# Quantum Entanglement Theory and Its Generic Searches in High Energy Physics

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The Fourth Developing Workshop on the High Energy Physics Theory and Experiment, Center for the High Energy Physics Theory and Experiment Study, Dalian, September 19-23, 2025

Junle Pei, Yaquan Fang, Lina Wu, Da Xu, Mustapha Biyabi, Tianjun Li, arXiv:2505.09280;
Junle Pei, Xiqing Hao, Xiaochuan Wang, Tianjun Li, arXiv:2505.09931.

# Outline

# The Cornerstones for Modern Physics

- Quantum Mechanics.
- Special Relativity.

# Two Most Genuine Features of Quantum Mechanics

- Quantum entanglement.
- Bell's theorem.

# Quantum Entanglement

- EPR Paradox: in 1935, Einstein-Podolsky-Rosen demonstrated the conflict between local realism and quantum mechanics by considering quantum entangled states . They claimed that quantum mechanics is incomplete.
- In 1935, Schrödinger called it quantum entanglement, a characteristic of quantum mechanics.
- In 1949, Wu-Shaknov provided the first photon entanglement experiment, and obtained a clearly spatially separated quantum entangled state.
- In 1957, Bohm-Aharonov found that Wu-Shaknov experiment achieved the photon polarization correlation, and showed that the non-entangled state could not give such results.

# Bell's Theorem

- In 1964, Bell proposed the inequality satisfied by local realism or local hidden-variable theory.
- Quantum entangled states might violate Bell's inequality, but polarization needs to be measured in a direction that is neither parallel nor perpendicular.
- It is not easy to measure the polarizations of high energy photons directly.
- Over decades, quantum entanglement and Bell non-locality have been rigorously confirmed through various experiments violating Bell inequalities and demonstrations of quantum teleportation, primarily in low-energy systems such as photons, ions, and solid-state qubits.

# Quantum Entanglement and Bell's Theorem

- Quantum entanglement and Bell non-locality are closely related. However, there is a subtle difference: quantum entanglement is a necessary condition for Bell non-locality, but not a sufficient condition.
- The exploration of entanglement in high-energy particle physics remains an emerging frontier, where the interplay between quantum correlations and relativistic dynamics opens new avenues to probe fundamental physics.

# High Energy Physics

- The foundation of the Stanard Model (SM) is quantum field theory, which is based on quantum mechanics and special relativity.
- We can probe the fundamental properties of quantum mechanics at various high energy physics experiments such as colliders.
- The quantum entanglement in top quark-antiquark ($t\bar{t}$) system has been observed by the ATLAS and CMS Collaborations [1].
- The polarization state of the $t\bar{t}$ system is encoded in its spin density matrix $\rho$

$$\rho = \frac{I_4 + \sum_i \left(B_i^+ \sigma^i \otimes I_2 + B_i^- I_2 \otimes \sigma^i\right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4} .$$

[1] G. Aad *et al.* [ATLAS], Nature **633**, no.8030, 542-547 (2024) [arXiv:2311.07288 [hep-ex]]; A. Hayrapetyan *et al.* [CMS], Rept. Prog. Phys. **87**, no.11, 117801 (2024) [arXiv:2406.03976 [hep-ex]].

# Spin Density Matrix

$$\rho = \frac{1}{4}\begin{bmatrix} 1+B_3^+ + B_3^- + C_{33} & B_1^- + C_{31} - i(B_2^- + C_{32}) & B_1^+ + C_{13} - i(B_2^+ + C_{23}) & C_{11} - C_{22} - i(C_{12} + C_{21}) \\ B_1^- + C_{31} + i(B_2^- + C_{32}) & 1 + B_3^+ - B_3^- - C_{33} & C_{11} + C_{22} + i(C_{12} - C_{21}) & B_1^+ - C_{13} - i(B_2^+ - C_{23}) \\ B_1^+ + C_{13} + i(B_2^+ + C_{23}) & C_{11} + C_{22} + i(C_{21} - C_{12}) & 1 - B_3^+ + B_3^- - C_{33} & B_1^- - C_{31} - i(B_2^- - C_{32}) \\ C_{11} - C_{22} + i(C_{21} + C_{12}) & B_1^+ - C_{13} + i(B_2^+ - C_{23}) & B_1^- - C_{31} + i(B_2^- - C_{32}) & 1 - B_3^+ - B_3^- + C_{33} \end{bmatrix}$$

$B_i^+$ and $B_i^-$ are respectively the spin polarizations of $t$ and $\bar{t}$, and $C_{ij}$ is the spin polarization correlation matrix. Under CP invariance, we obtain $B_i^+ = B_i^-$, and $C_{ij} = C_{ji}$.

# High Energy Physics

- For the decay processes $t \to e^+ + \nu_e + b$ and $\bar{t} \to e^- + \bar{\nu}_e + \bar{b}$, the angular distributions of the $e^+$ and $e^-$ in their parent particles' rest frames are

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_+ d\Omega_-} = \frac{1 + \mathbf{B}^+ \cdot \hat{\mathbf{q}}_+ - \mathbf{B}^- \cdot \hat{\mathbf{q}}_- - \hat{\mathbf{q}}_+ \cdot \mathbf{C} \cdot \hat{\mathbf{q}}_-}{(4\pi)^2} .$$

- The LHC experiments utilized the observable

$$D = \mathrm{tr}\left[\mathbf{C}\right]/3 = -3\langle \cos\theta_{e^-e^+}\rangle ,$$

and $\cos\theta_{e^-e^+} = \hat{\mathbf{q}}_+ \cdot \hat{\mathbf{q}}_-$.
- A value of $D < -\frac{1}{3}$ is a sufficient (though not necessary) condition for quantum entanglement.
- At the $e^-e^+$ colliders, the leading-order (LO) calculations predict $D \equiv \frac{1}{3}$, independent of beam energy, polarization, or the top quarks' emission angles.

## Quantum Entanglement Theory and Its Generic Searches

- The quantum entanglement and Bell's non-localities in the previous studies are all defined via the inequalities. Why?
- The criteria are not universal.

# Quantum Entanglement Theory and Its Generic Searches

- A great challenge question: can we propose a Quantum Entanglement Theory (QET) which can define the quantum entanglement exactly?
- We need to provide the solid foundation for quantum entanglement, and probe it via a fundamental approach at the exact level in general.
- Similarly, for any specific approach to probe the quantum entanglement, we need to define the corresponding quantum entanglement criterion exactly as well.

# Idea: Fundamental Approach

- Mathematician: think it geometrically, prove it algebraically.
- For a general quantum system, we define the quantum space as the total spin polarization parameter space, and the classical space as the spin polarization parameter space for classical theory.
- The quantum space and classical space are compact manifolds, and classical space is the hypersurface in quantum space and can be defined via algebraic equations (discriminants).
- The quantum entanglement space is the difference of these two spaces: quantum space minus classical space.

# Idea: Specific Approach

- For any specific approach, we factorize the spin polarization part, calculate the quantum range and classical range, and then the difference is the quantum entanglement range.
- Point: this quantum entanglement criteria usually are the sufficient condition, not the sufficient and necessary condition.

# Outline

# Complex Projective Space

- The complex projective space $\mathbb{CP}^n$ is the $(n+1)$-dimensional complex space $\mathbb{C}^{n+1}\backslash\{0\}$ modulo the following equivalent classes

$$z \sim w \;\; \text{iff} \;\; \exists\lambda \in \mathbb{C}\backslash\{0\}, \; w = \lambda z \; ,$$

$$(z_0, z_1, z_2, ..., z_n) \sim (\lambda z_0, \lambda z_1, \lambda z_2, ..., \lambda z_n) \;\; \text{for all} \;\; \lambda \in \mathbb{C}\backslash\{0\} \; .$$

- In physics, we assume $|z| = 1$, and then have $|\lambda| = 1$.
- The complex dimension of $\mathbb{CP}^n$ is $n$, or say the real dimension of $\mathbb{CP}^n$ is $2n$.
- We have $\mathbb{CP}^n \simeq S^{2n+1}/S^1$, where $S^n$ is $n$-dimensional sphere.
- In particular, we can prove that $\mathbb{CP}^1$ is diffeomorphic to $S^2$.

# Spin Polarization Space

- For a particle $P$ with spin $s$, the polarization state for its spin (or helicity) space is

$$|P\rangle = \sum_{i=-s}^{s} z_i |i\rangle \ , \quad \sum_{i=-s}^{s} |z_i|^2 = 1 \ .$$

- Two polarization states are equivalent if their coefficients $z_i$ and $z_i'$ satisfy the following equivalent relation

$$(z_{-s}, z_{-s+1}, ..., z_s) \sim (\lambda z'_{-s}, \lambda z'_{-s+1}, ..., \lambda z'_s)$$
$$\text{for all} \quad \lambda \in \mathbb{C} \quad \text{and} \quad |\lambda| = 1 \ .$$

**Therefore, we prove that the spin polarization space for a particle $P$ with spin $s$ is $\mathbb{CP}^{2s}$!**

# Spin Polarization Space

- For a general quantum system with $N$ particles with spin $s_1$, $s_2$, ..., and $s_n$, we obtain that the quantum space (the total spin polarization parameter space) is complex projective space $\mathbb{CP}^{J-1}$ with $J = (2s_1 + 1) \times (2s_2 + 1) \times ... \times (2s_n + 1)$.
- The classical space (the spin polarization parameter space for classical theory) is the cartesian product of the complex projective spaces $\mathbb{CP}^{2s_1} \times \mathbb{CP}^{2s_2} \times ... \times \mathbb{CP}^{2s_n}$.
- In mathematics, the classical space is the (generalized) Segre variety in the quantum space.
- The quantum entanglement space is the difference of these two spaces: quantum space minus classical space.

## Spin Polarization Space

- With the property of Cartesian product, we propose the discriminants $\Delta_i$, which are degree 2 homogeneous and holomorphic functions. Thus, the Number of Independent Discriminants (NID) is

$$\mathrm{NID} = J - 1 - \sum_i 2s_i = \Pi_i(2s_i + 1) - 1 - \sum_i 2s_i \ .$$

- We define the corresponding classical spaces as the discriminant locus $\Delta = 0$ for *ff* system, and the intersections of the discriminant loci $\Delta_i = 0$ for all the other systems in the quantum space.
- We define the quantum entanglement spaces as the quantum space with $\Delta \neq 0$ for *ff* system, and the quantum spaces without the intersections of the discriminant loci $\Delta_i = 0$ for all the other systems.

# High Energy Physics

- For high energy physics experiments, we can reconstruct the discriminants from various measurements, and probe the quantum entanglement spaces at exact level.
- We can perform such kind of studies in some two-fermion systems, but in general it might be very difficult.
- For classification, this kind of quantum entanglement search can be defined as the fundamental approach, or say kinematic approach.
- To probe the quantum no-locality, we just consider the space-like separated measurements.

