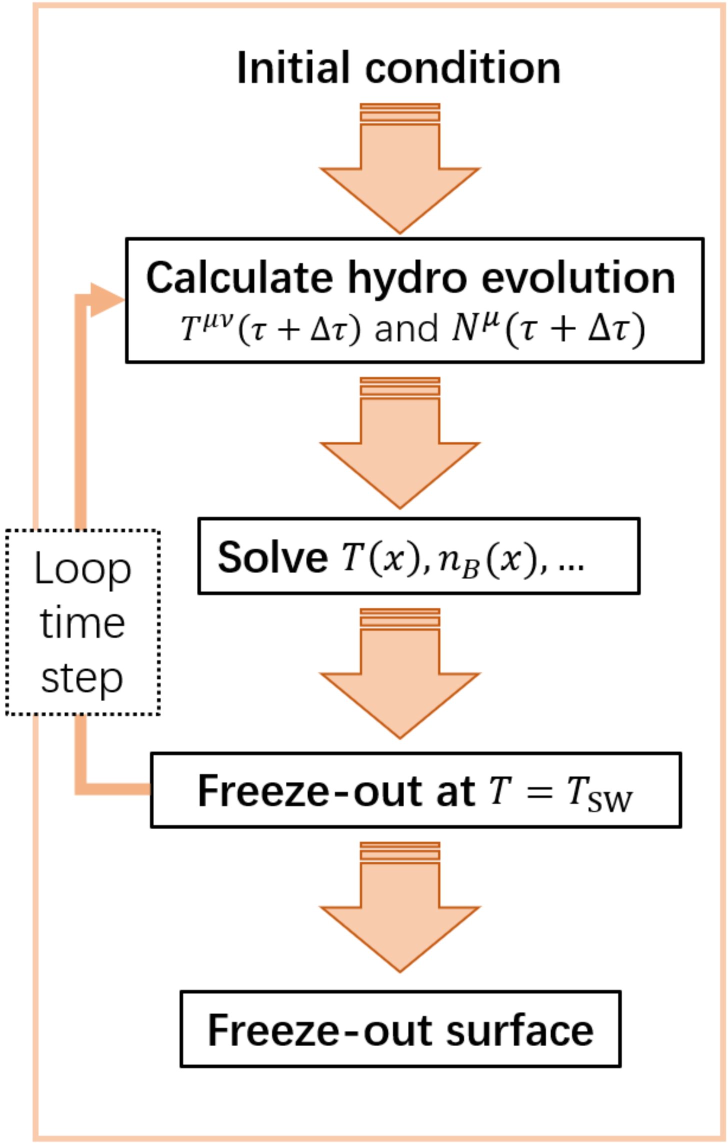
$$N_i^\mu = n_i u^\mu + q_i^\mu$$

+ EoS

+ Transport equations

$$\Delta^{\mu\nu} Dq_\nu = -\frac{1}{\tau_q}(q^\mu - q_{NS}^\mu)$$

Ordinary MUSIC

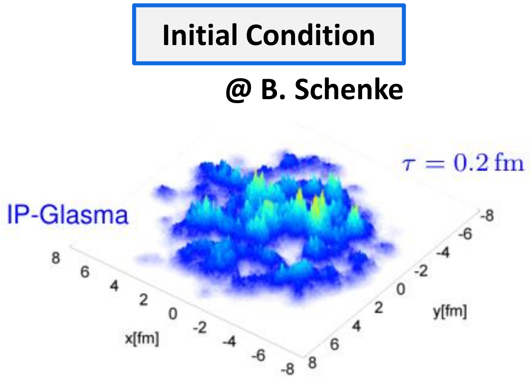


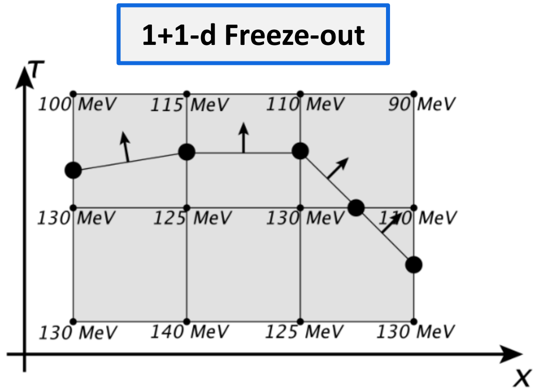
1. **Initial Condition:**
   - provide $T^{\mu\nu}(\tau_0, \boldsymbol{x})$ and $N^\mu(\tau_0, \boldsymbol{x})$
   - lots of IC models on market (Trento, etc.)
   - (in later comparison) 1+1-d boost invariant IC

$$e(\tau_0, \boldsymbol{x}) = \text{const.}\,,\; u^\mu(\tau_0, \boldsymbol{x}) = (1, \boldsymbol{0})$$

2. **Hydro evolution**
   - MUSIC: a 3+1-d viscous hydrodynamic code
   - input EoS and transport coefficients (η/s, κ_B, …)

