

Theory Overview of Rare b Decays: A personal perspective

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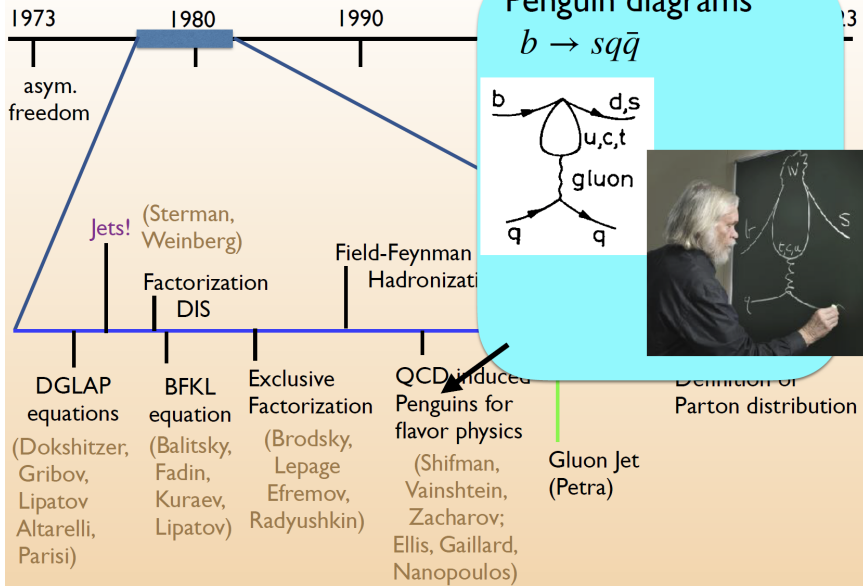
Thanks to Prof. Yu-Ming Wang

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The Strong Interaction at 50 Years [Iain Stewart, 2023]



Why rare decays?

- Crucial to understand the mechanism of CP violation.
Strong phase of $\mathcal{A}(B \rightarrow M_1 M_2)$ @ m_b scale at LP.
- Precision determinations of the CKM matrix elements such as $|V_{ub}|$.
Power corrections, QED corrections, BSM physics.
- An ideal place to search for hints of new physics beyond the standard model.
- Flavour puzzles in rare decays (continuously updated).
 - ▶ P'_5 and $R_{K^{(*)}}$ anomalies in $B \rightarrow K^{(*)} \ell^+ \ell^-$.
 - ▶ Penguin dominated hadronic B -meson decays: $B \rightarrow \pi K$ puzzle.
 - ▶ Color suppressed hadronic B -meson decays: $B \rightarrow \pi^0 \pi^0$ puzzle.
 - ▶ Polarization fractions of penguin dominated $B_{(s)} \rightarrow VV$ decays.
- Precision calculation in rare decays: Interesting to understand the strong interaction dynamics of heavy quark decays.
 - ▶ Factorization properties of the subleading-power amplitudes.
 - ▶ Perturbative properties of the (higher-twist) B -meson DAs (soft functions).

Theory tools for precision flavour physics

New Physics: $\mathcal{L}_{NP}$



EW scale (m_W): $\mathcal{L}_{SM} + \mathcal{L}_{D>4}$



Heavy-quark scale (m_b):

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} \sum_i C_i Q_i + \mathcal{L}_{eff, D>6}$$



QCD scale (Λ_{QCD})

- Aim: $\langle f | Q_i | \bar{B} \rangle = ?$
- QCD factorization
[Diagrammatic approach].
- SCET factorization
[Operator formalism].
- Lattice QCD.
- (Light-Cone) QCD sum rules
[Khodjamirian, Melić, YMW,
Eur. Phys. J. ST 233 (2024)
271].

- QCD is complicated (non)-perturbatively (Factorization, Resummation, Multi-loop techniques).

Reviews on Light-Cone QCD Sum Rules

QCD SUM RULES, A MODERN PERSPECTIVE

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An introduction to the method of QCD sum rules is given for those who want to learn how to use this method. Furthermore, we discuss various applications of sum rules, from the determination of quark masses to the calculation of hadronic form factors and structure functions. Finally, we explain the idea of the light-cone sum rules and outline the recent development of this approach.

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Review

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A guide to the QCD light-cone sum rules for b-quark decays

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Abstract We overview the current status and future perspective of the QCD-based method of light-cone sum rules. The two main versions of these sum rules, using light-meson and B -meson distribution amplitudes are introduced and the most important applications of the method are discussed. We also outline open problems and future perspective of this method.

1 Introduction

The QCD-based method of light-cone sum rules (LCSRs) has already a long history. It was originally developed in Refs. [1, 2] for the process of a hyperon radiative decay and, independently, in Ref. [3], where LCSRs were for the first time applied to B -meson exclusive decays. An application to the D -meson semileptonic decays followed then in Ref. [4]. The usefulness of this method for the calculation of B -meson transition form factors was recognized already in the early applications [3–5].

The LCSRs, in their different versions, offer a realistic possibility to calculate various form factors of semileptonic or radiative B -meson decays in the large recoil region of the final light hadron. The method can be extended to other hadronic amplitudes involved in B physics, such as strong couplings and nonperturbative decay amplitudes. It is also applicable to the decays containing two light hadrons in the final state and involving broad resonances. Neither the large recoil region, nor resonance final states are currently accessible to the lattice QCD calculations, and therefore LCSRs in several cases are the only available QCD-based method.

To describe the method in a nutshell: each LCSR for a hadronic transition form factor starts from a correlator which is a vacuum-to-hadron matrix element of two quark currents, chosen to match a form factor we are interested in. The correlator contains a quark-level transition current, and a current interpolating the other hadron involved in this transition. For this correlator, an operator product expansion (OPE) near the light-cone is applied, providing the external moments of quark currents are for off-shell. The OPE yields an analytical and factorizable expression for the correlator, written in a form of a convolution of the calculable hard-scattering kernel with universal light-cone distribution amplitudes (DAs) of the hadron entering the vacuum-to-hadron matrix element. These DAs absorb nonperturbative effects, and their parameters represent a universal, process-independent input. The factorized form of the OPE has its roots in the early studies of the QCD asymptotics of hadron form factors [6–12]. There is a crucial difference, though: in the LCSR approach factorization is applied at the correlator level. To access the form factor, the tools of conventional QCD (SVZ) sum rule method [13] are then applied to the same correlator, including the hadronic dispersion relation and the quark-hadron duality approximation. Combining these three main elements the light-cone OPE, dispersion relation and duality, leads to the resulting LCSR, which is an approximate analytical formula for the form factor depending on a set of universal inputs.

Without going into further details in this brief introduction, we only mention that the method of LCSRs allows us to calculate the so-called soft contribution to a form factor. It describes a transition between the initial and final

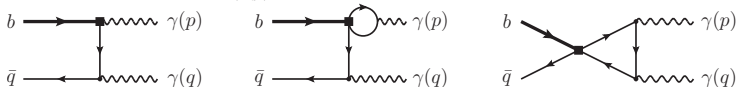
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Double Radiative B_q -Meson Decays

- Leading-order contributions at $\mathcal{O}(\alpha_s^0)$:



Kinematics:

$$p_\mu = \frac{n \cdot p}{2} \bar{n}_\mu \equiv \frac{m_{B_q}}{2} \bar{n}_\mu, \quad q_\mu = \frac{\bar{n} \cdot q}{2} n_\mu \equiv \frac{m_{B_q}}{2} n_\mu.$$

Interplay of the soft and collinear QCD dynamics!

B_q -meson photons

- Decay amplitude:

$$\mathcal{A}(\bar{B}_q \rightarrow \gamma\gamma) = -\frac{4G_F}{\sqrt{2}} \frac{\alpha_{\text{em}}}{4\pi} \varepsilon_1^{*\alpha}(p) \varepsilon_2^{*\beta}(q) \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^8 C_i T_{i,\alpha\beta}^{(p)}.$$

Hadronic tensors:

$$\begin{aligned} T_{7,\alpha\beta} &= 2\bar{m}_b(v) \int d^4x e^{iq \cdot x} \langle 0 | T \left\{ j_\beta^{\text{em}}(x), \bar{q}_L(0) \sigma_{\mu\alpha} p^\mu b_R(0) \right\} | \bar{B}_q(p+q) \rangle \\ &\quad + [p \leftrightarrow q, \alpha \leftrightarrow \beta], \\ T_{i,\alpha\beta}^{(p)} &= -(4\pi)^2 \int d^4x \int d^4y e^{ip \cdot x} e^{iq \cdot y} \langle 0 | T \left\{ j_\alpha^{\text{em}}(x), j_\beta^{\text{em}}(y), P_i^{(p)}(0) \right\} | \bar{B}_q(p+q) \rangle, \\ &\quad (i = 1, \dots, 6, 8). \end{aligned}$$

Main task: Construct the SCET factorization formulae beyond the leading power.

General aspects of $B_q \rightarrow \gamma\gamma$

- Parametrization:

$$T_{i,\alpha\beta}^{(p)} = im_{B_q}^3 \left[\left(g_{\alpha\beta}^\perp - i\varepsilon_{\alpha\beta}^\perp \right) F_{i,L}^{(p)} - \left(g_{\alpha\beta}^\perp + i\varepsilon_{\alpha\beta}^\perp \right) F_{i,R}^{(p)} \right].$$

- ▶ Only **two helicity form factors** due to the Ward identities and the transversity conditions.
- ▶ Similar decomposition for the **radiative leptonic B -meson decay** amplitude.

