Long-distance contributions to kaon decays on the lattice

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Flavor physics on the lattice

Introduction

- Lattice QCD: non-perturbative calculations of hadronic effects.
- Examples of lattice QCD application in flavor physics [FLAG '24]:
  - Form factors in D and B decays.
  - Nucleon matrix elements for the charges $g_{A,S,T}$.
- **Focus of this talk**:
  - Rare kaon decays: tests for CP-violation, low-energy probes for high-energy physics
    - $K \to \pi \nu \bar{\nu}$
    - $K \to \pi l^+ l^-$
    - $K \to l^+ l^-$
  - E&M corrections: precision tests of the Standard Model.
    - f_K / f_π

Flavor physics on the lattice

Lattice QCD

- **Goal**: computing matrix element $\langle f|\mathcal{O}(x)Q_i(0)|K\rangle$.

  Q_i: effective weak operator. Eg. $Q_1^{\Delta S=1} = (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}$.

- Euclidean formulation of QCD with finite lattice spacing a (UV) and extent L (IR).

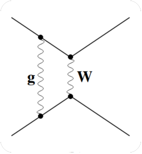
- Positive (semi-)definite Boltzmann weight + gauge-invariant path-integral measure $\Rightarrow$ suitable for Monte Carlo-based methods:

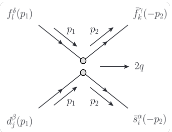
$$\langle \mathcal{O} \rangle \equiv \int \mathcal{D}[U] e^{-S[U]} \mathcal{O}[U] \approx \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}[U_n] , \quad U \in \mathrm{SU}(3)_{\mathrm{strong}} .$$

- Extrapolation to the physical point via $(a, L, M_\pi) \to (0, \infty, M_\pi^{\mathrm{Phys.}})$.

Flavor physics on the lattice

Operator mixing and NPR





- Effective weak Hamiltonian: four-quark operators after integrating out the heavy degrees of freedom.
- **Regularization-independent** renormalization schemes [Martinelli et al, Nucl. Phys. B '95]
  - Example (**RI/SMOM**): matching projected amputated Green's functions $\Gamma(p_1, p_2)_{p_1^2=p_2^2=(p_1-p_2)^2=\mu^2}$ to their tree-level values [Sturm *et al*, PRD '09].
- Recipe for non-perturbative renormalization:
  - Wilson coefficient at scale μ from perturbation theory (PT).
  - RI-renormalized effective operators in PT (eg. $\overline{\mathrm{MS}}$).
  - RI-renormalized effective operators from lattice calculations.
  - Match the two by demanding

$$\sum_{jkl} C_j^{\mathrm{cont}}(\mu) R_{jk}^{\mathrm{cont}\leftarrow\mathrm{RI}} Z_{kl}^{\mathrm{RI}\leftarrow\mathrm{lat}}(\mu, a) Q_l' \stackrel{!}{=} \sum_i C_i^{\mathrm{lat}}(\mu) Q_i \,.$$

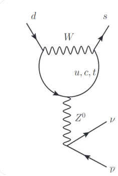
LD contributions in (semi-)leptonic decays

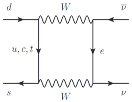
$K^+ \to \pi^+ \nu\bar{\nu}$

- *Long-distance* (LD): scale larger than $1/m_c$.
- $K \to \pi\nu\bar{\nu}$: GIM-suppressed charm contribution
  $\Rightarrow$ short-distance (SD) dominated, estimated from PT.
  - $K_{\mathrm{L}} \to \pi^0\nu\bar{\nu}$: CPV, dominated by the top-quark terms in the CKM matrix. [Buchalla et al, PLB '98]
  - $K_{\mathrm{S}} \to \pi^0\nu\bar{\nu}$: charm-enhanced but hard to observe in experiment.
  - $K^+ \to \pi^+\nu\bar{\nu}$: data available from experiment.