# High Energy Physics

- We shall study the discriminants in the $ff$, $AA$, $Af$, $fff$, and $ffA$ systems.
- To be general, we will not distinguish the fermion ($f$) and anti-fermion ($\bar{f}$).
- We only consider the massive gauge bosons since they need to decay.

# The Two-Fermion $f_1 f_2$ System

- The most general polarization state of a two-fermion system $f_1 f_2$ can be written as

$$|f_1 f_2\rangle = \sum_{k,j=\pm\frac{1}{2}} \alpha_{k,j} |k\rangle_{f_1} \otimes |j\rangle_{f_2} \ , \quad \sum_{k,j=\pm\frac{1}{2}} |\alpha_{k,j}|^2 = 1 \ .$$

- In the two-fermion system $f_1 f_2$, the quantum space is $\mathbb{CP}^3$ with complex dimension 3, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1$ with complex dimension 2. Thus, there is one discriminant with complex dimension 1

$$\Delta = \alpha_{\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2}} \ .$$

- We can prove that the range of the discriminant $\Delta$ is

$$-\frac{1}{2} \leq \Delta \leq \frac{1}{2} \ .$$

# The Two-Fermion $f_1 f_2$ System

- The classical space is the discriminant locus $\Delta = 0$ in quantum space.
- The quantum entanglement space is the quantum space with $\Delta \neq 0$.
- We can reconstruct $\Delta$ from the collider experiments, for example, the $\Lambda\bar{\Lambda}$ pair productions at the BES experiment. Thus, we can probe the quantum entanglement space via the fundamental approach.
- If the measurements are space-like separated, we can probe the quantum non-locality as well.

# The Two-Fermion $f_1 f_2$ System

- We study the relation between our discriminant criterion and the Peres-Horodecki criterion [2].
- In the basis $\left(|\frac{1}{2}\rangle_{f_1} \otimes |\frac{1}{2}\rangle_{f_2}, |\frac{1}{2}\rangle_{f_1} \otimes |-\frac{1}{2}\rangle_{f_2}, |-\frac{1}{2}\rangle_{f_1} \otimes |\frac{1}{2}\rangle_{f_2}, |-\frac{1}{2}\rangle_{f_1} \otimes |-\frac{1}{2}\rangle_{f_2}\right)$, we obtain the spin density matrix.
- Taking partial transpose of $\rho$, *i.e.*, the transposes of the four $2 \times 2$ sub-matrices of $\rho$, we obtain $\rho^{\mathrm{T}_2}$.

[2] A. Peres, Phys. Rev. Lett. **77**, 1413-1415 (1996) [arXiv:quant-ph/9604005 [quant-ph]]; P. Horodecki, Phys. Lett. A **232**, 333 (1997) [arXiv:quant-ph/9703004 [quant-ph]].

# Spin Density Matrix

$$\rho = |f_1 f_2\rangle\langle f_1 f_2| = \begin{pmatrix} |\alpha_{\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 & \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} & \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} & |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 \end{pmatrix}.$$

# Spin Density Matrix

$$\rho^{\mathrm{T}_2}=\begin{pmatrix} |\alpha_{\frac12,\frac12}|^2 & \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} & \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} & \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} \\ \alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} & |\alpha_{\frac12,-\frac12}|^2 & \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} & \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} \\ \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} & \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12} & |\alpha_{-\frac12,\frac12}|^2 & \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} \\ \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} & \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} & \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} & |\alpha_{-\frac12,-\frac12}|^2 \end{pmatrix}.$$

# The Two-Fermion $f_1 f_2$ System

- The original Peres-Horodecki criterion provides a sufficient and necessary condition for classical space: $\rho^{\mathrm{T}_2}$ is positive semi-definite.
- The four eigenvalues of $\rho^{\mathrm{T}_2}$ are

$$-|\Delta| \,, \quad |\Delta| \,, \quad \frac{1}{2}\left(1-\sqrt{1-4|\Delta|^2}\right) \,, \quad \frac{1}{2}\left(1+\sqrt{1-4|\Delta|^2}\right) \,.$$

- The original Peres-Horodecki criterion for classical space is $\Delta = 0$.

**We prove that our criterion for classical space is equivalent to the original Peres-Horodecki criterion.**

# The Two-Fermion $f_1 f_2$ System

- Bell inequality is an equation which distinguishes the Bell local quantum states (or general speaking Bell local states) and the Bell non-local quantum states.
- For a bipartite qubit system, it is the Clauser-Horne-Shimony-Holt (CHSH) inequality [3].
- Thus, we study the relation between our discriminant criterion and the CHSH inequality.
- For simplicity, we consider the equivalent definition of the CHSH inequality [4].

[3] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969)

[4] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A **200**, no.5, 340-344 (1995).

# $B_i^{\pm}$ and $C_{ij}$

$$
\begin{aligned}
B_1^+ &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} \ , \\
B_2^+ &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12}) \ , \\
B_3^+ &= 2\alpha_{\frac12,\frac12}\alpha^*_{\frac12,\frac12} + 2\alpha_{\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} - 1 \ , \\
B_1^- &= \alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} \ , \\
B_2^- &= i(\alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12}) \ , \\
B_3^- &= 2\alpha_{\frac12,\frac12}\alpha^*_{\frac12,\frac12} + 2\alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - 1 \ , \\
C_{11} &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12} \ , \\
C_{12} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12}) \ , \\
C_{13} &= \alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} \ , \\
C_{21} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12}) \ , \\
C_{22} &= -\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} + \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,\frac12} \ , \\
C_{23} &= i(\alpha_{\frac12,\frac12}\alpha^*_{-\frac12,\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{\frac12,\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{\frac12,-\frac12}) \ , \\
C_{31} &= \alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} + \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} - \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12} \ , \\
C_{32} &= i(\alpha_{\frac12,\frac12}\alpha^*_{\frac12,-\frac12} - \alpha_{\frac12,-\frac12}\alpha^*_{\frac12,\frac12} - \alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,-\frac12} + \alpha_{-\frac12,-\frac12}\alpha^*_{-\frac12,\frac12}) \ , \\
C_{33} &= 1 - 2\alpha_{\frac12,-\frac12}\alpha^*_{\frac12,-\frac12} - 2\alpha_{-\frac12,\frac12}\alpha^*_{-\frac12,\frac12} \ .
\end{aligned}
$$

# The Two-Fermion $f_1 f_2$ System

- If $\alpha_{k,j}$ are all real for real case, we obtain

$$C_{12} = C_{21} = C_{23} = C_{32} = 0 \ .$$

- Defining the matrix $C$ as the matrix with elements $C_{ij}$, we obtain the eigenvalues of $C^{\mathrm{T}} C$

$$\lambda_1 = 1 \ , \quad \lambda_2 = \lambda_3 = 4 \left|\Delta\right|^2 \ .$$

- Note that $0 \leq |\Delta| \leq \frac{1}{2}$, the ranges of $\lambda_{2,3}$ are given by

$$0 \leq \lambda_{2,3} \leq 1 \ .$$

- Thus, we obtain the Bell variable for the CHSH inequality

$$\mathcal{B} \equiv 2\sqrt{1 + 4|\Delta|^2} \ .$$

# The Two-Fermion $f_1 f_2$ System

- Because $0 \leq |\Delta| \leq \frac{1}{2}$, we prove

$$2 \leq \mathcal{B} \leq 2\sqrt{2}~.$$

- The CHSH inequality is $\mathcal{B} \leq 2$, and thus for the Bell local parameter space the CHSH inequality becomes the CHSH criterion $\mathcal{B} = 2$, *i.e.*, $\Delta = 0$. The CHSH inequality is violated if and only if $\mathcal{B} > 2$, *i.e.*, $\Delta \neq 0$.
- Thus, our discriminant criterion for classical space is the same as the CHSH criterion for the Bell local parameter space, and our discriminant criterion for quantum entanglement space is the same as the CHSH criterion for the Bell non-local parameter space.

# The Two-Fermion $f_1 f_2$ System

- Therefore, we prove that our classical space is the same as the Bell local parameter space, and our quantum entanglement space is the same as the Bell non-local parameter space.
- In particular, our quantum entanglement space is Bell non-local in high energy physics.
- To distinguish the classical space and Bell local parameter space, or distinguish the quantum entanglement space and Bell non-local parameter space, we consider the Werner states.
- Because our discriminant criterion is the same as the CHSH criterion, we need to prove that the Werner state, which satisfies the Peres-Horodecki criterion for quantum entanglement space and the CHSH criterion for Bell local parameter space, does not exist in our quantum entanglement space.

# The Two-Fermion $f_1 f_2$ System

The Werner states with a free parameter $w$ for the spin density matrix are

$$\rho_{\rm W} = \begin{pmatrix} \frac{1-w}{4} & 0 & 0 & 0 \\ 0 & \frac{1+w}{4} & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & \frac{1+w}{4} & 0 \\ 0 & 0 & 0 & \frac{1-w}{4} \end{pmatrix} ~,~ w \in \left[-\frac{1}{3},~1\right]~.$$

# The Two-Fermion $f_1 f_2$ System

- Using the Peres-Horodecki criterion, we can prove that the Werner states are classical or separable for $w \leq \frac{1}{3}$, and quantum entangled for $w > \frac{1}{3}$.
- The Bell variable for the CHSH inequality for a Werner state is $2\sqrt{2}\,|w|$, *i.e.*, $\mathcal{B} = 2\sqrt{2}\,|w|$. And thus we can realize Bell non-locality for $w > \frac{1}{\sqrt{2}}$.
- For the Werner states, the quantum entanglement range is $w \in \left(\frac{1}{3},\ 1\right]$, and the Bell non-locality range is $w \in \left(\frac{1}{\sqrt{2}},\ 1\right]$.

# The Two-Fermion $f_1 f_2$ System

- Comparing with our spin density matrix, we can easily show that the Werner states with $w \in \left(\frac{1}{3}, \frac{1}{\sqrt{2}}\right]$ cannot be realized.
- We can only achieve the Werner state with $w = 1$.
- We prove that the Werner state, which satisfies the Peres-Horodecki criterion for quantum entanglement space and the CHSH criterion for Bell local parameter space, does not exist in our quantum entanglement space.
- By the way, if $\alpha_{k,j}$ are all real for real case, we obtain

$$\mathcal{B} \equiv 2\sqrt{1 + C_{22}^2} \ .$$

# The Two-Gauge Boson $AA$ System

- The most general polarization state of a physical system with two gauge bosons $A_1 A_2$ can be written as

$$|A_1 A_2\rangle = \sum_{j,k=1,0,-1} \alpha_{j,k} |j\rangle_{A_1} \otimes |k\rangle_{A_2} , \quad \sum_{j,k=1,0,-1} |\alpha_{j,k}|^2 = 1 .$$

- In the two-gauge boson system $A_1 A_2$, the quantum space is $\mathbb{CP}^8$ with complex dimension 8, and the classical space is $\mathbb{CP}^2 \otimes \mathbb{CP}^2$ with complex dimension 4. Thus, there are four independent discriminants with complex dimension 1.