- Hierarchy structure due to the chiral weak interactions and helicity conservation:

$$F_{i,L}^{(p)} : F_{i,R}^{(p)} = 1 : \left(\frac{\Lambda_{\text{QCD}}}{m_b} \right).$$

In analogy to the two-body nonleptonic $B \rightarrow VV$ decays [Beneke, Rohrer, Yang, 2007].

- Transversity form factors:

$$F_{i,\parallel}^{(p)} = F_{i,L}^{(p)} - F_{i,R}^{(p)}, \quad F_{i,\perp}^{(p)} = F_{i,L}^{(p)} + F_{i,R}^{(p)}.$$

The two-photon final states as the **CP eigenstates** with the eigenvalues $+1$ and -1 .

Current status of $B_q \rightarrow \gamma\gamma$

- QCD factorization at leading power in Λ/m_b and at NLO in α_s [Descotes-Genon, Sachrajda, 2003].
 - ▶ No collinear strong interaction dynamics at LP ($\Rightarrow$ **no need of the hadronic photon LCDAs**).
 - ▶ The two-loop $b \rightarrow q \gamma$ matrix elements of QCD penguin operators NOT included.
 $\Rightarrow$ **A complete factorization-scale independence at NLO is absent!**
- Subleading power contributions from the **weak annihilation** diagrams [Bosch, Buchalla, 2002].
 - ▶ Complex perturbative hard functions evaluated at one loop.
 - ▶ Diagrammatic factorization established at two loops.
- Previous model-dependent calculations introduce additional systematic uncertainties.
- The new (technical)-ingredients from [Shen, YMW, Wei, 2020]:
 - ▶ A complete NLL calculation for the LP contribution $\Rightarrow$ 2-loop evolution of ϕ_B^+ .
 - ▶ The NLP factorization for **the energetic photon radiation off the light quark**.
The so-called soft form factor introduced by Beneke and Rohrwild is calculable!
 - ▶ The NLP factorization for the light-quark mass effect.
 - ▶ The NLP factorization for the SCET current $J^{(A2)} \supset (\frac{\bar{\epsilon}}{S_{hc}} W_{hc}) \gamma_\alpha^\perp P_L \left(\frac{i \vec{D}_\perp}{2m_b} \right) h_v$.
 - ▶ The NLP factorization for the subleading twist B -meson LCDAs (**2-particle $\oplus$ 3-particle**).
 - ▶ The resolved photon contribution with the dispersion technique.
- QCD factorization for the long-distance penguin contribution to $B_{d,s} \rightarrow \gamma\gamma$ decays by introducing **a novel generalized bottom-meson distribution amplitude** [Qin, Shen, Wang, YMW, 2023].

SCET factorization at leading power

- QCD $\rightarrow$ SCET_I matching at LP:

$$\sum_{i=1}^8 C_i T_{i,\alpha\beta}^{(p)} = \sum_{i=1}^8 C_i H_i^{(p)} \left\{ \int d^4x e^{iq \cdot x} \langle 0 | T \left\{ j_{\beta, \text{SCET}_I}^{\text{em}}(x), \left[(\bar{\xi}_{\text{hc}} W_{\text{hc}}) \gamma_{\alpha}^{\perp} P_L h_v \right] (0) \right\} | \bar{B}_q \rangle + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \right\}.$$

Universal SCET_I correlation function independent of the weak-operator index!

- Perturbative matching coefficients at NLO:

$$\sum_{i=1}^8 C_i H_i^{(p)} = (-2i) \bar{m}_b(v) m_{B_q} V_{7,\text{eff}}^{(p)},$$

$$V_{7,\text{eff}}^{(p)} = C_7^{\text{eff}} C_{T_1}^{(\text{A0})} + \sum_{i=1,\dots,6,8} \frac{\alpha_s(\mu)}{4\pi} C_i^{\text{eff}} F_{i,7}^{(p)}.$$

- The hard function $C_{T_1}^{(\text{A0})}$ from matching the heavy-light tensor current onto SCET_I.
- The hard functions $F_{i,7}^{(p)}$ ($i = 1, \dots, 6, 8$) from perturbative matching of the $b \rightarrow q\gamma$ matrix elements [Buras, Czarnecki, Misiak, Urban, 2002].
- $V_{7,\text{eff}}^{(p)}$ due to both the electro-weak scale and the heavy-quark mass scale fluctuations.

SCET factorization at leading power

- SCET_I → SCET_{II} matching in coordinate space:

$$\begin{aligned}\mathcal{T}_{\alpha\beta} &= \int d^4x e^{iq \cdot x} \langle 0 | T \left\{ J_{\beta, \text{SCET}_I}^{\text{em}}(x), \left[(\bar{\xi}_{\text{hc}} W_{\text{hc}}) \gamma_{\alpha}^{\perp} P_L h_v \right] (0) \right\} | \bar{B}_q \rangle \\ &= \int dt \mathcal{J} \left(\frac{\bar{n} \cdot q}{\mu^2 t} \right) \langle 0 | (\bar{q}_s Y_s)(tn) \gamma_{\beta}^{\perp} \not{n} \gamma_{\alpha}^{\perp} P_L (Y_s^{\dagger} h_v)(0) | \bar{B}_q \rangle.\end{aligned}$$

- The explicit form of the SCET_I electromagnetic current [Lunghi, Pirjol, Wyler, 2003]:

$$\begin{aligned}j_{q, \text{SCET}_I}^{\mu}(x) &= \sum_q Q_q \left[\bar{q}_{\text{hc}} \left(\gamma_{\perp}^{\mu} \frac{1}{in_+ D_{\text{hc}}} i \not{D}_{\text{hc}\perp} + i \not{D}_{\text{hc}\perp} \frac{1}{in_+ D_{\text{hc}}} \gamma_{\perp}^{\mu} \right) \frac{\not{n}_+}{2} q_{\text{hc}} \right] (x) \\ &\quad + \sum_q Q_q [\bar{q}_s(x_-) \gamma_{\perp}^{\mu} q_{\text{hc}}(x)].\end{aligned}$$

- Only need the one-loop jet function [2-loop result by Liu and Neubert, 2020].

$$J = 1 + \frac{\alpha_s(\mu) C_F}{4\pi} \left[\ln^2 \left(\frac{\mu^2}{m_b \omega} \right) - \frac{\pi^2}{6} - 1 \right] + \mathcal{O}(\alpha_s^2).$$

- The soft dynamics encoded in the twist-two HQET *B-meson LCDA* [Grozin, Neubert, 1997].
- The resulting LP factorization formula:

$$\begin{aligned}\mathcal{A}_{\text{LP}}(\bar{B}_q \rightarrow \gamma\gamma) &= i \frac{4G_F}{\sqrt{2}} \frac{\alpha_{\text{em}}}{4\pi} \varepsilon_1^{*\alpha}(p) \varepsilon_2^{*\beta}(q) \left[g_{\alpha\beta}^{\perp} - i \varepsilon_{\alpha\beta}^{\perp} \right] e_q f_{B_q} m_{B_q}^2 K^{-1}(m_b, \mu) \\ &\quad \left[\sum_{p=u,c} V_{pb} V_{pq}^* \bar{m}_b(v) V_{7, \text{eff}}^{(p)}(m_b, \mu, v) \right] \int_0^{\infty} \frac{d\omega}{\omega} \phi_B^+(\omega, \mu) J(m_b, \omega, \mu).\end{aligned}$$

SCET factorization at leading power

- No common choice of the factorization scale to avoid the parametrically large logarithms.
 $\Rightarrow$ QCD rsummation for the enhanced logarithms with the standard RG formalism.
- RG evolution functions for the hard functions at NLL [Beneke, Rohrwild, 2011]:

$$\begin{aligned} V_{7,\text{eff}}^{(p)}(m_b, \mu, \nu) &= \hat{U}_1(m_b, \mu_h, \mu) V_{7,\text{eff}}^{(p)}(m_b, \mu_h, \nu), \\ K^{-1}(m_b, \mu) &= \hat{U}_2(m_b, \mu_h, \mu) K^{-1}(m_b, \mu_h). \end{aligned}$$

[The NNLL evolution functions are actually also available.]

- Taking the factorization scale of order $\sqrt{m_b \Lambda_{\text{QCD}}} \Rightarrow$ **No resummation for the jet function.**
- RG evolution of $\phi_B^+(\omega, \mu)$ at two loops [Braun, Ji, Manashov, 2019]:

$$\begin{aligned} \frac{d\phi_B^+(\omega, \mu)}{d\ln\mu} = & \left[\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\omega}{\mu} - \gamma_\eta(\alpha_s) \right] \phi_B^+(\omega, \mu) + \Gamma_{\text{cusp}}(\alpha_s) \int_0^\infty dx \Gamma(1, x) \phi_B^+(\omega/x, \mu) \\ & + \left(\frac{\alpha_s}{2\pi} \right)^2 C_F \int_0^1 \frac{dx}{1-x} h(x) \phi_B^+(\omega/x, \mu). \end{aligned}$$

The last missing element for the NLL predictions of exclusive B -meson decay observables!