3. **Freeze-out**
   - mapping the field info onto constant T hyper-surface





Theory vs. Experiments

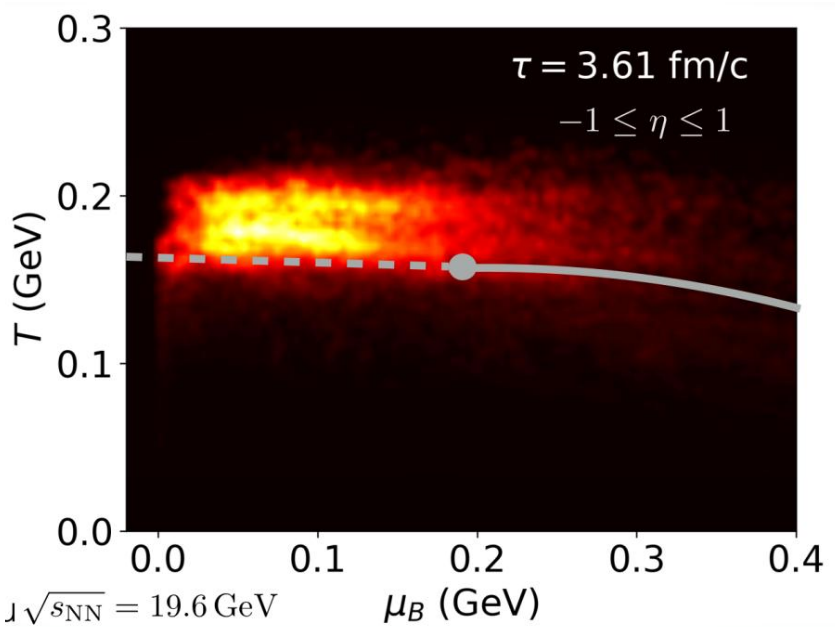
Theory

- Long lived and spatially infinite
- Uniform, particle heat bath
- And static

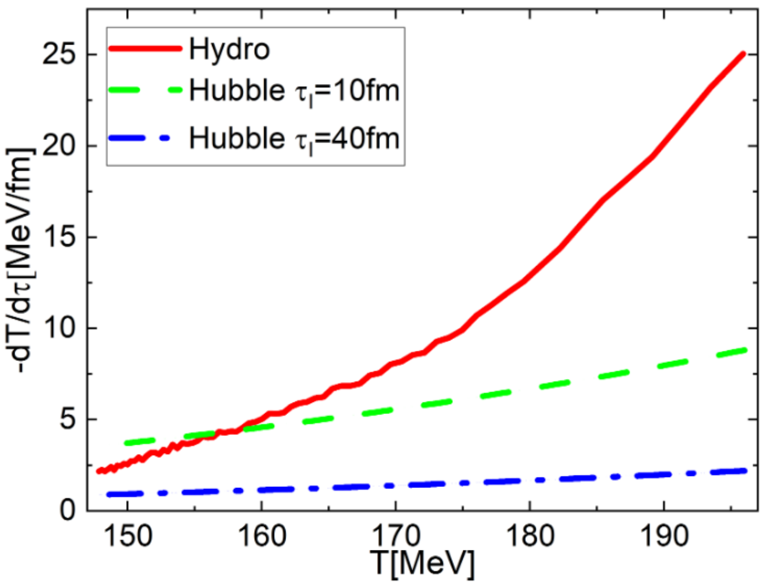
Experiments

- Short lived and spatially small (~10 fm)
- Inhomogeneous
- And highly dynamical

QGP fireball trajectory



QGP cooling rate

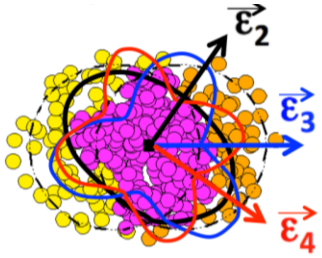


The dynamical description of fluctuations is necessary!

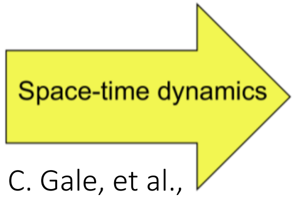
C. Shen,
HENPIC-EVO, 2018
S. Tang, S, Wu, H. Song,
Phys.Rev.C 106 (2022) 3, 034905

Fluctuations in Heavy-Ion Collisions

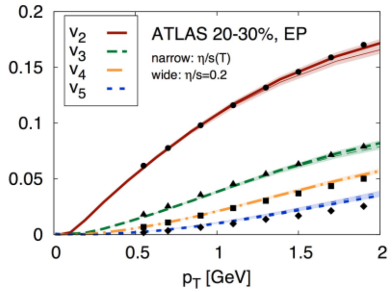
1. **Initial State Fluctuation**
   - nucleon distribution + sub-nucleon structure
   - recent: nuclei deformation, mode decomposition, …



