Inclusive and exclusive semileptonic decays of heavy mesons on lattice

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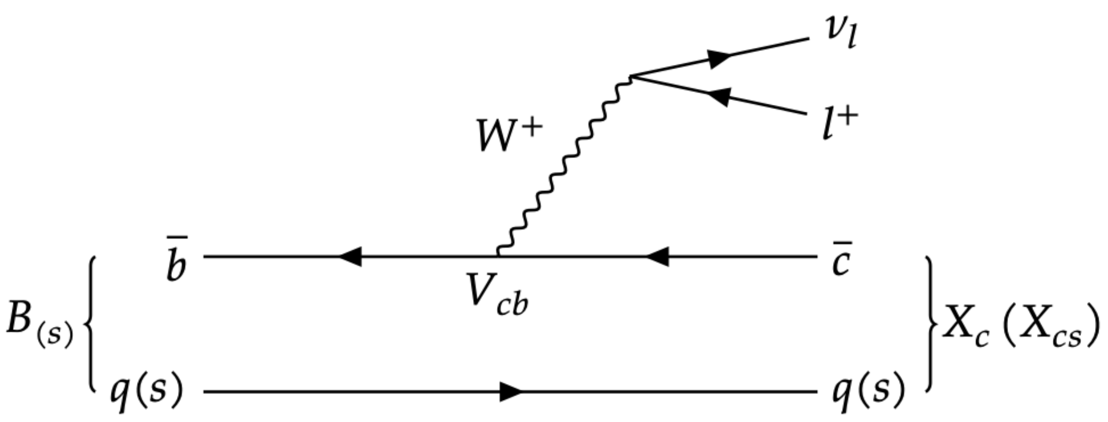
Based on *JHEP* **07** (2023) 145, *PoS* **LATTICE2024** (2025) 241, *PRD* **112** (2025) 014501

Semileptonic decays of $B_{(s)}$-meson

- in the quark level it corresponds to $\boldsymbol{b \to c l \nu_l}$
- the golden channel to measure $\boldsymbol{V_{cb}}$
- we work in the rest frame of the initial meson
- exclusive: decay into one specific hadronic state, we concentrate on final state containing only one meson
  $X_c(X_{cs}) = D_{(s)}, D^*_{(s)}, D^*_{(s)0}, D_{(s)1}, D'_{(s)1}, D^*_{(s)2}$
- inclusive: sum over all possible $X_c(X_{cs})$
- **the long-standing V_{cb} tension**



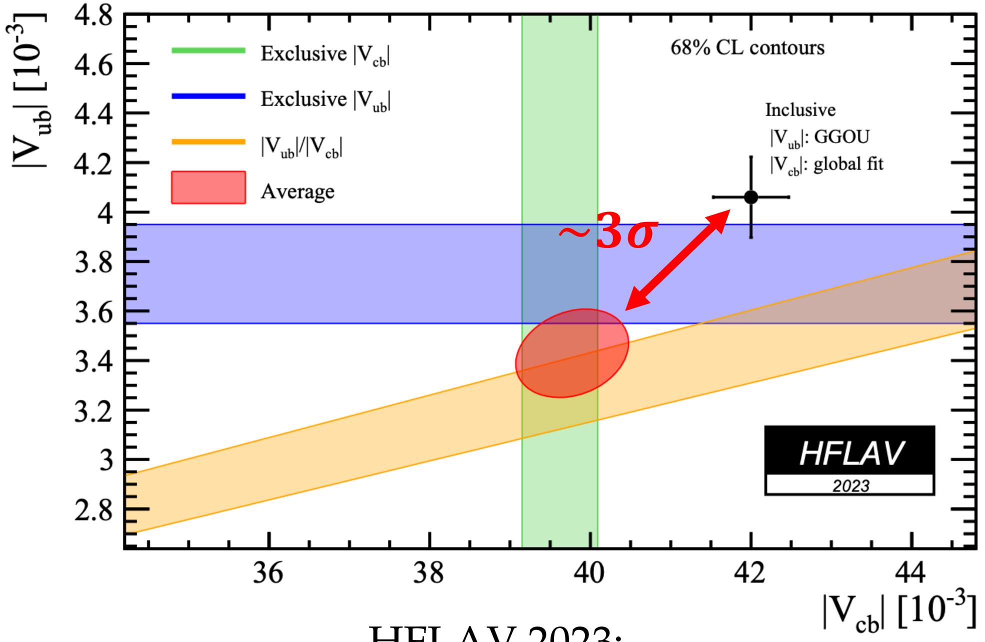
$$B_{(s)} \to X_c(X_{cs}) + l + \nu_l$$

$$v = \frac{p_{B_{(s)}}}{M_{B_{(s)}}} = (1,0,0,0) \quad v' = \frac{p_{X_{c(s)}}}{M_{X_{c(s)}}} \quad q$$

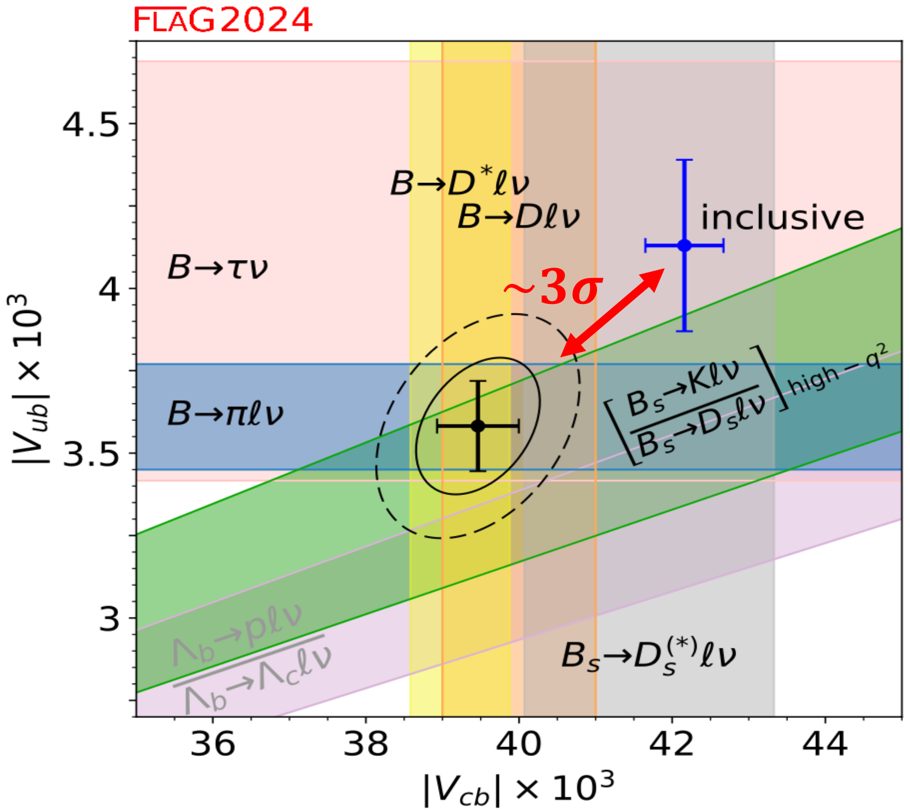
for exclusive: $w = v \cdot v' = \sqrt{1 + \frac{\vec{q}^2}{M^2_{X_{c(s)}}}}$

for inclusive: $\omega = M_{B_s} - q^0$

V_{cb} tension



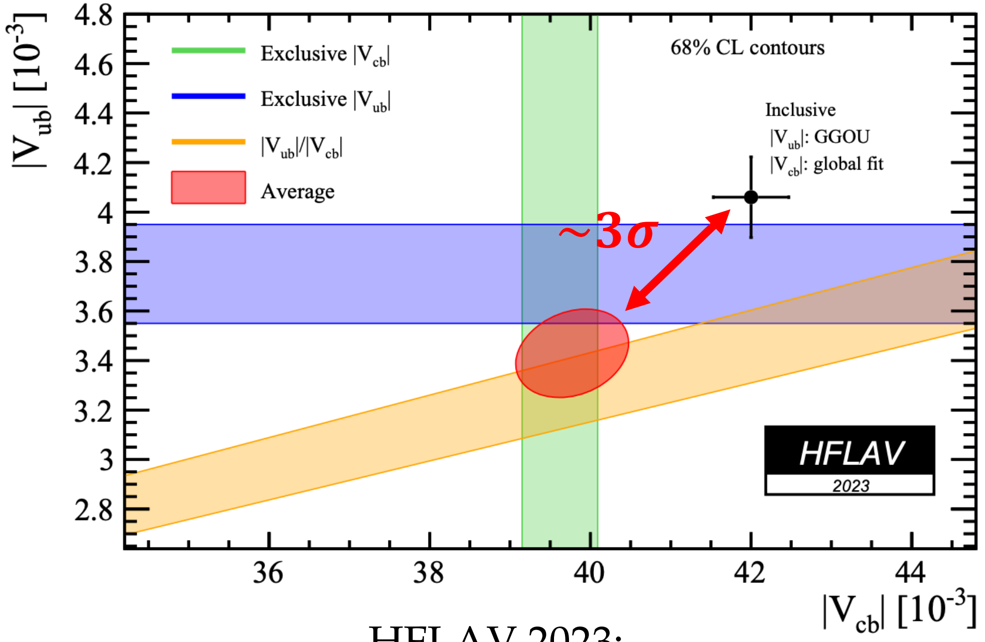
HFLAV 2023:
[2411.18639]



FLAG 2024:
[2411.04268]

V_{cb} tension

experimental measurements



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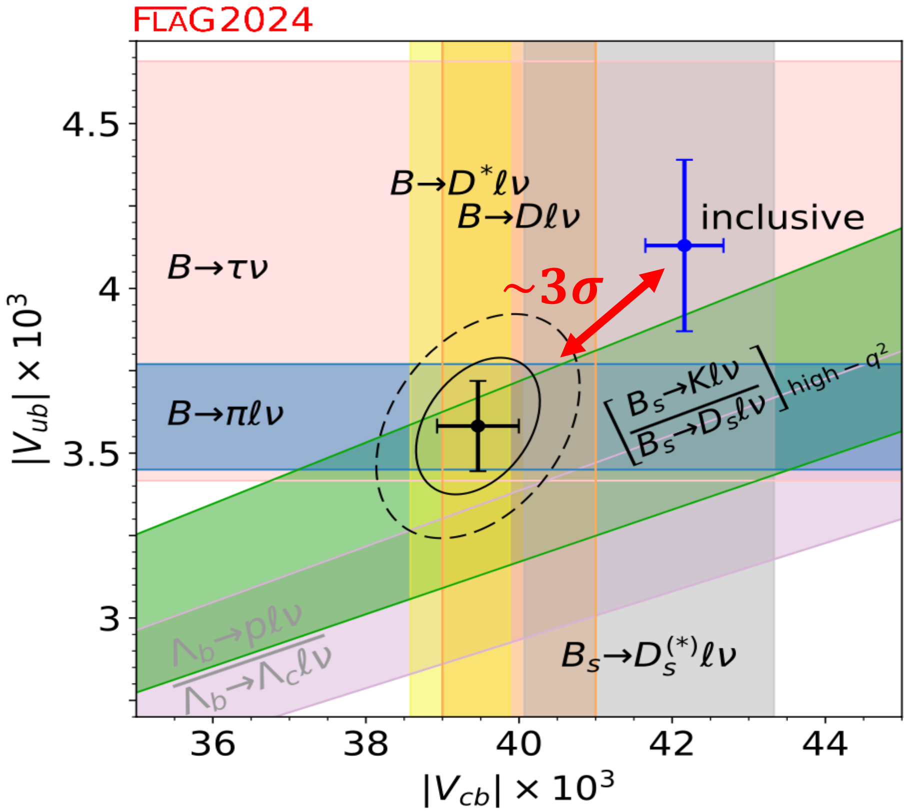

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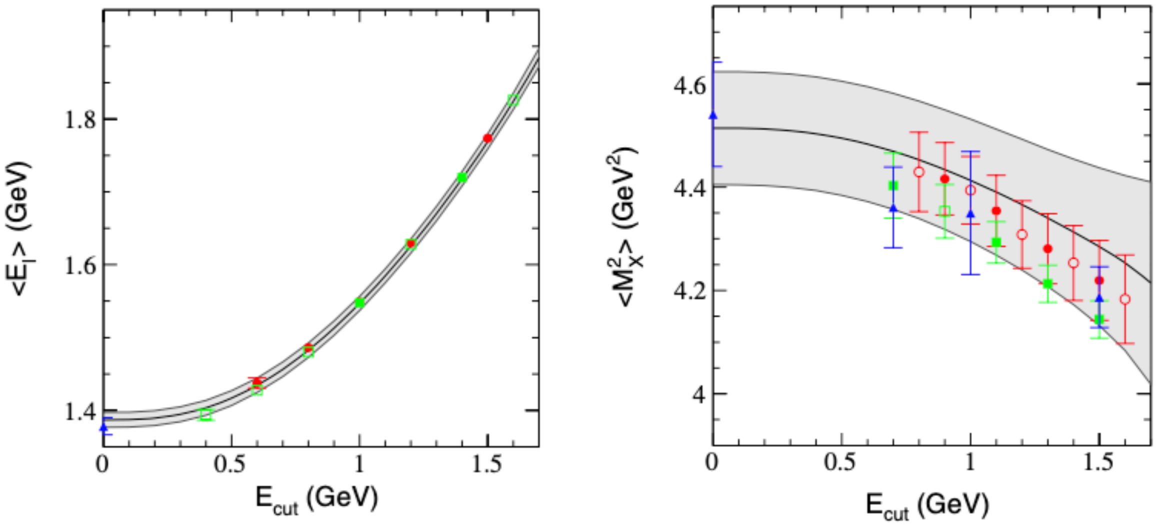
V_{cb} tension

OPE for inclusive decay widths and moments

experimental measurements



$$X = X^{(0)} + \frac{\alpha_s}{\pi} X^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 X^{(2)} + \left(\frac{\mu_\pi}{m_b}\right)^2 X^{(\pi)}$$
$$+ \left(\frac{\mu_G}{m_b}\right)^2 X^{(G)} + \left(\frac{\mu_D}{m_b}\right)^3 X^{(D)} + \left(\frac{\mu_{LS}}{m_b}\right)^2 X^{(LS)} + \ldots$$

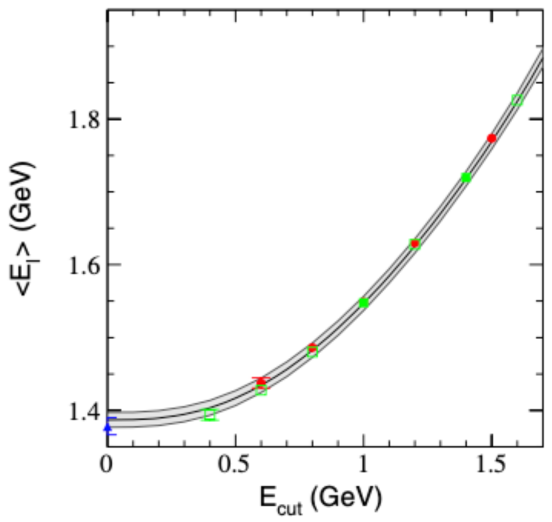
[*PRD* **89** (2014) 014022, *PLB* **822** (2021) 136679, *JHEP* **10** (2022) 068]

