Application of stochastic fluids in model H

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October 11, 2025@DUT, Dalian

Outline and Motivations

- To provide a field-theoretical justification for stochastic hydrodynamics
- To determine the magnitude of the multiplicative noise and study its effect.
- To reveal some unusual behavior (critical phenomena) through the framework

Fluid Dynamics for Relativistic QCD Matter

Fluid dynamics is a universal effective field theory (EFT) of non-equilibrium many-body systems with a stable equation of state and

- Conservation of charge: $\partial_\mu J^\mu = 0$
- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

$$
\begin{aligned}
J^\mu &= n\, u^\mu + \nu^\mu \\
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \qquad \nu^\mu = -\kappa T \Delta^{\mu\nu}\partial_\nu \left(\frac{\mu}{T}\right) \\
\pi^{ij} &= -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}
\end{aligned}
$$

The dissipation terms are described by the shear viscosity η, bulk viscosity ζ and charge conductivity κ

Dynamical Model in RHIC Beam Energy Scan

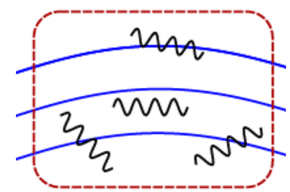
Additional factors must be considered, such as:

- Finite size and finite expansion rate effects
- Freeze-out, resonances, global charge conservation, and others
- Non-dissipation effects
  - The role of fluctuations is enhanced in nearly perfect fluids, i.e., long time tails
  - Fluctuations are dominant near critical points

FLUCTUATIONS IN HYDRO

- The deterministic hydro equations do not lead to spontaneous fluctuations
- The occurrence of fluctuations is a consequence of the microscopic dynamics and must persist at the coarse-grained hydro-level

Introducing non-linear dissipation with density dependent transport coefficients and random noises:



$$J^\mu \rightarrow J^\mu + \theta^\mu$$
$$T^{\mu\nu} \rightarrow T^{\mu\nu} + \theta^{\mu\nu}$$

$$\langle \theta^\mu \rangle = 0 \quad \left\langle (\theta^\mu)^2 \right\rangle \sim L_J(x)\delta(x-x')(t-t')$$
$$\langle \theta^{\mu\nu} \rangle = 0 \quad \left\langle (\theta^{\mu\nu})^2 \right\rangle \sim L_T(x)\delta(x-x')(t-t')$$

Representation in MSRJD Field Theory

In terms of the slow variable (a conserved density), the free energy of the fluid:

$$\mathcal{F}[\psi] = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\psi)^2 + \frac{r}{2}\psi(x,t)^2 + \frac{\lambda}{3!}\psi(x,t)^3 + \dots + h(x,t)\psi(x,t) \right\}$$

The diffusion equation:

$$\partial_t \psi(x,t) = \vec{\nabla}\left\{ \kappa(\psi)\,\vec{\nabla}\left(\frac{\delta\mathcal{F}[\psi]}{\delta\psi}\right)\right\} + \theta(x,t)$$

where the Gaussian noise term $\theta(x,t)$ has a distribution

$$P[\theta] \sim \exp\left(-\frac{1}{4}\int d^3x\,dt\,\theta(x,t)L(\psi)^{-1}\theta(x,t)\right)$$

Representation in MSRJD Field Theory, cont.

The conductivity, $\kappa(\psi)$, is field-dependent: $\kappa(\psi) = \kappa_0 \left(1 + \lambda_D \psi\right)$

The partition function is given as: Martin, Siggia and Rose, PhysRevA.8:423(1973)

$$
\begin{aligned}
Z &= \int \mathscr{D}\psi \, P[\theta] \exp\left(-i\tilde{\psi}\,(\text{e.o.m}\,[\psi,\,\theta\,])\right) \\
&= \int \mathscr{D}\psi \, \mathscr{D}\tilde{\psi}\ \exp\left(-\int d^3x\,dt\,\mathscr{L}(\psi,\tilde{\psi})\right)
\end{aligned}
$$

The effective Lagrangian of this theory is:

$$
\mathscr{L}(\psi,\tilde{\psi}) = \tilde{\psi}\left(\partial_t - D_0\nabla^2\right)\psi - \frac{D_0\lambda'}{2}\left(\nabla^2\tilde{\psi}\right)\psi^2 - \tilde{\psi}L(\psi)\tilde{\psi}
$$

Note: $D_0 = r\kappa_0$ and $\lambda' = \lambda/r + \lambda_D$.

The noise kernel is still unknown.

Time Reversal Symmetry

Stochastic theories must describe the detailed balance condition:

$$\frac{P\left(\psi_1 \longrightarrow \psi_2\right)}{P\left(\psi_2 \longrightarrow \psi_1\right)} = e^{-\Delta \mathcal{F}/k_B T}$$

Definie the time-reversal symmetry: Janssen, ZPhyB.23:377(1976)

$$\psi(t) \longrightarrow \psi(-t) \quad \tilde{\psi}(t) \longrightarrow -\left[\tilde{\psi}(-t) + \frac{\delta F}{\delta \psi}\right]$$

$$\mathcal{L} \longrightarrow \mathcal{L} + \frac{d}{dt}F$$

if we choose the noise kernel: $L(\psi) = \overleftarrow{\nabla}\left[k_B T \kappa(\psi)\right]\overrightarrow{\nabla}$

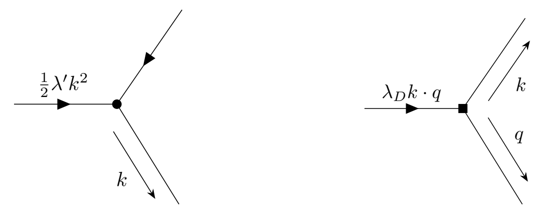
Such TSR implies the fluctuation-dissipation relation

$$\left\langle \psi(x_1, t_1)\left[\overleftarrow{\nabla}\kappa(\psi)\overrightarrow{\nabla}\tilde{\psi}\right](x_2, t_2)\right\rangle = \Theta(t_2 - t_1)\left\langle \psi(x_1, t_1)\dot{\psi}(x_2, t_2)\right\rangle$$

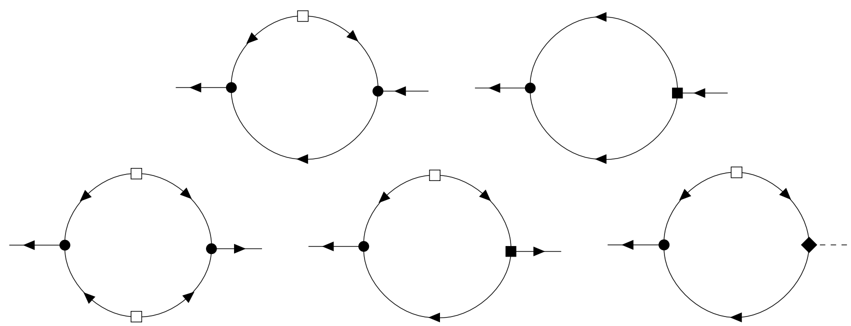
Simpler Example of Model B

- Linearized propagator:

- Vertex and new vertices:



- Loop contributions:



We find that multiplicative noise contributes to the long-time tails of the density correlation functions at leading order. JC and Schefer, JHEP01(2021)071

1PI Effective Action

Consider the generating functional with local source J, $\tilde{J}$:

$$W[J,\tilde{J}] = -\ln \int \mathscr{D}\psi \mathscr{D}\tilde{\psi}\; e^{-\int dt\, d^3x \left\{\mathscr{L} + J\psi + \tilde{J}\tilde{\psi}\right\}}$$