SCET factorization at leading power

- Solving the **integro-differential equation** governing the ϕ_B^+ evolution **nontrivial but interesting**.
 - ▶ **One-loop eigenfunctions** first derived in [Bell, Feldmann, YMW, Yip, 2013].
 - ▶ Collinear conformal symmetry for the Lange-Neubert kernel at one loop
 $\Rightarrow$ Commutation relations [Braun, Manashov, 2014].
 - ▶ **The two-loop eigenfunctions** depend on the strong coupling α_s [Braun, Ji, Manashov, 2019]
 - ▶ The explicit solution to the **momentum-space** evolution equation at 2-loops [Galda, Neubert, Wang, 2022].
- Applying the Laplace transformation of the LCDA [Galda, Neubert, 2020]

$$\tilde{\phi}_B^+(\eta, \mu) = \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega, \mu) \left(\frac{\omega}{\bar{\omega}}\right)^{-\eta},$$

$\Rightarrow$ the general solution to the two-loop RGE of $\phi_B^+(\omega, \mu)$

$$\begin{aligned} \tilde{\phi}_B^+(\eta, \mu) &= \exp \left[S(\mu_0, \mu) + a_\gamma(\mu_0, \mu) + 2 \gamma_E a_\Gamma(\mu_0, \mu) \right] \left(\frac{\bar{\omega}}{\mu_0} \right)^{-a_\Gamma(\mu_0, \mu)} \\ &\times \frac{\Gamma(1 + \eta + a_\Gamma(\mu_0, \mu)) \Gamma(1 - \eta)}{\Gamma(1 - \eta - a_\Gamma(\mu_0, \mu)) \Gamma(1 + \eta)} \exp \left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \mathcal{G}(\eta + a_\Gamma(\mu_\alpha, \mu), \alpha) \right] \\ &\times \tilde{\phi}_B^+(\eta + a_\Gamma(\mu_0, \mu), \mu_0). \end{aligned}$$

$\Rightarrow$ The NLL resummation improved expression for the LP decay amplitude.

Historical remarks on the B -meson LCDA

Renormalization group evolution of the B meson light cone distribution amplitude

Bjorn O. Lange (Cornell U., LNS), Matthias Neubert (Cornell U., LNS) (Mar, 2003)

Published in: *Phys.Rev.Lett.* 91 (2003) 102001 • e-Print: [hep-ph/0303082](#) [hep-ph]

 pdf  DOI  cite  claim

 reference search  180 citations

Light-Cone Distribution Amplitudes for Heavy-Quark Hadrons

Guido Bell (Oxford U., Theor. Phys.), Thorsten Feldmann (Siegen U.), Yu-Ming Wang (RWTH Aachen U. and Munich, Tech. U.), Matthew W Y Yip (Durham U., IPPP) (Aug 28, 2013)

Published in: *JHEP* 11 (2013) 191 • e-Print: [1308.6114](#) [hep-ph]

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 reference search  126 citations

Conformal symmetry of the Lange-Neubert evolution equation

V.M. Braun (Regensburg U.), A.N. Manashov (Regensburg U. and St. Petersburg State U.) (Feb 24, 2014)

Published in: *Phys.Lett.B* 731 (2014) 316-319 • e-Print: [1402.5822](#) [hep-ph]

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 reference search  37 citations

Two-loop evolution equation for the B -meson distribution amplitude

V.M. Braun (Regensburg U.), Yao Ji (Regensburg U.), A.N. Manashov (Regensburg U. and Hamburg U., Inst. Theor. Phys. II and Steklov Math. Inst., St. Petersburg) (May 11, 2019)

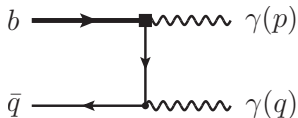
Published in: *Phys.Rev.D* 100 (2019) 1, 014023 • e-Print: [1905.04498](#) [hep-ph]

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QCD factorization beyond leading power

- The NLP factorization from the hard-collinear propagator:



Heavy quark expansion (the soft quark momentum k):

$$\frac{i(\not{q} - \not{k})}{(q-k)^2} = \frac{i\not{q}}{(q-k)^2} - \underbrace{\frac{i\not{k}}{(q-k)^2}}_{\text{power suppressed}}.$$

power suppressed

- The yielding NLP amplitude:

$$T_{7,\alpha\beta}^{\text{hc,NLP}} = 2e_q \bar{m}_b(v) m_{B_q} \int d^4x \int \frac{d^4\ell}{(2\pi)^4} \exp[i(q-\ell) \cdot x] \frac{1}{\ell^2 + i0} \\ \times \langle 0 | \bar{q}(x) \gamma_\beta^\perp (\not{q} - \not{\ell}) \not{n} \gamma_\alpha^\perp P_R h_v(0) | \bar{B}_q \rangle.$$

- Light-cone decomposition of the four-momentum $(\not{q} - \not{\ell})$:

$$\not{q} - \not{\ell} = \underbrace{\frac{n \cdot (q - \ell)}{2} \not{n}}_{\text{vanishes obviously}} + \underbrace{\frac{\bar{n} \cdot (q - \ell)}{2} \not{\bar{n}}}_{\text{the only relevant term}} + (\not{q}_\perp - \not{\ell}_\perp).$$

vanishes obviously the only relevant term

The third term does not contribute to the matrix element (hint: symmetry with $\alpha \leftrightarrow \beta$).

- Applying the IBP reduction leads to

$$T_{7,\alpha\beta}^{\text{hc,NLP}} = [ie_q \bar{m}_b(v) m_{B_q}] \int d^4x \int \frac{d^4\ell}{(2\pi)^4} \exp[i(q-\ell) \cdot x] \frac{\bar{n}_\mu}{\ell^2 + i0} \\ \times \frac{\partial}{\partial x_\mu} \langle 0 | \bar{q}(x) \gamma_\beta^\perp \not{n} \not{\bar{n}} \gamma_\alpha^\perp P_R h_v(0) | \bar{B}_q \rangle.$$

QCD factorization beyond leading power

- Performing the integration over ℓ leads to (note that $q_\mu = (\bar{n} \cdot q/2) n_\mu$)

$$T_{7,\alpha\beta}^{\text{hc,NLP}} = \left[-\frac{e_q \bar{m}_b(v) m_{B_q}}{4\pi^2} \right] \int d^4x \frac{e^{iq \cdot x}}{x^2} (2v_\mu - n_\mu) \times \frac{\partial}{\partial x_\mu} \langle 0 | \bar{q}(x) \gamma_\beta^\perp \not{n} \gamma_\alpha^\perp P_R h_v(0) | \bar{B}_q \rangle.$$

The second term $n_\mu (\partial/\partial x_\mu) [\dots] = 2 (\partial/\partial \bar{n} \cdot x) [\dots]$ can be computed directly.

- The equation of motion for the non-local operator [Kawamura, Kodaira, Qiao, Tanaka, 2001]:

$$v_\mu \frac{\partial}{\partial x_\mu} [\bar{q}(x) \Gamma h_v(0)] = i \int_0^1 du \bar{u} \bar{q}(x) g_s G_{\alpha\beta}(ux) x^\alpha v^\beta \Gamma h_v(0) + (v \cdot \partial) [\bar{q}(x) \Gamma h_v(0)].$$

- Translation invariance of the HQET matrix element (note the total translation operator ∂_α)

$$(v \cdot \partial) \langle 0 | [\bar{q}(x) \Gamma h_v(0)] | \bar{B}_q(v) \rangle = -i \bar{\Lambda} \langle 0 | [\bar{q}(x) \Gamma h_v(0)] | \bar{B}_q(v) \rangle.$$

- The yielding NLP factorization formula (preserving the large-recoil symmetry)

$$T_{7,\alpha\beta}^{\text{hc,NLP}} = \left[-2i e_q \bar{m}_b(v) f_{B_q} m_{B_q} \right] \left[g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right] \left\{ \frac{1}{2} - \left(\frac{\bar{\Lambda}}{\lambda_{B_q}} \right) + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \frac{1}{\omega_2} \left[\frac{1}{\omega_2} \ln \frac{\omega_1}{\omega_1 + \omega_2} + \frac{1}{\omega_1} \right] \underbrace{\Psi_4(\omega_1, \omega_2, \mu)}_{\propto \omega_1 \omega_2 \text{ at small } \omega_i} \right\}.$$

QCD factorization beyond leading power

- The NLP correction from the non-vanishing quark mass:

$$T_{7,\alpha\beta}^{m_q,\text{NLP}} = [-i e_q \bar{m}_b(v) m_q f_{B_q} m_{B_q}] \left[g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right] \int_0^\infty d\omega \frac{\phi_B^-(\omega, \mu)}{\omega}.$$

The asymptotic behaviour of $\phi_B^-(\omega, \mu_0) \rightarrow \text{constant}$ at small ω .

- Rapidity divergence implies the **breakdown of the naive soft-collinear factorization**.
- Nonperturbative parametrization of the convolution integral [BBNS, 2001]:

$$\begin{aligned} \int_0^\infty d\omega \frac{\phi_B^-(\omega, \mu)}{\omega} &= \left[\int_0^{\Lambda_{\text{UV}}} + \int_{\Lambda_{\text{UV}}}^\infty \right] d\omega \frac{\phi_B^-(\omega, \mu)}{\omega} \\ &= \left[\phi_B^-(0, \mu) X_{\text{NLP}} + \int_0^{\Lambda_{\text{UV}}} d\omega \frac{\phi_B^-(\omega, \mu) - \phi_B^-(0, \mu)}{\omega} \right] + \int_{\Lambda_{\text{UV}}}^\infty d\omega \frac{\phi_B^-(\omega, \mu)}{\omega}. \end{aligned}$$

↓

UV and IR Finite!

$$X_{\text{NLP}} = [1 + \rho_S \exp(i\varphi_S)] \ln \left(\frac{\Lambda_{\text{UV}}}{\Lambda_h} \right).$$

- ▶ Alternative estimate from the LCSR method also possible [Li², Lü, Shen, 2022].
- ▶ The complete SCET factorization in demand but rather challenging.
Refactorization-based subtraction scheme? [Beneke, Neubert, Feldmann, Liu, Wang², ...]