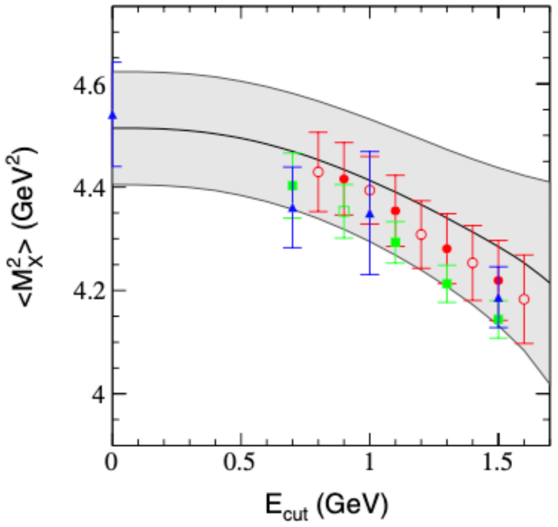
V_{cb} tension

OPE for inclusive decay widths and moments

experimental measurements

lattice three-point correlators for exclusive form factors





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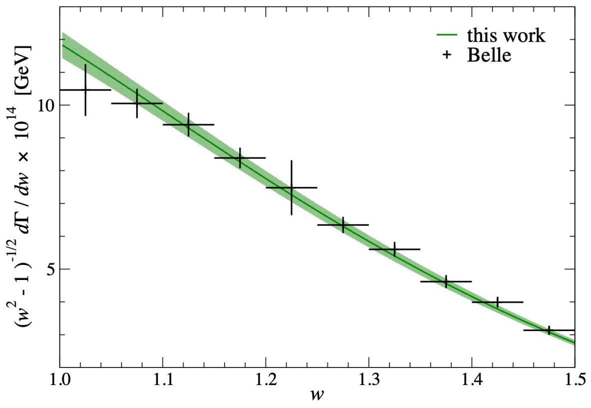

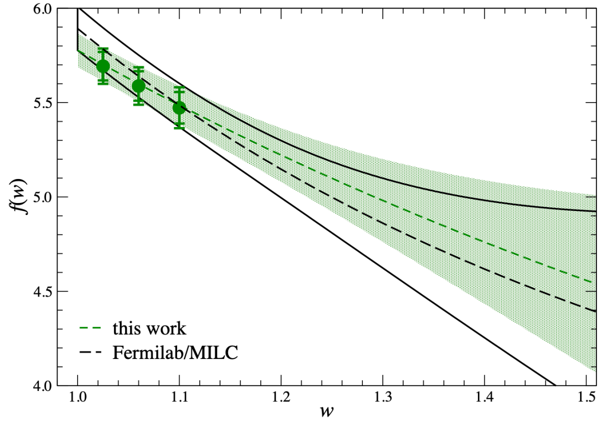

$$\frac{d\Gamma}{d\omega} = G(f, \mathcal{F}_1, \mathcal{F}_2)$$

$$F(z) = \frac{1}{P_F(z)\phi_F(z)} \sum_{k=0}^{N_F} a_{F,k} z^k, \quad z(\omega) = \frac{\sqrt{\omega+1} - \sqrt{2}}{\sqrt{\omega+1} + \sqrt{2}}$$

[Fermilab: *PRD* **92** (2015) 034506,
HPQCD: *PRD* **92** (2015) 054510,
JLQCD: *PRD* **109** (2024) 074503]

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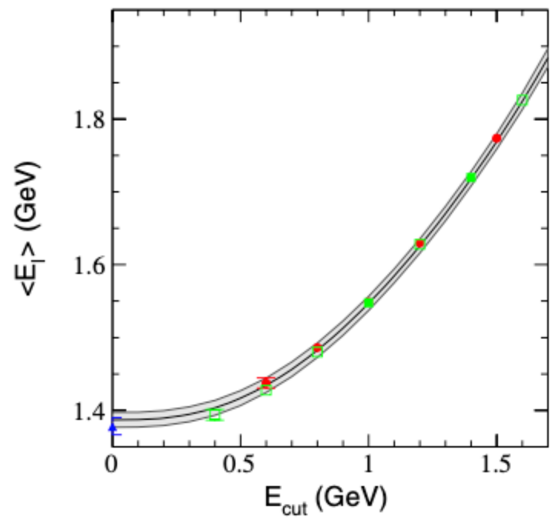

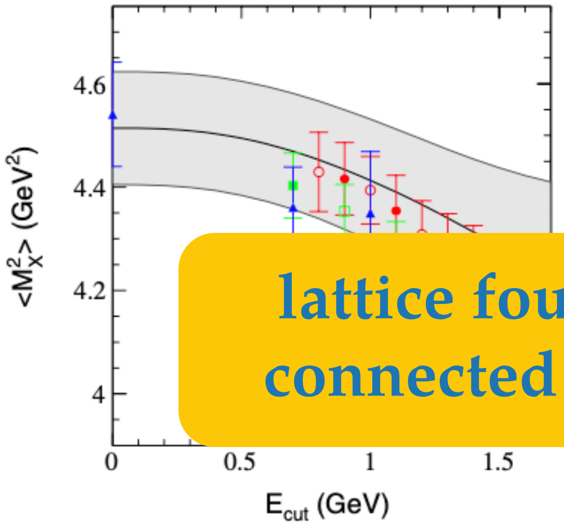

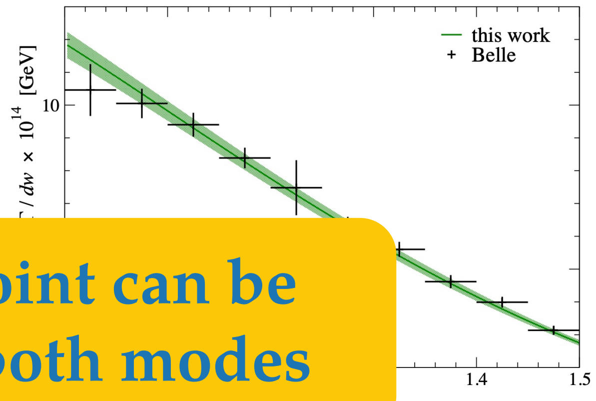

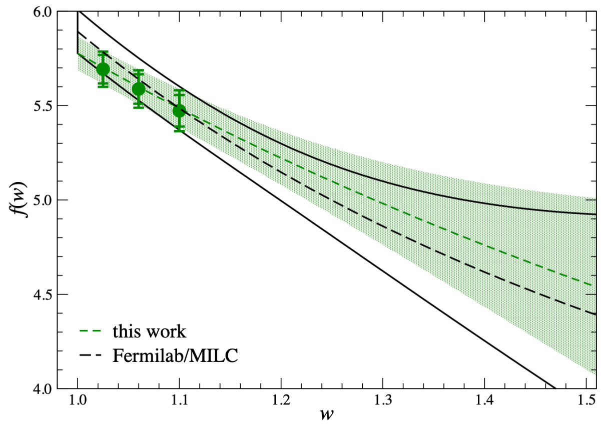

lattice four-point can be connected to both modes

$$X = X^{(0)} + \frac{\alpha_s}{\pi} X^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 X^{(2)} + \left(\frac{\mu_\pi}{m_b}\right)^2 X^{(\pi)}$$
$$+ \left(\frac{\mu_G}{m_b}\right)^2 X^{(G)} + \left(\frac{\mu_D}{m_b}\right)^3 X^{(D)} + \left(\frac{\mu_{LS}}{m_b}\right)^2 X^{(LS)} + \ldots$$

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Four-point-correlator methodology

a long, but not exhaustive, line of development

Hashimoto, *PTEP* **2017** (2017) 053B03

Hansen *et al.*, *PRD* **96** (2017) 094513

Bailas *et al.*, *PoS* **LATTICE2019** (2019) 148

Gambino and Hashimoto, *PRL* **125** (2020) 032001

Barone *et al.*, *JHEP* **07** (2023) 145

Hu *et al.*, *PoS* **LATTICE2024** (2025) 241

De Santis *et al.*, arXiv: 2504.06063

Kellermann *et al.*, *PRD* **112** (2025) 014501

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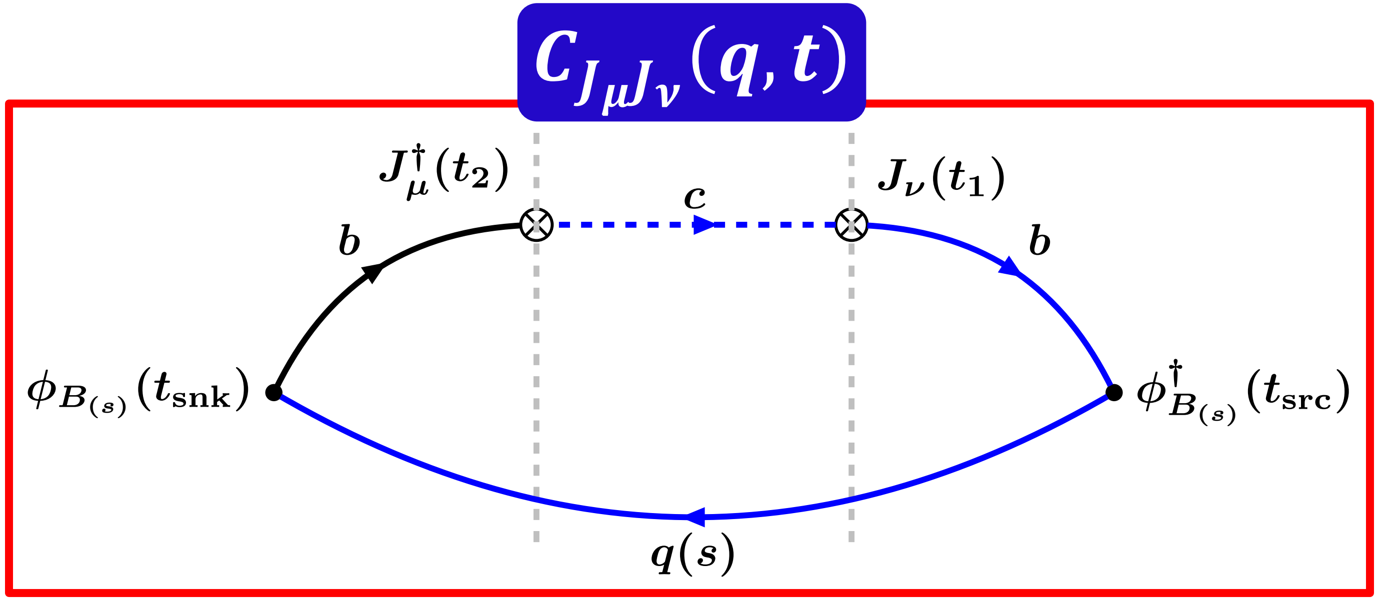
Four-point correlators on the lattice



$$C_{J_\mu J_\nu}(q,t^{\mathrm{E}}) \equiv \int \mathrm{d}^3\boldsymbol{x}\, \frac{e^{i\,\boldsymbol{q}\cdot\boldsymbol{x}}}{2M_{B_s}} \langle\, B_s \,|\, J_\mu^\dagger(\boldsymbol{x},t^{\mathrm{E}})\, J_\nu(0) \,|\, B_s \,\rangle$$

Simulation of 4-pt correlators

- $t_{\mathrm{snk}} - t_{\mathrm{src}}$ and $t_2 - t_{\mathrm{src}}$ fixed
- vary $t_1 \Longrightarrow$ vary $t^{\mathrm{E}} = t_2 - t_1$, can only take discrete value, multiple of lattice spacing a
- $J_\mu \equiv V_\mu - A_\mu$, $V_\mu = \bar{b}\gamma^\mu c$, $A_\mu = \bar{b}\gamma^\mu\gamma^5 c$

Four-point correlators and inclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2\mathrm{d}\omega} \propto |V_{cb}|^2\, W_{\mu\nu}\, K^{\mu\nu}$$

known leptonic tensor and kinematical factors

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$$\frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2\mathrm{d}\omega} \propto |V_{cb}|^2\, W_{\mu\nu}\, K^{\mu\nu}$$

known leptonic tensor and kinematical factors

hadronic tensor $W^{\mu\nu}$ encodes all the non-perturbative dynamics

$$W_{\mu\nu}(\boldsymbol{q},\omega) \equiv \frac{1}{2M_{B_s}(2\pi)} \int \mathrm{d}^4x\, e^{i\,q\cdot x} \langle B_s \,|\, J_\mu^\dagger(x)\, J_\nu(0) \,|\, B_s \rangle$$

$$C_{J_\mu J_\nu}(\boldsymbol{q},t^{\mathrm{E}}) \equiv \int \mathrm{d}^3\boldsymbol{x}\, \frac{e^{i\,\boldsymbol{q}\cdot\boldsymbol{x}}}{2M_{B_s}} \langle B_s \,|\, J_\mu^\dagger(\boldsymbol{x},t^{\mathrm{E}})\, J_\nu(0) \,|\, B_s \rangle$$

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$$= \sum_{X_{cs}} \frac{(2\pi)^3}{2M_{B_s}}\, \delta^{(3)}(\boldsymbol{q}+\boldsymbol{p}_{X_{cs}})\, \langle B_s|J_\mu^\dagger(0)|X_{cs}\rangle\langle X_{cs}|J_\nu(0)|B_s\rangle\, \boxed{\delta(\omega - E_{X_{cs}})}$$

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$$C_{J_\mu J_\nu}(t^{\mathrm{E}}) = \int_0^\infty d\omega \, W_{\mu\nu}(\omega) \, e^{-\omega t^{\mathrm{E}}}$$

Laplace transform
spectral representation

$$C_{J_\mu J_\nu} \longleftarrow W_{\mu\nu}$$

$$\begin{aligned} &W_{\mu\nu}(\boldsymbol{q}, \omega) \\ &\equiv \frac{1}{2M_{B_s}(2\pi)} \int d^4x \, e^{i\, q\cdot x} \langle \, B_s \, | \, J_\mu^\dagger(x) \, J_\nu(0) \, | \, B_s \, \rangle \\ &= \sum_{X_{cs}} \frac{(2\pi)^3}{2M_{B_s}} \, \delta^{(3)}(\boldsymbol{q} + \boldsymbol{p}_{X_{cs}}) \, \langle B_s | J_\mu^\dagger(0) | X_{cs} \rangle \langle X_{cs} | J_\nu(0) | B_s \rangle \boxed{\delta(\omega - E_{X_{cs}})} \end{aligned}$$

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Laplace transform
spectral representation

$$C_{J_\mu J_\nu} \rightleftarrows W_{\mu\nu}$$

ill-posedness in Hadamard's terms
the infamous inverse problem

Four-point correlators and inclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2\mathrm{d}\omega} \propto |V_{cb}|^2\, W_{\mu\nu}\, K^{\mu\nu}$$

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$$C_{J_\mu J_\nu}(t^{\mathrm{E}}) = \int_0^\infty \mathrm{d}\omega\, W_{\mu\nu}(\omega)\, e^{-\omega t^{\mathrm{E}}}$$

- luckily, for inclusive observables, what we need is the smearing of the spectral

$$\int d\omega\, W_{\mu\nu}(\omega)\, K^{\mu\nu}(\omega)$$

- thus, if we could approximate the smearing kernel as

$$K^{\mu\nu}(\omega) = \sum_k a_k^{\mu\nu}(e^{-\omega a})^k$$

Four-point correlators and inclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2\mathrm{d}\omega} \propto |V_{cb}|^2\, W_{\mu\nu}\, K^{\mu\nu}$$

known leptonic tensor and kinematical factors

hadronic tensor $W^{\mu\nu}$ encodes all the non-perturbative dynamics

$$W_{\mu\nu}(\boldsymbol{q},\omega) \equiv \frac{1}{2M_{B_s}(2\pi)}\int \mathrm{d}^4x\, e^{i\,q\cdot x}\,\langle\, B_s \,|\, J_\mu^\dagger(x)\, J_\nu(0)\,|\, B_s\,\rangle$$

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$$\int d\omega\, W_{\mu\nu}(\omega)\, K^{\mu\nu}(\omega)$$

- thus, if we could approximate the smearing kernel as

$$K^{\mu\nu}(\omega) = \sum_k a_k^{\mu\nu}(e^{-\omega a})^k$$

- inclusive decay widths and moments become

$$\frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2} \propto \sum_k a_k^{\mu\nu}\, C_{J_\mu J_\nu}(t^{\mathrm{E}} = ka)$$

KEK

Four-point correlators and exclusive decays

$$\frac{\mathrm{d}\Gamma_{X_{cs}}^{\mathrm{exc}}}{\mathrm{d}w} \propto |V_{cb}|^2\, \mathcal{F}_{X_{cs}}(w)\, \boxed{\otimes\, \mathcal{K}_{X_{cs}}(w)}$$

known functional forms and kinematical factors

Four-point correlators and exclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{exc}}_{X_{cs}}}{\mathrm{d}w} \propto |V_{cb}|^2\, \mathcal{F}_{X_{cs}}(w) \boxed{\otimes \mathcal{K}_{X_{cs}}(w)}$$