Hydro-response

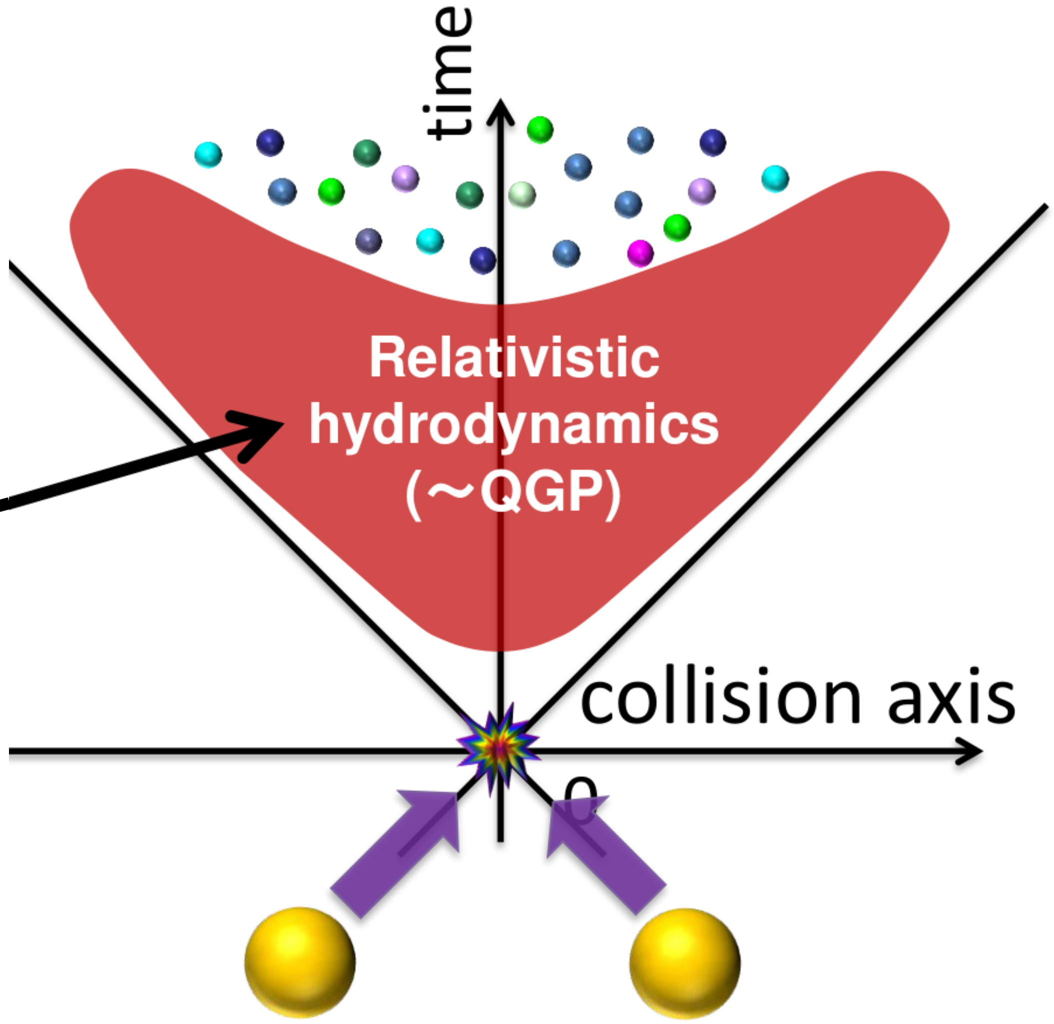


C. Gale, et al.,
PRL 110(2013) 012302



2. **Hydrodynamic Fluctuation**
   - **Thermal fluctuation (including critical fluc.)**
   - jets, core-corona, etc.

3. **Freeze-out & Hadronic Fluctuation**
4. **Imperfect detection efficiency**



IS: STAR, Nature 635 (2024) 8037, 67-72
R. Krupczak, et al., arxiv: 2508.05336
Hydro: Gokce Basar, PPNP 143 014175 (2024)
Frz: J. Karthein, et al., arxiv: 2508.19237
HRG: J. Hammelmann, et al., PRC 110 (2024) 5, 054910

Dynamical Approaches for Hydro Fluctuation

Hydro-kinetics

- Deterministic
- **Hydrodynamic + correlation modes**
- Cut-off independent
- One sample, millions of equations (3+3+1)

Linearized Stochastic baryon transport

Background ($T^{\mu\nu}$) + Noise Transport (N^{μ})

- Application in 3+1-d MUSIC
- Stable and cut-off independent solution
- Save ~98% computation time
- Towards mixed BQS-charge transport

Stochastic Hydro

- **Hydrodynamic + stochastic noise**

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{\text{viscous}} + S^{\mu\nu}_{\text{noise}},$$
$$\partial_\mu J^{\mu} = 0, \quad J^{\mu} = J^{\mu}_{\text{ideal}} + J^{\mu}_{\text{viscous}} + I^{\mu}_{\text{noise}}.$$

- Fluctuation-Dissipation Relation (FDR)

$$\langle I^\mu(x_1) I^\nu(x_2)\rangle = 2T\sigma\Delta^{\mu\nu}\delta^{(4)}(x_1 - x_2)$$

- Sensitive on cut-off and grid spacing
- Instabilities and challenges for PDE solver
- One equation, millions of samples
- **Advantages: Observables, IC models, hard probes, etc.**

A. De, C. Shen, J. Kapusta, PRC 106 (2022) 5, 054903
A. Sakai, K. Murase, T. Hirano, PRC 102 (2020) 6, 064903
M. Singh, et al., Nucl.Phys.A 982 (2019) 319