Performing a Legendre transform to the 1PI effective action via background field method with $\psi = \Psi + \delta\psi$:

$$\Gamma[\Psi,\tilde{\Psi}] = W[J,\tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi}\right)$$

Taking the derivative of the 1PI effective action w.r.t. the classical field Ψ yields the e.o.m. encoded the long time tails in the evolution of the density:

$$(\partial_t - D\nabla^2)\Psi - \frac{\kappa\lambda_3^2}{2}\nabla^2\Psi^2 + \int d^3x'\, dt'\, \Psi(x',t')\Sigma(x,t;x',t') = 0$$

DOUBLE LEGENDRE TRANSFORMATION

☞ nPI effective action $\Longrightarrow$ e.o.m. for n-point functions

✓ Couple a bi-local source $\frac{1}{2}\psi_a K_{ab} \psi_b$ to the system 🔗 Cornwall, Jackiw and Tomboulis, PhysRevD.10:2428 (1974)

✓ Plug in the 1-loop 1PI effective action

✓ Sum beyond 1-loop terms

✓ Apply the stationary conditions:

$$\frac{\delta W}{\delta J_a} = \langle \psi_a \rangle = \Psi_a \,, \qquad \frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle \psi_a \psi_b \rangle = \frac{1}{2}[\Psi_a \Psi_b + G_{ab}]$$

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A \Psi_A - \frac{1}{2} K_{AB} [\Psi_A \Psi_B + G_{AB}]$$

2PI Effective Action

The 2PI effective action is given by:

$$\Gamma[\Psi_a, G_{ab}] = S[\Psi_a] + \frac{1}{2}\frac{\delta^2 S}{\delta\Psi_A \delta\Psi_B} G_{AB} - \frac{1}{2}\mathrm{Tr}\left[\log(G)\right] + \Gamma_F[\Psi_a, G_{ab}]$$

The higher order fluctuations are:

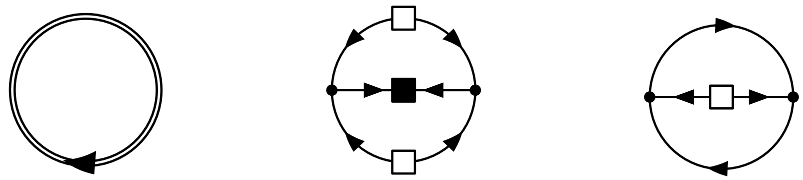
$$\begin{aligned}\exp(-\Gamma_F[\Psi_a, G_{ab}]) \quad = \quad & \frac{1}{\sqrt{\det(G)}} \int D(\delta\psi_a)\, \exp\Big\{ -\frac{1}{2}\delta\psi_A (G^{-1})_{AB}\delta\psi_B \\ & - \Big[S_3[\Psi_a, \delta\psi_a] - \bar{J}_A\delta\psi_A - \bar{K}_{AB}(\delta\psi_A\delta\psi_B - G_{AB})\Big] \Big\}\end{aligned}$$

with

$$\bar{J}_a = \frac{1}{2}\frac{\delta^3 S}{\delta\Psi_a\delta\Psi_B\delta\Psi_C} G_{BC} + \frac{\delta\Gamma_F}{\delta\Psi_a}, \quad \bar{K}_{ab} = \frac{\delta\Gamma_F}{\delta G_{ab}}$$

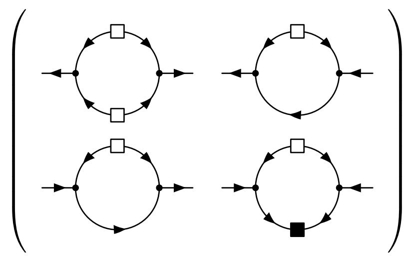
DSE in Mixed Representation

The loop diagrams generated by Γ_F use the full propagator G_{ab}:



Taking the derivative w.r.t G, obtain the DS equation ($\tilde{\psi}$, ψ = 1, 2):

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$

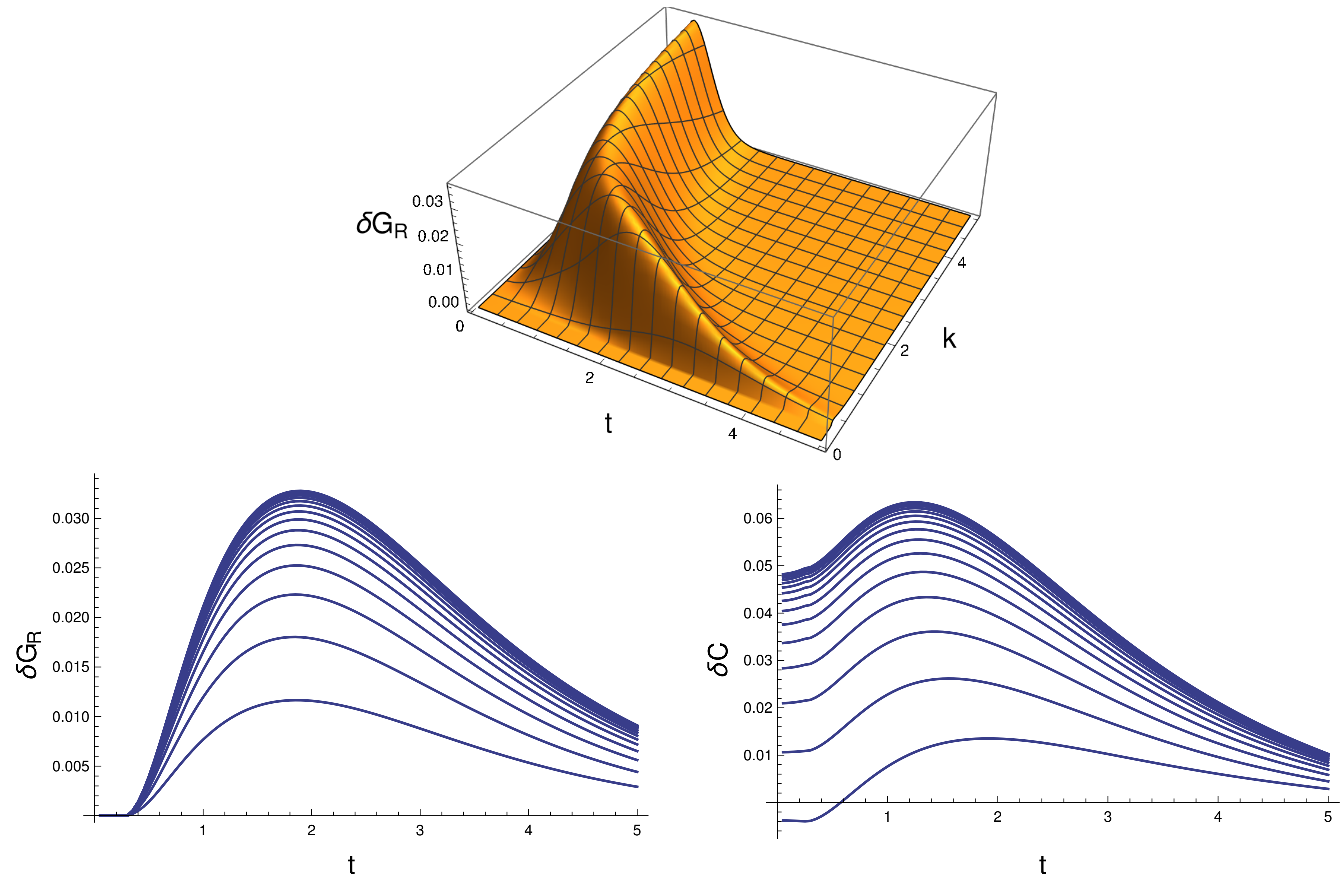


In time-momentum mixed representation

$$\Sigma(t, k^2) = (\kappa\lambda_3)^2 \int d^3k'\, k^2 (k+k')^2\, C(t, k')\, G_R(t, k+k')\,,$$

$$\delta D(t, k^2) = \frac{(\kappa\lambda_3)^2}{2} \int d^3k'\, k^4 C(t, k')\, C(t, k+k')$$