$\otimes \mathcal{K}_{X_{cs}}(w)$ → known functional forms and kinematical factors

$\mathcal{F}_{X_{cs}}(w)$ → form factors, non-perturbative information, defined from three-point correlators

- **S wave**
  - $D_s\,(0^-)$

$$\langle B_s, v | V^\dagger_\mu | D_s, v' \rangle = \left[h_+(v_\mu + v'_\mu) + h_-(v_\mu - v'_\mu) \right] \sqrt{M_{B_s} M_{D_s}}$$
$$\langle B_s, v | A^\dagger_\mu | D_s, v' \rangle = 0$$

  - $D^*_s\,(1^-)$

$$\langle B_s, v | V^\dagger_\mu | D^*_s, v', \sigma \rangle = \left[h_V \varepsilon_{\mu\alpha\beta\gamma} \epsilon^\alpha v'^{,\beta} v^\gamma \right] \sqrt{M_{B_s} M_{D^*_s}}$$
$$\langle B_s, v | A^\dagger_\mu | D^*_s, v', \sigma \rangle = i \left[(\omega+1) h_{A1} \epsilon_\mu - (\epsilon \cdot v) \left(h_{A2} v_\mu + h_{A3} v'_\mu \right) \right] \sqrt{M_{B_s} M_{D^*_s}}$$

- **P wave, $j = \frac{1}{2}$**
  - $D^*_{s0}\,(0^+)$

$$\langle B_s, v | V^\dagger_\mu | D^*_{s0}, v' \rangle = 0$$
$$\langle B_s, v | A^\dagger_\mu | D^*_{s0}, v' \rangle = \left[g_+(v_\mu + v'_\mu) + g_-(v_\mu - v'_\mu) \right] \sqrt{M_{B_s} M_{D^*_{s0}}}$$

  - $D'_{s1}\,(1^+)$

$$\langle B_s, v | V^\dagger_\mu | D'_{s1}, v', \sigma \rangle = \left[g_{V1} \epsilon_\mu + (\epsilon \cdot v) \left(g_{V2} v_\mu + g_{V3} v'_\mu \right) \right] \sqrt{M_{B_s} M_{D'_{s1}}},$$
$$\langle B_s, v | A^\dagger_\mu | D'_{s1}, v', \sigma \rangle = i \left[g_A \varepsilon_{\mu\alpha\beta\gamma} \epsilon^\alpha v'^{,\beta} v^\gamma \right] \sqrt{M_{B_s} M_{D'_{s1}}}.$$

Four-point correlators and exclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{exc}}_{X_{cs}}}{\mathrm{d}w} \propto |V_{cb}|^2\, \mathcal{F}_{X_{cs}}(w) \boxed{\otimes \mathcal{K}_{X_{cs}}(w)}$$

known functional forms and kinematical factors

form factors, non-perturbative information, defined from three-point correlators

$$\begin{aligned} &C_{J_\mu J_\nu}(\boldsymbol{q}, t^{\mathrm{E}}) \\ &\equiv \int \mathrm{d}^3\boldsymbol{x}\, \frac{e^{i\,\boldsymbol{q}\cdot\boldsymbol{x}}}{2M_{B_s}} \langle\, B_s \,|\, J^\dagger_\mu(\boldsymbol{x}, t^{\mathrm{E}})\, J_\nu(0) \,|\, B_s \,\rangle \\ &= \sum\!\!\!\!\!\!\int_{X_{cs}} \frac{(2\pi)^3}{2M_{B_s}}\, \delta^{(3)}(\boldsymbol{q} + \boldsymbol{p}_{X_{cs}}) \boxed{\langle B_s | J^\dagger_\mu(0) | X_{cs}\rangle} \langle X_{cs} | J_\nu(0) | B_s\rangle\, e^{-E_{X_{cs}} t^{\mathrm{E}}} \end{aligned}$$

- **S wave**
  - D_s (0^-)

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$$\langle B_s, v | A^\dagger_\mu | D_s, v'\rangle = 0$$

  - D_s^* (1^-)

$$\langle B_s, v | V^\dagger_\mu | D^*_s, v', \sigma\rangle = \left[h_V \varepsilon_{\mu\alpha\beta\gamma}\epsilon^\alpha v'^{,\beta} v^\gamma\right]\sqrt{M_{B_s} M_{D^*_s}}$$
$$\langle B_s, v | A^\dagger_\mu | D^*_s, v', \sigma\rangle = i\left[(\omega+1) h_{A1}\epsilon_\mu - (\epsilon\cdot v)\left(h_{A2} v_\mu + h_{A3} v'_\mu\right)\right]\sqrt{M_{B_s} M_{D^*_s}}$$

- **P wave, $j = \frac{1}{2}$**
  - D^*_{s0} (0^+)

$$\langle B_s, v | V^\dagger_\mu | D^*_{s0}, v'\rangle = 0$$
$$\langle B_s, v | A^\dagger_\mu | D^*_{s0}, v'\rangle = \left[g_+(v_\mu + v'_\mu) + g_-(v_\mu - v'_\mu)\right]\sqrt{M_{B_s} M_{D^*_{s0}}}$$

  - D'_{s1} (1^+)

$$\langle B_s, v | V^\dagger_\mu | D'_{s1}, v', \sigma\rangle = \left[g_{V1}\epsilon_\mu + (\epsilon\cdot v)\left(g_{V2} v_\mu + g_{V3} v'_\mu\right)\right]\sqrt{M_{B_s} M_{D'_{s1}}},$$
$$\langle B_s, v | A^\dagger_\mu | D'_{s1}, v', \sigma\rangle = i\left[g_A \varepsilon_{\mu\alpha\beta\gamma}\epsilon^\alpha v'^{,\beta} v^\gamma\right]\sqrt{M_{B_s} M_{D'_{s1}}}.$$

Four-point correlators and exclusive decays

$$\frac{\mathrm{d}\Gamma^{\mathrm{exc}}_{X_{cs}}}{\mathrm{d}w} \propto |V_{cb}|^2\, \mathcal{F}_{X_{cs}}(w) \boxed{\otimes \mathcal{K}_{X_{cs}}(w)}$$

known functional forms and kinematical factors

form factors, non-perturbative information, defined from three-point correlators

$$C_{J_\mu J_\nu}(\boldsymbol{q}, t^{\mathrm{E}})$$
$$\equiv \int \mathrm{d}^3\boldsymbol{x}\, \frac{e^{i\,\boldsymbol{q}\cdot\boldsymbol{x}}}{2M_{B_s}} \langle\, B_s \,|\, J^\dagger_\mu(\boldsymbol{x}, t^{\mathrm{E}})\, J_\nu(0) \,|\, B_s \,\rangle$$
$$= \sum_{X_{cs}} \frac{(2\pi)^3}{2M_{B_s}} \delta^{(3)}(\boldsymbol{q} + \boldsymbol{p}_{X_{cs}}) \boxed{\langle B_s | J^\dagger_\mu(0) | X_{cs}\rangle} \langle X_{cs} | J_\nu(0) | B_s\rangle\, e^{-E_{X_{cs}} t^{\mathrm{E}}}$$

$$C_{J_\mu J_\nu}(t^{\mathrm{E}}) = \sum_{X_{cs}} [\mathcal{F}_{X_{cs}}]^2 e^{-E_{X_{cs}} t^{\mathrm{E}}}$$

- **S wave**
  - D_s (0^-)

$$\langle B_s, v | V^\dagger_\mu | D_s, v'\rangle = \left[h_+(v_\mu + v'_\mu) + h_-(v_\mu - v'_\mu)\right]\sqrt{M_{B_s} M_{D_s}}$$
$$\langle B_s, v | A^\dagger_\mu | D_s, v'\rangle = 0$$

  - D^*_s (1^-)

$$\langle B_s, v | V^\dagger_\mu | D^*_s, v', \sigma\rangle = \left[h_V \varepsilon_{\mu\alpha\beta\gamma} \epsilon^\alpha v'^{,\beta} v^\gamma\right]\sqrt{M_{B_s} M_{D^*_s}}$$
$$\langle B_s, v | A^\dagger_\mu | D^*_s, v', \sigma\rangle = i\left[(\omega+1) h_{A1}\epsilon_\mu - (\epsilon\cdot v)\left(h_{A2} v_\mu + h_{A3} v'_\mu\right)\right]\sqrt{M_{B_s} M_{D^*_s}}$$

- **P wave, $j = \frac{1}{2}$**
  - D^*_{s0} (0^+)

$$\langle B_s, v | V^\dagger_\mu | D^*_{s0}, v'\rangle = 0$$
$$\langle B_s, v | A^\dagger_\mu | D^*_{s0}, v'\rangle = \left[g_+(v_\mu + v'_\mu) + g_-(v_\mu - v'_\mu)\right]\sqrt{M_{B_s} M_{D^*_{s0}}}$$

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$$\langle B_s, v | V^\dagger_\mu | D'_{s1}, v', \sigma\rangle = \left[g_{V1}\epsilon_\mu + (\epsilon\cdot v)\left(g_{V2} v_\mu + g_{V3} v'_\mu\right)\right]\sqrt{M_{B_s} M_{D'_{s1}}},$$
$$\langle B_s, v | A^\dagger_\mu | D'_{s1}, v', \sigma\rangle = i\left[g_A \varepsilon_{\mu\alpha\beta\gamma}\epsilon^\alpha v'^{,\beta} v^\gamma\right]\sqrt{M_{B_s} M_{D'_{s1}}}.$$

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known functional forms and kinematical factors

form factors, non-perturbative information, defined from three-point correlators

$$\begin{aligned} C_{J_\mu J_\nu}(\boldsymbol{q}, t^{\mathrm{E}}) &\equiv \int \mathrm{d}^3\boldsymbol{x}\, \frac{e^{i\,\boldsymbol{q}\cdot\boldsymbol{x}}}{2M_{B_s}} \langle\, B_s \,|\, J^\dagger_\mu(\boldsymbol{x}, t^{\mathrm{E}})\, J_\nu(0) \,|\, B_s \,\rangle \\ &= \sum_{X_{cs}} \frac{(2\pi)^3}{2M_{B_s}} \delta^{(3)}(\boldsymbol{q} + \boldsymbol{p}_{X_{cs}}) \boxed{\langle B_s | J^\dagger_\mu(0) | X_{cs} \rangle} \langle X_{cs} | J_\nu(0) | B_s \rangle\, e^{-E_{X_{cs}} t^{\mathrm{E}}} \end{aligned}$$

$$C_{J_\mu J_\nu}(t^{\mathrm{E}}) = \sum_{X_{cs}} [\mathcal{F}_{X_{cs}}]^2 e^{-E_{X_{cs}} t^{\mathrm{E}}}$$

multiple-exp fitting of $C_{J_\mu J_\nu}(t^{\mathrm{E}})$

- **S wave**
  - $D_s\,(0^-)$

$$\langle B_s, v | V^\dagger_\mu | D_s, v' \rangle = [h_+(v_\mu + v'_\mu) + h_-(v_\mu - v'_\mu)] \sqrt{M_{B_s} M_{D_s}}$$
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$$\langle B_s, v | V^\dagger_\mu | D^*_s, v', \sigma \rangle = [h_V \varepsilon_{\mu\alpha\beta\gamma} \epsilon^\alpha v'^\beta v^\gamma] \sqrt{M_{B_s} M_{D^*_s}}$$
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$$\langle B_s, v | V^\dagger_\mu | D^*_{s0}, v' \rangle = 0$$
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$$\langle B_s, v | V^\dagger_\mu | D'_{s1}, v', \sigma \rangle = [g_{V1} \epsilon_\mu + (\epsilon \cdot v)(g_{V2} v_\mu + g_{V3} v'_\mu)] \sqrt{M_{B_s} M_{D'_{s1}}},$$
$$\langle B_s, v | A^\dagger_\mu | D'_{s1}, v', \sigma \rangle = i[g_A \varepsilon_{\mu\alpha\beta\gamma} \epsilon^\alpha v'^\beta v^\gamma] \sqrt{M_{B_s} M_{D'_{s1}}}.$$

Numerical considerations and results

Lattice setup

RBC/UKQCD on B_s to X_{cs}

- gauge configuration from [*PRD* **78** (2008) 114509, *PRD* **107** (2023) 114512]
  - gauge field: Iwasaki gauge action
  - sea quarks: 2+1-flavor DWF action
  - lattice size: $24^3 \times 64$
  - lattice spacing: $a \approx 0.11$ fm, energy unit: $a^{-1} \approx 1.785$ GeV
- correlator simulation
  - DWF action for s, c at near-to-physical mass
  - Relativistic Heavy Quark (RHQ) actions [*PRD* **55** (1997) 3933, *PRD* **76** (2007) 074505, *PRD* **76** (2007) 074506] for b at physical mass
  - Twisted BC to induce 10 different momenta along the (1,1,1) direction

JLQCD on D_s to $X_{s\bar{s}}$

- gauge configuration from [*PRD* **78** (2008) 114509, *PRD* **107** (2023) 114512]
  - gauge field: tree-level Symanzik improved
  - sea quarks: 2+1-flavor Möbius DWF action
  - lattice size: $48^3 \times 96$
  - lattice spacing: $a \approx 0.055$ fm, energy unit: $a^{-1} \approx 3.610$ GeV
- correlator simulation
  - Möbius DWF action for all valence quarks
  - s, c quark simulated at near-physical values
  - four momenta: (0,0,0), (0,0,1), (0,1,1), (1,1,1)

currently only proof-of-concept investigation using one ensemble for each

Inclusive: numerical subtleties

$a = 1$

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, W_{\mu\nu}(\omega)\, k^{\mu\nu}(\omega)$$

kinemactical constraints

$$\bar{X} \equiv \frac{\mathrm{d}\Gamma^{\mathrm{inc}}}{\mathrm{d}\boldsymbol{q}^2}$$

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$$= \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W_{\mu\nu}(\omega)\, k^{\mu\nu}(\omega)\, \theta(\omega_{\max} - \omega)$$

step function to cut off unphysicsl contributions

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step function to cut off unphysicsl contributions

$$\approx \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega)\, \underbrace{k^{\mu\nu}(\omega)\, \theta_\sigma(\omega_{\max} - \omega)}_{\equiv K_\sigma^{\mu\nu}(\omega)}$$

smeared step function, need $\sigma \to 0$

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$$K_\sigma^{\mu\nu}(\omega, N) = \sum_{k=0}^{N} a_{\sigma,k}^{\mu\nu}\, (e^{-\omega})^k$$

the highest order accessible, N, is determined by the time range T of lattice simulation, need $N/T \to \infty$

$$\Longrightarrow \bar{X}(\sigma, N; \qquad) = \sum_{k=0}^{N} a_{\sigma,k}^{\mu\nu}\, C_{J_\mu J_\nu}(k; \qquad)$$

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$$\Longrightarrow \bar{X}(\sigma, N; L, a, m) = \sum_{k=0}^{N} a_{\sigma,k}^{\mu\nu}\, C_{J_\mu J_\nu}(k; V = L^3, a, m)$$

continuum, infinite-volume, and chiral extrapolations and error controls

Inclusive: kernel approximation

$$K_{\sigma}^{\mu\nu}(\omega, N) = \sum_{k=0}^{N} \boxed{a_{\sigma,k}^{\mu\nu}} (e^{-\omega})^k \implies \bar{X}(\sigma, N; L, a, m) = \sum_{k=0}^{N} \boxed{a_{\sigma,k}^{\mu\nu}} C_{J_\mu J_\nu}(k; V = L^3, a, m)$$

Inclusive: kernel approximation

Method 1: Chebyshev method

- shifted Chebyshev polynomial

$$\tilde{T}_j(\omega) = T_j\boxed{\left(h(\omega)\right)} : \ [\omega_0, \infty] \to [-1,1]$$

to change the domain

$$h(\omega) = Ae^{-\omega} + B$$

$$\tilde{T}_j(\omega) = \sum_{k=0}^{j} \tilde{t}_j^k (e^{-\omega})^k$$

- approximate the kernel as

$$K_\sigma^{\mu\nu}(\omega) = \sum_{j=0}^{N} \boxed{c_{\sigma,j}^{\mu\nu}} \tilde{T}_j(\omega)$$

analytically calculable

$$c_{\sigma,j}^{\mu\nu} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, K_\sigma^{\mu\nu}(\omega) \, \tilde{T}_j(\omega) \, \tilde{\Omega}(\omega)$$