QCD factorization beyond leading power

- The NLP contribution from the subleading SCET current [Beneke, Feldmann, 2002]

$$J^{(A2)} \supset (\bar{\xi}_{\text{hc}} W_{\text{hc}}) \gamma_{\alpha}^{\perp} P_L \left(\frac{i \vec{D}_{\perp}}{2m_b} \right) h_v + \dots, \quad D_{\perp}^{\mu} \equiv D^{\mu} - (v \cdot D) v^{\mu}.$$

Arise from the HQET representation of the QCD b -quark field.

- The resulting non-local hadronic matrix element

$$T_{7,\alpha\beta}^{A2,\text{NLP}} = \left[-\frac{ie_q m_{B_q}^2}{2} \right] \int d^4x \int \frac{d^4\ell}{(2\pi)^4} \exp[i(q-\ell) \cdot x] \frac{1}{\ell^2 + i0} \\ \times \langle 0 | \bar{q}(x) \gamma_{\beta}^{\perp} \not{n} \gamma_{\alpha}^{\perp} \vec{D} P_L h_v(0) | \bar{B}_q \rangle.$$

⇒ Applying the HQET equation of motion at the the classical level

$$\bar{q}(x) \Gamma \vec{D}_{\rho} h_v(0) = \partial_{\rho} [\bar{q}(x) \Gamma h_v(0)] + i \int_0^1 du \bar{u} \bar{q}(x) g_s G_{\lambda\rho}(ux) x^{\lambda} \Gamma h_v(0) - \frac{\partial}{\partial x^{\rho}} \bar{q}(x) \Gamma h_v(0),$$

leads to the soft-collinear factorization formula

$$T_{7,\alpha\beta}^{A2,\text{NLP}} = \left[ie_q f_{B_q} m_{B_q}^2 \right] \left[g_{\alpha\beta}^{\perp} - i \varepsilon_{\alpha\beta}^{\perp} \right] \left\{ \frac{1}{2} \left(\frac{\bar{\Lambda}}{\lambda_{B_q}} \right) - 1 \int_0^{\infty} d\omega_1 \int_0^{\infty} d\omega_2 \frac{\Phi_3(\omega_1, \omega_2, \mu)}{\omega_1(\omega_1 + \omega_2)} \right\}.$$

- ▶ The convolution integral **free of the end-point divergence** (because $\Phi_3 \sim \omega_1 \omega_2^2$ at small ω_i).
- ▶ In agreement with the counterpart contribution to $B \rightarrow \gamma \ell \nu$ [Beneke, Braun, Ji, Wei, 2018].

QCD factorization beyond leading power

- The **two-particle higher-twist** B -meson LCDAs up to the $\mathcal{O}(x^2)$ accuracy:

$$\begin{aligned} & \langle 0 | (\bar{q}_s Y_s)_\beta(x) (Y_s^\dagger h_\nu)_\alpha(0) | \bar{B}_q \rangle \\ &= -\frac{i\tilde{f}_{B_q}(\mu) m_{B_q}}{4} \int_0^\infty d\omega e^{-i\omega v \cdot x} \left[\frac{1+\not{v}}{2} \left\{ 2 \left[\phi_B^+(\omega, \mu) + x^2 g_B^+(\omega, \mu) \right] \right. \right. \\ & \quad \left. \left. - \frac{1}{v \cdot x} \left[(\phi_B^+(\omega, \mu) - \phi_B^-(\omega, \mu)) + x^2 (g_B^+(\omega, \mu) - g_B^-(\omega, \mu)) \right] \not{x} \right\} \gamma_5 \right]_{\alpha\beta}. \end{aligned}$$

- The **three-particle higher-twist** B -meson LCDAs up to twist-6 [Braun, Ji, Manashov, 2017].

- ▶ The EOM relation between the two-particle and three-particle LCDAs.

$$\begin{aligned} -2 \frac{d^2}{d\omega^2} g_B^-(\omega, \mu) &= \left[\frac{3}{2} + (\omega - \bar{\Lambda}) \frac{d}{d\omega} \right] \phi_B^-(\omega, \mu) - \frac{1}{2} \phi_B^+(\omega, \mu) \\ &+ \int_0^\infty \frac{d\omega_2}{\omega_2} \left[\frac{d}{d\omega} - \frac{1}{\omega_2} \right] \Psi_5(\omega, \omega_2, \mu) + \int_0^\omega \frac{d\omega_2}{\omega_2^2} \Psi_5(\omega - \omega_2, \omega_2, \mu). \end{aligned}$$

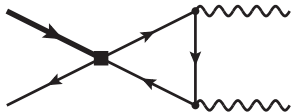
- ▶ The systematic parametrization requires **8 independent LCDAs**.

- The higher-twist factorization formula at tree level:

$$\begin{aligned} T_{7,\alpha\beta}^{\text{HT,NLP}} &\simeq [-i e_q \bar{m}_b(v) f_{B_q} m_{B_q}] \left[g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right] \left\{ -1 + 2 \int_0^\infty d\omega \ln \omega \underbrace{\Delta \phi_B^-(\omega, \mu)}_{\equiv \left[-\omega \frac{d}{d\omega} \phi_B^-(\omega, \mu) - \phi_B^+(\omega, \mu) \right]} \right. \\ &\quad \left. - 2 \int_0^\infty d\omega_2 \frac{1}{\omega_2} \Phi_4(\omega_1=0, \omega_2, \mu) \right\}. \end{aligned}$$

QCD factorization beyond leading power

- The NLP contribution from the **weak annihilation diagram**:



The resulting helicity form factors:

$$\sum_{i=1}^6 C_i F_{i,L}^{(p),\text{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathcal{F}_V^{(p),\text{WA}} - \mathcal{F}_A^{(p),\text{WA}} \right],$$

$$\sum_{i=1}^6 C_i F_{i,R}^{(p),\text{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathcal{F}_V^{(p),\text{WA}} + \mathcal{F}_A^{(p),\text{WA}} \right].$$

- ▶ The weak-annihilation effect will **spoil the large-recoil symmetry** (non-vanishing $F_{i,R}^{(p)}$).
- ▶ **Massive quark loops generate the non-trivial strong phase.**
 $\Leftrightarrow$ Dual to the final-state rescattering $\bar{B}_q \rightarrow H_c H'_c \rightarrow \gamma\gamma$ at hadronic level.
- ▶ Tree-operator contributions consistent with [Bosch, Buchalla, 2002].

- The NLP contribution from the (anti)-collinear photon radiation off **the bottom-quark**:

$$T_{7,\alpha\beta}^{e_b,\text{NLP}} = \left[-i e_q f_{B_q} m_{B_q}^2 \right] \left[g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right].$$

- ▶ Local correction preserves the large-recoil symmetry!
- ▶ In analogy to the P_7 contribution to $B_q \rightarrow \gamma \ell \bar{\ell}$ with the B -type insertion [Beneke, Bobeth, YMW, 2020].

The resolved photon contribution

- The NLP contribution from the “**hadronic**” component of the on-shell photon state [Ball, Braun, Kivel, 2003].
- The dispersion technique [Khodjamirian, 1999; Braun, Khodjamirian, 2013; YMW, 2016]:

$$\begin{aligned}
 \tilde{T}_{7,\alpha\beta} &= 2\bar{m}_b(v) \int d^4x e^{iq \cdot x} \langle 0 | T \left\{ j_\beta^{\text{em}}(x), \bar{q}_L(0) \sigma_{\mu\alpha} p^\mu b_R(0) \right\} | \bar{B}_q(p+q) \rangle \Big|_{q^2 < 0} \\
 &\quad + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \\
 &= -\frac{ie_q \bar{m}_b(v) m_{B_q}^2}{2} \left\{ \left(g_{\alpha\beta}^\perp - i\varepsilon_{\alpha\beta}^\perp \right) \tilde{F}_{7,L}(\bar{n} \cdot q, \mathbf{n} \cdot \mathbf{q}) + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \right\}.
 \end{aligned}$$

Power counting scheme for the four-momentum q :

$$\bar{n} \cdot q \sim \mathcal{O}(m_b), \quad \mathbf{n} \cdot \mathbf{q} \sim \mathcal{O}(\Lambda).$$

- The hadronic dispersion relation:

$$\begin{aligned}
 \tilde{T}_{7,\alpha\beta} &= -i\bar{m}_b(v) m_{B_q}^2 \left(g_{\alpha\beta}^\perp - i\varepsilon_{\alpha\beta}^\perp \right) \left\{ \sum_V \frac{c_V f_V m_V T_1^{B_q \rightarrow V}(0)}{\bar{n} \cdot q (m_V^2/\bar{n} \cdot q - n \cdot q - i0)} \right. \\
 &\quad \left. + \int_{\omega_s}^\infty d\omega' \frac{\rho^{\text{had}}(\bar{n} \cdot q, \omega')}{\omega' - n \cdot q - i0} \right\}.
 \end{aligned}$$

The constant c_V determined by the flavour factor and the electric charge of the QED quark-current.