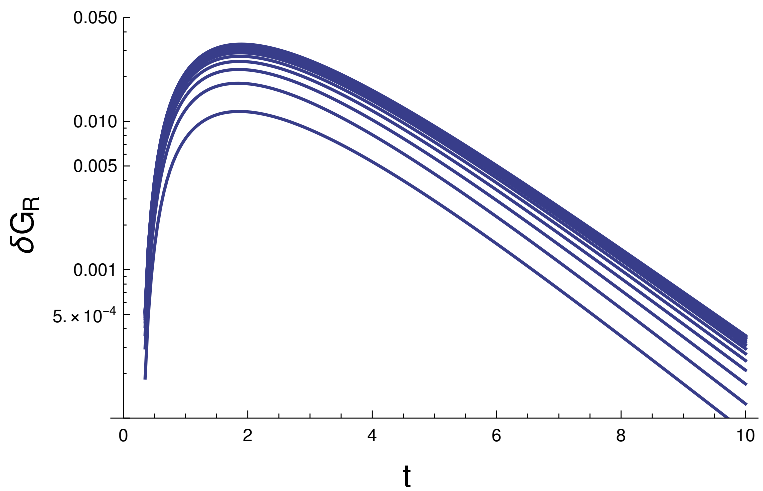
Numerical Simulations in Hydro Limit



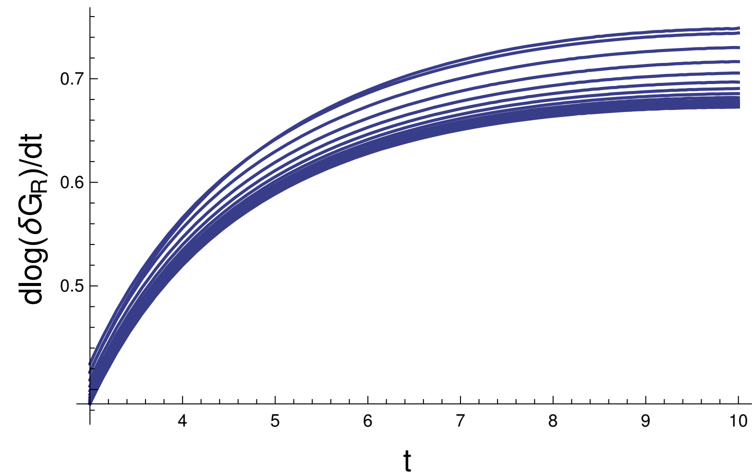
3D curve of $\delta G(t, k)$ and the iterative solutions of DSE

JC and Schefer, JHEP06(2023)057

LONG-TIME BEHAVIOR



A logarithmic plot of the loop corrections to the retarded $\delta G_R(k, t)$



The logarithmic derivative of $\delta G_R(k, t)$ w.r.t t

The long-time behavior of the diffusion cascade is conjectured to be $\sim n!\exp(-Dk^2t/n)$ because of the n-loop terms (shown but not reached).

Delacretaz, SciPostPhys.9:034(2020)

Mode Coupling Theory (MCT)

- For non-critical fluids, the gradient expansion method is used with $k\xi \ll 1$
- For critical fluids, their behaviors are characterized by the transport coefficients in the MCT (= Poisson bracket terms $+$ the critical transport coefficients)

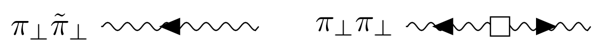
By applying a naive approximation within the MCT, the well known retarded function $G^{-1}(\omega, k) = i\omega - \Gamma_k$ of the diffusion mode is modified to:

$$\Gamma_k = \frac{T}{6\pi\eta_0\xi^3}K(k\xi) \quad \text{with} \quad K(k\xi = x) = \frac{3}{4}\left[1 + x^2 + (x^3 - x^{-1})\arctan(x)\right]$$

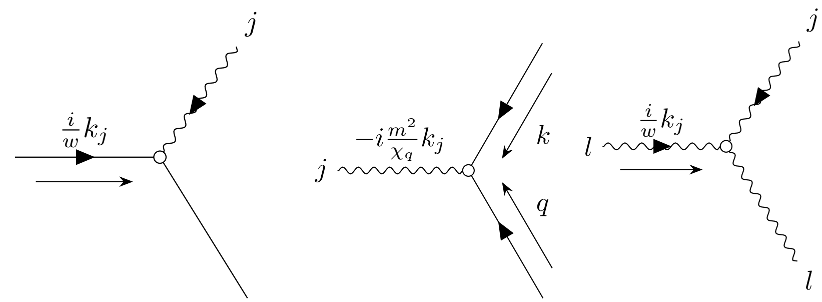
η_0 is the bare shear viscosity. Kawasaki, AnnPhys.61:1(1970)

Model H

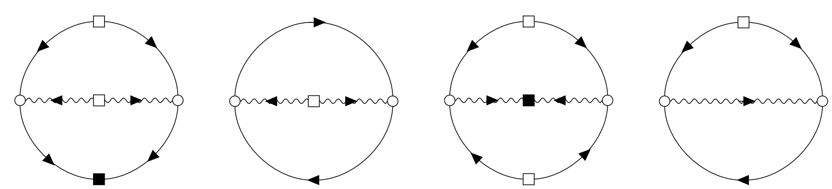
- Linearized propagator:



- Vertices and new vertices:



- Mode-coupling loop contributions:



☞ The contribution of multiplicative noise is sub-leading order compared to that induced by mode couplings.

Scaling Forms of the Transport Coefficients

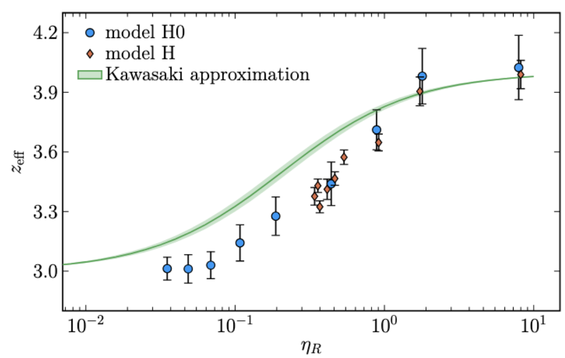
Four modified critical transport coefficients:

$$
\begin{aligned}
D \to D^c(\omega, k, \xi) &= D\,(k\xi_0)^{x_D} F_D\,(\omega\xi^z, k\xi) \\
\kappa \to \kappa^c(\omega, k, \xi) &= \kappa\,(k\xi_0)^{x_\kappa} F_\kappa\,(\omega\xi^z, k\xi) \\
\eta \to \eta^c(\omega, k, \xi) &= \eta\,(k\xi_0)^{x_\eta} F_\eta\,(\omega\xi^z, k\xi) \\
\gamma \to \gamma^c(\omega, k, \xi) &= \gamma\,(k\xi_0)^{x_\gamma} F_\gamma\,(\omega\xi^z, k\xi)
\end{aligned}
$$

- These coefficients differ from the hydrodynamic limit, in which $D = \kappa m^2$ and $\gamma = \eta/w$ no longer hold, where w is enthalpy.
- The dynamical exponent z is determined as $z = 4 - \tilde{\eta} + x_D$ for the diffusion mode in the regime where $k \gg \xi^{-1}$.