- finally we have

$$\bar{X} = \sum_{j=0}^{N} c_{\sigma,j}^{\mu\nu} \boxed{\langle \tilde{T}_j \rangle_{\mu\nu}}$$

Chebyshev matrix element

$$\langle \tilde{T}_j \rangle_{\mu\nu} \propto \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W_{\mu\nu}(\omega) \, \tilde{T}_j(\omega)$$

$$\Leftrightarrow \bar{X} = \sum_{k=0}^{N} \boxed{a_{\sigma,k}^{\mu\nu}} C_{\mu\nu}(k)$$

$$a_{\sigma,k}^{\mu\nu} = \sum_{j=0}^{k} c_{\sigma,j}^{\mu\nu} \, \tilde{t}_j^k$$

Inclusive: kernel approximation

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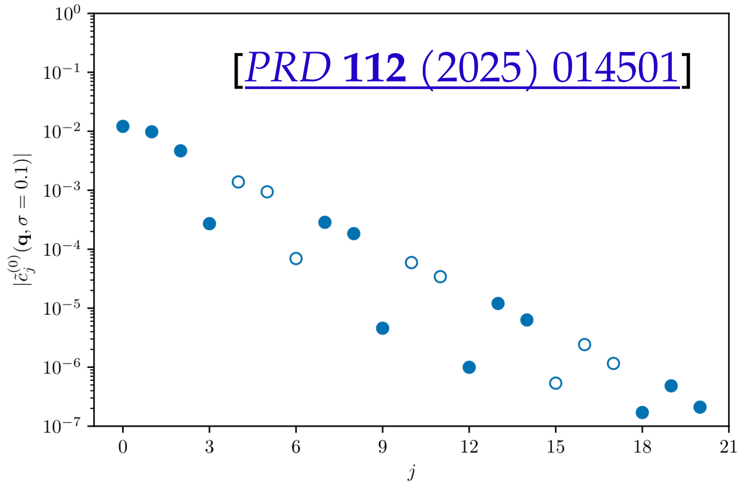
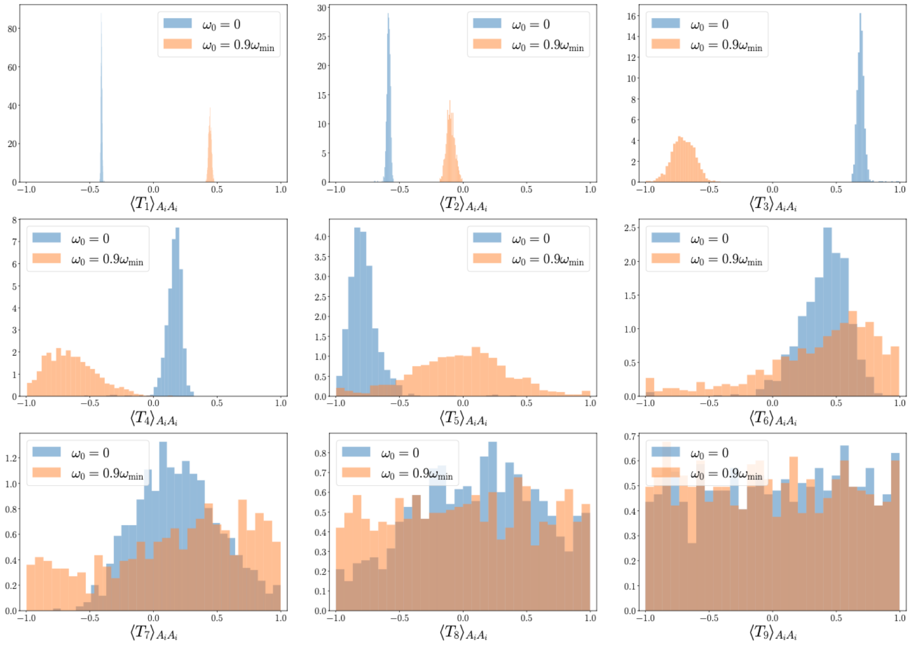
- **Chebyshev matrix elements are bounded by ± 1**
- **exponentially decreasing coefficients $c_{\sigma,j}^{\mu\nu}$**





[JHEP 07 (2023) 145]

Inclusive: kernel approximation

Method 2: Backus-Gilbert (HLT) method

- functional representing the systematic errors from kernel approximation

$$A[\boldsymbol{a}] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[K(\omega) - \sum_k a_k\, e^{-\omega k}\right]^2$$

- functional representing the statistical errors from lattice simulation

$$B[\boldsymbol{a}] = \sum_{k,l} a_k\, \mathrm{Cov}\left[C_{\mu\nu}(k), C_{\mu\nu}(l)\right] a_l$$

- compond functional

$$W_\lambda[\boldsymbol{a}] = (1-\lambda)\frac{A[\boldsymbol{a}]}{A[\boldsymbol{0}]} + \lambda B[\boldsymbol{a}]$$

- for each λ, minimize $W_\lambda\,[\boldsymbol{a}]$, resulting in $\boldsymbol{a}^*(\lambda)$
- find the λ^* to maximize $W_\lambda[\boldsymbol{a}^*]$, and use the corresponding $\boldsymbol{a}^*(\lambda^*)$

Inclusive: kernel approximation

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balance between systematical and statistical errors

Inclusive: kernel approximation

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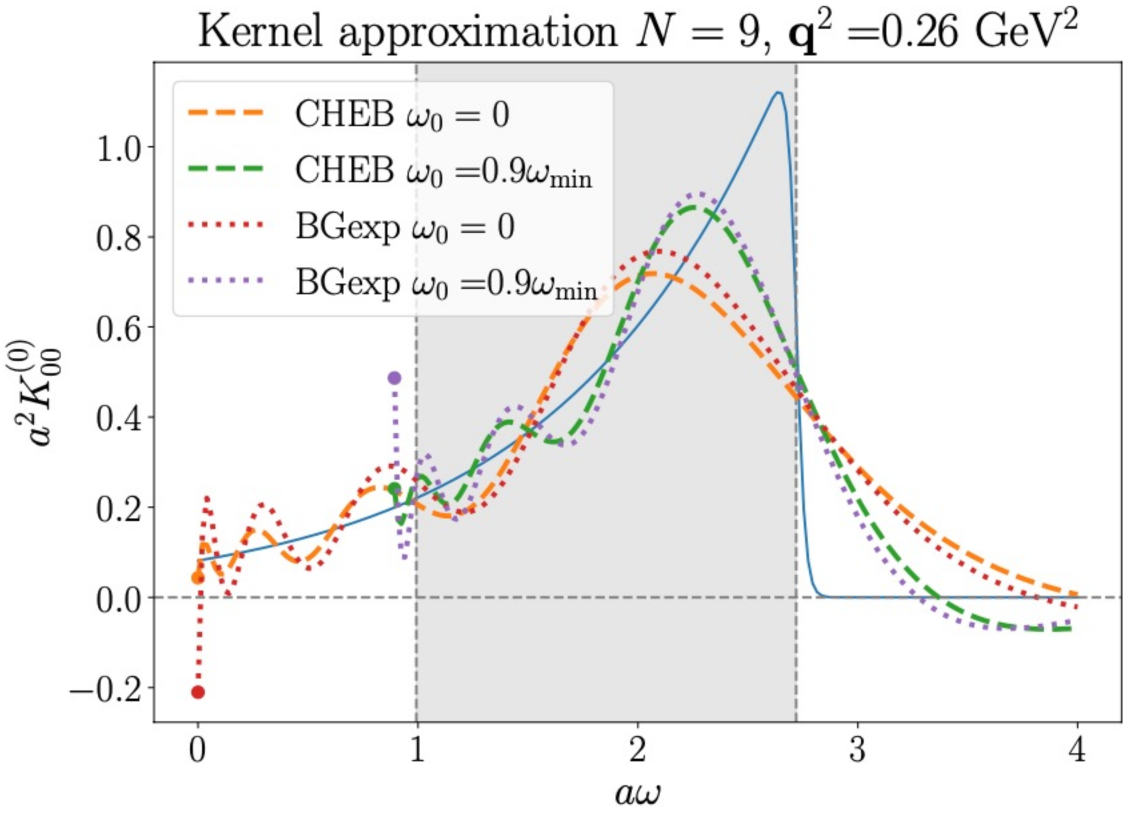


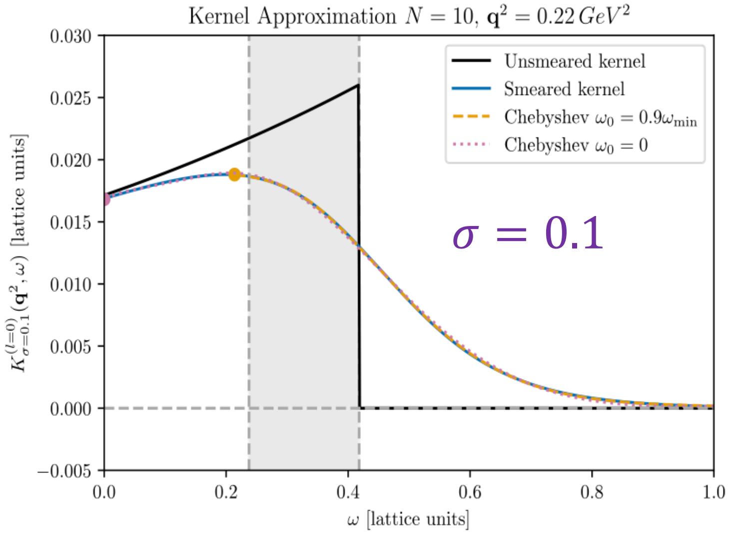
- comparisons between Backus-Gilbert and Chebyshev methods [*JHEP* **07** (2023) 145]
- for more results on BG/HLT method, see [2504.06063, 2504.06064]

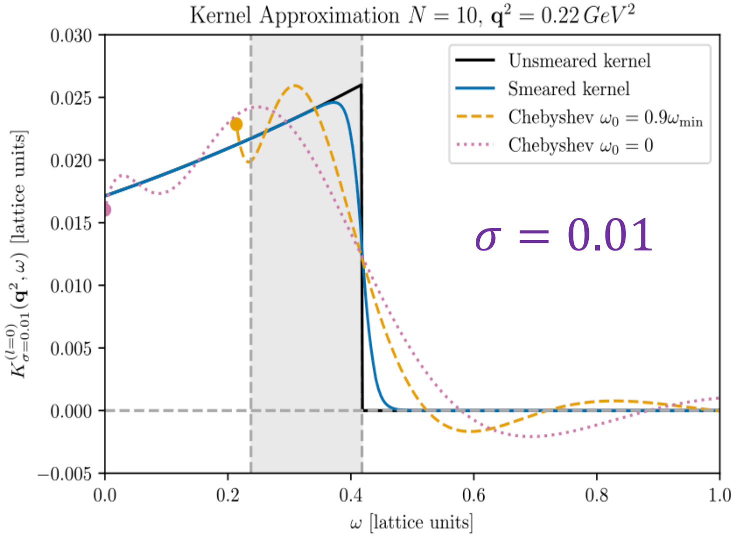
Inclusive: error from kernel approximation

σ

N

Inclusive: error from kernel approximation



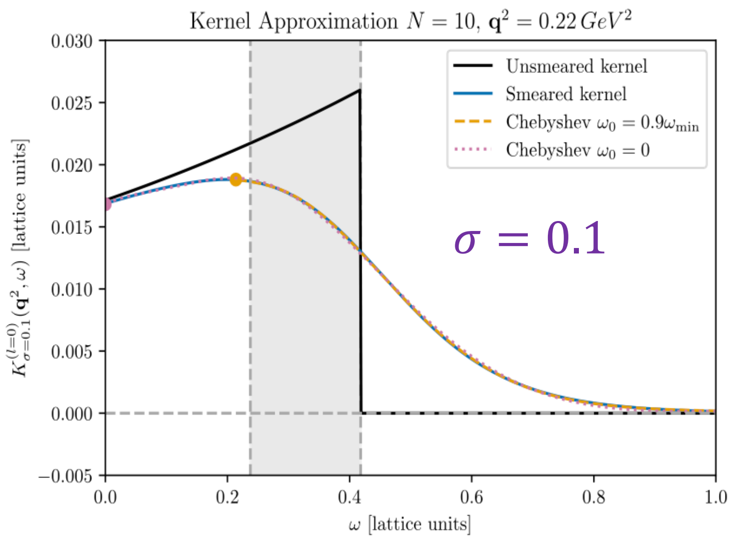


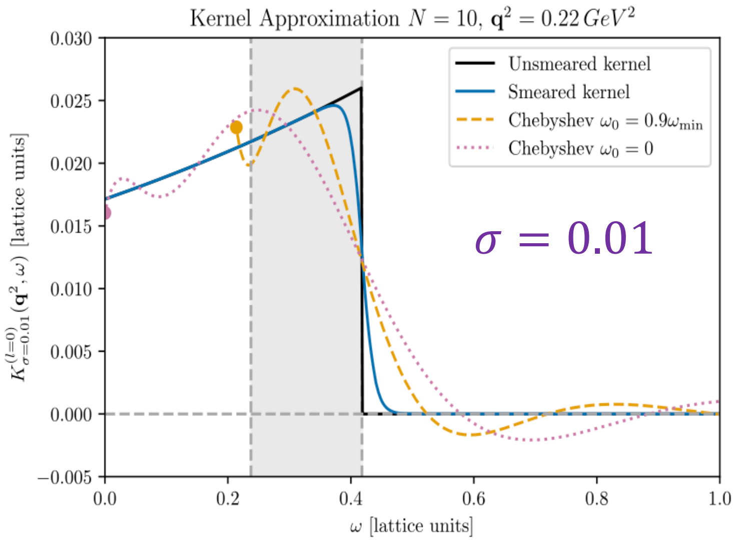
smaller $\sigma \Longrightarrow$ smaller smearing width
$\Longrightarrow$ more high frequency component

[*PRD* **112** (2025) 014501]

N

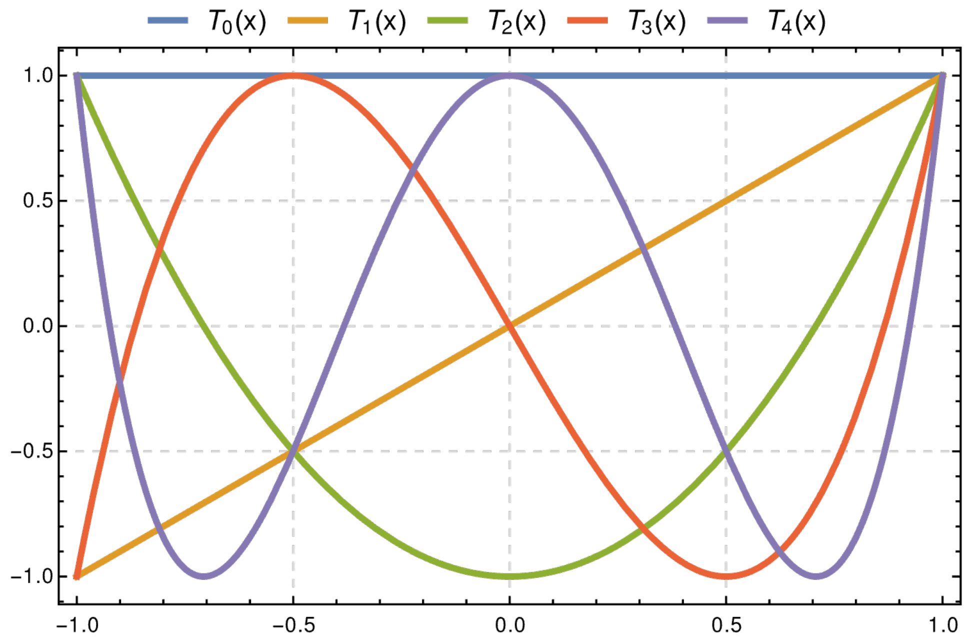
Inclusive: error from kernel approximation





smaller $\sigma \Longrightarrow$ smaller smearing width $\Longrightarrow$ more high frequency component