- LCSR for the tensor $B \rightarrow V$ form factors:

$$\sum_V \frac{c_V f_V m_V}{\bar{n} \cdot q} \exp \left[-\frac{m_V^2}{\bar{n} \cdot q \omega_M} \right] T_1^{B_q \rightarrow V}(0) = \frac{e_q}{2} \frac{1}{\pi} \int_0^{\omega_s} d\omega' e^{-\omega'/\omega_M} \text{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n} \cdot q, \omega').$$

The resolved photon contribution

- Improved dispersion relations (setting $n \cdot q = 0$) [Master formula]:

$$T_{7,\alpha\beta} = -\frac{ie_q \bar{m}_b(v) m_{B_q}^2}{2} \left(g_{\alpha\beta}^\perp - i \varepsilon_{\alpha\beta}^\perp \right) \underbrace{\left\{ \frac{1}{\pi} \int_0^\infty \frac{d\omega'}{\omega'} \text{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n} \cdot q, \omega') \right\}}_{\text{LP factorization formulae}} \\ + \underbrace{\frac{1}{\pi} \int_0^{\omega_s} d\omega' \left[\frac{\bar{n} \cdot q}{m_V^2} \exp\left(\frac{m_V^2 - \bar{n} \cdot q \omega'}{\bar{n} \cdot q \omega_M}\right) - \frac{1}{\omega'} \right] \text{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n} \cdot q, \omega')}_{\text{nonperturbative modification}} \} + [p \leftrightarrow q, \alpha \leftrightarrow \beta].$$

- Power counting scheme for the sum-rule parameters:

$$\omega_s = \frac{s_0}{\bar{n} \cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_{B_q}}\right), \quad \omega_M = \frac{M^2}{\bar{n} \cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{m_{B_q}}\right).$$

$\Rightarrow$ **Nonperturbative modification** yields the soft non-factorizable contribution.

- Spectral density at tree level:

$$\frac{1}{\pi} \text{Im}_{\omega'} \tilde{F}_{7,L}(\bar{n} \cdot q, \omega') = f_{B_q} \underbrace{\phi_B^+(\omega', \mu)}_{\text{of } \mathcal{O}(1/\Lambda)} + \mathcal{O}(\alpha_s, \Lambda/m_b). \\ \text{of } \mathcal{O}(1/\Lambda) [\mathcal{O}(1/m_b)] \text{ for } \omega' \sim \mathcal{O}(\Lambda) [\omega' \sim \mathcal{O}(\Lambda^2/m_b)]$$

Power suppressed soft contribution!

- Alternative LCSR calculation with the subleading-twist photon LCDAs [Shen, YMW, 2018].

Summary for the helicity amplitudes of $B_q \rightarrow \gamma\gamma$

- Final expressions for the factorized NLP corrections:

$$\begin{aligned} \sum_{i=1}^8 C_i F_{i,L}^{(p), \text{fac}, \text{NLP}} &= \textcolor{red}{C}_7^{\text{eff}} \left[F_{7,L}^{\text{hc}, \text{NLP}} + F_{7,L}^{m_q, \text{NLP}} + F_{7,L}^{A2, \text{NLP}} + F_{7,L}^{\text{HT}, \text{NLP}} + F_{7,L}^{e_b, \text{NLP}} \right] \\ &\quad + \frac{f_{B_q}}{m_{B_q}} \left[\mathcal{F}_V^{(p), \text{WA}} - \mathcal{F}_A^{(p), \text{WA}} \right], \\ \sum_{i=1}^8 C_i F_{i,R}^{(p), \text{fac}, \text{NLP}} &= \frac{f_{B_q}}{m_{B_q}} \left[\mathcal{F}_V^{(p), \text{WA}} + \mathcal{F}_A^{(p), \text{WA}} \right]. \end{aligned}$$

Large-recoil symmetry violation from the weak annihilation correction completely.

- Final expressions for the two helicity amplitudes:

$$\begin{aligned} \tilde{\mathcal{A}}_L &= \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^8 C_i \left[F_{i,L}^{(p), \text{LP}} + F_{i,L}^{(p), \text{fac}, \text{NLP}} + F_{i,L}^{(p), \text{soft}, \text{NLP}} \right], \\ \tilde{\mathcal{A}}_R &= \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^8 C_i \left[F_{i,R}^{(p), \text{LP}} + F_{i,R}^{(p), \text{fac}, \text{NLP}} + F_{i,R}^{(p), \text{soft}, \text{NLP}} \right]. \end{aligned}$$

- The fundamental nonperturbative functions: **HQET B -meson LCDAs**.

- Key hadronic inputs for exclusive B -meson decay phenomenologies, for instance $B \rightarrow \pi \ell \nu$, $B \rightarrow D^{(*)} \ell \nu$, $B \rightarrow K^* \gamma$, $B \rightarrow K^{(*)} \ell \ell$, $B \rightarrow \pi \pi$ [Lü, Shen, Wang, YMW, 2023].

B -meson distribution amplitudes in HQET

- The LP contribution depends on λ_{B_q} and the inverse-logarithmic moments.
- Applying the general ansatz of the 2- and 3- particle B -meson LCDAs

$$\phi_B^+(\omega, \mu) = \omega f(\omega), \quad \Phi_3(\omega_1, \omega_2, \mu_0) = -\frac{1}{2} \kappa(\mu_0) \left[\lambda_E^2(\mu_0) - \lambda_H^2(\mu_0) \right] \omega_1 \omega_2^2 f'(\omega_1 + \omega_2),$$

$$\Phi_4(\omega_1, \omega_2, \mu_0) = \frac{1}{2} \kappa(\mu_0) \left[\lambda_E^2(\mu_0) + \lambda_H^2(\mu_0) \right] \omega_2^2 f(\omega_1 + \omega_2),$$

$$\Psi_4(\omega_1, \omega_2, \mu_0) = \kappa(\mu_0) \lambda_E^2(\mu_0) \omega_1 \omega_2 f(\omega_1 + \omega_2).$$

The factorized NLP corrections can then be parameterized by the local HQET parameters.

- The NLP soft contribution sensitive to the precise shape of $\phi_B^+(\omega, \mu)$.

$$\phi_B^+(\omega, \mu_0) = \int_0^\infty ds \sqrt{ws} J_1(2\sqrt{ws}) \eta_+(s, \mu_0), \quad \eta_+(s, \mu_0) = {}_1F_1(\alpha; \beta; -s\omega_0).$$

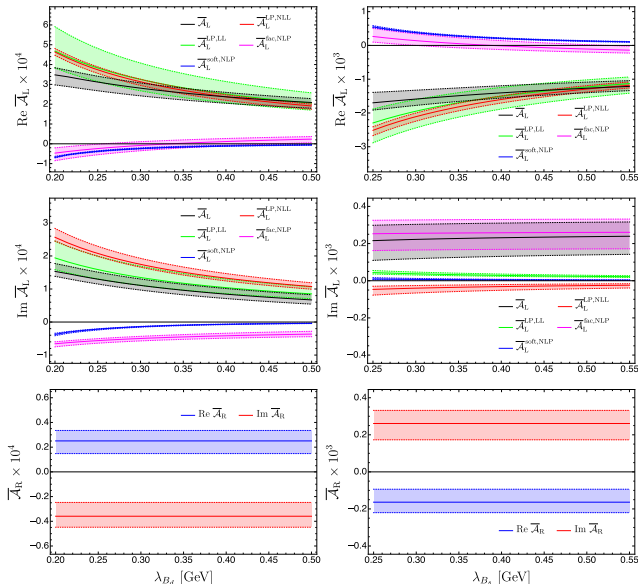
- ▶ Such three-parameter ansatz [Beneke, Braun, Ji, Wei, 2018] is advantageous, since the resulting RG evolution can be done **analytically in terms of ${}_2F_2$ functions**.
- ▶ An alternative parametrization of the twist-two B -meson DA in **Laplace space** [Galda, Neubert, Wang, 2022]:

$$\tilde{\phi}_B^+(\eta, \mu) = \int_0^\infty \frac{d\omega}{\omega} \left(\frac{\omega}{\bar{\omega}} \right)^\eta \phi_B^+(\omega, \mu) \stackrel{|\eta| \ll 1}{=} \frac{1}{\lambda_B(\mu)} \left[1 + \sum_{n \geq 1} \frac{\eta^n}{n!} \sigma_n^B(\mu) \right].$$

- ▶ New parametrization of the momentum-space DA in terms of **associated Laguerre polynomials** [Feldmann, Lügghausen, van Dyk, 2022; Feldmann, Lügghausen, Seitz, 2023].

Theory predictions for the helicity amplitudes

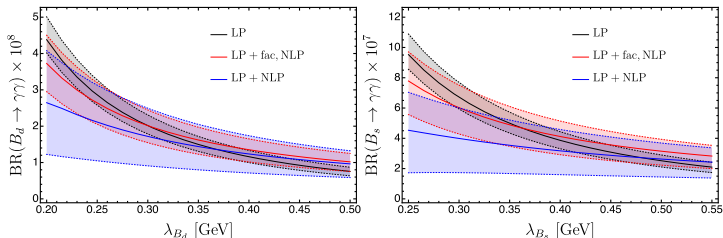
● Breakdown of the various QCD mechanisms:



- NLL effects stabilize the factorization-scale dependence.
- Factorizable NLP effects around $\mathcal{O}(30\%)$.
- Destructive effects from the NLP soft corrections.
- Strong phase from the 2-loop matrix element of P_2^c and the weak annihilation.