Critical Dynamics in Model H

- self-consistent calculation with ansatz: $z = 3.054$ Ohta and Kawasaki, (1976)
- renormalization group (RG) theory: $z = 3.036$ Siggia, Halperin, and Hohenberg (1976)
- real-time fRG approach:
  $z = 3.0507$ for $d = 3$ Chen, Tan and Fu [2406.00679]
  $z = 3.051$ Roth, Ye, Schlichting, and von Smekal [2409.14470]
- Metropolis algorithm: z is function of shear viscosity Chattopadhyay, Ott, Schaefer and Skokov [2403.10608]



Self-Consistent Equations

Re-scale the frequency and the momentum as $(s, r) = (\omega\xi^z, \omega'\xi^z)$, $(x, y) = (k\xi, k'\xi)$ and $[\Sigma, \Delta](\omega, k, \xi) = \xi^{-z}x^2[\bar{\Sigma}, \bar{\Delta}](s, x)$

$$
\begin{aligned}
\bar{\Sigma}^c_{12}(s,x) \quad = \quad & \xi^{2z-7}\int_{r,y}\left\{\frac{y_-^2\,\bar{\Delta}^c_{11}(r_-,y_-)}{\left|-ir_- + y_-^2\,\bar{\Delta}^c_{12}(r_-,y_-)\right|^2}\frac{\xi^2}{w^2y_-^2}\frac{y^2(1-\cos^2\theta)}{i\,r_+ + y_+^2\,\bar{\Sigma}^c_{12}(-r_+,y_+)}\right.\\
& \left. -\frac{y_-^2\,\bar{\Sigma}^c_{11}(r_-,y_-)}{\left|-ir_- + y_-^2\,\bar{\Sigma}^c_{12}(r_-,y_-)\right|^2}\frac{(y_-^2+1)-(x^2+1)}{wy_+^2}\frac{y^2(1-\cos^2\theta)}{i\,r_+ + y_+^2\,\bar{\Delta}^c_{12}(-r_+,y_+)}\right\}
\end{aligned}
$$

$$
\bar{\Sigma}^c_{11}(s,x) \quad = \quad \xi^{2z-7}\int_{r,y}\left\{\frac{y_-^2\,\bar{\Delta}^c_{11}(r_-,y_-)}{\left|-ir_- + y_-^2\,\bar{\Delta}^c_{12}(r_-,y_-)\right|^2}\frac{\xi^2y^2}{w^2y_-^2}\frac{\left(1-\cos^2\theta\right)\,y_+^2\,\bar{\Sigma}^c_{11}(r_+,y_+)}{\left|-ir_+ + y_+^2\,\bar{\Sigma}^c_{12}(r_+,y_+)\right|^2}\right\}
$$

Obviously, Kawasaki's approximation of correlation is $\bar{\Sigma}^c_{11} = \xi^2\bar{\Sigma}^c_{12}/(x^2+1)$

DOMINANT KINETIC CONTRIBUTION

In the one-loop approach, the main contribution comes from the region in which the external momentum of order parameter, represented by the vector x, is parallel to the inner momentum of momentum density, represented by the vector y_-.

$$\bar{\Sigma}^c_{12}(0, x) \sim \int_0^x y^4 \, dy \int_{1-\epsilon^2}^1 d\cos\theta \, \frac{1-\cos^2\theta}{y_-^2 \left(-i0 + y_-^2\right)}$$

In the momentum region, where $y_- \sim \epsilon x$ ($y_+ \sim \epsilon x$ for anti-parallel), the loop contribution can be estimated as x. This recovers the Kawasaki approximation of $\Sigma_{12} \sim x^2 \bar{\Sigma}^c_{12} \sim x^3$ for large x.

SELF-CONSISTENT EQUATIONS, CONT.

$$\bar{\Delta}^c_{12}(s,x) \quad = \quad \xi^{2z-7}\int_{r,y} \frac{2y_-^2\,\bar{\Sigma}^c_{11}(r_-,y_-)}{\left|-ir_- + y_-^2\,\bar{\Sigma}^c_{12}(r_-,y_-)\right|^2}\,\frac{y^3}{wx}\,\frac{\cos\theta - \cos^3\theta}{i\,r_+ + y_+^2\,\bar{\Sigma}^c_{12}(-r_+,y_+)}$$

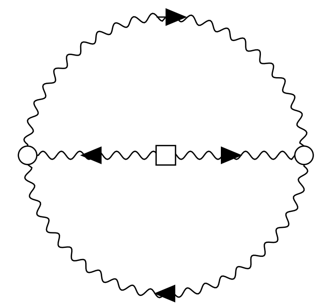
$$\bar{\Delta}^c_{11}(s,x) \quad = \quad \xi^{2z-7}\int_{r,y} \frac{2y_-^2\,\bar{\Sigma}^c_{11}(r_-,y_-)}{\left|-ir_- + y_-^2\,\bar{\Sigma}^c_{12}(r_-,y_-)\right|^2}\,\frac{y^3y_+^2}{\xi^2x}\,\frac{\left(\cos\theta - \cos^3\theta\right)y_+^2\,\bar{\Sigma}^c_{11}(r_+,y_+)}{\left|-ir_+ + y_+^2\,\bar{\Sigma}^c_{12}(r_+,y_+)\right|^2}$$

$$\int_{r,y} = \int_{-\omega_\Lambda}^{\omega_\Lambda} dr\int_0^\Lambda \frac{y^2}{(2\pi)^3}\,dy\int_0^\pi \sin\theta\,d\theta$$

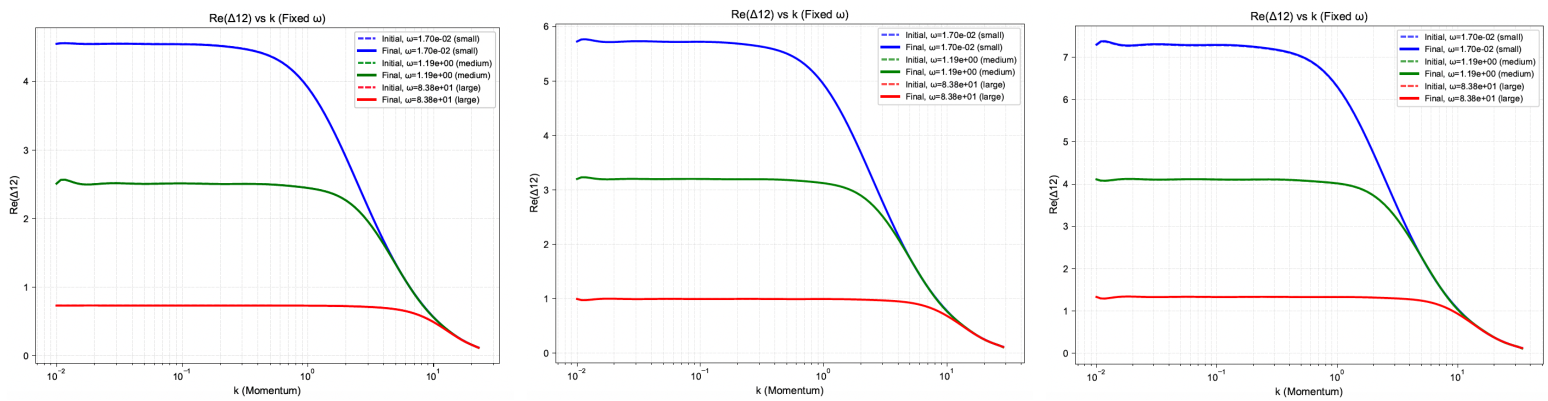
Note that the UV cutoffs, ω_Λ and Λ, are introduced. Both are proportional to the microscopic scale of ξ_0^{-1}.