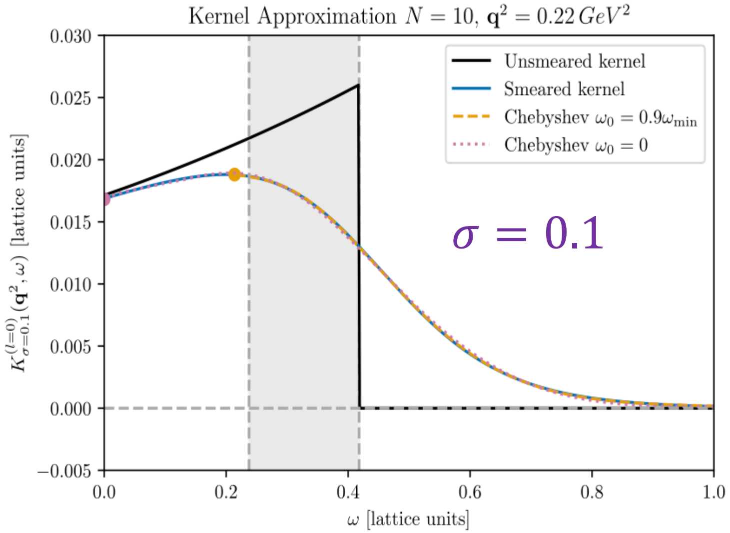
[*PRD* **112** (2025) 014501]

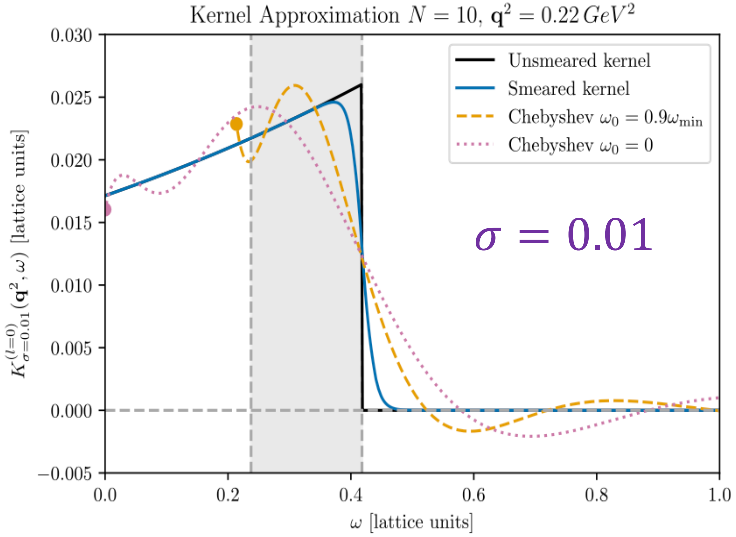


N controls the highest frequency in the Chebyshev approximation

[Wikipedia for Chebyshev Polynomial]

Inclusive: error from kernel approximation

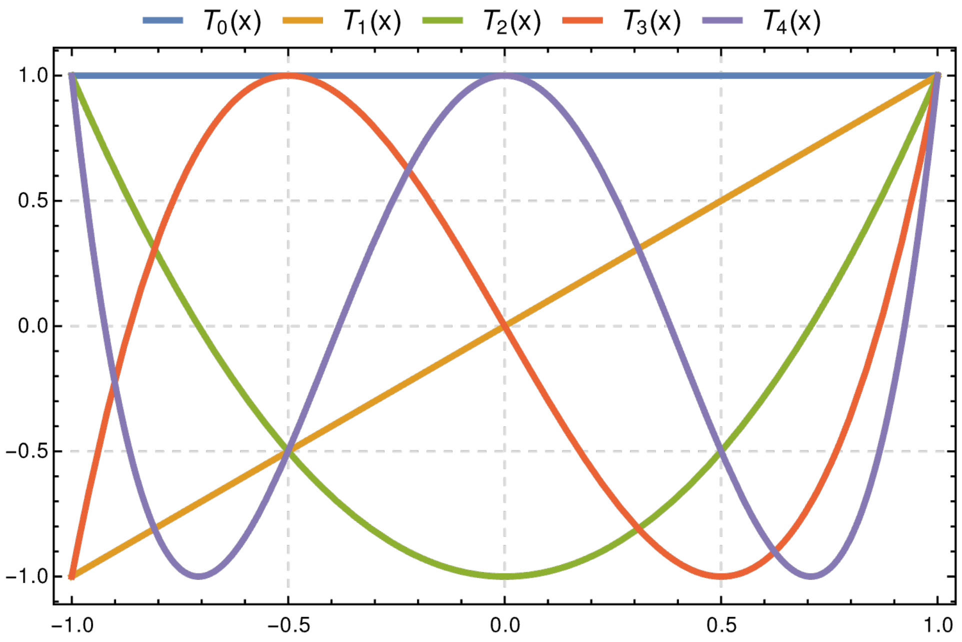




smaller $\sigma \Longrightarrow$ smaller smearing width
$\Longrightarrow$ more high frequency component

$\sigma \to 0$ and $N \to \infty$ limits should be taken together
we set $\sigma = \frac{1}{N}$

[*PRD* **112** (2025) 014501]



N controls the highest frequency in the Chebyshev approximation

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Inclusive: error from kernel approximation

$$\bar{X} = \sum_{j=0}^{N} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu} = \sum_{j=0}^{N_{\mathrm{cut}}-1} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu} + \sum_{j=N_{\mathrm{cut}}}^{N} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu}$$

- T limits the largest N_{cut} we could access

Inclusive: error from kernel approximation

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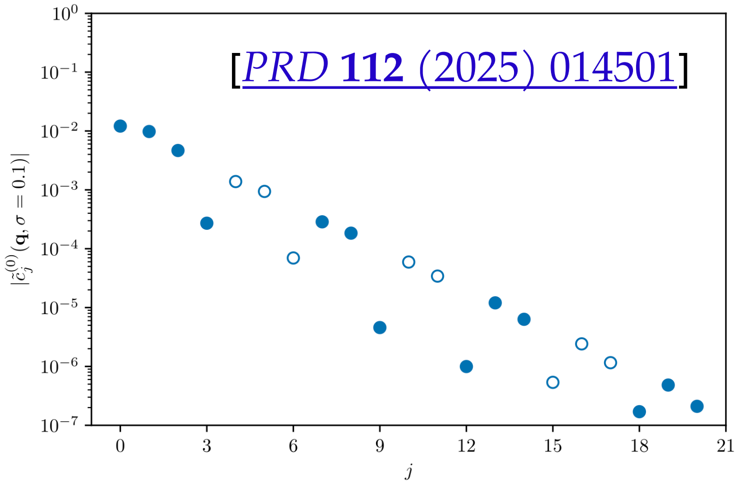
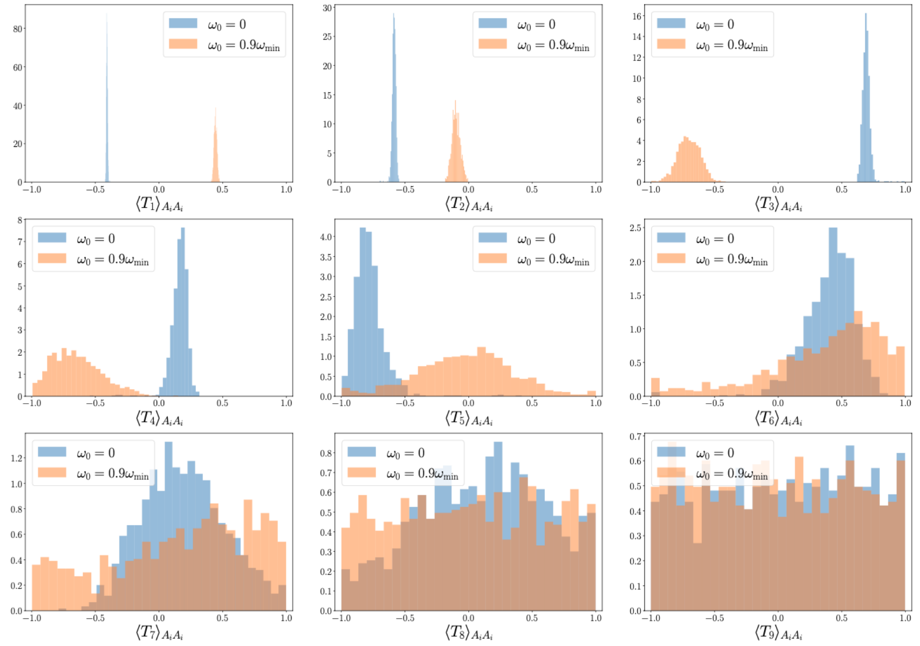
- T limits the largest N_{cut} we could access
- boundedness of Chebyshev matrix element and exponentially decresing coefficients



[*PRD* **112** (2025) 014501]

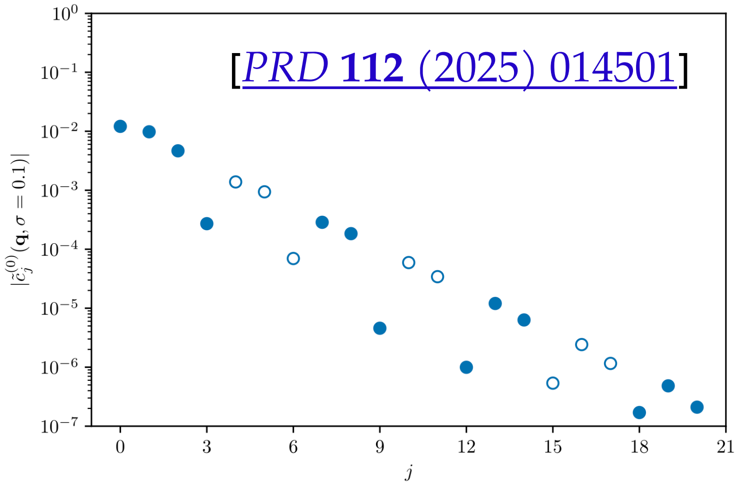


[*JHEP* **07** (2023) 145]

Inclusive: error from kernel approximation

$$\bar{X} = \sum_{j=0}^{N} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu} = \sum_{j=0}^{N_{\text{cut}}-1} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu} + \sum_{j=N_{\text{cut}}}^{N} c_{\sigma,j}^{\mu\nu} \langle \tilde{T}_j \rangle_{\mu\nu}$$

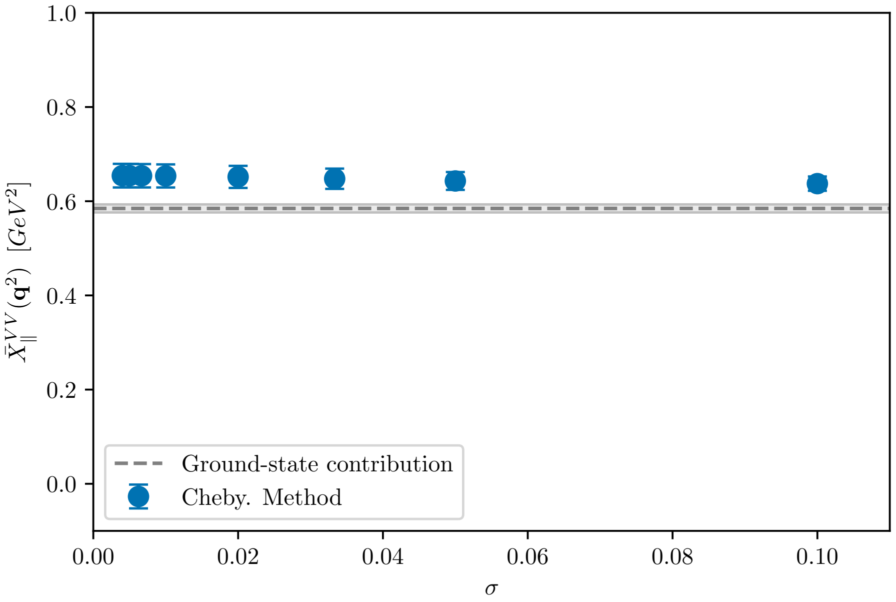
- T limits the largest N_{cut} we could access
- boundedness of Chebyshev matrix element and exponentially decresing coefficients

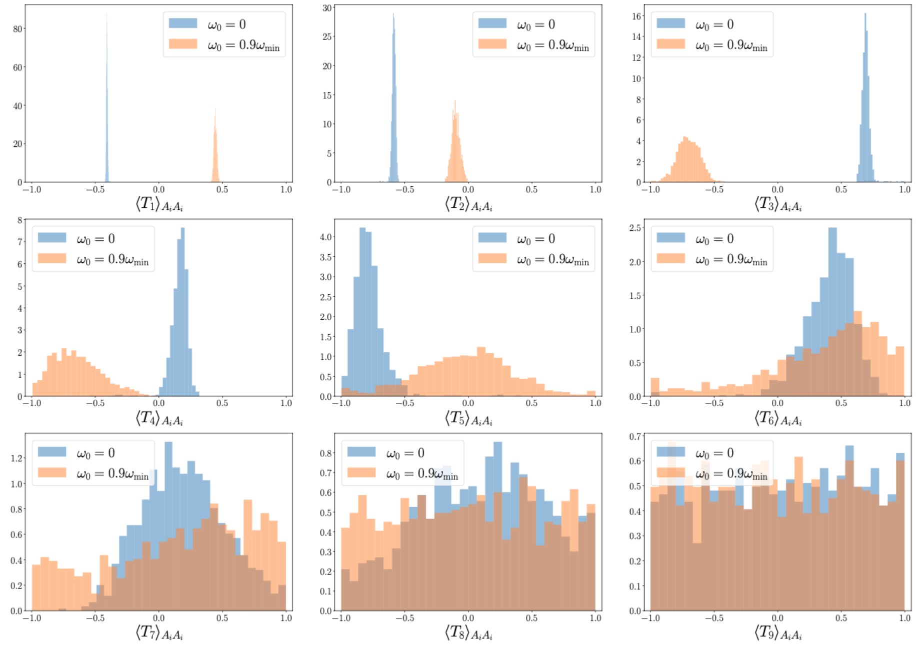


[PRD **112** (2025) 014501]

$$\delta_{\max}^2 = \sum_{j=N_{cut}}^{N} |c_j|^2$$

$$N = \frac{1}{\sigma}$$

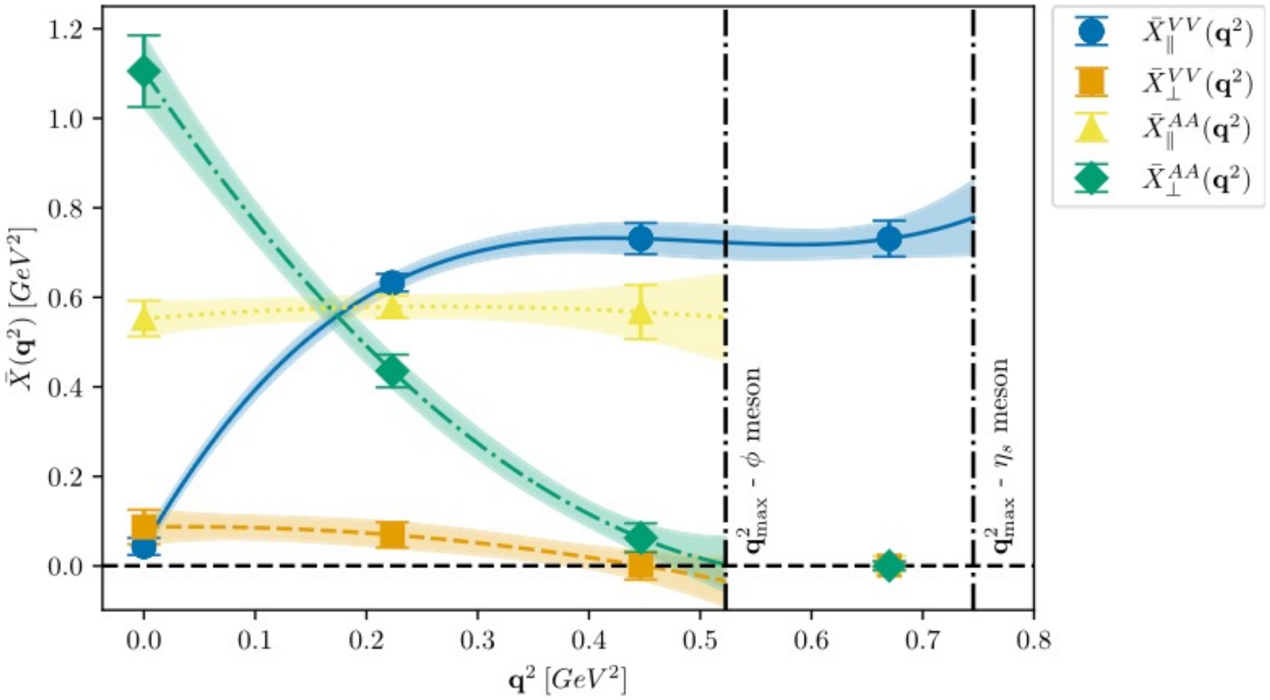




[JHEP **07** (2023) 145]