# The Two-Gauge Boson *AA* System

We define the general discriminants as

$$\Delta_1 = \alpha_{1,1}\alpha_{0,0} - \alpha_{1,0}\alpha_{0,1} \ ,$$
$$\Delta_2 = \alpha_{1,1}\alpha_{0,-1} - \alpha_{1,-1}\alpha_{0,1} \ ,$$
$$\Delta_3 = \alpha_{1,1}\alpha_{-1,0} - \alpha_{1,0}\alpha_{-1,1} \ ,$$
$$\Delta_4 = \alpha_{1,1}\alpha_{-1,-1} - \alpha_{1,-1}\alpha_{-1,1} \ ,$$
$$\Delta_5 = \alpha_{-1,-1}\alpha_{0,0} - \alpha_{-1,0}\alpha_{0,-1} \ ,$$
$$\Delta_6 = \alpha_{-1,-1}\alpha_{0,1} - \alpha_{-1,1}\alpha_{0,-1} \ ,$$
$$\Delta_7 = \alpha_{-1,-1}\alpha_{1,0} - \alpha_{-1,0}\alpha_{1,-1} \ ,$$
$$\Delta_8 = \alpha_{0,0}\alpha_{1,-1} - \alpha_{0,-1}\alpha_{1,0} \ ,$$
$$\Delta_9 = \alpha_{0,0}\alpha_{-1,1} - \alpha_{0,1}\alpha_{-1,0} \ .$$

# The Two-Gauge Boson *AA* System

- We can prove that the ranges of the discriminants $\Delta_i$ are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- We can prove that the four independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space by removing the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

# The Gauge Boson-Fermion $Af$ System

- The most general polarization state of a physics system with one gauge boson $A$ and one fermion $f$ can be written as

$$|Af\rangle = \sum_{j=1,0,-1} \sum_{k=\pm\frac{1}{2}} \alpha_{j,k} |j\rangle_A \otimes |k\rangle_f \ , \quad \sum_{j=1,0,-1} \sum_{k=\pm\frac{1}{2}} |\alpha_{j,k}|^2 = 1 \ .$$

- In the $Af$ system, the quantum space is $\mathbb{CP}^5$ with complex dimension 5, and the classical space is $\mathbb{CP}^2 \otimes \mathbb{CP}^1$ with complex dimension 3. Thus, there are two independent discriminants with complex dimension 1.
- We define the general discriminants

$$\Delta_1 = \alpha_{1,\frac{1}{2}}\alpha_{0,-\frac{1}{2}} - \alpha_{1,-\frac{1}{2}}\alpha_{0,\frac{1}{2}} \ ,$$

$$\Delta_2 = \alpha_{1,\frac{1}{2}}\alpha_{-1,-\frac{1}{2}} - \alpha_{1,-\frac{1}{2}}\alpha_{-1,\frac{1}{2}} \ ,$$

$$\Delta_3 = \alpha_{0,\frac{1}{2}}\alpha_{-1,-\frac{1}{2}} - \alpha_{0,-\frac{1}{2}}\alpha_{-1,\frac{1}{2}} \ .$$

# The Gauge Boson-Fermion *Af* System

- we can prove that the ranges of the discriminants $\Delta_i$ are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} ~.$$

- The two independent discriminants can be chosen as $\{\Delta_1, \Delta_2\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space without the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

# The Three Fermion *fff* System

- The most general polarization state of a physics system with three fermions $f_1 f_2 f_3$ can be written as

$$|f_1 f_2 f_3\rangle = \sum_{j,k,l=\pm\frac{1}{2}} \alpha_{j,k,l} |j\rangle_{f_1} \otimes |k\rangle_{f_2} \otimes |l\rangle_{f_2} \ , \quad \sum_{j,k,l=\pm\frac{1}{2}} |\alpha_{j,k,l}|^2 = 1 \ .$$

- In the physics system with three fermions $f_1 f_2 f_3$, the quantum space is $\mathbb{CP}^7$ with complex dimension 7, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^1$ with complex dimension 3. Thus, there are four independent discriminants with complex dimension 1.
- The strategy to construct the discriminants of $N+1$ particles is that we fix the spin (helicity) of one particle and construct the corresponding discriminants of $N$ particles, thus, we have $N+1$ kinds. Next, we consider the new discriminants where all the particles have different spins (helicities).

# The Three Fermion *fff* System

We define the general discriminants

$$\Delta_1 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta'_1 = \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta_2 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta'_2 = \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta_3 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta'_3 = \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} ~,$$

$$\Delta_4 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} ~,$$

$$\Delta'_4 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} ~,$$

$$\Delta''_4 = \alpha_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}\alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} ~.$$

# The Three Fermion *fff* System

- We can prove that the ranges of the discriminants $\Delta_i$ are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} \ .$$

- The four independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space by removing the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

# The System with Two Fermions and One Gauge Boson $ffA$

- The most general polarization state of a physics system with two fermions and one gauge boson $f_1 f_2 A$ can be written as

$$|f_1 f_2 A\rangle = \sum_{j,k=\pm\frac{1}{2}} \sum_{l=1,0,-1} \alpha_{j,k,l} |j\rangle_{f_1} \otimes |k\rangle_{f_2} \otimes |l\rangle_A ~,$$

$$\sum_{j,k=\pm\frac{1}{2}} \sum_{l=1,0,-1} |\alpha_{j,k,l}|^2 = 1 ~.$$

- In the physics system with two fermions and one gauge boson $f_1 f_2 A$, the quantum space is $\mathbb{CP}^{11}$ with complex dimension 11, and the classical space is $\mathbb{CP}^1 \otimes \mathbb{CP}^1 \otimes \mathbb{CP}^2$ with complex dimension 4. Thus, there are seven independent discriminants with complex dimension 1.
- The strategy to construct the discriminants is similar to the three fermion system.

# The System with Two Fermions and One Gauge Boson *ffA*

We define the general discriminants

$$
\begin{aligned}
\Delta_1 &= \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},1} ~, \\
\Delta_1' &= \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ~, \\
\Delta_2 &= \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},1} ~, \\
\Delta_2' &= \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ~, \\
\Delta_3 &= \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},0} ~, \\
\Delta_3' &= \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} ~, \\
\Delta_4 &= \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},1} ~, \\
\Delta_4' &= \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ~,
\end{aligned}
$$

## The System with Two Fermions and One Gauge Boson *ffA*

$$\Delta_5 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},1} \ ,$$

$$\Delta_5' = \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} \ ,$$

$$\Delta_6 = \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},0} \ ,$$

$$\Delta_6' = \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} \ ,$$

$$\Delta_7 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},1} \ ,$$

$$\Delta_7' = \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},0} \ ,$$

$$\Delta_7'' = \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},-1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} \ ,$$

# The System with Two Fermions and One Gauge Boson *ffA*

$$\Delta_8 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},\frac{1}{2},0}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ,$$

$$\Delta_8' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} ,$$

$$\Delta_8'' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},0} - \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} ,$$

$$\Delta_9 = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},\frac{1}{2},-1}\alpha_{-\frac{1}{2},-\frac{1}{2},1} ,$$

$$\Delta_9' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},-1} ,$$

$$\Delta_9'' = \alpha_{\frac{1}{2},\frac{1}{2},1}\alpha_{-\frac{1}{2},-\frac{1}{2},-1} - \alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} .$$

- We can prove that the ranges of the discriminants $\Delta_i$ are

$$-\frac{1}{2} \leq \Delta_i \leq \frac{1}{2} .$$

- The seven independent discriminants can be chosen as $\{\Delta_1, \Delta_2, \Delta_4, \Delta_5, \Delta_7, \Delta_8, \Delta_9\}$.
- The classical space is the intersection of the discriminant loci $\Delta_i = 0$ in quantum space, and the quantum entanglement space is the quantum space without the intersection of the discriminant loci $\Delta_i = 0$, *i.e.*, the quantum space minus the classical space.

# Quantum Entanglement Theory (QET)

- The criterion on the sufficient and necessary condition for classical space might be only found for the simplest two fermion system in 1996 or 1997, *i.e.*, the original Peres-Horodecki criterion [5], 32 or 33 years after Bell's original paper. Thus, it might be a highly non-trivial problem.
- We propose the Quantum Entanglement Theory (QET), and can achieve the criteria on the sufficient and necessary conditions for any generic physics systems.

[5] A. Peres, Phys. Rev. Lett. **77**, 1413-1415 (1996) [arXiv:quant-ph/9604005 [quant-ph]]; P. Horodecki, Phys. Lett. A **232**, 333 (1997) [arXiv:quant-ph/9703004 [quant-ph]].

# Outline

# Theoretical framework: two-particle systems

Two-particle systems:

$$|AB\rangle = \sum_{k,j} \alpha_{k,j} |k\rangle_A |j\rangle_B \ .$$

- The existence of quantum entanglement (QE) is independent of the choice of reference frame. So we adopt the center-of-mass (c.m.) frame of the $AB$ system for our analysis without loss of generality.
- The spin projection quantum numbers $k$ for particle $A$ and $j$ for particle $B$ are defined along their respective momentum directions $\hat{e}_A$ and $\hat{e}_B$.

# Theoretical framework: two-particle systems

In the decay processes $A \to A_1 + A_2 + \ldots$ and $B \to B_1 + B_2 + \ldots$, the angular distributions of the decay products $A_i$ and $B_i$ are characterized in their respective parent rest frames using spherical coordinates: $(\theta_{A_i}, \phi_{A_i})$ for $A_i$ in the $A$-rest frame, and $(\theta_{B_i}, \phi_{B_i})$ for $B_i$ in the $B$-rest frame.