Renormalized Transport Coefficients

The renormalization of η: 🔗 Kovtun, Moore and Romatschke [1104.1586]

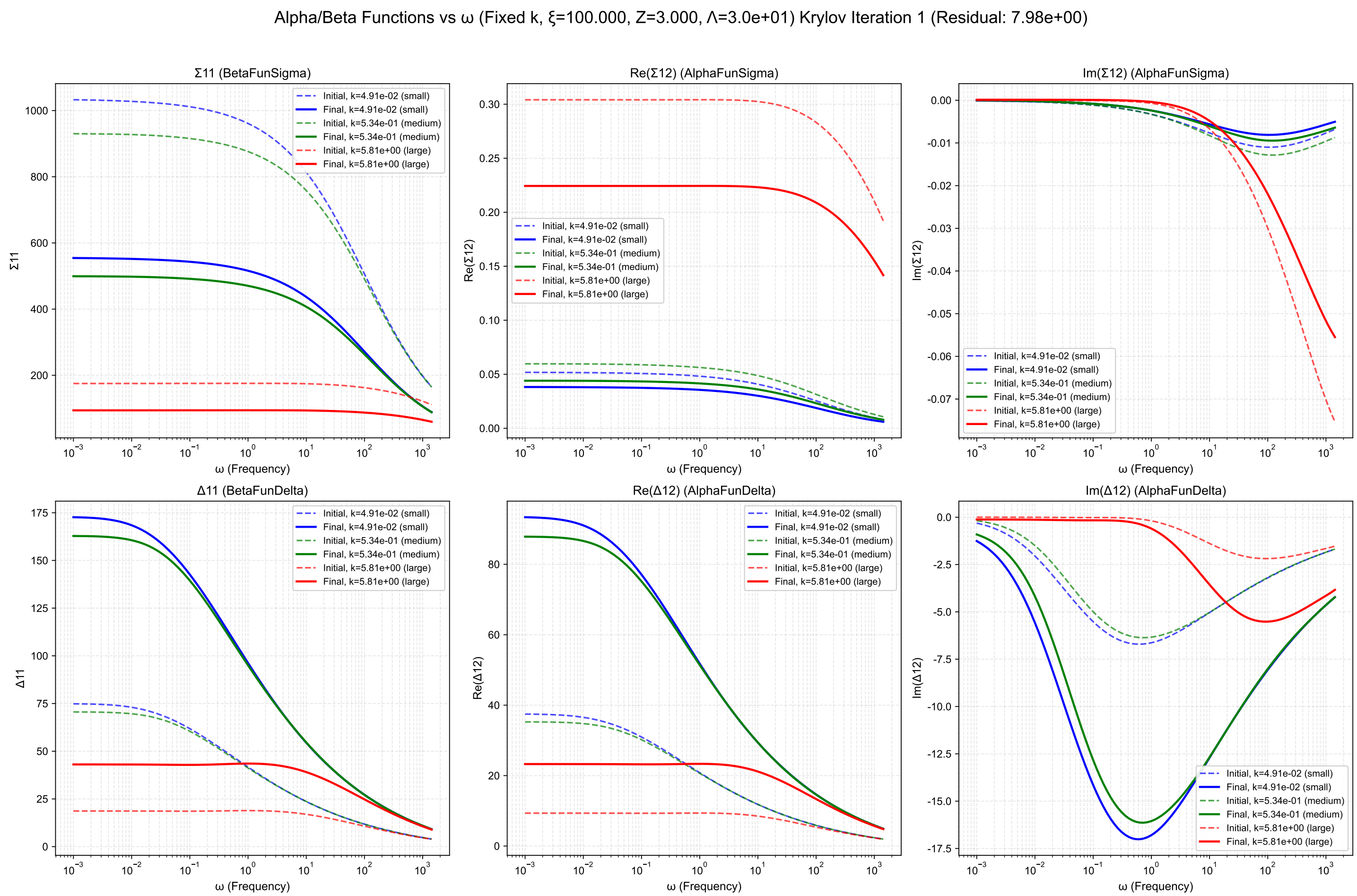


$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}$$



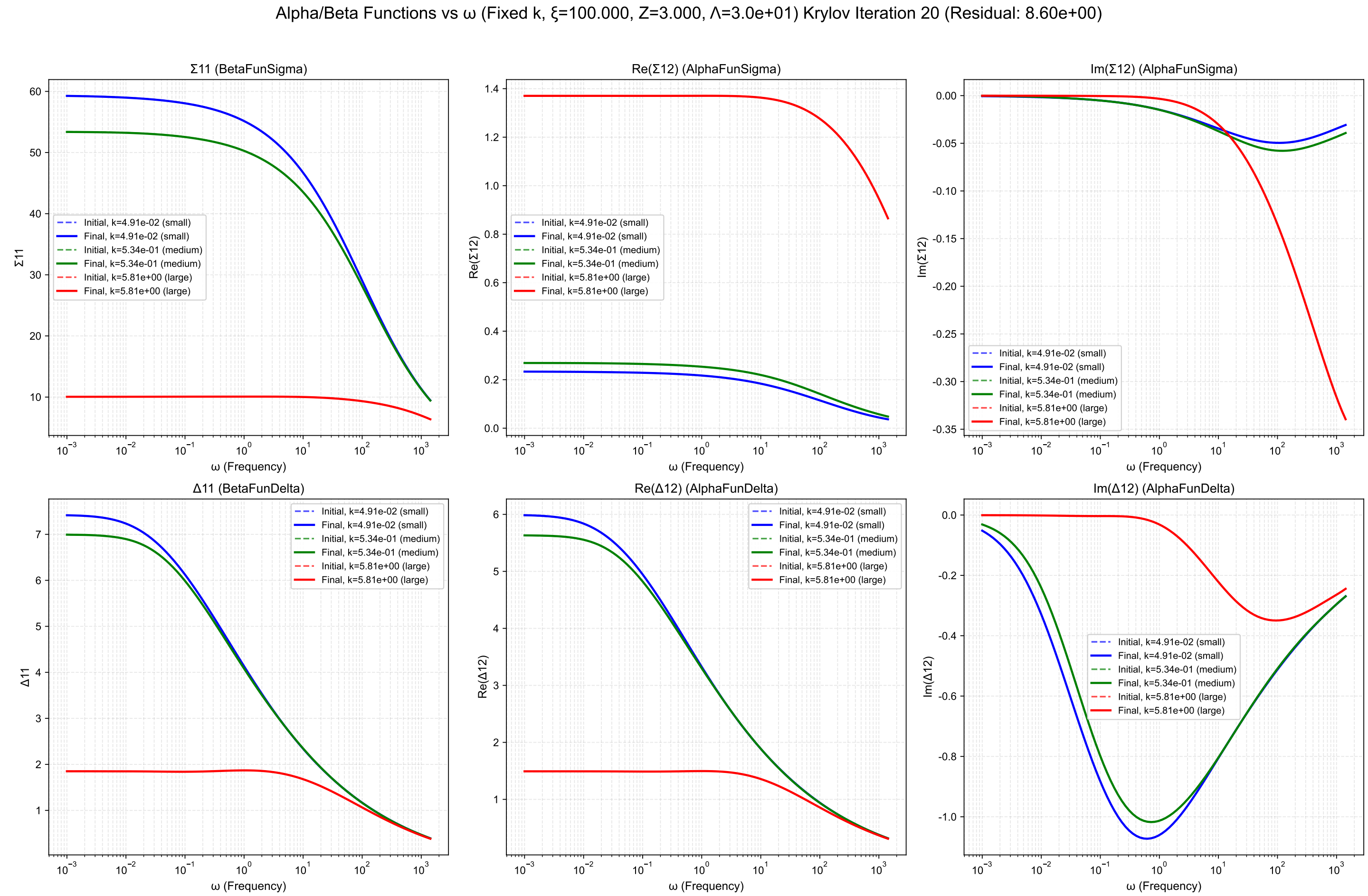
From left to right are the corrected shear modes with UV cutoffs of 24, 30, and 36.