Phenomenological observables for $B_q \rightarrow \gamma\gamma$

- Time-integrated branching fraction (for the flavour-tagged measurement):



- The yielding theory predictions

$$\mathcal{BR}(B_d \rightarrow \gamma\gamma) = \left(1.44_{-0.80}^{+1.35}\right) \times 10^{-8}, \quad \mathcal{BR}(B_s \rightarrow \gamma\gamma) = \left(3.17_{-1.74}^{+1.96}\right) \times 10^{-7}.$$

with the dominant uncertainties from λ_{B_q} , $\hat{\sigma}_{B_q}^{(1)}$, $\hat{\sigma}_{B_q}^{(2)}$ and the QCD renormalization scale v .

- Both the factorizable and soft NLP corrections are numerically important.
- The ratio of the two branching ratios for $B_{d,s} \rightarrow \gamma\gamma$

$$\frac{\mathcal{BR}(B_s \rightarrow \gamma\gamma)}{\mathcal{BR}(B_d \rightarrow \gamma\gamma)} = 33.80 \left(\frac{\lambda_{B_d}}{\lambda_{B_s}}\right)^2 + \mathcal{O}\left(\frac{\Lambda}{m_b}, \alpha_s\right).$$

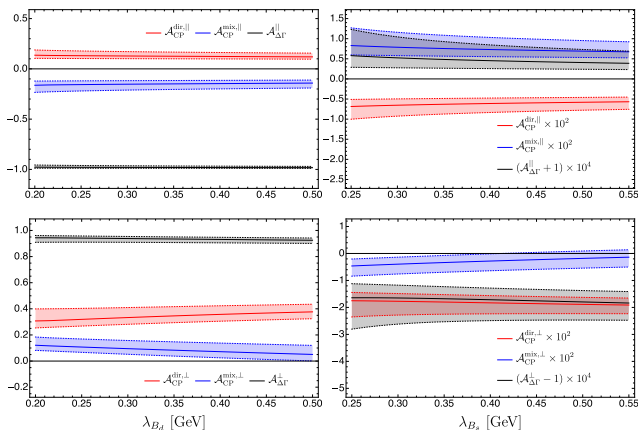
The λ_{B_q} -scaling violation effect due to the NLP contributions approximately (10 – 20) %.

- The ratio $\lambda_{B_s} : \lambda_{B_d} = 1.19 \pm 0.14$ from QCDSR [Khodjamirian, Mandal, Mannel, 2020].

Phenomenological observables for $B_q \rightarrow \gamma\gamma$

- Time-dependent CP asymmetries:

$$A_{\text{CP}}^{\chi}(t) = \frac{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \rightarrow \gamma\gamma) - \Gamma^{\chi}(B_q(t) \rightarrow \gamma\gamma)}{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \rightarrow \gamma\gamma) + \Gamma^{\chi}(B_q(t) \rightarrow \gamma\gamma)} = - \frac{\mathcal{A}_{\text{CP}}^{\text{dir},\chi} \cos(\Delta m_q t) + \mathcal{A}_{\text{CP}}^{\text{mix},\chi} \sin(\Delta m_q t)}{\cosh(\Delta\Gamma_q t/2) + \mathcal{A}_{\Delta\Gamma}^{\chi} \sinh(\Delta\Gamma_q t/2)}.$$

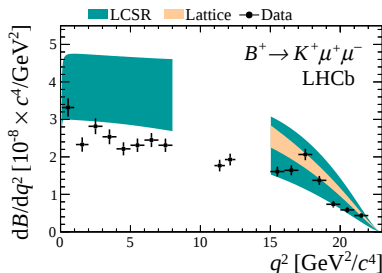
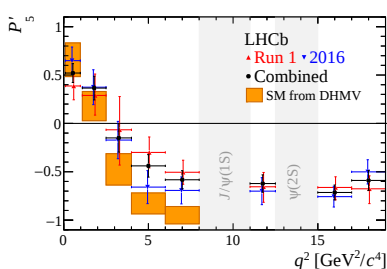


- $|\mathcal{A}_{\text{CP}}^{\text{dir},\chi}|$ and $|\mathcal{A}_{\text{CP}}^{\text{mix},\chi}|$ around $(10 - 40)\%$ for $B_d \rightarrow \gamma\gamma$.
- Tiny CP asymmetries for $B_s \rightarrow \gamma\gamma$ as expected.
- The difference between $\mathcal{A}_{\text{CP}}^{\text{dir},\parallel}$ and $\mathcal{A}_{\text{CP}}^{\text{dir},\perp}$ due to the NLP corrections.

Exclusive $b \rightarrow s \ell \bar{\ell}$ anomalies

- Several LHCb measurements deviate from the Standard Model (SM) predictions by $2 - 3 \sigma$.

- Angular observables in $B \rightarrow K^* \mu^+ \mu^-$ [JHEP 02 (2016) 104, 1005 citations].
- Branching fractions of $B \rightarrow K \mu^+ \mu^-$, $B \rightarrow K^* \mu^+ \mu^-$, and $B_s \rightarrow \phi \mu^+ \mu^-$.



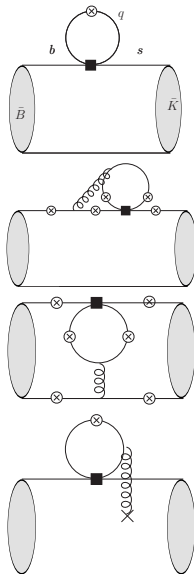
- Better understanding of the local and non-local hadronic effects in high demand.

A longstanding charming-penguin story

● (Non)-factorizable quark loops in $B \rightarrow K\ell\ell$:

- ▶ OPE works far below the **charm-quark** threshold!
- ▶ QCD factorization valid for $1 \text{ GeV}^2 < q^2 \ll 4m_c^2$.
- ▶ Factorizable charm loops reduced to $B \rightarrow K$ form factors.
- ▶ Charm-loop induced spectator effects calculable in QCD factorization [Beneke, Feldmann, Seidel, 2001, 2005].
- ▶ Soft charm loop computed in the OPE controlled approach with the dispersion relation [Khodjamirian, Mannel, Pivovarov, YMW, 2010].
- ▶ Parametrization of the hadronic-operator effects:

$$\Delta C_9^{(BK)}(q^2) = \frac{16\pi^2 \mathcal{H}^{(BK)}(q^2)}{f_{BK}^+(q^2)}.$$



A longstanding charming-penguin story

Forward backward asymmetry of dilepton angular distribution in the decay $b \rightarrow s l^+ l^-$

Ahmed Ali (DESY), T. Mannel (DESY), T. Morozumi (Rockefeller U.) (Sep, 1991)

Published in: *Phys.Lett.B* 273 (1991) 505-512

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 reference search  370 citations

Lepton polarization in the decays $b \rightarrow X(s) \mu^+ \mu^-$ and $B \rightarrow X(s) \tau^+ \tau^-$

F. Kruger (Aachen, Tech. Hochschule.), L.M. Sehgal (Aachen, Tech. Hochschule.) (Mar, 1996)

Published in: *Phys.Lett.B* 380 (1996) 199-204 • e-Print: [hep-ph/9603237](#) [hep-ph]

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 reference search  305 citations

Charm-loop effect in $B \rightarrow K^{(*)} \ell^+ \ell^-$ and $B \rightarrow K^* \gamma$

A. Khodjamirian (Siegen U.), Th. Mannel (Siegen U.), A.A. Pivovarov (Siegen U.), Y.-M. Wang (Siegen U.) (Jun, 2010)

Published in: *JHEP* 09 (2010) 089 • e-Print: [1006.4945](#) [hep-ph]

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 reference search  534 citations

$B \rightarrow K \ell^+ \ell^-$ decay at large hadronic recoil

A. Khodjamirian (Siegen U.), Th. Mannel (Siegen U.), Y.M. Wang (Siegen U.) (Nov, 2012)

Published in: *JHEP* 02 (2013) 010 • e-Print: [1211.0234](#) [hep-ph]

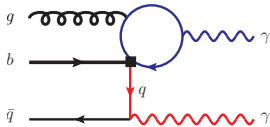
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 reference search  220 citations

Charming-penguin effect in $B_q \rightarrow \gamma\gamma$

- Soft-gluon radiation off the factorizable quark loop [Qin, Shen, Wang, YMW, 2023]:

The resulting helicity form factors:



$$\sum_{i=1}^8 C_i F_{i,L}^{(p), \text{soft}4q} = -\frac{Q_q f_{B_q}}{m_{B_q}} \int_{-\infty}^{+\infty} \frac{d\omega_1}{\omega_1 - i0} \int_{-\infty}^{+\infty} \frac{d\omega_2}{\omega_2 - i0} \left\{ \left(C_2 - \frac{C_1}{2N_c} \right) Q_p [F(z_p) - 1] + 6 C_6 Q_c [F(z_c) - 1] - \left[\left(C_3 - \frac{C_4}{2N_c} \right) + 16 \left(C_5 - \frac{C_6}{2N_c} \right) \right] Q_q \right\} \Phi_G(\omega_1, \omega_2, \mu).$$

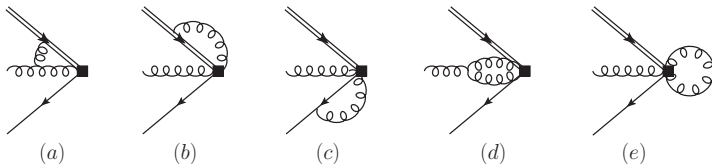
- Adopt the favored power-counting scheme $m_b \gg m_c \sim \mathcal{O}(\sqrt{\Lambda_{\text{QCD}} m_b}) \gg \Lambda_{\text{QCD}}$.
- The long-distance penguin contribution is indeed **power-suppressed**.
- The subleading distribution amplitude Φ_G defined by the HQET matrix element of the non-local operator with quark-gluon fields **localized on distinct light-cone directions**:

$$\begin{aligned} & \langle 0 | (\bar{q}_s S_n) (\tau_1 n) (S_n^\dagger S_{\bar{n}})(0) (S_{\bar{n}}^\dagger g_s G_{\mu\nu} S_{\bar{n}}) (\tau_2 \bar{n}) \bar{n}^\nu \not{n} \gamma_\perp^\mu \gamma_5 (S_{\bar{n}}^\dagger h_\nu)(0) | \bar{B}_\nu \rangle \\ &= 2 \tilde{f}_B(\mu) m_B \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \exp[-i(\omega_1 \tau_1 + \omega_2 \tau_2)] \Phi_G(\omega_1, \omega_2, \mu). \end{aligned}$$

$\Rightarrow$ **Non-trivial RG evolution** of this soft function in analogy to the QED-generalized bottom-meson distribution amplitude [Beneke, Böer, Toelstede, Vos, 2022].