- $\hat{e}_A$ and $\hat{e}_B$ are used as the polar axes for $\theta_{A_i}$ and $\theta_{B_i}$, respectively.
- Azimuthal angle reference protocol:
  -**Construct orthogonal bases**: Choose auxiliary axes $\hat{e}'_A$ and $\hat{e}'_B$ orthogonal to $\hat{e}_A$ and $\hat{e}_B$, respectively
  -**Define zero azimuth**: Align $\phi_{A_i} = 0$ with $\hat{e}'_A$ and $\phi_{B_i} = 0$ with $\hat{e}'_B$
  -**Angular measurement**: $\phi_{A_i/B_i} \in [0, 2\pi]$ increases following the right-handed coordinate system about $\hat{e}_{A/B}$

# Theoretical framework: two-particle systems

## Decay amplitudes:

$$\mathcal{M} = \langle f_A f_B | AB \rangle = \sum_{k,j} \alpha_{k,j} \langle f_A | k \rangle_A \langle f_B | j \rangle_B \ ,$$

$$\langle f_A | k \rangle_A = \sqrt{\frac{2S_A + 1}{4\pi}} e^{i(k - \tilde{\lambda}_{f_A})\phi_{A_1}} d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) H_A(\lambda_{f_A}) \ ,$$

$$\langle f_B | j \rangle_B = \sqrt{\frac{2S_B + 1}{4\pi}} e^{i(j - \tilde{\lambda}_{f_B})\phi_{B_1}} d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) H_B(\lambda_{f_B}) \ .$$

- $S_{A/B}$ denote the spin quantum numbers of particles $A/B$. $k/j$ represent the spin projection quantum numbers along $A/B$'s momentum direction in the $AB$ c.m. frame.
- We collectively denote the final-state particles as $f_A \equiv (A_1, A_2, \ldots)$ and $f_B \equiv (B_1, B_2, \ldots)$. $\lambda_{f_{A/B}}$ encode polarization configurations: $\lambda_{f_A} = (\lambda_{A_1}, \lambda_{A_2}, \ldots)$, $\lambda_{f_B} = (\lambda_{B_1}, \lambda_{B_2}, \ldots)$ , where $\lambda_{A_i}$ and $\lambda_{B_i}$ are spin projections defined relative to directions of $(\theta_{A_1}, \phi_{A_1})$ and $(\theta_{B_1}, \phi_{B_1})$, respectively. The helicity summation rules are defined as $\tilde{\lambda}_{f_A} = \sum_i \lambda_{A_i}$ and $\tilde{\lambda}_{f_B} = \sum_i \lambda_{B_i}$.
- $H_A(\lambda_{f_A})/H_B(\lambda_{f_B})$ remains independent of both the angular variables $(\theta_{A_1}, \phi_{A_1})/(\theta_{B_1}, \phi_{B_1})$ and the parent particle spin projections $k/j$. The Wigner $d$-functions satisfy the normalization conditions:

$$\int_{-1}^{1} d\cos\theta_{A_1} \left( d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) \right)^2 = \frac{2}{2S_A + 1} \ , \int_{-1}^{1} d\cos\theta_{B_1} \left( d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) \right)^2 = \frac{2}{2S_B + 1} \ .$$

# Theoretical framework: two-particle systems

$$\Gamma' = \iint d\pi_{f_A} d\pi_{f_B} \left|\mathcal{M}\right|^2 = \int d\pi'_{f_A} \left|H_A(\lambda_{f_A})\right|^2 \int d\pi'_{f_B} \left|H_B(\lambda_{f_B})\right|^2 \ ,$$

$$d\pi_{f_A} = d\pi'_{f_A} d\phi_{A_1} d\cos\theta_{A_1} \ ,$$
$$d\pi_{f_B} = d\pi'_{f_B} d\phi_{B_1} d\cos\theta_{B_1} \ .$$

- $d\pi_{f_A}$ and $d\pi_{f_B}$ correspond to the phase space volume elements for the decay products of particles $A$ and $B$, respectively.
- $\Gamma'$ remains independent not only of the polarization coefficients $\alpha_{k,j}$, but also of the invariant mass squared $s$ characterizing the $AB$ system.

# Theoretical framework: two-particle systems

Given that $\theta_{A_1}$, $\theta_{B_1}$, $\phi_{A_1}$, and $\phi_{B_1}$ represent measurable quantities, they naturally serve as building blocks for constructing composite observables $\mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1})$.

$$\langle \mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1}) \rangle = \sum_{k,j,m,n} \mathcal{O}_{k,j;m,n} \alpha_{k,j} \alpha^*_{m,n} \ .$$

$$\mathcal{O}_{k,j;m,n} = \frac{(2S_A+1)(2S_B+1)}{16\pi^2} \sum_{\lambda_{f_A}, \lambda_{f_B}} w_{\lambda_{f_A}, \lambda_{f_B}} \left( \int_0^{2\pi} d\phi_{A_1} \int_{-1}^{1} d\cos\theta_{A_1} \int_0^{2\pi} d\phi_{B_1} \int_{-1}^{1} d\cos\theta_{B_1} \right.$$
$$\left. \mathcal{O}(\theta_{A_1}, \theta_{B_1}, \phi_{A_1}, \phi_{B_1}) e^{i(k-m)\phi_{A_1}} e^{i(j-n)\phi_{B_1}} d^{S_A}_{k,\tilde{\lambda}_{f_A}}(\theta_{A_1}) d^{S_A}_{m,\tilde{\lambda}_{f_A}}(\theta_{A_1}) d^{S_B}_{j,\tilde{\lambda}_{f_B}}(\theta_{B_1}) d^{S_B}_{n,\tilde{\lambda}_{f_B}}(\theta_{B_1}) \right) ,$$

$$w_{\lambda_{f_A}, \lambda_{f_B}} = \frac{\int d\pi'_{f_A} \left| H_A(\lambda_{f_A}) \right|^2 \int d\pi'_{f_B} \left| H_B(\lambda_{f_B}) \right|^2}{\sum_{\lambda'_{f_A}, \lambda'_{f_B}} \int d\pi'_{f_A} \left| H_A(\lambda'_{f_A}) \right|^2 \int d\pi'_{f_B} \left| H_B(\lambda'_{f_B}) \right|^2} ,$$

$$\sum_{\lambda_{f_A}, \lambda_{f_B}} w_{\lambda_{f_A}, \lambda_{f_B}} = 1 \ .$$

# Theoretical framework: two-particle systems

## Some specific results:

$$\begin{aligned}&\langle f(\theta_{A_1},\theta_{B_1})\cos(d_A\phi_{A_1}+d_B\phi_{B_1})\rangle\\ =&\frac{(2S_A+1)(2S_B+1)}{8}\sum_{k,j}\left(\alpha_{k,j}\alpha^*_{k+d_A,j+d_B}+\alpha_{k+d_A,j+d_B}\alpha^*_{k,j}\right)\sum_{\lambda_{f_A},\lambda_{f_B}}w_{\lambda_{f_A},\lambda_{f_B}}\\ &\left(\int_{-1}^{1}d\cos\theta_{A_1}\int_{-1}^{1}d\cos\theta_{B_1}\; f(\theta_{A_1},\theta_{B_1})d^{S_A}_{k,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_A}_{k+d_A,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_B}_{j,\tilde\lambda_{f_B}}(\theta_{B_1})d^{S_B}_{j+d_B,\tilde\lambda_{f_B}}(\theta_{B_1})\right)\ ,\end{aligned}$$

$$\begin{aligned}&\langle f(\theta_{A_1},\theta_{B_1})\sin(d_A\phi_{A_1}+d_B\phi_{B_1})\rangle\\ =&\frac{(2S_A+1)(2S_B+1)}{8i}\sum_{k,j}\left(\alpha_{k,j}\alpha^*_{k+d_A,j+d_B}-\alpha_{k+d_A,j+d_B}\alpha^*_{k,j}\right)\sum_{\lambda_{f_A},\lambda_{f_B}}w_{\lambda_{f_A},\lambda_{f_B}}\\ &\left(\int_{-1}^{1}d\cos\theta_{A_1}\int_{-1}^{1}d\cos\theta_{B_1}\; f(\theta_{A_1},\theta_{B_1})d^{S_A}_{k,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_A}_{k+d_A,\tilde\lambda_{f_A}}(\theta_{A_1})d^{S_B}_{j,\tilde\lambda_{f_B}}(\theta_{B_1})d^{S_B}_{j+d_B,\tilde\lambda_{f_B}}(\theta_{B_1})\right)\ .\end{aligned}$$

# Theoretical framework: multi-particle systems

Multi-particle systems:

$$|P_1 P_2 \ldots P_N\rangle = \sum_{k_1,k_2,\ldots,k_N} \alpha_{k_1,k_2,\ldots,k_N} |k_1\rangle_{P_1} |k_2\rangle_{P_2} \ldots |k_N\rangle_{P_N} \ ,$$

$$\sum_{k_1,k_2,\ldots,k_N} |\alpha_{k_1,k_2,\ldots,k_N}|^2 = 1 \ .$$

# Theoretical framework: multi-particle systems

## Multi-particle systems:

- $k_i$ $(i = 1, 2, \ldots, N)$ denote spin projection quantum numbers for respective particles $P_i$, with quantization axes $\hat{e}_i$ aligned to each particle's momentum direction in the system's c.m. frame.
- For each parent particle $P_i$, we identify a corresponding daughter particle $D_i$ within its decay final-state products. Within the rest frame of $P_i$, the momentum orientation of daughter particle $D_i$ is parameterized by polar angle $\theta_i$ and azimuthal angle $\phi_i$.
- The polar angle $\theta_i$ is defined with respect to the quantization axis $\hat{e}_i$. The azimuthal angle $\phi_i$ is established through the following coordinate convention:
  - Select an arbitrary fixed auxiliary axis $\hat{e}'_i$ orthogonal to $\hat{e}_i$
  - Define the reference direction $\phi_i = 0$ via $\hat{e}'_i$
  - The angular parameter $\phi_i \in [0, 2\pi]$ increases following the right-hand rule about $\hat{e}_i$

# Theoretical framework: multi-particle systems

Any physical observable $\mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N)$ constructed from angular parameters:

$$\langle \mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N) \rangle = \sum_{k_1, k_2, \ldots, k_N, k_1', k_2', \ldots, k_N'} \mathcal{O}_{k_1, k_2, \ldots, k_N; k_1', k_2', \ldots, k_N'} \alpha_{k_1, k_2, \ldots, k_N} \alpha^*_{k_1', k_2', \ldots, k_N'} ,$$

$$\mathcal{O}_{k_1, k_2, \ldots, k_N; k_1', k_2', \ldots, k_N'} = \frac{\prod_{i=1}^N \left(2S_{P_i} + 1\right)}{(4\pi)^N} \sum_{\lambda_{f_{P_1}}, \lambda_{f_{P_2}}, \ldots, \lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}}, \lambda_{f_{P_2}}, \ldots, \lambda_{f_{P_N}}} \times \left( \int_0^{2\pi} d\phi_1 \int_{-1}^1 d\cos\theta_1 \int_0^{2\pi} d\phi_2 \int_{-1}^1 d\cos\theta_2 \ldots \int_0^{2\pi} d\phi_N \int_{-1}^1 d\cos\theta_N \right.$$
$$\mathcal{O}(\theta_1, \theta_2, \ldots, \theta_N, \phi_1, \phi_2, \ldots, \phi_N) e^{i\left(k_1 - k_1'\right)\phi_1} e^{i\left(k_2 - k_2'\right)\phi_2} \ldots e^{i\left(k_N - k_N'\right)\phi_N} \times$$
$$\left. d^{S_{P_1}}_{k_1, \tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_1}}_{k_1', \tilde{\lambda}_{f_{P_1}}}(\theta_1) d^{S_{P_2}}_{k_2, \tilde{\lambda}_{f_{P_2}}}(\theta_2) d^{S_{P_2}}_{k_2', \tilde{\lambda}_{f_{P_2}}}(\theta_2) \ldots d^{S_{P_N}}_{k_N, \tilde{\lambda}_{f_{P_N}}}(\theta_N) d^{S_{P_N}}_{k_N', \tilde{\lambda}_{f_{P_N}}}(\theta_N) \right) ,$$