THE ONE-LOOP RESULTS



The iteration results maintain the shape.

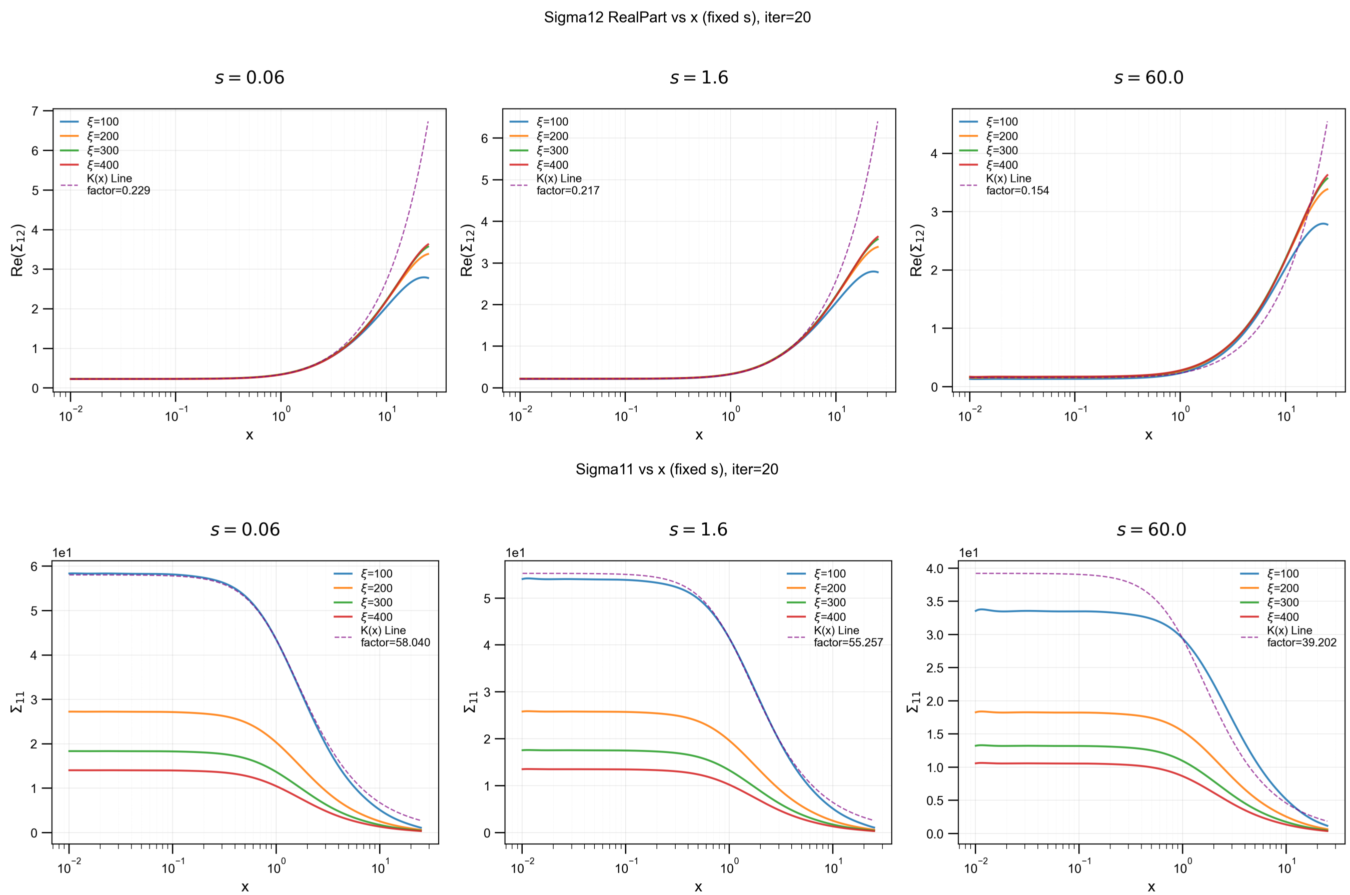
THE SELF-CONSISTENT RESULTS



Eventually, the self-consistent results show a significant change in scale but only a slight modification in shape.