	Th. [Buras, EPJC '22]	Exp.
$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})$	$8.60(42) \times 10^{-11}$	$13.0^{+3.3}_{-3.0} \times 10^{-11}$ [NA62, JHEP '25]
$\mathrm{Br}(K_{\mathrm{L}} \to \pi^0\nu\bar{\nu})$	$2.94(15) \times 10^{-11}$	$\leq 2.2 \times 10^{-9}$ (90% C.L.) [KOTO, PRL '25]

- LD charm effects to the theoretical branching ratio at O(6%), comparable to the total quoted error [Isidori et al, Nucl. Phys. B '05].





LD contributions in (semi-)leptonic decays

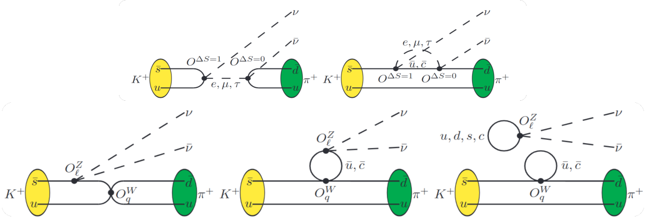
$K^+ \to \pi^+ \nu \bar{\nu}$

- LD operators mix with the SD eff. Hamiltonian $\mathcal{H}_{\text{eff},0} \sim (\bar{s}d)_{V-A}(\bar{\nu}_l \nu_l)_{V-A}$ through renormalization.

- Necessary to remove unphysical exponential growth caused by intermediate states lighter than the kaon in a lattice calculation. [Christ et al, PRD '16]

$$\sim -\sum_n \frac{\langle f|O^{\Delta S=0}|n\rangle\langle n|O^{\Delta S=1}|K\rangle}{E_K - E_n}\left(1 - e^{(E_K - E_n)T_a}\right) .$$

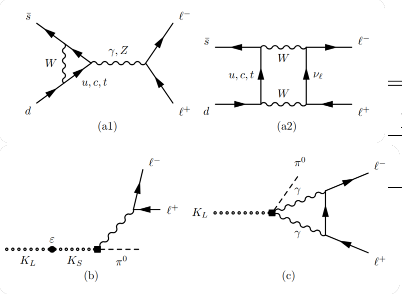
- State-of-the-art calculation at $(m_\pi, M_K) \approx (170, 493)$ MeV [Christ et al, PRD '19]:
  - Mild momentum dependence.
  - Negligible unphysical 2π contribution and finite-volume effects (FSE).



LD contributions in (semi-)leptonic decays

$K \to \pi\mu^+\mu^-$

- $K^+/K_{\rm S}$: dominated by LD $K \to \pi\gamma^*$ ("soft-GIM").
- $K_{\rm L}$: SD-dominated, probe for CPV (indirect CPV from $K_{\rm S} \to \pi^0 l^+ l^-$).
- Structure of the $K \to \pi l^+ l^-$ amplitude one-photon exchange
$$A_i = -\frac{G_{\rm F}\alpha}{4\pi} V_i(z)(k+p)^\mu \bar{u}_l(p_-)\gamma_\mu v_l(p_+)\,, \quad V_i(z) = a_i + b_i z + V^{\pi\pi}(z)\,.$$
- Lattice result at $M_\pi = 139$ MeV and $a^{-1} = 1.73$ GeV [Boyle et al, PRD '23]
  - Analog to $K^+ \to \pi^+\nu\bar{\nu}$, require removal of unphysical terms [Christ et al, PRD '15].
  - $V_+(0.013(2)) = -0.87(4.44)$, large statistical error, improvement on the way. [Hodgson, Lattice'24]