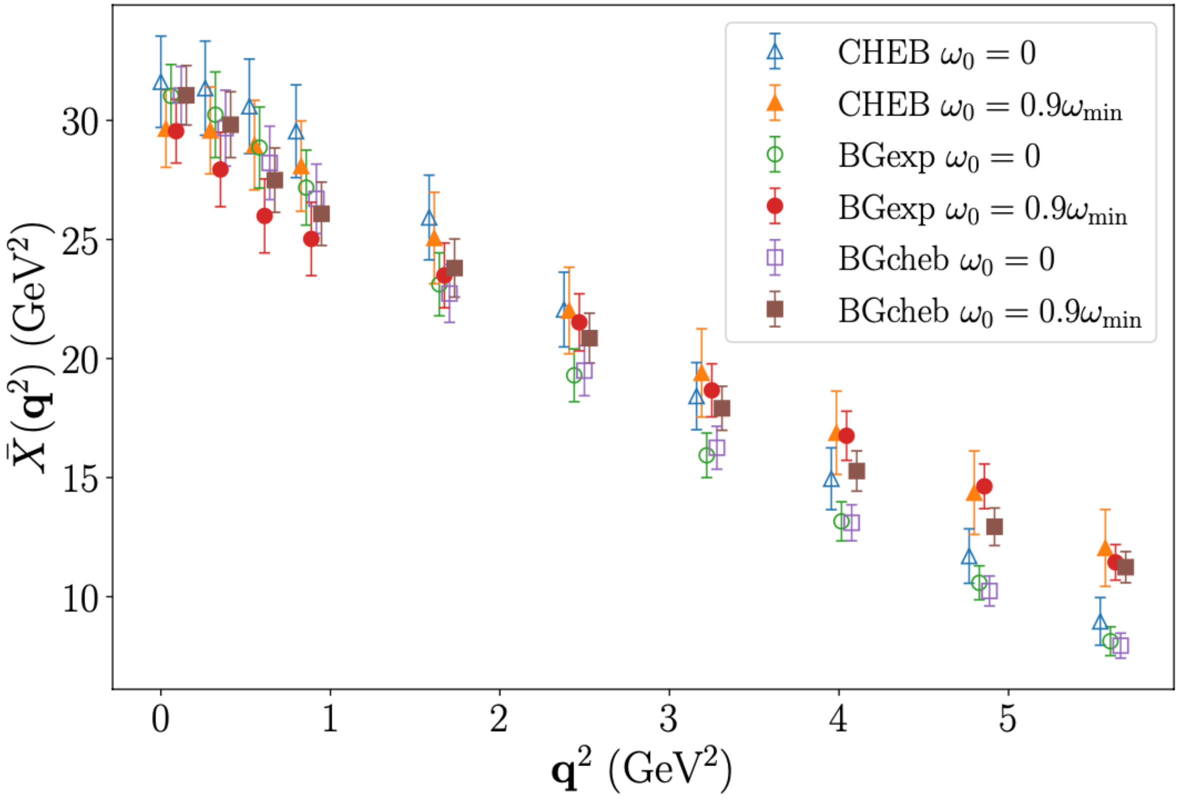
Inclusive: $\bar{X}$ before taking physical limits

JLQCD ensemble on $D_s \to X_{s\bar{s}}\, l\nu_l$



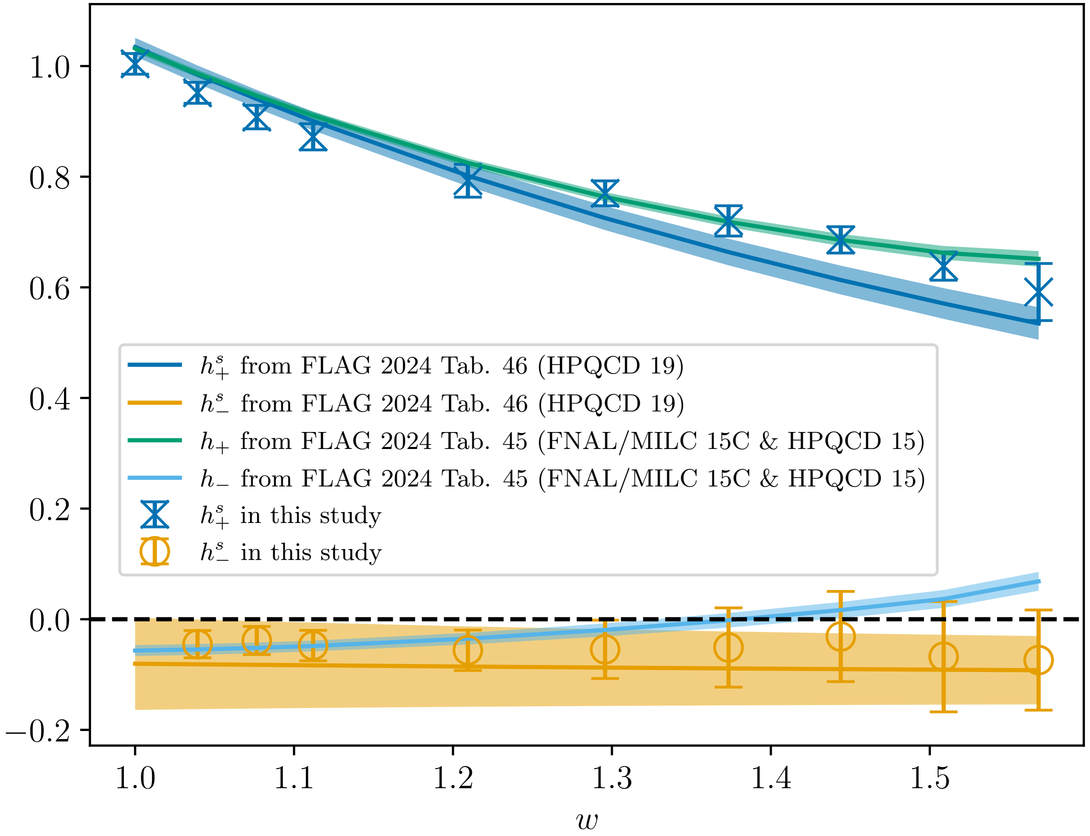
[*PRD* **112** (2025) 014501]

RBC/UKQCD ensemble on $B_s \to X_{cs}\, l\nu_l$



[*JHEP* **07** (2023) 145]

Exclusive: $B_s \to D_s$ form factors befor limits



preliminary results from our study

- heavy-quark convention for form factors

$$\langle B_s \,|\, V_\mu^\dagger \,|\, D_s \rangle = \left[h_+^s(v_\mu + v'_\mu) + h_-^s(v_\mu - v'_\mu)\right]\sqrt{M_{B_s} M_{D_s}}$$

- different conventions

$$f_+^s(q^2) = \frac{1+r}{2\sqrt{r}}\left[h_+^s(w) - \frac{1-r}{1+r}\, h_-^s(w)\right]$$

$$f_0^s(q^2) = \sqrt{r}\left[\frac{1+w}{1+r}\, h_+^s(w) + \frac{1-w}{1-r}\, h_-^s(w)\right]$$

- BCL parameterizations

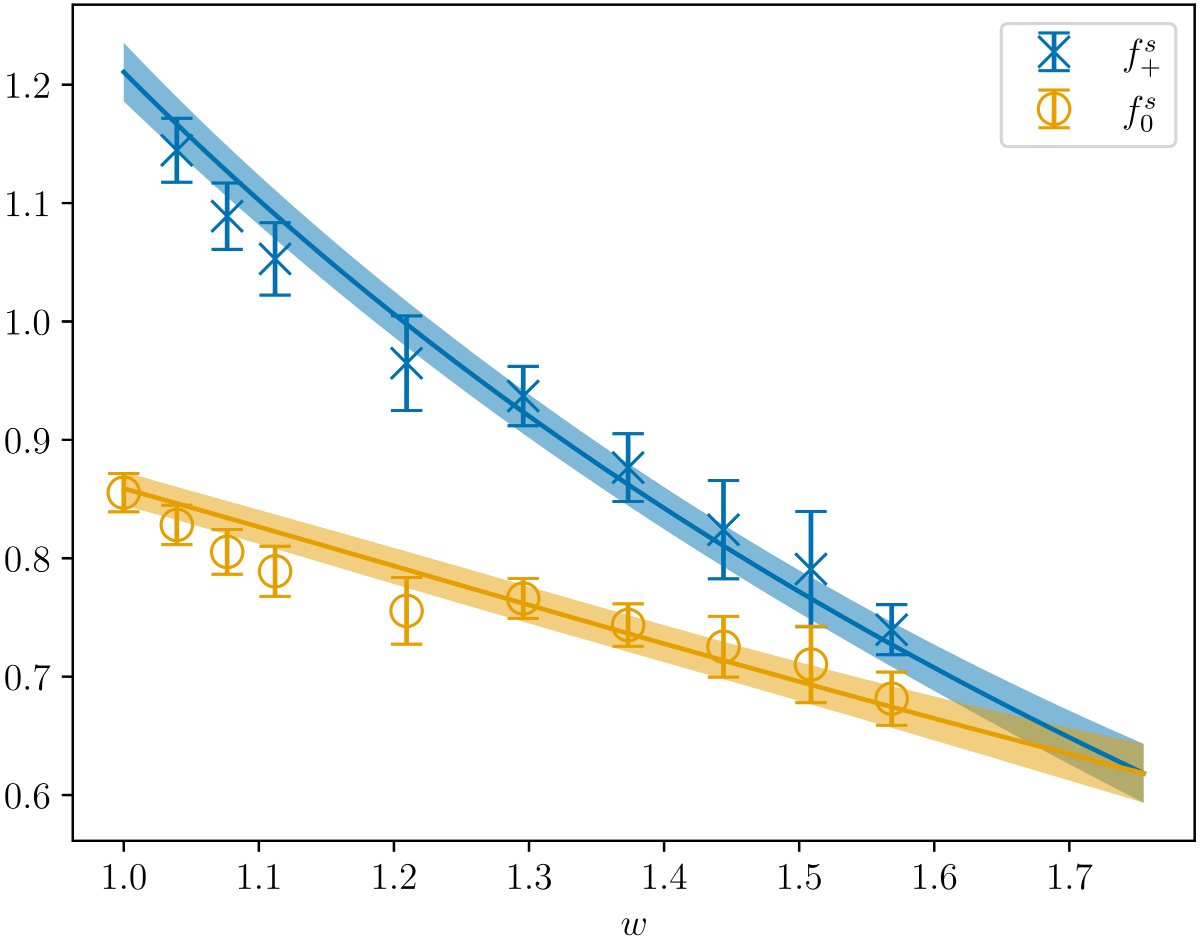
$$f_+^s(q^2) = \sum_{n=0}^{N^+-1} a_n^+\left[z^n - (-1)^{n-N^+}\frac{n}{N^+} z^{N^+}\right]$$

$$f_0^s(q^2) = \sum_{n=0}^{N^0-1} a_n^0 z^n$$

$$r = M_{D_s}/M_{B_s}$$

$$z \equiv \frac{\sqrt{1+w} - \frac{1+r}{\sqrt{2r}}}{\sqrt{1+w} + \frac{1+r}{\sqrt{2r}}}$$

Exclusive: $B_s \to D_s$ form factors before limits



preliminary results from our study

preliminary from 4pt

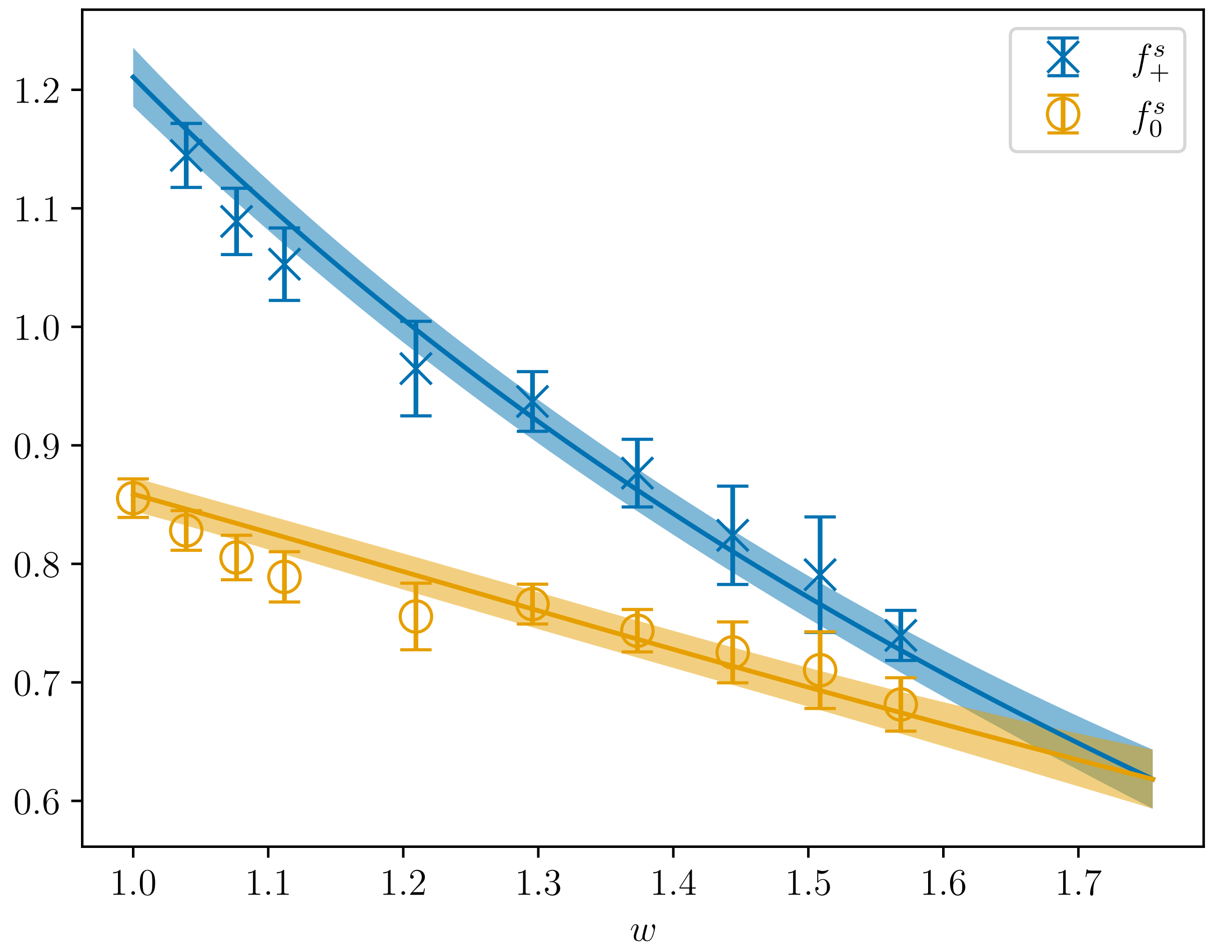
a_1^0	a_2^0	$a_0^+ = a_0^0$	a_1^+	a_2^+
-0.24248	-9.61441	0.61836	-2.58817	-10.4371
0.69609	6.20441	0.01669	0.65081	5.66506
6.20441	58.6077	0.12821	5.84714	54.5366
0.01669	0.12821	0.00062	0.01541	0.11655
0.65081	5.84714	0.01541	0.66499	6.11029
5.66506	54.5366	0.11655	6.11029	63.8852