# Theoretical framework: multi-particle systems

$$w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} = \frac{\prod_{i=1}^{N}\int d\pi'_{f_{P_i}} \left|H_{P_i}(\lambda_{f_{P_i}})\right|^2}{\sum_{\lambda'_{f_{P_1}},\lambda'_{f_{P_2}},\ldots,\lambda'_{f_{P_N}}} \left(\prod_{i=1}^{N}\int d\pi'_{f_{P_i}} \left|H_{P_i}(\lambda'_{f_{P_i}})\right|^2\right)} ,$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} = 1 .$$

# Theoretical framework: multi-particle systems

## Some epecific results:

$$\left\langle f(\theta_1,\theta_2,\ldots,\theta_N)\cos\left(\sum_{i=1}^{N} d_i\phi_i\right)\right\rangle = \frac{\prod_{i=1}^{N}\left(2S_{P_i}+1\right)}{2^{N+1}}\times$$

$$\sum_{k_1,k_2,\ldots,k_N}\left(\alpha_{k_1,k_2,\ldots,k_N}\alpha^*_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}+\alpha_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}\alpha^*_{k_1,k_2,\ldots,k_N}\right)\times$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}}\left(\int_{-1}^{1} d\cos\theta_1\int_{-1}^{1} d\cos\theta_2\ldots\int_{-1}^{1} d\cos\theta_N f(\theta_1,\theta_2,\ldots,\theta_N)\right.$$

$$\left. d^{S_{P_1}}_{k_1,\tilde{\lambda}_{f_{P_1}}}(\theta_1)d^{S_{P_1}}_{k_1+d_1,\tilde{\lambda}_{f_{P_1}}}(\theta_1)d^{S_{P_2}}_{k_2,\tilde{\lambda}_{f_{P_2}}}(\theta_2)d^{S_{P_2}}_{k_2+d_2,\tilde{\lambda}_{f_{P_2}}}(\theta_2)\ldots d^{S_{P_N}}_{k_N,\tilde{\lambda}_{f_{P_N}}}(\theta_N)d^{S_{P_N}}_{k_N+d_N,\tilde{\lambda}_{f_{P_N}}}(\theta_N)\right),$$

# Theoretical framework: multi-particle systems

## Some epecific results:

$$\left\langle f(\theta_1,\theta_2,\ldots,\theta_N)\sin\left(\sum_{i=1}^{N} d_i\phi_i\right)\right\rangle = \frac{\prod_{i=1}^{N}\left(2S_{P_i}+1\right)}{2^{N+1}i}\times$$

$$\sum_{k_1,k_2,\ldots,k_N}\left(\alpha_{k_1,k_2,\ldots,k_N}\alpha^*_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N} - \alpha_{k_1+d_1,k_2+d_2,\ldots,k_N+d_N}\alpha^*_{k_1,k_2,\ldots,k_N}\right)\times$$

$$\sum_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}} w_{\lambda_{f_{P_1}},\lambda_{f_{P_2}},\ldots,\lambda_{f_{P_N}}}\left(\int_{-1}^{1} d\cos\theta_1 \int_{-1}^{1} d\cos\theta_2 \ldots \int_{-1}^{1} d\cos\theta_N f(\theta_1,\theta_2,\ldots,\theta_N)\right.$$

$$\left. d^{S_{P_1}}_{k_1,\tilde\lambda_{f_{P_1}}}(\theta_1) d^{S_{P_1}}_{k_1+d_1,\tilde\lambda_{f_{P_1}}}(\theta_1) d^{S_{P_2}}_{k_2,\tilde\lambda_{f_{P_2}}}(\theta_2) d^{S_{P_2}}_{k_2+d_2,\tilde\lambda_{f_{P_2}}}(\theta_2)\ldots d^{S_{P_N}}_{k_N,\tilde\lambda_{f_{P_N}}}(\theta_N) d^{S_{P_N}}_{k_N+d_N,\tilde\lambda_{f_{P_N}}}(\theta_N)\right) .$$

# $t\bar{t}$

## The polarization state of $t\bar{t}$:

$$|t\bar{t}\rangle = \sum_{k,j=\pm\frac{1}{2}} \alpha_{k,j} |k\rangle_t |j\rangle_{\bar{t}} \ .$$

- For a given production channel, described as *initial states* $\to t\bar{t}$, we define the amplitude of producing the $t\bar{t}$ pair in the state $|k\rangle_t |j\rangle_{\bar{t}}$ as $\tilde{\mathcal{M}}_{k,j}$.
- The coefficients $\alpha_{k,j}$ can be calculated using the expression

$$\alpha_{k,j} = \tilde{\mathcal{M}}_{k,j} / \sqrt{\sum_{k,j=\pm\frac{1}{2}} |\tilde{\mathcal{M}}_{k,j}|^2} \ .$$

- The Sufficient and Necessary Condition for indicating QE in the $t\bar{t}$ system:

$$\alpha_{\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2}} \neq 0 \ .$$

# $t\bar{t}$: observable I

The decay processes of $t \to e^+ + \nu_e + b$ and $\bar{t} \to e^- + \bar{\nu}_e + \bar{b}$ after the production of on-shell $t$ and $\bar{t}$:

$$e^+ : (\theta_{e^+}, \phi_{e^+}) \ , \quad e^- : (\theta_{e^-}, \phi_{e^-}) \ .$$

- Observable I: $D = -3 \cdot \langle \cos\theta_{e^+e^-} \rangle$

$$\cos\theta_{e^+e^-} = -\cos\theta_{e^+}\cos\theta_{e^-} + \sin\theta_{e^+}\sin\theta_{e^-}\cos\left(\phi_{e^+} + \phi_{e^-}\right) \ ,$$

$$\langle \cos\theta_{e^+e^-} \rangle = \frac{1}{9}\left( |\alpha_{\frac{1}{2},\frac{1}{2}}|^2 + |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2 \right) + \frac{1}{9}\left( 2\alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + 2\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} \right) \ .$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D$ | $[-1, 1/3]$ | $[-1/3, 1/3]$ | $[-1, -1/3)$ |

# $t\bar{t}$: observable I

The LO results at the $e^+e^-$ collider satisfy the relation

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}}\,, \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 \leq \frac{1}{4}\,.$$

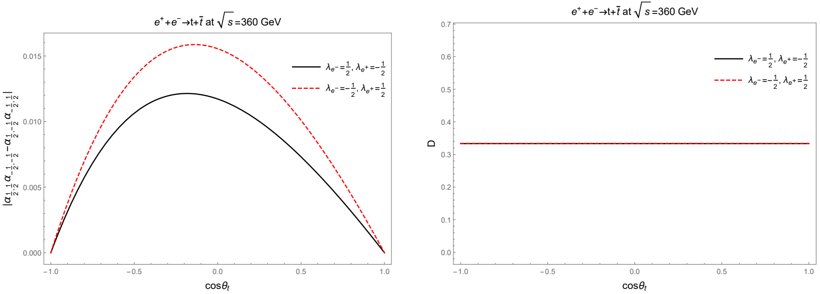



Figure: The LO predictions of $|\alpha_{\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2}}|$ and $D = -3 \cdot \langle \cos\theta_{e^+e^-} \rangle$ for $t\bar{t}$ pairs produced at an $e^+e^-$ collider operating at a c.m. energy of $\sqrt{s} = 360$ GeV. Here, $\theta_t$ denotes the polar angle of the top quark $t$ in the laboratory frame. The symbols $\lambda_{e^\pm}$ represent the helicities of the $e^+$ and $e^-$ beams, defined along their respective momentum directions in the laboratory frame.

# $t\bar{t}$: observable II

$$\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle = \frac{\pi^2}{32}\left(\alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}}\right) \;,\quad D' = \frac{32}{\pi^2}\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle \;. \tag{1}$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ |

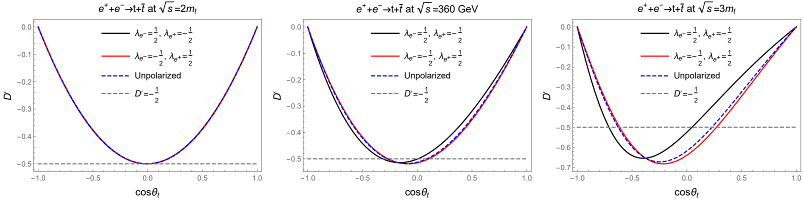



Figure: The LO predictions of $D' = \frac{32}{\pi^2}\langle \cos\left(\phi_{e+} - \phi_{e-}\right)\rangle$ for $t\bar{t}$ pairs produced at $e^+e^-$ collider with $\sqrt{s} = 2m_t$, 360 GeV, and $3m_t$, respectively. The angle $\theta_t$ represents the polar angle of the top quark $t$ in the laboratory frame. The symbols $\lambda_{e\pm}$ indicate the helicities of the $e^+$ and $e^-$ beams, defined along their respective momentum directions in the laboratory frame.

# $t\bar{t}$: observable III

$$D'' = \frac{9}{2}\langle \sin\theta_{e^+} \sin\theta_{e^-} \cos\left(\phi_{e^+} + \phi_{e^-}\right)\rangle = \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} \ .$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D''$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ |

- Within the SM at $e^+e^-$ colliders, we obtain

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}} \ , \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 \leq \frac{1}{4} \Longrightarrow D'' \in [-\tfrac{1}{2}, 0) \ .$$

- Considering beyond-Standard-Model scenarios with a Higgs-like particle exhibiting Yukawa coupling:

$$\propto h' t\bar{t} \ .$$

Direct calculation reveals that $t\bar{t}$ pairs from $h' \to t + \bar{t}$ decays satisfy

$$\alpha_{\frac{1}{2},\frac{1}{2}} = -\alpha_{-\frac{1}{2},-\frac{1}{2}} \ , \quad \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 = \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 = \frac{1}{2} \ .$$

For such $h'$-mediated $t\bar{t}$ production, we obtain $D'' = -1$.