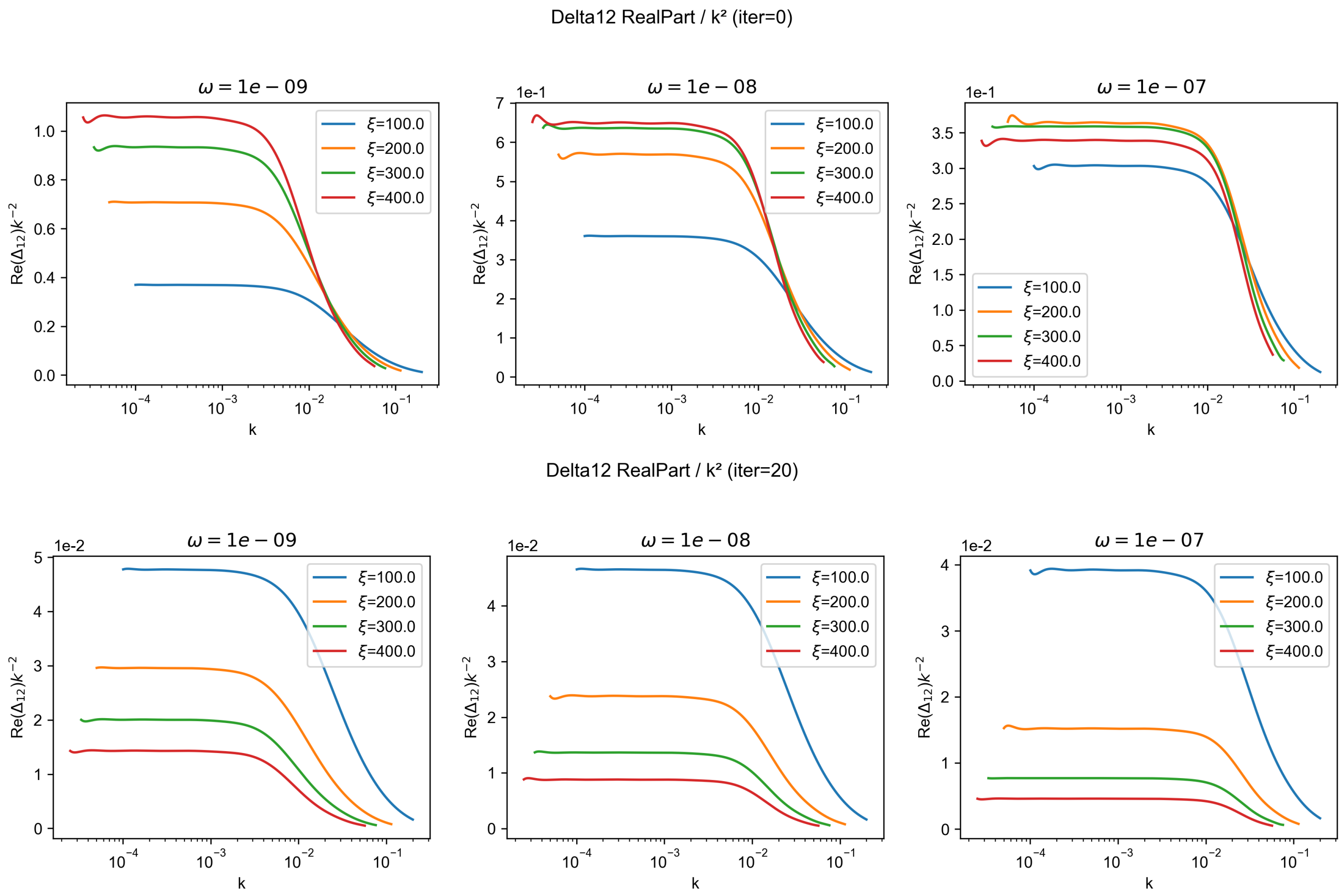
Comparison with Kawasaki Function (preliminary)

The order parameter sector has a curve similar to the Kawasaki approximation.



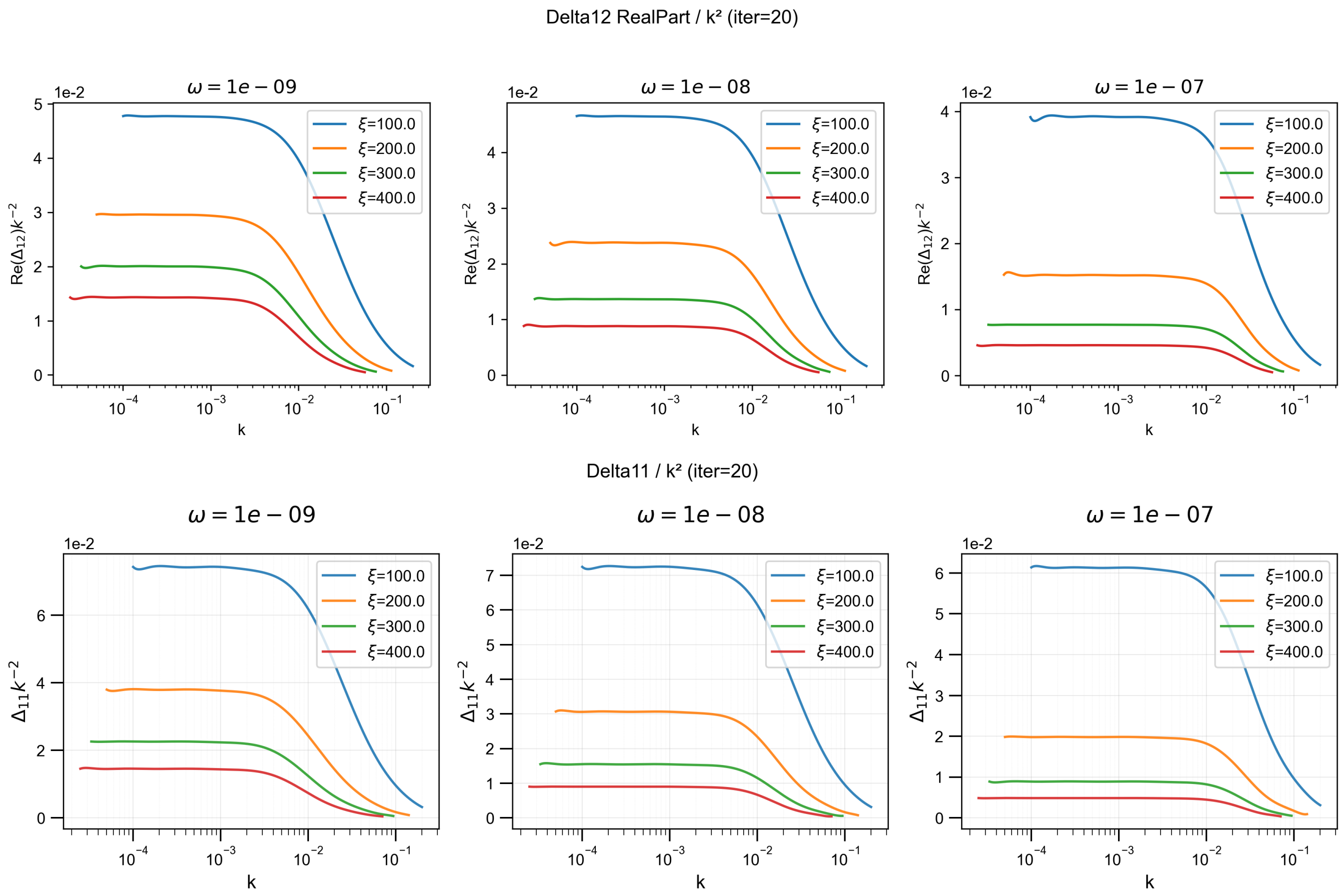
ORDER SHIFTED FOR $\eta(\xi)$ (PRELIMINARY)

Under Kawasaki's approximation, $\eta = \eta_0(1 + \frac{8}{15\pi^2}\ln\xi)$. However, the order is inverted at the end. Perl and Ferrell, PhysRevA.6.2358 (1972)