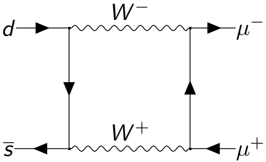


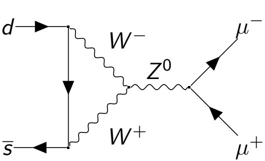
	Exp. [PDG '24]
$\mathrm{Br}(K^+ \to \pi^+ e^+ e^-)$	$3.00(9) \times 10^{-7}$
$\mathrm{Br}(K^+ \to \pi^+ \mu^+ \mu^-)$	$9.17(14) \times 10^{-8}$
$\mathrm{Br}(K_{\rm S} \to \pi^0 e^+ e^-)$	$3.0^{+1.5}_{-1.2} \times 10^{-9}$
$\mathrm{Br}(K_{\rm S} \to \pi^0 \mu^+ \mu^-)$	$2.9^{+1.5}_{-1.2} \times 10^{-9}$
$\mathrm{Br}(K_{\rm L} \to \pi^0 e^+ e^-)$	$\leq 2.8 \times 10^{-10}$
$\mathrm{Br}(K_{\rm L} \to \pi^0 \mu^+ \mu^-)$	$\leq 3.8 \times 10^{-10}$

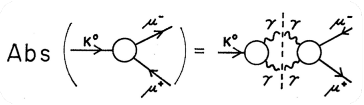
LD contributions in (semi-)leptonic decays

$K \to \mu^+\mu^-$

- $\mathrm{Br}(K_\mathrm{L} \to \mu^+\mu^-)_{\mathrm{exp.}} = 6.84(11) \times 10^{-9}$ [E871, PRL '00], probe for FCNC processes.
- Sizable $\mathrm{O}(G_\mathrm{F}\alpha^2_{\mathrm{QED}})$ 2γ-exchange interferes with the $\mathrm{O}(G^2_\mathrm{F})$ SD contribution
- A lattice framework and related systematic errors [Christ et al, Lattice '19]:
  - QED_∞, treated in the continuum and infinite volume [Green et al, PRL '15; Blum et al, PRD '17].
  - Systematic uncertainty from $2\pi\gamma$ estimated $\lesssim 10\%$ [Chao et al, PRD'24].







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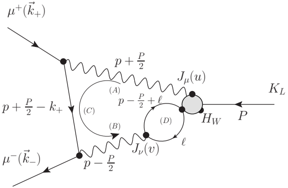
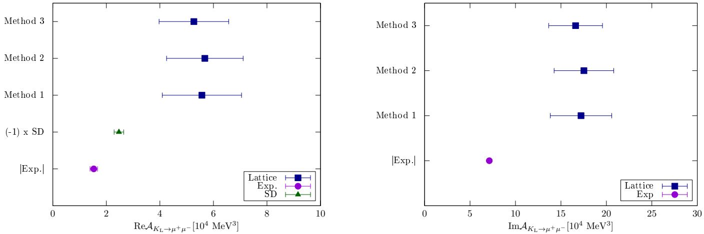


LD contributions in (semi-)leptonic decays

$K \to \mu^+\mu^-$ (cont'd)

- First lattice result from a $24^3 \times 64$ ensemble at physical quark masses and $a^{-1} = 1.023$ GeV [Boyle et al, 2509.04346]



- Combined results:

	$\mathrm{Re}\mathcal{A} \times 10^{-4}$ [MeV^3]	$\mathrm{Im}\mathcal{A} \times 10^{-4}$ [MeV^3]
Th. [Boyle et al, 2509.04346]	$2.80(1.30)_{\mathrm{stat}}(18)_{\mathrm{SD}}$	$16.61(2.98)_{\mathrm{stat}}$
exp.	$\pm 1.53(14)$	$\pm 7.10(3)$

Discrepancy in $\mathrm{Im}\mathcal{A} \Rightarrow$ uncontrolled systematic errors?

- Ongoing: physical $48^3 \times 96$ ($a^{-1} = 1.73$ GeV) and $64^3 \times 128$ ($a^{-1} = 2.38$ GeV).