# $\tau^+\tau^-$: observable I

The decay processes of $\tau^+ \to e^+ + \nu_e + \bar{\nu}_\tau$ and $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$ after the production of on-shell $\tau^+$ and $\tau^-$:

$$e^+ : (\theta_{e^+}, \phi_{e^+}) \ . \ e^- : (\theta_{e^-}, \phi_{e^-}) \ .$$

- Observable I:

$$\begin{aligned}
\cos\theta_{e^+e^-} =& -\cos\theta_{e^+}\cos\theta_{e^-} + \sin\theta_{e^+}\sin\theta_{e^-}\cos(\phi_{e^+}+\phi_{e^-}) \ , \\
\langle\cos\theta_{e^+e^-}\rangle =& 0.01254\left(|\alpha_{\frac{1}{2},\frac{1}{2}}|^2 + |\alpha_{-\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{\frac{1}{2},-\frac{1}{2}}|^2 - |\alpha_{-\frac{1}{2},\frac{1}{2}}|^2\right. \\
& \left. +2\alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} + 2\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}}\right) \ , \\
=& 0.01254 \times (-1) \ .
\end{aligned}$$

# $\tau^+\tau^-$: observable II

$$\langle\cos\left(\phi_{e^+}-\phi_{e^-}\right)\rangle = 0.03426\left(\alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}}+\alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}}\right)\ ,\quad D'_{\tau^+\tau^-}=\frac{1}{0.03426}\langle\cos\left(\phi_{e^+}-\phi_{e^-}\right)\rangle\ .$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'_{\tau^+\tau^-}$ | $[-1,1]$ | $[-1/2,1/2]$ | $[-1,-\frac{1}{2})\cup(\frac{1}{2},1]$ |

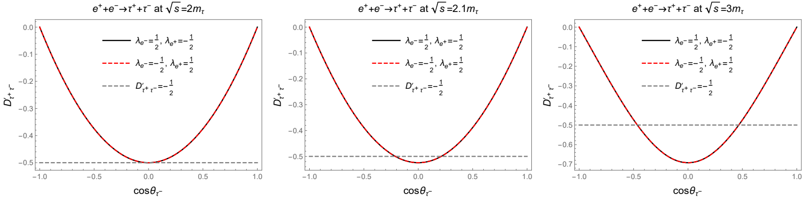



**Figure:** The LO predictions of $D'_{\tau^+\tau^-}=\langle\cos\left(\phi_{e^+}-\phi_{e^-}\right)\rangle/0.03426$ for $\tau^+\tau^-$ pairs produced at $e^+e^-$ collider with $\sqrt{s}=2m_\tau$, $2.1m_\tau$, and $3m_\tau$, respectively. The angle $\theta_{\tau^-}$ represents the polar angle of $\tau^-$ in the laboratory frame. The symbols $\lambda_{e\pm}$ indicate the helicities of the $e^+$ and $e^-$ beams, defined along their respective momentum directions in the laboratory frame.

# $W^-W^+$: observable I

The decay processes of $W^- \to e^- + \bar{\nu}_e$ and $W^+ \to e^+ + \nu_e$ after the production of on-shell $W^-$ and $W^+$:

$$e^- : (\theta_{e^-}, \phi_{e^-}) \,, \quad e^+ : (\theta_{e^+}, \phi_{e^+}) \,.$$

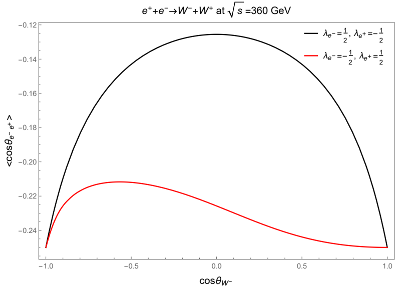
- Observable I:

$$\cos\theta_{e^+e^-} = -\cos\theta_{e^+}\cos\theta_{e^-} + \sin\theta_{e^+}\sin\theta_{e^-}\cos\left(\phi_{e^+} + \phi_{e^-}\right) \,,$$

$$\langle\cos\theta_{e^-e^+}\rangle = \frac{1}{4} \times \Big( |\alpha_{-1,-1}|^2 + |\alpha_{1,1}|^2 - |\alpha_{-1,1}|^2 - |\alpha_{1,-1}|^2 - \alpha_{-1,-1}\alpha^*_{0,0} - \alpha_{0,0}\alpha^*_{-1,-1} - \alpha_{1,1}\alpha^*_{0,0} - \alpha_{0,0}\alpha^*_{1,1} - \alpha_{-1,0}\alpha^*_{0,1} - \alpha_{0,1}\alpha^*_{-1,0} - \alpha_{0,-1}\alpha^*_{1,0} - \alpha_{1,0}\alpha^*_{0,-1} \Big) \,.$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $\langle\cos\theta_{e^-e^+}\rangle$ | $[-1/4, 1/2]$ | $[-1/4, 1/4]$ | $(1/4, 1/2]$ |

# $W^-W^+$: observable I



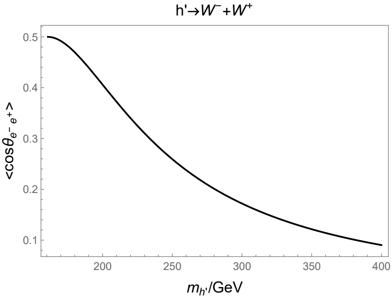




Figure: Left: The LO predictions of $\langle\cos\theta_{e^-e^+}\rangle$ for the $W^-W^+$ pair produced at $e^+e^-$ collider with $\sqrt{s} = 360$ GeV. The angle $\theta_{W^-}$ represents the polar angle of $W^-$ in the laboratory frame. The symbols $\lambda_{e^\pm}$ indicate the helicities of the $e^+$ and $e^-$ beams, defined along their respective momentum directions in the laboratory frame. Right: The LO predictions of $\langle\cos\theta_{e^-e^+}\rangle$ for the $W^-W^+$ pair produced from $h'$ decay ($\propto h' g^{\mu\nu} W^-_\mu W^+_\nu$).

# $W^-W^+$: observable II

$$\langle \cos(2\phi_{e^+} - 2\phi_{e^-})\rangle = \frac{1}{8}\left(\alpha_{-1,1}\alpha^*_{1,-1} + \alpha_{1,-1}\alpha^*_{-1,1}\right) ,$$
$$D'_{W^-W^+} = 8\langle \cos(2\phi_{e^+} - 2\phi_{e^-})\rangle .$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'_{W^-W^+}$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ |

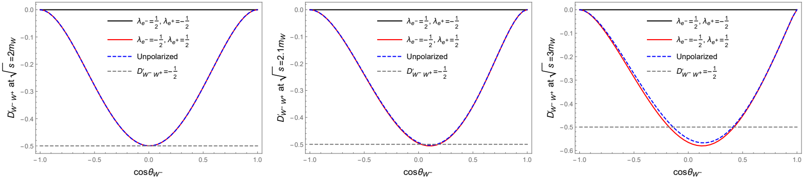



Figure: The LO predictions of $D'_{W^-W^+} = 8\langle \cos\left(2\phi_{e^+} - 2\phi_{e^-}\right)\rangle$ for $W^+W^-$ pairs produced at $e^+e^-$ collider with $\sqrt{s} = 2m_W$, $2.1m_W$ GeV, and $3m_W$, respectively. The angle $\theta_{W^-}$ represents the polar angle of the top quark $W^-$ in the laboratory frame. The symbols $\lambda_{e^\pm}$ indicate the helicities of the $e^+$ and $e^-$ beams, defined along their respective momentum directions in the laboratory frame.

# $W^- t$

The decay processes of $W^- \to e^- + \bar{\nu}_e$ and $t \to W^+ + b$ after the production of on-shell $W^-$ and $t$:

$$e^- : (\theta_{e^-}, \phi_{e^-}) \;, \quad W^+ : (\theta_{W^+}, \phi_{W^+}) \;.$$

- Observable:

$$\begin{aligned}
\cos\theta_{e^- W^+} =& -\cos\theta_{W^+}\cos\theta_{e^-} + \sin\theta_{W^+}\sin\theta_{e^-}\cos\left(\phi_{W^+} + \phi_{e^-}\right) \;, \\
\langle\cos\theta_{e^- W^+}\rangle =& 0.0658 \times \Bigg( |\alpha_{-1,-\frac{1}{2}}|^2 + |\alpha_{1,\frac{1}{2}}|^2 - |\alpha_{-1,\frac{1}{2}}|^2 - |\alpha_{1,-\frac{1}{2}}|^2 \\
& + \sqrt{2}\left(\alpha_{-1,-\frac{1}{2}}\alpha^*_{0,\frac{1}{2}} + \alpha_{0,\frac{1}{2}}\alpha^*_{-1,-\frac{1}{2}} + \alpha_{0,-\frac{1}{2}}\alpha^*_{1,\frac{1}{2}} + \alpha_{1,\frac{1}{2}}\alpha^*_{0,-\frac{1}{2}}\right)\Bigg) \;, \\
D'_{W^- t} =& \langle\cos\theta_{e^- W^+}\rangle / 0.0658 \;.
\end{aligned}$$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'_{W^- t}$ | $[-1, 2]$ | $[-1, 1]$ | $(1, 2]$ |

# $W^- t$

Supposing a BSM bottom-like quark, denoted as $b'$, the interaction of $b'$ with the $W$ boson and the top quark $t$ is analogous to that of the SM bottom quark $b$: $\propto W^+_\mu \bar{t}_L \gamma^\mu b'_L + h.c.$

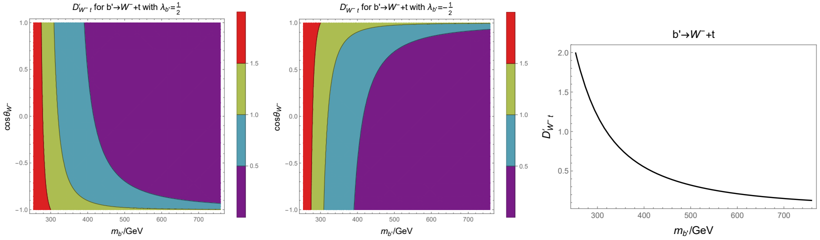



**Figure:** Left and middle: The LO predictions of $D'_{W^-t} = \langle \cos\theta_{e^-W^+} \rangle / 0.0658$ for the $W^- t$ pair produced by $b'$ decay. Here, $\theta_{W^-}$ represents the angle between the momentum direction of the $W^-$ in the $b'$ rest frame and the direction of $b'$ motion in the laboratory frame. The symbols $\lambda_{b'}$ denote the helicities of $b'$, defined along the momentum direction of $b'$ in the laboratory frame. Right: The LO predictions of $D'_{W^-t} = \langle \cos\theta_{e^-W^+} \rangle / 0.0658$ for the $W^- t$ produced by $b'$ decay, averaged over $\cos\theta_{W^-}$.