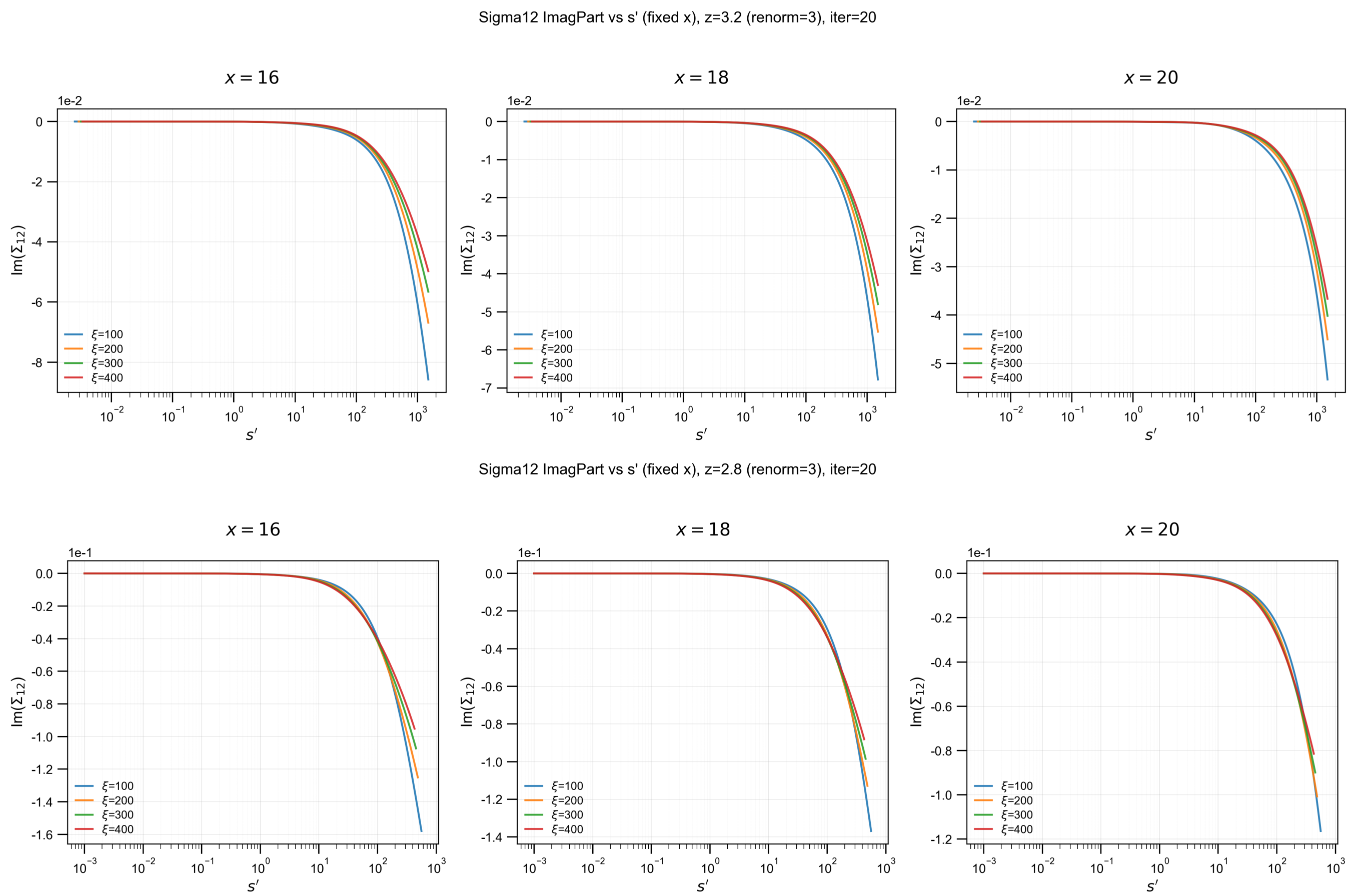
Decoupled Shear Term (preliminary)

It is believed that the shear mode is decoupled and behaves as $\omega \sim \eta_R k^2$ in the small momemtum region.



Dynamic Scaling For z_{prime} (preliminary)

To determine the value of "z_{prime}", one applies the dimensionless quantity.



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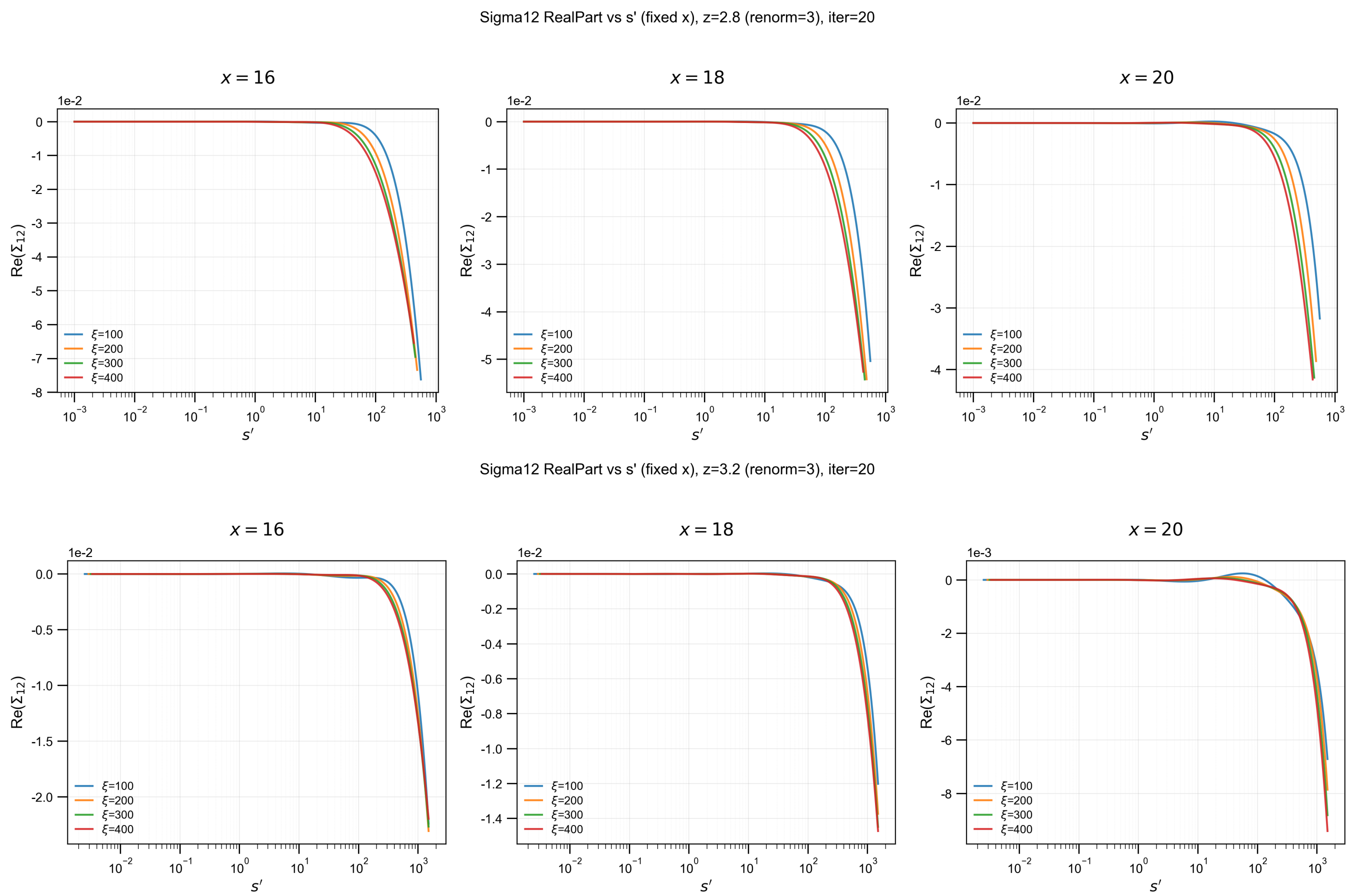

Summary and Outlooks

- By comparing the one-loop approach, we gain more insight into the evolution of the order parameter and the momentum density.

- By considering the magnitude of η, the order parameter relaxation rate is
$$\Gamma_k = \frac{D_0}{\xi^4}(k\xi)^2\left(1 + (k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$
For $k\xi \gg 1$, $\tau_R \sim \xi^4$ for large η_R, and $\tau_R \sim \xi^3$ for small η_R

- By extending our model to include expanding systems, we can better understand the dynamical nature of phase transitions.

Thank You for Your Attention!