RG evolution of the B -meson soft function

- Sample Feynman diagrams at one loop [Huang, Ji, Shen, Wang, YMW, 2024]:

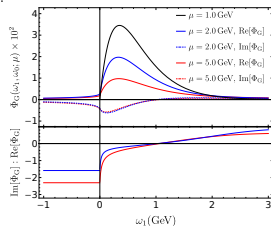


- Novel feature of the UV renormalization due to the diagram (e):

- Generate the (unexpected?) mixing from $\omega'_i > 0$ to $\omega_i < 0$, independent of the initial condition of the soft function Φ_G .
- The Wilson-line structure differs from the one in the QED-generalized B -meson soft function, on account of the absence of the semi-infinite Wilson lines.

- An exact solution to the RG equation can be derived with the Laplace transform technique.

Numerics for the RG evolution:



The **constant phase** of the RG-evolved soft function with the default initial condition.

$B_q \rightarrow \gamma\gamma$ versus $B_q \rightarrow V\ell\ell$

	$B_q \rightarrow V\ell\ell$	$B_q \rightarrow \gamma\gamma$
Factorization properties	Incomplete @ LP (FFs, WA) [see PRL 134 (2025) 091901 for WA]	Complete @ LP
Perturbative corrections	Incomplete @ NLO [penguin amplitude, WA]	Complete @ NLL
Nonperturbative inputs	LCDAs of B_q - and vector-mesons @ LP $B_q \rightarrow V$ form factors	B_q -meson LCDA @ LP
NLP improvement	Less studied (LCSR)	better studied (QCDF, LCSR)
NNLO improvement	Rather difficult (?)	In active progress (Misiak et al)
QCD uncertainties	Strong dynamical cancellation @ LP [photon-pole enhancement]	No strong cancellation @ LP

NLO Weak Annihilation Correction to $B \rightarrow \{K, \pi\} \ell^+ \ell^-$



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Next-to-Leading-Order Weak-Annihilation Correction to Rare $B \rightarrow \{K, \pi\} \ell^+ \ell^-$ Decays

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We accomplish for the first time the next-to-leading-order computation of the weak annihilation contribution to the electroweak penguin decays $B \rightarrow \{K, \pi\} \ell^+ \ell^-$, which is an essential missing piece of the complete QCD correction to the matrix elements of hadronic operators in the weak effective Hamiltonian. Both the hard functions and the jet functions in the perturbative factorization formulas from the short-distance fluctuations at the two distinct scales m_b and $\sqrt{m_b \Lambda}$ are determined at $\mathcal{O}(\alpha_s)$. We then demonstrate that the one-loop weak annihilation correction can bring about the noticeable impacts on theory predictions for the CP asymmetries and the isospin asymmetry in the $B \rightarrow \pi^+ \pi^- \ell^+ \ell^-$ decays.

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Systematic approach to exclusive $B \rightarrow V \ell^+ \ell^-$, $V \gamma$ decays

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Abstract

We show — by explicit computation of first-order corrections — that the QCD factorization approach previously applied to hadronic two-body decays and to form factor ratios also allows us to compute non-factorizable corrections to exclusive, radiative B meson decays in the heavy quark mass limit. This removes a major part of the theoretical uncertainty in the region of small invariant mass of the photon. We discuss in particular the decays $B \rightarrow K^* \gamma$ and $B \rightarrow K^{*0} \gamma$, and complete the calculation of corrections to the forward-backward asymmetry zero. The new correction shifts the asymmetry zero by 30%, but the result confirms our previous conclusion that the asymmetry zero provides a clean phenomenological determination of the Wilson coefficient C_9 . © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Radiative B meson decays are interesting because they proceed entirely through loop effects. The chiral nature of weak interactions implies additional suppression factors thus enhancing the sensitivity of radiative decays to virtual effects in theories beyond the standard theory of flavour violation, if these new effects violate chirality. The observation of $b \rightarrow s \gamma$ transitions [1] with a branching ratio of about 10^{-4} (in agreement with the standard theory) has since provided significant constraints on extensions of the standard model. The semileptonic decay $b \rightarrow s \ell^+ \ell^-$ is expected to occur with a branching fraction of a few times 10^{-6} and will probably be first observed in the near future.

Radiative flavour-changing neutral current transitions can be measured inclusively over the hadronic final state (containing strangeness in the particular case considered here) or exclusively by tagging a particular light hadron, typically a kaon. The inclusive

Introduction—It is widely recognized that the exclusive electroweak penguin B -meson decays are of extraordinary importance to explore the intricate dynamics of the quark and lepton sectors in the standard model (SM) and to probe sensitively the nonstandard effects with clear experimental signatures. The prominent flavor anomalies observed in the semileptonic $b \rightarrow s \ell^+ \ell^-$ decays include the $\mathcal{O}(\alpha_s)$ deviations of the measured $R(B^* \rightarrow K^* \mu^+ \mu^-)$ in the low- q^2 bins with respect to the SM predictions and the $\mathcal{O}(2\sigma)$ tensions between the LHCb measurements of the angular observable $P_1(B^* \rightarrow K^* \mu^+ \mu^-)$ and the state-of-the-art SM computations in the two anomalous bins (see [1] for an overview). In addition, the first LHCb measurement of the integrated isospin asymmetry $A_1(B \rightarrow K^* \mu^+ \mu^-)$ in the entire q^2 spectrum deviates from zero at the level of 4σ [2], while the updated LHCb measurement of this asymmetry favors the lower significance of 1.5σ [3]. Precision measurements of the isospin asymmetries in the exclusive $b \rightarrow \{d, s\} \ell^+ \ell^-$ decays will be highly beneficial for constraining Wilson coefficients of the $\Delta B = 1$ four-quark operators and for unveiling the ultimate nature of the spectator-scattering mechanism in hard exclusive

reactions [4–8] (see also [9,10]). As a consequence, the in-depth exploration of the flavor structure of the weak effective theory from the exclusive electroweak penguin B -meson decay constitutes one of the primary physics goals of the LHCb and Belle II experiments, thus triggering the intense theoretical activities toward controlling better hadronic uncertainties of the SM predictions (see for instance [11–20]).

The field-theoretical framework to systematically address the hadronic matrix elements of the electroweak penguin B -meson decays at large recoil has been constructed in the heavy quark mass limit with the perturbative factorization formalism [21,22] by Beneke, Feldmann and Seidel (BFS) [see also [23,24] for an equivalent formulation in soft-collinear effective theory (SCET)]. In particular, both the weak annihilation and spectator-scattering topologies due to the hard-collinear photon radiation off the spectator quark yield vanishing contributions to the transverse amplitude of $B \rightarrow V \ell^+ \ell^-$ at leading power in the Λ/m_b expansion [21,25]. This interesting observation motivates further factorization analysis of the particular subleading power corrections to the transverse amplitude dependent of the electric charge of the spectator quark [4,9,22]. However, the factorization-scale invariance of both the transverse and longitudinal amplitudes of $B \rightarrow V \ell^+ \ell^-$ was not completely achieved at the next-to-leading order (NLO) in QCD even in the leading power approximation, owing to the unknown two-loop matrix elements of QCD penguin operators and the absence of the one-loop weak annihilation contribution in the longitudinal amplitude.