# *ttt*

For the decay process $t_i \to W_i^+ + b_i$ for each top quark $(i = 1, 2, 3)$, we define the spherical coordinates $(\theta_i, \phi_i)$ for the $W_i^+$ momentum direction in respective $t_i$ rest frames. The momentum unit vectors are expressed as

$$\hat{e}_{W_i^+} = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i) \ .$$

The triple product correlation observable:

$$
\begin{aligned}
&\left(\hat{e}_{W_1^+} \times \hat{e}_{W_2^+}\right) \cdot \hat{e}_{W_3^+} \\
=& - \sin\theta_1 \sin\theta_2 \cos\theta_3 \sin(\phi_1 - \phi_2) - \sin\theta_2 \sin\theta_3 \cos\theta_1 \sin(\phi_2 - \phi_3) - \sin\theta_3 \sin\theta_1 \cos\theta_2 \sin(\phi_3 - \phi_1) \\
=& 0.004557 \ i \ \times ( \\
& - \alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \\
& - \alpha_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} + \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} - \alpha^*_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \\
& - \alpha_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha^*_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha^*_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} - \alpha^*_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} \alpha_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \Big) \ , \\
& D'_{3t} = \left\langle \left(\hat{e}_{W_1^+} \times \hat{e}_{W_2^+}\right) \cdot \hat{e}_{W_3^+} \right\rangle / 0.004557 \ .
\end{aligned}
$$

# *ttt*

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'_{3t}$ | $[-\sqrt{3}, \sqrt{3}]$ | $[-1/2, 1/2]$ | $\left[-\sqrt{3}, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \sqrt{3}\right]$ |

# $t\bar{t}W^-$

Considering the decay channels $t \to W^+ b$, $\bar{t} \to W^- \bar{b}$, and $W^- \to e^- \bar{\nu}_e$, we analyze the angular correlations of final-state particles ($W^+$, $W^-$, and $e^-$) in their respective parent particle rest frames.

$$\hat{e}_{W^+} = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \ , \quad \hat{e}_{W^-} = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2) \ ,$$
$$\hat{e}_{e^-} = (\sin\theta_3 \cos\phi_3, \sin\theta_3 \sin\phi_3, \cos\theta_3) \ .$$

The triple product correlation observable:

$$\begin{aligned}
&\left(\hat{e}_{W^+} \times \hat{e}_{W^-}\right) \cdot \hat{e}_{e^-} \\
=& -\sin\theta_1 \sin\theta_2 \cos\theta_3 \sin(\phi_1 - \phi_2) - \sin\theta_2 \sin\theta_3 \cos\theta_1 \sin(\phi_2 - \phi_3) - \sin\theta_3 \sin\theta_1 \cos\theta_2 \sin(\phi_3 - \phi_1) \\
=& 0.0122433 \ i \times ( \\
& -\alpha_{-\frac{1}{2},-\frac{1}{2},0}\alpha^*_{-\frac{1}{2},\frac{1}{2},-1} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},0}\alpha_{-\frac{1}{2},\frac{1}{2},-1} - \alpha_{-\frac{1}{2},-\frac{1}{2},1}\alpha^*_{-\frac{1}{2},\frac{1}{2},0} + \alpha^*_{-\frac{1}{2},-\frac{1}{2},1}\alpha_{-\frac{1}{2},\frac{1}{2},0} \\
& + \alpha_{-\frac{1}{2},-\frac{1}{2},0}\alpha^*_{\frac{1}{2},-\frac{1}{2},-1} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},0}\alpha_{\frac{1}{2},-\frac{1}{2},-1} - \sqrt{2}\alpha_{-\frac{1}{2},\frac{1}{2},-1}\alpha^*_{\frac{1}{2},-\frac{1}{2},-1} + \sqrt{2}\alpha^*_{-\frac{1}{2},\frac{1}{2},-1}\alpha_{\frac{1}{2},-\frac{1}{2},-1} \\
& + \alpha_{-\frac{1}{2},-\frac{1}{2},1}\alpha^*_{\frac{1}{2},-\frac{1}{2},0} - \alpha^*_{-\frac{1}{2},-\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},0} + \sqrt{2}\alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha^*_{\frac{1}{2},-\frac{1}{2},1} - \sqrt{2}\alpha^*_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},-\frac{1}{2},1} \\
& - \alpha_{-\frac{1}{2},\frac{1}{2},0}\alpha^*_{\frac{1}{2},\frac{1}{2},-1} + \alpha^*_{-\frac{1}{2},\frac{1}{2},0}\alpha_{\frac{1}{2},\frac{1}{2},-1} + \alpha_{\frac{1}{2},-\frac{1}{2},0}\alpha^*_{\frac{1}{2},\frac{1}{2},-1} - \alpha^*_{\frac{1}{2},-\frac{1}{2},0}\alpha_{\frac{1}{2},\frac{1}{2},-1} \\
& -\alpha_{-\frac{1}{2},\frac{1}{2},1}\alpha^*_{\frac{1}{2},\frac{1}{2},0} + \alpha^*_{-\frac{1}{2},\frac{1}{2},1}\alpha_{\frac{1}{2},\frac{1}{2},0} + \alpha_{\frac{1}{2},-\frac{1}{2},1}\alpha^*_{\frac{1}{2},\frac{1}{2},0} - \alpha^*_{\frac{1}{2},-\frac{1}{2},1}\alpha_{\frac{1}{2},\frac{1}{2},0}\Big) \\
& D'_{t\bar{t}W^-} = \left\langle \left(\hat{e}_{W^+} \times \hat{e}_{W^-}\right) \cdot \hat{e}_{e^-} \right\rangle / 0.0122433 \ .
\end{aligned}$$

# $t\bar{t}W^-$

| Observables | Quantum Range | Classical Range | criteria for entanglement |
|---|---|---|---|
| $D'_{3t}$ | $[-2, 2]$ | $\left[-\sqrt{2}/2, \sqrt{2}/2\right]$ | $\left[-2, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 2\right]$ |

# $\Lambda\bar{\Lambda}$

arXiv:2505.09931

- The weak decay processes:

$$\Lambda \to p + \pi^- \ , \quad \bar{\Lambda} \to \bar{p} + \pi^+ \ .$$

- The relevant couplings:

$$W_\mu^+ \bar{p} \left(g_V \gamma^\mu + g_A \gamma^\mu \gamma_5\right) \Lambda + W_\mu^- \bar{\Lambda} \left(g_V' \gamma^\mu + g_A' \gamma^\mu \gamma_5\right) p \ ,$$
$$\propto W_\mu^- \partial^\mu \pi^+ + \mathrm{H.C.} \ .$$

The independent coupling parameters $g_V, g_A$ and $g_V', g_A'$ allow for potential CP-violating effects.

- In the rest frames of $\Lambda$ and $\bar{\Lambda}$, the polar angle distributions of final-state $p$ and $\bar{p}$ satisfy

$$\frac{1}{\Gamma_{\Lambda \to p + \pi^-}} \frac{d\Gamma_{\Lambda \to p + \pi^-}}{d \cos\theta_p} = \frac{1}{2}\left(1 + \alpha_\Lambda \cos\theta_p\right) \ ,$$
$$\frac{1}{\Gamma_{\bar{\Lambda} \to \bar{p} + \pi^+}} \frac{d\Gamma_{\bar{\Lambda} \to \bar{p} + \pi^+}}{d \cos\theta_{\bar{p}}} = \frac{1}{2}\left(1 + \alpha_{\bar{\Lambda}} \cos\theta_{\bar{p}}\right) \ .$$

# $\Lambda\bar{\Lambda}$: decay processes

We parameterize the momentum directions of $p$ in the $\Lambda$ rest frame and $\bar{p}$ in the $\bar{\Lambda}$ rest frame using spherical coordinates:

$$p : (\theta_1, \phi_1) \to \hat{e}_p = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \ ,$$
$$\bar{p} : (\theta_2, \phi_2) \to \hat{e}_{\bar{p}} = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2) \ .$$

- The polar angles $\theta_1$ and $\theta_2$ are defined with respect to the $\Lambda$ momentum direction $\hat{e}_\Lambda$ in the $\Lambda\bar{\Lambda}$ center-of-mass (c.m.) frame.
- The azimuthal angles $\phi_1$ and $\phi_2$ are measured from an arbitrary reference axis orthogonal to $\hat{e}_\Lambda$, increasing in the right-handed screw direction about $\hat{e}_\Lambda$ with $\phi \in [0, 2\pi]$.
- We define the opening angle $\theta_{p\bar{p}}$ between the proton momenta as:

$$\cos\theta_{p\bar{p}} = \hat{e}_p \cdot \hat{e}_{\bar{p}} = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) \ .$$

# $\Lambda\bar{\Lambda}$: decay processes

Helicity amplitudes:

$$\langle p, \pi^- | k \rangle_\Lambda = \frac{1}{\sqrt{2\pi}} e^{i(k-\lambda_p)\phi_1} d^{1/2}_{k,\lambda_p}(\theta_1) H_\Lambda(\lambda_p) \ , \ ,$$

$$\langle \bar{p}, \pi^+ | j \rangle_{\bar{\Lambda}} = \frac{1}{\sqrt{2\pi}} e^{-i(j+\lambda_{\bar{p}})\phi_2} d^{1/2}_{\lambda_{\bar{p}},j}(\pi - \theta_2) H_{\bar{\Lambda}}(\lambda_{\bar{p}}) \ .$$

- $k/j$ represent the spin projection quantum numbers along the momentum direction of $\Lambda/\bar{\Lambda}$ in the $\Lambda\bar{\Lambda}$ c.m. frame.
- $\lambda_p$ and $\lambda_{\bar{p}}$ ($\lambda_p, \lambda_{\bar{p}} = \pm\frac{1}{2}$) are spin projections of $p$ and $\bar{p}$ defined relative to directions of $\hat{e}_p$ and $\hat{e}_{\bar{p}}$, respectively.
- $H_\Lambda(\lambda_p)/H_{\bar{\Lambda}}(\lambda_{\bar{p}})$ remain independent of both the angular variables $(\theta_1, \phi_1)/(\theta_2, \phi_2)$ and the parent particle spin projections $k/j$.
- The Wigner $d$-functions are

$$d^{1/2}_{\frac{1}{2},\frac{1}{2}}(\theta) = d^{1/2}_{-\frac{1}{2},-\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \ , \quad d^{1/2}_{-\frac{1}{2},\frac{1}{2}}(\theta) = -d^{1/2}_{\frac{1}{2},-\frac{1}{2}}(\theta) = \sin\frac{\theta}{2} \ .$$

# $\Lambda\bar{\Lambda}$: observables

For any physical observable $\mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2)$ constructed from angular variables, its statistical average can be expressed as:

$$\langle \mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) \rangle = \sum_{k,j,m,n} \mathcal{O}_{k,j;m,n} \alpha_{k,j} \alpha^*_{m,n} \ . \qquad (2)$$

$$\mathcal{O}_{k,j;m,n} = \frac{1}{4\pi^2} \sum_{\lambda_p, \lambda_{\bar{p}}} \left( w_{\lambda_p, \lambda_{\bar{p}}} \int_0^{2\pi} d\phi_1 \int_{-1}^{1} d\cos\theta_1 \int_0^{2\pi} d\phi_2 \int_{-1}^{1} d\cos\theta_2 \right.$$
$$\left. \mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) e^{i(k-m)\phi_1} e^{i(n-j)\phi_2} d^{1/2}_{k,\lambda_p}(\theta_1) d^{1/2}_{m,\lambda_p}(\theta_1) d^{1/2}_{\lambda_{\bar{p}},j}(\pi - \theta_2) d^{1/2}_{\lambda_{\bar{p}},n}(\pi - \theta_2) \right) \ .$$

$$w_{\lambda_p, \lambda_{\bar{p}}} = \frac{\left| H_\Lambda(\lambda_p) \right|^2 \left| H_{\bar{\Lambda}}(\lambda_{\bar{p}}) \right|^2}{\sum_{\lambda'_p, \lambda'_{\bar{p}}} \left| H_\Lambda(\lambda'_p) \right|^2 \left| H_{\bar{\Lambda}}(\lambda'_{\bar{p}}) \right|^2} \ .$$