QED corrections and radiative decays

Including QED on the lattice

- Obstacle for a lattice formulation of QED: periodic boundary conditions in conflict with Gauß' Law.

$$Q = \int_{\mathbb{T}^3} d^3x\, j_0(t, \mathbf{x}) = \int_{\mathbb{T}^3} d^3x\, \partial_k E_k(t, \mathbf{x}) = 0$$

- $\mathbf{QED}_\mathrm{L}$ [Mayakawa and Uno, Prog.Theor.Phys. '08]: remove the $\mathbf{p} = 0$ photon mode from each time slice.
  $\Rightarrow$ Power-law-suppressed finite-size effects (FSEs).
- $\mathbf{QED}_\mathrm{m}$ [Endres et al, PRL '16]: introduce massive photons.
  $\Rightarrow$ Local formulation but several m_γ needed for the $1/L, m_\gamma \to 0$ limits.
- $\mathbf{QED}_\mathrm{C}$ [Lucini et al, JHEP '16]: C^*-boundary conditions.
  $\Rightarrow$ Dedicated simulations. Peculiar spectrum.
- $\mathbf{QED}_\infty$ [Asmussen et al, Lattice '16; Blum et al, PRD '17]: QED in infinite-volume.
  $\Rightarrow$ Exponentially-suppressed FSEs but unusual features while combining a finite-volume (QCD) and infinite-volume formalisms.

QED corrections and radiative decays

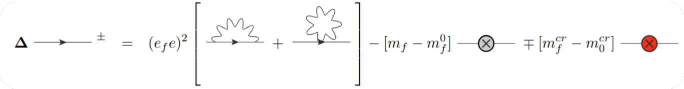
$K_{\ell 2}$ and $\pi_{\ell 2}$

- $|V_{us}|$ from $\Gamma(K^{\pm} \to \mu^{\pm}\nu)/\Gamma(\pi^{\pm} \to \mu^{\pm}\nu)$ with f_K/f_π from theory.
- $O(\alpha_{\rm QED}, (m_u - m_d)\Lambda^{-1}_{\rm QCD})$ corr. for subpercent accuracy [Cirigliano et al PLB '11]
- **RM123 Method**: series-expansion in the above parameters. [de Divitiis et al, PRD '13]
- QED IR divergences in Γ_0 cancelled with radiative corr. Γ_1 (Bloch-Nordsieck)
  $\Rightarrow$ lattice calculation regulated with a point-like approximation [Carrasco et al, PRD '15]

$$\Gamma(P_{l2}) = \lim_{L\to\infty}\left[\Gamma_0(L) - \Gamma^{\rm pt}(L)\right] + \lim_{\mu_\gamma\to 0}\left[\Gamma_0^{\rm pt}(\mu_\gamma) + \Gamma_1^{\rm pt}(\Delta E_\gamma, \mu_\gamma)\right] .$$

- Estimates for the corr. in $\Gamma(K_{\ell 2})/\Gamma(\pi_{\ell 2})$:

ChPT [Cirigliano et al, PLB '11]	-0.0112(21)
RM123S [Di Carlo et al, PRD '19]	-0.0126(14)
RBC/UKQCD [Boyle et al, JHEP '23]	-0.0086(13)(39)$_{\rm vol}$



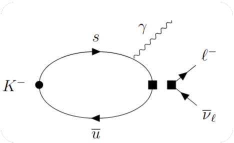
QED corrections and radiative decays

$K_{\ell 2}$ and $\pi_{\ell 2}$

Further developments:

- Replace point-like interaction with structure. [Di Palma et al, PRD '25]
- Improved FSE from the external state based on QED_∞ and the Infinite Volume Reconstruction method [Feng and Jin, PRD '19] [Christ et al, PRD '23]

  $\Rightarrow$ see Xin-Yu Tuo's talk at Kaon 2025 for an update



Concluding remarks

- Precision tests of the SM and opportunities to probe BSM physics from kaon decays.
- Remarkable progress from lattice QCD, helping pin down systematic uncertainties from non-perturbative long-distance hadronic effects.
- Applications can be envisaged for promising planned measurements to provide further tests of the SM.
  Eg. 2γ contribution to $K_{\mathrm{L}} \to \pi^0\mu^+\mu^-$.