In order to probe the weak annihilation mechanism of the electroweak penguin bottom-meson decays in a

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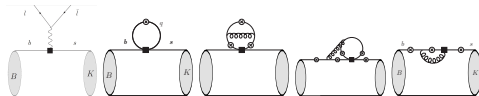
$B \rightarrow \{K, \pi\} \ell^+ \ell^-$: Leading power amplitudes at large recoil

- the contribution from semileptonic operators $\mathcal{O}_{9,10}$

$$\mathcal{A}|_{\mathcal{O}_{9,10}} = \frac{G_F}{\sqrt{2}} \frac{\alpha_{em}}{\pi} V_{tb} V_{ts}^* \left[\bar{\ell} \gamma_\mu \ell p^\mu C_9 f_{BK}^+(q^2) + \bar{\ell} \gamma_\mu \gamma_5 \ell p^\mu C_{10} f_{BK}^+(q^2) \right]$$

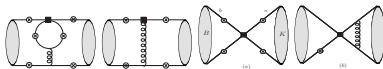

- the contribution from electro-magnetic penguin and four-quark operators $B \rightarrow (K, \pi) \gamma^* \gamma^* \rightarrow \ell^+ \ell^-$

$$\langle P(p') \gamma^*(q, \mu) | \mathcal{H}_{\text{eff}}^{(t,u)} | \bar{B}(p) \rangle = -\frac{g_{\text{em}} m_b}{4\pi^2} \frac{\mathcal{T}_P^{(t,u)}(q^2)}{m_B} \left[q^2 (p_\mu + p'_\mu) - (m_B^2 - m_P^2) q_\mu \right]$$



- the non-factorizable contribution: hard spectator scattering and annihilation

$$\mathcal{T}_P^{(t,u)}|_{\text{anni}} = -\frac{\pi^2}{N_c} \frac{\mathcal{F}_B f_P}{m_B} \sum_{m=\pm} \int_0^\infty \frac{d\omega}{\omega} \int_0^1 du T_{P,m}^{(t,u)}(\omega, u, \mu) \phi_{B,m}(\omega, \mu) \phi_P(u, \mu),$$



- weak annihilation diagrams contribute at leading power for $B \rightarrow P \ell^+ \ell^-$ and the longitudinal polarization of $B \rightarrow V \ell^+ \ell^-$

NLO WA correction to $B \rightarrow \{K, \pi\} \ell^+ \ell^-$: Analytic results

- The hard function and jet function can be obtained by a two-step matching
- the explicit expressions of the NLO weak annihilation corrections to the short-distance matching functions $T_{P,m}^{(t,u)}$

$$\begin{aligned} \left\{ T_{P,+}^{(u)}, T_{P,+}^{(t)} \right\} &\supset -\frac{2m_B Q_q N_c}{m_b} \int_0^1 d\tau \mathbb{J}_{\parallel,+}(\omega, \tau) \left\{ -\sum_{i=1}^2 C_i \left[\mathbb{H}_i^{\text{II}}(u, \tau) \delta_{qu} - \tilde{\mathbb{H}}_i^{\text{II}}(u, \tau) \delta_{qd} \delta_{P\pi} \right], \right. \\ &\quad \left. \sum_{i=3}^6 C_i \tilde{\mathbb{H}}_i^{\text{II}}(u, \tau) \right\}, \\ \left\{ T_{P,-}^{(u)}, T_{P,-}^{(t)} \right\} &\supset \frac{4m_B Q_q N_c}{m_b} \frac{\mathbb{J}_{\parallel,-}(n \cdot q, \bar{n} \cdot q, \omega)}{\hat{q}^2 - 1 + i0} \left\{ -\sum_{i=1}^2 C_i \left[\mathbb{H}_i^{\text{I}}(u) \delta_{qu} - \tilde{\mathbb{H}}_i^{\text{I}}(u) \delta_{qd} \delta_{P\pi} \right], \right. \\ &\quad \left. \sum_{i=3}^6 C_i \tilde{\mathbb{H}}_i^{\text{I}}(u) \right\}, \end{aligned}$$

- the asymptotic behavior of B meson LCDA

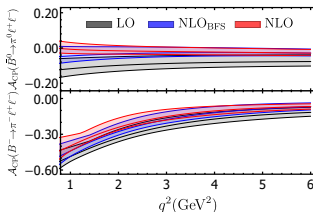
$$\begin{aligned} \phi_B^+(\omega) &\rightarrow \omega, \quad \phi_B^-(\omega) \rightarrow 1, \quad \mathbb{J}_+(\omega) \rightarrow 1, \quad \mathbb{J}_-(\omega) \rightarrow 1, \quad \omega \rightarrow 0 \quad \omega \rightarrow 0 \\ \phi_B^\pm(\omega) &\rightarrow \frac{1}{\omega} \ln \frac{\mu}{\omega}, \quad \mathbb{J}_+(\omega) \rightarrow \frac{1}{\omega} \ln \frac{\omega}{\bar{n} \cdot q}, \quad \mathbb{J}_-(\omega) \rightarrow \ln^2 \frac{\omega}{\bar{n} \cdot q}, \quad \omega \rightarrow \infty \end{aligned}$$

- convergence of the convolution

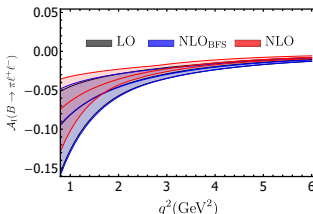
$$\int d\omega \mathbb{J}_+(\omega) \phi_B^+(\omega) \rightarrow \text{finite}, \quad \int \frac{d\omega}{\bar{n} \cdot q - \omega} \mathbb{J}_-(\omega) \phi_B^-(\omega) \rightarrow \text{finite}$$

NLO WA correction to $B \rightarrow \{K, \pi\} \ell^+ \ell^-$: Phenomenology

- CP asymmetry in $B \rightarrow \pi \ell^+ \ell^-$: about 15% corrections to $\mathcal{A}_{\text{CP}}(B \rightarrow \pi \ell^+ \ell^-)$

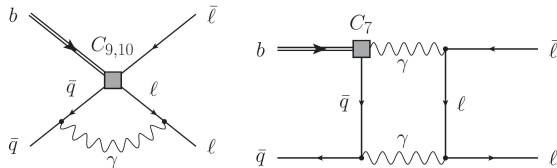


- Isospin asymmetry in $B \rightarrow \pi \ell^+ \ell^-$: about 25% corrections to $\mathcal{A}_1(B \rightarrow \pi \ell^+ \ell^-)$



Pure Leptonic Decays and QED corrections

- Theoretical uncertainty of pure leptonic decays is very small:
 $f_B = (189.4 \pm 1.4) \text{ MeV}$
- Excellent place to determine $|V_{ub}|$ and $|V_{tb}V_{ts}^*|$
- Power enhancement in structure-dependent QCD corrections: comparable to theoretical uncertainties. [Beneke, Bobeth, Szafron, 2018]



- Endpoint singularity in the contribution from electro-magnetic penguin operators: Refactorization. [Feldmann, Gubernari, Huber, Seitz, 2023]
- QED corrections in $B \rightarrow \tau^+ \tau^-$ [Huang, Shen, Zhao, Zhou, 2023]:
 - ▶ Power enhanced, no endpoint singularity: much reliable theoretically
 - ▶ No large logarithmic enhancement, about 0.04% contribution

Current status of $B \rightarrow \gamma \ell \bar{\nu}_\ell$ (for Belle II)

- **Factorization properties at leading power** [Korchensky, Pirjol, Yan, 2000; Descotes-Genon, Sachrajda, 2002; Lunghi, Pirjol, Wyler, 2003; Bosch, Hill, Lange, Neubert, 2003].
- Leading power contributions at NLL and **(partial)-subleading power corrections at tree level** [Beneke, Rohrwild, 2011].
- **Subleading power corrections from the dispersion technique:**
 - ▶ Soft two-particle correction **at tree level** [Braun, Khodjamirian, 2013].
 - ▶ Soft two-particle correction **at one loop** [YMW, 2016].
 - ▶ **Three-particle B -meson DA's contribution** at tree level [Beneke, Braun, Ji, Wei, 2018].
 - ▶ Subleading effective current and twist-5 and 6 corrections (in the factorization limit) at tree level [Beneke, Braun, Ji, Wei, 2018].
- **Subleading power corrections from the direct QCD approach:**
 - ▶ Hadronic photon corrections **at tree level** up to the twist-4 accuracy [Khodjamirian, Stoll, Wyler, 1995; Ali, Braun, 1995; Eilam, Halperin, Mendel, 1995].
 - ▶ Hadronic photon corrections of **twist-two at one loop** and of **higher-twist at tree level** [Ball, Kou, 2003; Wang, Shen, 2018].
- **Leading power contributions at NNLL and the updated NLP corrections:**
 - ▶ Two-loop RG evolution of $\phi_B^+(\omega, \mu)$ derived in [Braun, Ji, Manashov, 2019].
 - ▶ Two-loop jet function obtained in [Liu, Neubert, 2020].
 - ▶ Updated (approximate) NNLL analysis at LP [Galda, Neubert, Wang, 2022].
 - ▶ Further improvement in progress [Shen, Wang, YMW, Wei, 2025].

Theoretical wishlist

- **Systematic understanding of the (higher-twist) B -meson distribution amplitudes.**
 - ▶ Renormalization properties **beyond the one-loop** approximation [conformal symmetry].
 - ▶ Perturbative constraints at large ω_i [OPE technique].
 - ▶ **Rapidity/Ultraviolet subtraction** to get rid of the radiative tail (short-distance effect).
 - ▶ Nonperturbative determinations from the lattice QCD simulation.
- **QCD factorization for the subleading power corrections.**
 - ▶ SCET analysis for the pion-photon form factor as the first step [operator structures, symmetry constraints, etc].
 - ▶ General treatment of the **rapidity divergences** in the (naïve)-factorization formulae.
 - ▶ Rigorous factorization proof taking into account the **Glauber gluons**.
 - ▶ Novel resummation techniques for collecting enhanced logarithms.
- **Future phenomenological applications in preparation.**
 - ▶ Subleading power corrections to the radiative leptonic $B \rightarrow \gamma \ell \bar{\nu}_\ell$ decays.
 - ▶ Nonfactorizable quark-loop effects for $B \rightarrow V \ell \ell$ and $B \rightarrow V \gamma$.
 - ▶ QED factorization for the exclusive bottom-meson decays.
 - ▶ Factorization for the subleading power contributions to nonleptonic bottom-meson decays.