HPQCD 2019 using 3pt

[*PRD 101 (2020) 074513*]

a_0^0	a_1^0	a_2^0	a_0^+	a_1^+	a_2^+
0.66574	−0.25944	−0.10636	0.66574	−3.23599	−0.07478
0.00015	0.00188	0.00070	0.00015	0.00022	0.00003
	0.06129	0.16556	0.00188	0.01449	0.00001
		3.29493	0.00070	0.18757	−0.00614
			0.00015	0.00022	0.00003
				0.20443	0.10080
					4.04413

Exclusive: $B_s \to D_s$ form factors before limits



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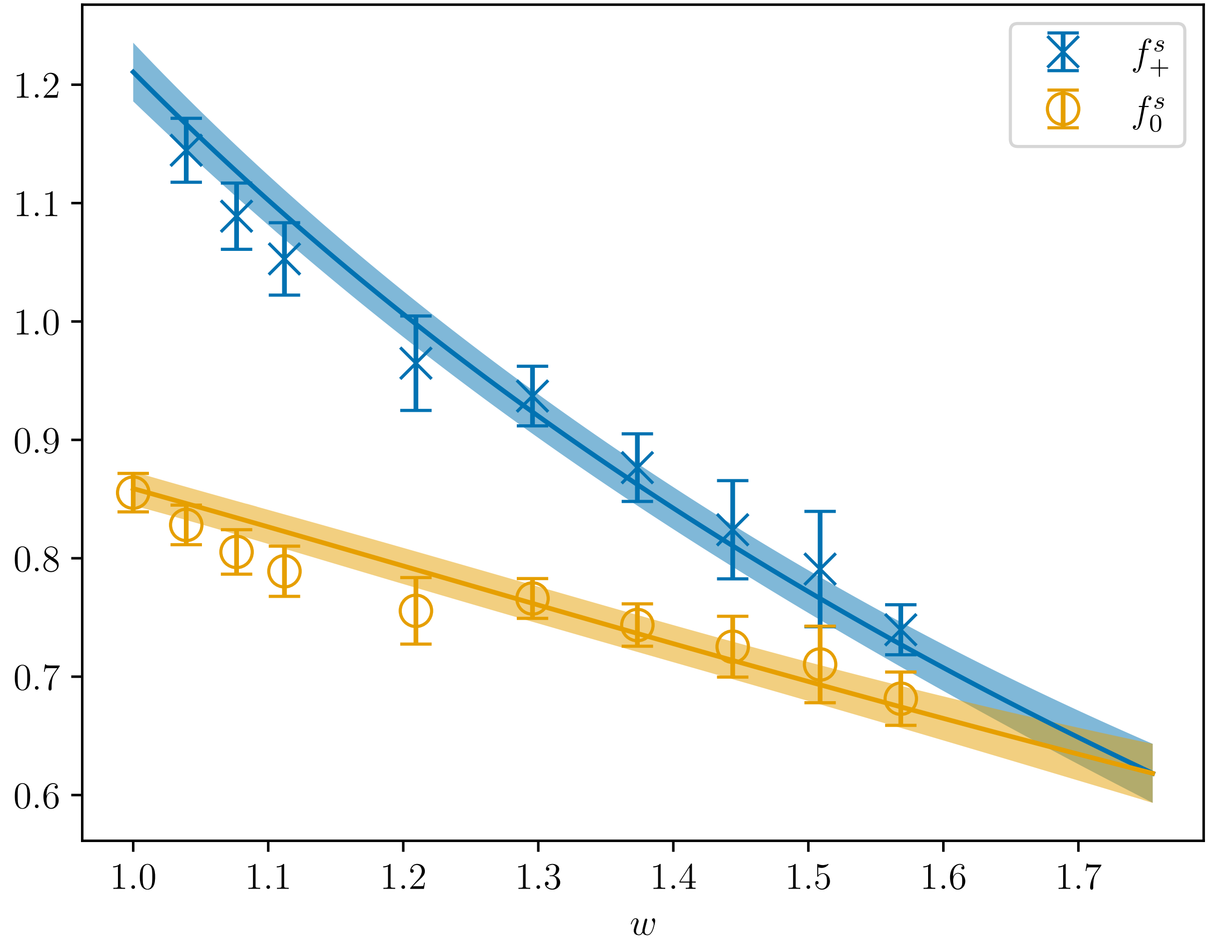

preliminary results from our study

preliminary from 4pt

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HPQCD 2019 using 3pt

[[PRD 101 (2020) 074513]]

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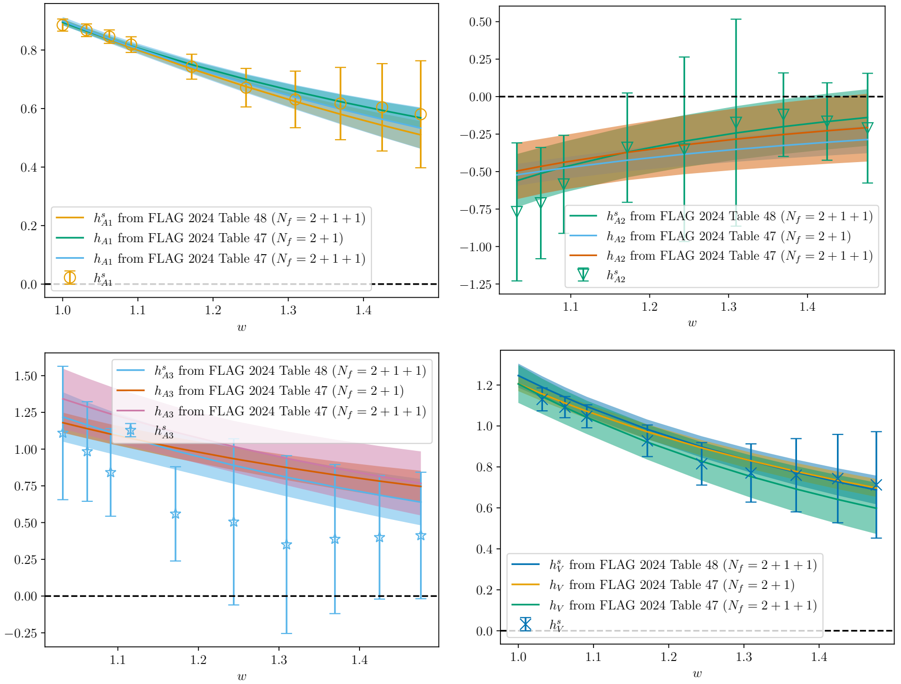
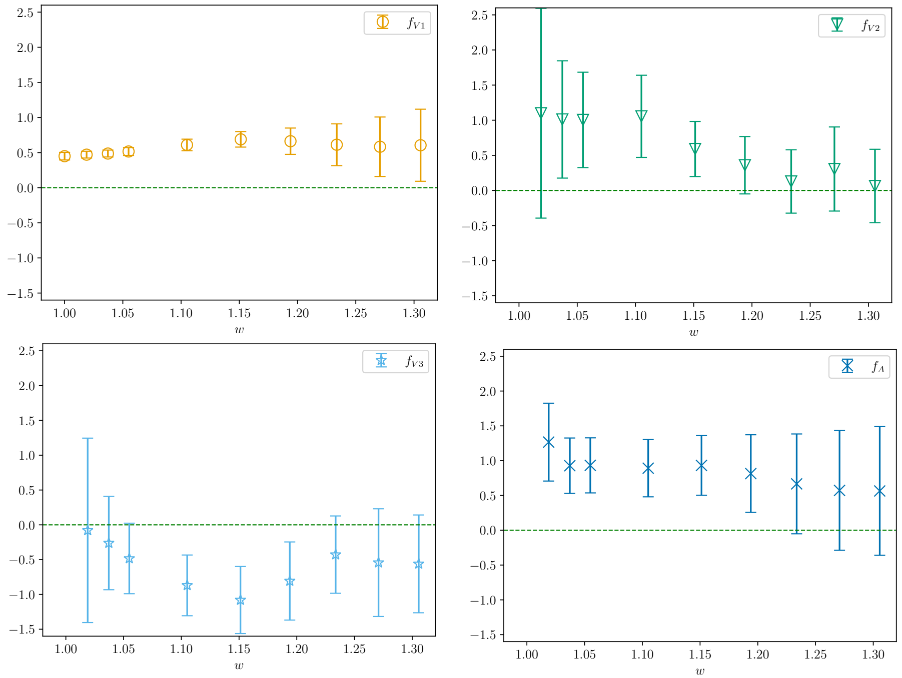
			0.00015	0.00022	0.00003
				0.20443	0.10080
					4.04413

Exclusive: excited-state channels

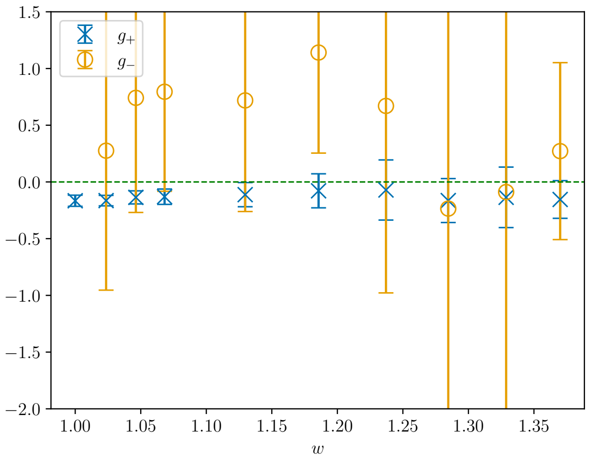


$B_s \to D_s^*$
pseudo scalar to vector

$B_s \to D_{s1}$
pseudo scalar to axial vector



$B_s \to D_{s0}^*$
pseudo scalar to scalar



all from one single set of four-point correlators

Conclusions and perspectives

- a systematic methodology to simultaneously extract inclusive and exclusive observables from lattice four-point correlators
- lattice four-point correlators are expensive, better make most use of them
- infinite-volume, physical-mass and continuum limits

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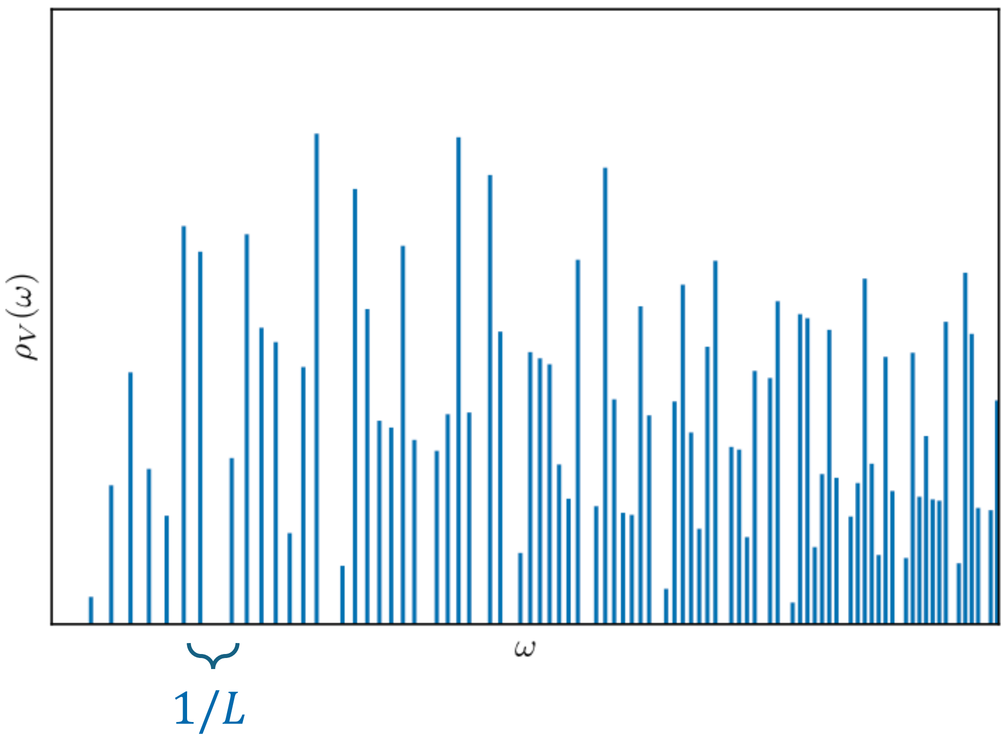
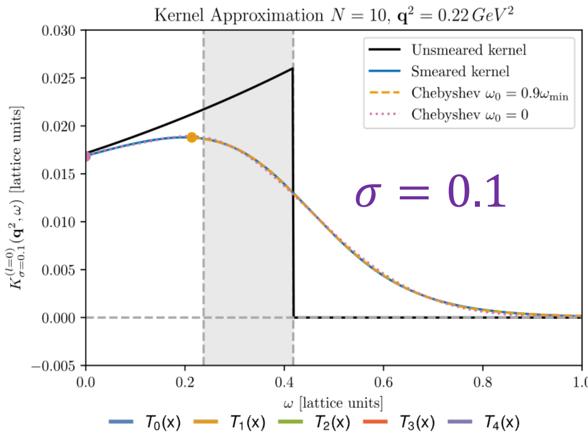


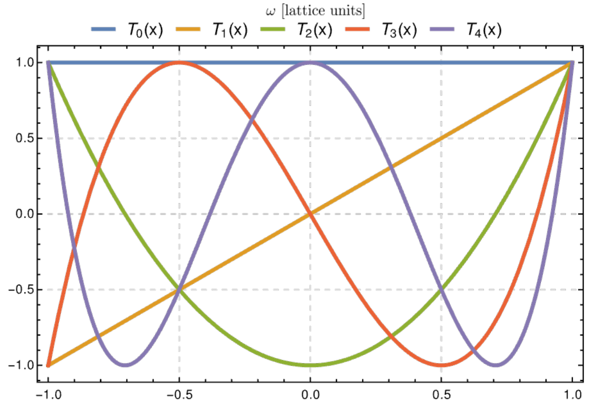
$$\bar{X} = \sum_{j=0}^{N} c_{\sigma,j}^{\mu\nu} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W_{\mu\nu}(\omega) \, \tilde{T}_j(\omega)$$

physical contents

numerical probes

ideally, infinite-volume limit before $N - \sigma$ limit





Thank you very much!

KEK

Backups

Four-point correlators and exclusive decays

$$\frac{d\Gamma^{\rm exc}_{X_{cs}}}{dw} \propto |V_{cb}|^2\, \mathcal{F}_{X_{cs}}(w) \boxed{\otimes \mathcal{K}_{X_{cs}}(w)}$$

known functional forms and kinematical factors

form factors, non-perturbative information, defined from three-point correlators

in the $m_c, m_b \to \infty$ limit, spectra of heavy-light mesons can be clarified by two quantum numbers of **the light degree of freedom**

(n) : radial excitation

j : angular momentum

j^P	J^P	
$(1/2)^- \equiv S$	0^-	D_s
	1^-	D_s^*
$(1/2)^+ \equiv P_{1/2}$	0^+	D_{s0}^*
	1^+	D_{s1}'
$(3/2)^+ \equiv P_{3/2}$	1^+	D_{s1}
	2^+	D_{s2}^*

$$j = L \oplus s_{\rm light}$$
$$J = j \oplus s_{\rm heavy}$$

- **S wave**
  - $D_s\,(0^-)$

$$\langle B_s, v | V_\mu^\dagger | D_s, v' \rangle = \left[h_+(v_\mu + v'_\mu) + h_-(v_\mu - v'_\mu)\right]\sqrt{M_{B_s}M_{D_s}}$$
$$\langle B_s, v | A_\mu^\dagger | D_s, v' \rangle = 0$$

  - $D_s^*\,(1^-)$

$$\langle B_s, v | V_\mu^\dagger | D_s^*, v', \sigma \rangle = \left[h_V \varepsilon_{\mu\alpha\beta\gamma}\epsilon^\alpha v'^\beta v^\gamma\right]\sqrt{M_{B_s}M_{D_s^*}}$$
$$\langle B_s, v | A_\mu^\dagger | D_s^*, v', \sigma \rangle = i\left[(\omega+1)h_{A1}\epsilon_\mu - (\epsilon\cdot v)\left(h_{A2}v_\mu + h_{A3}v'_\mu\right)\right]\sqrt{M_{B_s}M_{D_s^*}}$$

- **P wave, $j = \frac{1}{2}$**
  - $D_{s0}^*\,(0^+)$

$$\langle B_s, v | V_\mu^\dagger | D_{s0}^*, v' \rangle = 0$$
$$\langle B_s, v | A_\mu^\dagger | D_{s0}^*, v' \rangle = \left[g_+(v_\mu + v'_\mu) + g_-(v_\mu - v'_\mu)\right]\sqrt{M_{B_s}M_{D_{s0}^*}}$$

  - $D_{s1}'\,(1^+)$

$$\langle B_s, v | V_\mu^\dagger | D_{s1}', v', \sigma \rangle = \left[g_{V1}\epsilon_\mu + (\epsilon\cdot v)\left(g_{V2}v_\mu + g_{V3}v'_\mu\right)\right]\sqrt{M_{B_s}M_{D_{s1}'}}\,,$$
$$\langle B_s, v | A_\mu^\dagger | D_{s1}', v', \sigma \rangle = i\left[g_A \varepsilon_{\mu\alpha\beta\gamma}\epsilon^\alpha v'^\beta v^\gamma\right]\sqrt{M_{B_s}M_{D_{s1}'}}\,.$$

$C_{J_\mu J_\nu}$ receive contributions from different final states

parallel or perpendicular to final-hadron momentum

	S-channel		$P_{1/2}$-channel		$P_{3/2}$-channel	
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D'_{s1}(1^+)$	$D_{s1}(1^+)$	$D_{s2}^*(2^+)$
$C_{V_0V_0}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{V_\parallel V_\parallel}$	h_+, h_-			g_{V1}, g_{V3}	f_{V1}, f_{V3}	
$C_{V_0V_\parallel}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			$\cdots$
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-			$\cdots$
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			$\cdots$
$C_{V_\perp V_\perp}$		h_V		g_{V1}	f_{V1}	k_V
$C_{A_\perp A_\perp}$		h_{A1}		g_A	f_A	$\cdots$
$C_{V_\perp A_\perp}$		h_{A1}, h_V		g_A, g_{V1}	f_A, f_{V1}	$\cdots$

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	S-channel		$P_{1/2}$-channel		$P_{3/2}$-channel	
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$C_{V_0V_0}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{V_\parallel V_\parallel}$	h_+, h_-			g_{V1}, g_{V3}	f_{V1}, f_{V3}	
$C_{V_0V_\parallel}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			...
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-			...
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			...
$C_{V_\perp V_\perp}$		h_V		g_{V1}	f_{V1}	k_V
$C_{A_\perp A_\perp}$		h_{A1}		g_A	f_A	...
$C_{V_\perp A_\perp}$		h_{A1}, h_V		g_A, g_{V1}	f_A, f_{V1}	...

the current lattice data in hand is coarse

$C_{J_\mu J_\nu}$ receive contributions from different final states

parallel or perpendicular to final-hadron momentum

	S-channel		$P_{1/2}$-channel		$P_{3/2}$-channel	
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D'_{s1}(1^+)$	$D_{s1}(1^+)$	$D_{s2}^*(2^+)$
$C_{V_0V_0}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{V_\parallel V_\parallel}$	h_+, h_-			g_{V1}, g_{V3}	f_{V1}, f_{V3}	
$C_{V_0V_\parallel}$	h_+, h_-			g_{V1}, g_{V2}, g_{V3}	f_{V1}, f_{V2}, f_{V3}	
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			...
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-			...
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-			...
$C_{V_\perp V_\perp}$		h_V		g_{V1}	f_{V1}	k_V
$C_{A_\perp A_\perp}$		h_{A1}		g_A	f_A	...
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the current lattice data in hand is coarse

[Leibovich et al., *PRD* **57** (1998):308–330]
[Bernlochner et al., *PRD* **95**.1 (2017)]

$\boxed{\text{from HQET}}$

$\left(\frac{1}{m_c} - \frac{3}{m_b}\right)^n$ $\left(\frac{1}{m_c}\right)^n$

$C_{J_\mu J_\nu}$ receive contributions from different final states

	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D_{s1}(1^+)$
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$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-	
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

How many fitting parameters?

	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D_{s1}(1^+)$
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$C_{V_\parallel V_\parallel}$	h_+, h_-			f_{V1}, f_{V3}
$C_{V_0V_\parallel}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-	
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

$$C_{J_\mu J_\nu}(w,t) = \mathcal{A}_S^{J_\mu J_\nu}(w)e^{-E_S t} + \mathcal{A}_P^{J_\mu J_\nu}(w)e^{-E_P t}$$

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	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D_{s1}(1^+)$
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$C_{V_0V_\parallel}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-	
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

$$C_{J_\mu J_\nu}(w,t) = \mathcal{A}_S^{J_\mu J_\nu}(w)e^{-E_S t} + \mathcal{A}_P^{J_\mu J_\nu}(w)e^{-E_P t}$$

- some of the correlators receive contributions from the same set of final states $\Longrightarrow$ simultanesout fit and 2 consistent energies

KEK

How many fitting parameters?

	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
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$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

$$C_{J_\mu J_\nu}(w,t) = \mathcal{A}_S^{J_\mu J_\nu}(w)e^{-E_S t} + \mathcal{A}_P^{J_\mu J_\nu}(w)e^{-E_P t}$$

- some of the correlators receive contributions from the same set of final states $\Longrightarrow$ simultanesout fit and 2 consistent energies
- $\mathcal{A}_{S,P}^{V_0V_\parallel} = \sqrt{\mathcal{A}_{S,P}^{V_\parallel V_\parallel} \times \mathcal{A}_{S,P}^{V_0V_0}} \Longrightarrow$ 4 but not 6 amplitudes

How many fitting parameters?

	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D_{s1}(1^+)$
$C_{V_0V_0}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{V_\parallel V_\parallel}$	h_+, h_-			f_{V1}, f_{V3}
$C_{V_0V_\parallel}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-	
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

in total 5 free parameters at every non-zero $\vec{q}^2$ to describe $C_{V_0V_0}(t)$, $C_{V_0V_\parallel}(t)$, $C_{V_\parallel V_\parallel}(t)$

$$C_{J_\mu J_\nu}(w,t) = \mathcal{A}_S^{J_\mu J_\nu}(w)e^{-E_S t} + \mathcal{A}_P^{J_\mu J_\nu}(w)e^{-E_P t}$$

- some of the correlators receive contributions from the same set of final states $\Longrightarrow$ simultanesout fit and 2 consistent energies
- $\mathcal{A}_{S,P}^{V_0V_\parallel} = \sqrt{\mathcal{A}_{S,P}^{V_\parallel V_\parallel} \times \mathcal{A}_{S,P}^{V_0V_0}} \Longrightarrow$ 4 but not 6 amplitudes
- we have a quite coarse data set $\Longrightarrow$ lattice dispersion relation to better constrain P-channel energy $\Longrightarrow$ only S-channel energy totally free

$$E^P = \cosh^{-1}\left(\cosh M^P + \sum_i (1 - \cos p_i)\right)$$

How many fitting parameters?

	S-channel		$P_{1/2}$-channel	$P_{3/2}$-channel
	$D_s(0^-)$	$D_s^*(1^-)$	$D_{s0}^*(0^+)$	$D_{s1}(1^+)$
$C_{V_0V_0}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{V_\parallel V_\parallel}$	h_+, h_-			f_{V1}, f_{V3}
$C_{V_0V_\parallel}$	h_+, h_-			f_{V1}, f_{V2}, f_{V3}
$C_{A_0A_0}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{A_\parallel A_\parallel}$		h_{A1}, h_{A3}	g_+, g_-	
$C_{A_0A_\parallel}$		h_{A1}, h_{A2}, h_{A3}	g_+, g_-	
$C_{V_\perp V_\perp}$		h_V		f_{V1}
$C_{A_\perp A_\perp}$		h_{A1}		f_A
$C_{V_\perp A_\perp}$		h_{A1}, h_V		f_A, f_{V1}

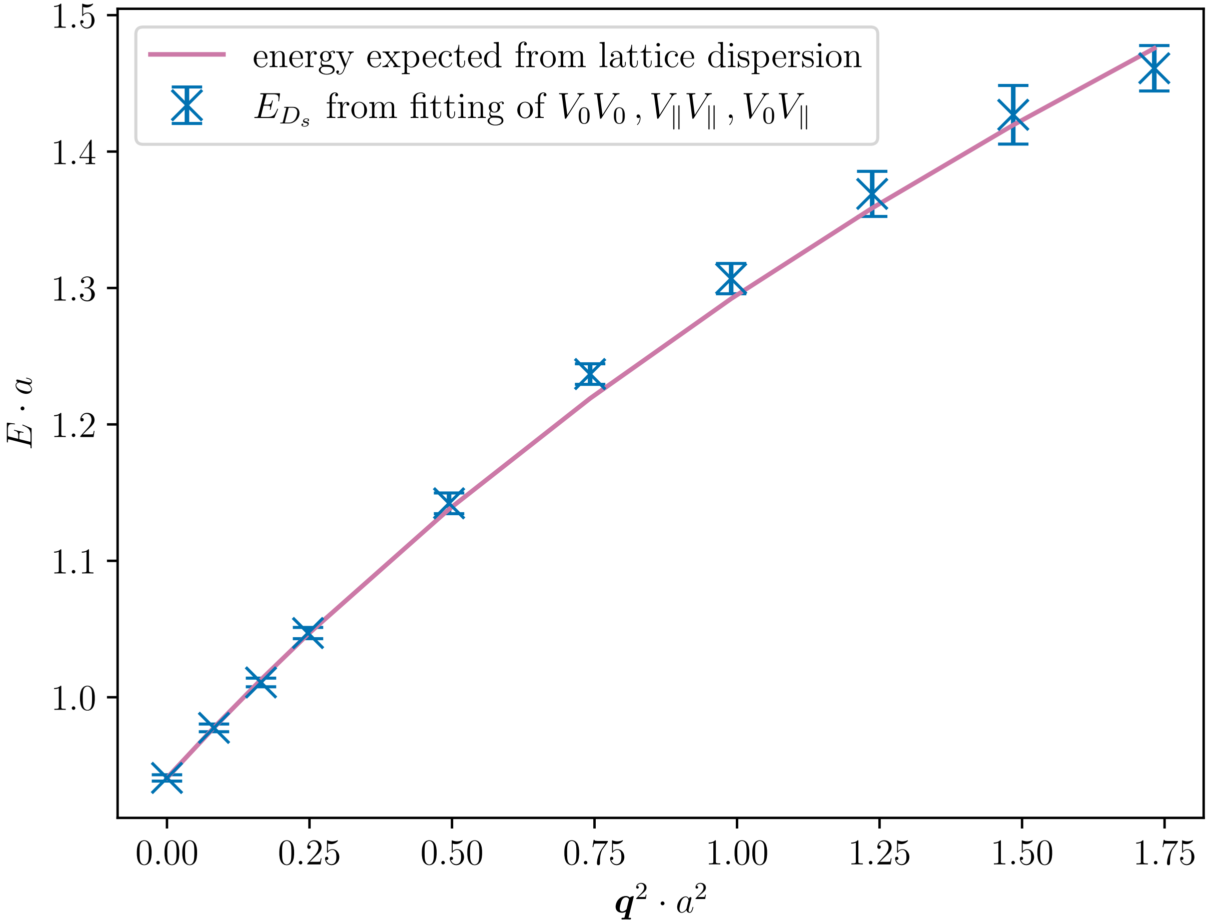
in total 5 free parameters at every non-zero $\vec{q}^2$ to describe each set of three correlators

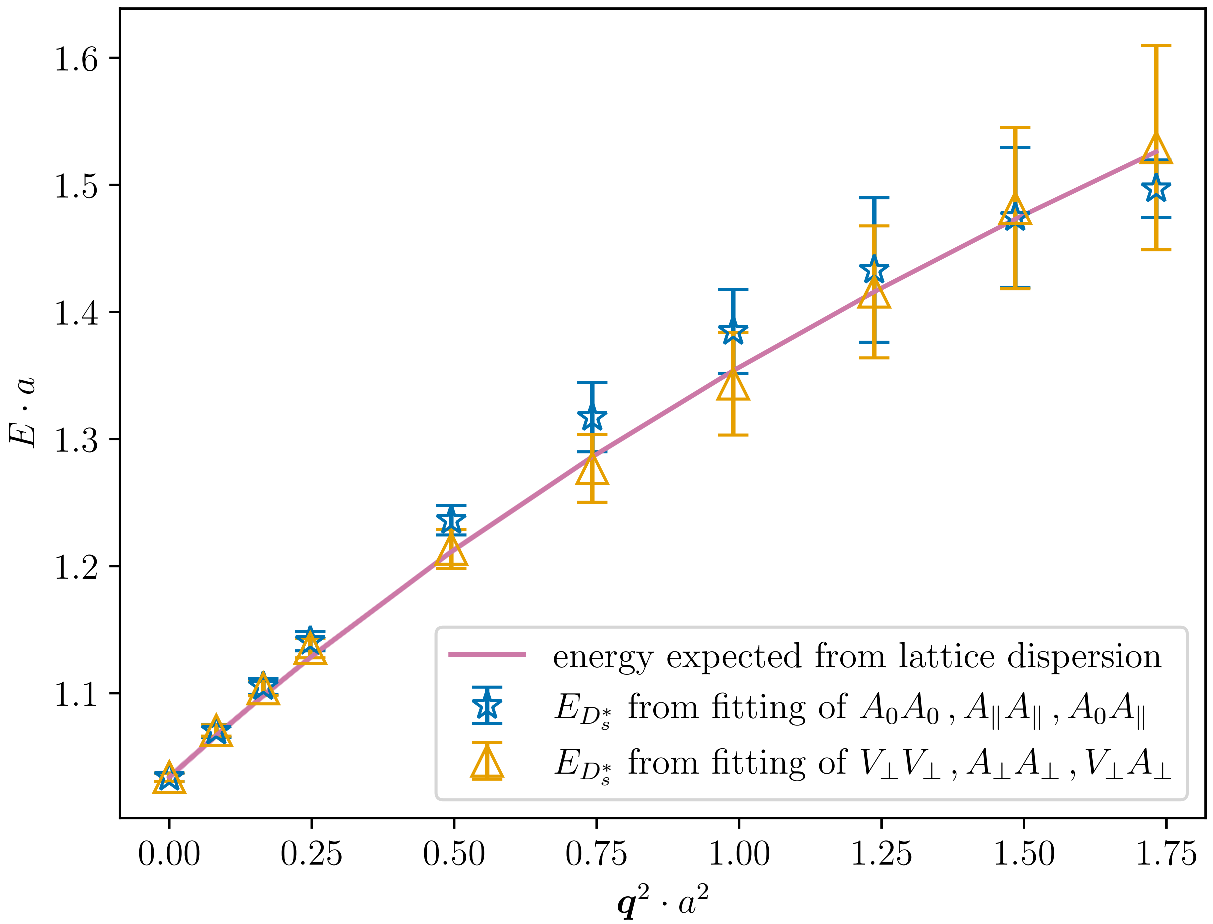
$$C_{J_\mu J_\nu}(w,t) = \mathcal{A}_S^{J_\mu J_\nu}(w)e^{-E_S t} + \mathcal{A}_P^{J_\mu J_\nu}(w)e^{-E_P t}$$

- some of the correlators receive contributions from the same set of final states $\Longrightarrow$ simultanesout fit and 2 consistent energies
- $\mathcal{A}_{S,P}^{V_0V_\parallel} = \sqrt{\mathcal{A}_{S,P}^{V_\parallel V_\parallel} \times \mathcal{A}_{S,P}^{V_0V_0}} \Longrightarrow$ 4 but not 6 amplitudes
- we have a quite coarse data set $\Longrightarrow$ lattice dispersion relation to better constrain P-channel energy $\Longrightarrow$ only S-channel energy totally free

$$E^P = \cosh^{-1}\left(\cosh M^P + \sum_i (1 - \cos p_i)\right)$$

Fitted S-channel energies at different $\vec{q}^2$





- **consistent with expectation within errors**
- **discrepancy as possible indicator of systematic effects**