# $\Lambda\bar{\Lambda}$: observables

- The hyperon decay parameters:

$$\alpha_{\Lambda/\bar{\Lambda}} = \frac{\left|H_{\Lambda/\bar{\Lambda}}(-\frac{1}{2})\right|^2 - \left|H_{\Lambda/\bar{\Lambda}}(\frac{1}{2})\right|^2}{\left|H_{\Lambda/\bar{\Lambda}}(\frac{1}{2})\right|^2 + \left|H_{\Lambda/\bar{\Lambda}}(-\frac{1}{2})\right|^2} \ .$$

- The weight factors:

$$w_{\frac{u_1}{2},\frac{u_2}{2}} = \frac{1}{4}(1 - u_1\alpha_\Lambda)(1 - u_2\alpha_{\bar{\Lambda}}) \ , \quad u_1, u_2 = \pm 1 \ .$$

# $\Lambda\bar{\Lambda}$: observables

$$\mathcal{O}_0 = \langle \cos\theta_{p\bar{p}} \rangle / \left(-\frac{1}{9}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right) = \left|\alpha_{\frac{1}{2},\frac{1}{2}}\right|^2 + \left|\alpha_{-\frac{1}{2},-\frac{1}{2}}\right|^2 - \left|\alpha_{\frac{1}{2},-\frac{1}{2}}\right|^2 - \left|\alpha_{-\frac{1}{2},\frac{1}{2}}\right|^2 ,$$
$$+ 2\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} + 2\alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} ,$$
$$\mathcal{O}_1 = \langle \cos(\phi_1+\phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right) = \alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} + \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}} ,$$
$$\mathcal{O}_2 = \langle \cos(\phi_1-\phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right) = \alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} + \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}} ,$$
$$\mathcal{O}_3 = \langle \sin(\phi_1+\phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right) = \frac{1}{i}\left(\alpha_{-\frac{1}{2},\frac{1}{2}}\alpha^*_{\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha^*_{-\frac{1}{2},\frac{1}{2}}\right) ,$$
$$\mathcal{O}_4 = \langle \sin(\phi_1-\phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right) = \frac{1}{i}\left(\alpha_{-\frac{1}{2},-\frac{1}{2}}\alpha^*_{\frac{1}{2},\frac{1}{2}} - \alpha_{\frac{1}{2},\frac{1}{2}}\alpha^*_{-\frac{1}{2},-\frac{1}{2}}\right) .$$

# $\Lambda\bar{\Lambda}$: observables

| Observables | Quantum Ranges | Classical Ranges | criteria for entanglement |
|---|---|---|---|
| $\mathcal{O}_0$ | $[-1, 3]$ | $[-1, 1]$ | $(1, 3]$ |
| $\mathcal{O}_1$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -1/2) \cup (1/2, 1]$ |
| $\mathcal{O}_2$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -1/2) \cup (1/2, 1]$ |
| $\mathcal{O}_3$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -1/2) \cup (1/2, 1]$ |
| $\mathcal{O}_4$ | $[-1, 1]$ | $[-1/2, 1/2]$ | $[-1, -1/2) \cup (1/2, 1]$ |

# $\Lambda\bar{\Lambda}$: production

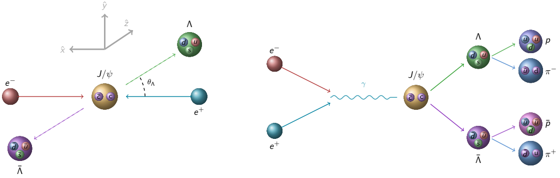



Figure: The processes of
$e^+ + e^- \to J/\psi \to \Lambda(\to p + \pi^-) + \bar{\Lambda}(\to \bar{p} + \pi^+)$.

# $\Lambda\bar{\Lambda}$: production

- The $J/\psi - \Lambda\bar{\Lambda}$ interaction is conventionally parameterized as:

$$\psi_\mu \bar{\Lambda} \left( G_M \gamma^\mu + \frac{2m_\Lambda}{m_\psi^2 - 4m_\Lambda^2} \left( G_M - G_E \right) \left( p_\Lambda^\mu - p_{\bar{\Lambda}}^\mu \right) \right) \Lambda \,,$$
$$G_E / G_M = R e^{i\Delta\phi} \,.$$

- In the $J/\psi$ rest frame, the $\Lambda$ polar angle distribution follows:

$$\frac{1}{\Gamma_{\psi\to\Lambda+\bar{\Lambda}}} \frac{d\Gamma_{\psi\to\Lambda+\bar{\Lambda}}}{d\cos\theta_\Lambda} = \frac{1}{2 + \frac{2}{3}\alpha_\psi} \left( 1 + \alpha_\psi \cos^2\theta_\Lambda \right) \,.$$

# $\Lambda\bar{\Lambda}$: production

- For the $J/\psi \to \Lambda + \bar{\Lambda}$ process, we derive:

$$\mathcal{P}(\Delta s^2 < 0) = \sqrt{1 - 4m_\Lambda^2/m_\psi^2} \ .$$

This result implies that 69.3% of the decay events involving $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$ are spacelike-separated.

- For $\Lambda\bar{\Lambda}$ pairs produced through $e^+ + e^- \to J/\psi \to \Lambda + \bar{\Lambda}$, theoretical calculations under decoherence-free conditions yield:

$$\alpha_{\frac{1}{2},\frac{1}{2}}\alpha_{-\frac{1}{2},-\frac{1}{2}} - \alpha_{\frac{1}{2},-\frac{1}{2}}\alpha_{-\frac{1}{2},\frac{1}{2}} = \frac{1-\alpha_\psi}{4}\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi\cos^2\theta_\Lambda}\left(\frac{1+\alpha_\psi}{1-\alpha_\psi} - e^{2i\Delta\phi}\right) \ .$$

# $\Lambda\bar{\Lambda}$: measurements

The complete angular distribution for the cascade decay of $e^+ + e^- \to J/\psi \to \Lambda(\to p + \pi^-) + \bar{\Lambda}(\to \bar{p} + \pi^+)$:

$$\begin{aligned}\mathcal{W} =& \mathcal{T}_0 + \alpha_\psi \mathcal{T}_5 \\ & + \alpha_\Lambda \alpha_{\bar{\Lambda}} \left( \mathcal{T}_1 + \sqrt{1-\alpha_\psi^2} \cos(\Delta\phi) \mathcal{T}_2 + \alpha_\psi \mathcal{T}_6 \right) \\ & + \sqrt{1-\alpha_\psi^2} \sin(\Delta\phi) \left( \alpha_\Lambda \mathcal{T}_3 + \alpha_{\bar{\Lambda}} \mathcal{T}_4 \right) \ ,\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_0 &= 1 \ , \\ \mathcal{T}_1 &= \sin\theta_\Lambda^2 \sin\theta_1 \sin\theta_2 \cos\phi_1 \cos\phi_2 + \cos^2\theta_\Lambda \cos\theta_1 \cos\theta_2 \ , \\ \mathcal{T}_2 &= \sin\theta_\Lambda \cos\theta_\Lambda \left( \sin\theta_1 \cos\theta_2 \cos\phi_1 + \cos\theta_1 \sin\theta_2 \cos\phi_2 \right) \ , \\ \mathcal{T}_3 &= \sin\theta_\Lambda \cos\theta_\Lambda \sin\theta_1 \sin\phi_1 \ , \\ \mathcal{T}_4 &= \sin\theta_\Lambda \cos\theta_\Lambda \sin\theta_2 \sin\phi_2 \ , \\ \mathcal{T}_5 &= \cos^2\theta_\Lambda \ , \\ \mathcal{T}_6 &= \cos\theta_1 \cos\theta_2 - \sin^2\theta_\Lambda \sin\theta_1 \sin\theta_2 \sin\phi_1 \sin\phi_2 \ .\end{aligned}$$

# $\Lambda\bar{\Lambda}$: measurements

$$\langle \mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) \rangle = \frac{1}{N} \int \mathcal{O}(\theta_1, \theta_2, \phi_1, \phi_2) \ \mathcal{W} \ d\Omega_p \ d\Omega_{\bar{p}} \ ,$$

$$N = \int \mathcal{W} \ d\Omega_p \ d\Omega_{\bar{p}} \ , \quad d\Omega_{p/\bar{p}} = d \cos \theta_{1/2} d\phi_{1/2} \ .$$

- This leads to the observables:

$$\mathcal{O}_0 = -1 \ ,$$
$$\mathcal{O}_1 = -\frac{1}{2}(1+\alpha_\psi)\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi \cos^2\theta_\Lambda} \ ,$$
$$\mathcal{O}_2 = -\frac{1}{2}(1-\alpha_\psi)\frac{1-\cos^2\theta_\Lambda}{1+\alpha_\psi \cos^2\theta_\Lambda} \ ,$$
$$\mathcal{O}_3 = 0 \ ,$$
$$\mathcal{O}_4 = 0 \ .$$

- $\mathcal{O}_0$, $\mathcal{O}_3$, and $\mathcal{O}_4$ all do not violate their respective separable state boundaries, thus can not demonstrate quantum entanglement in $\Lambda\bar{\Lambda}$.

# $\Lambda\bar{\Lambda}$: measurements

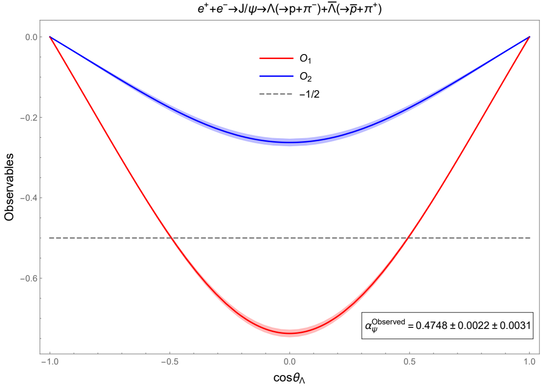



Figure: The $\mathcal{O}_1 = \langle \cos(\phi_1 + \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right)$ and $\mathcal{O}_2 = \langle \cos(\phi_1 - \phi_2) \rangle / \left(-\frac{\pi^2}{32}\alpha_\Lambda \alpha_{\bar{\Lambda}}\right)$ for $\Lambda\bar{\Lambda}$ pairs produced at $e^+e^-$ collider by using $\alpha_\psi^{\text{Observed}} = 0.4748 \pm 0.0022 \pm 0.0031$ from the BESIII experiment. The solid lines are given by the central value of $\alpha_\psi^{\text{Observed}}$, while the shaded bands correspond to the $5\sigma$ confidence region of $\alpha_\psi^{\text{Observed}}$.

# Summary

- We propose a new formalism for quantum entanglement, and study its generic searches at the colliders.
- We show that the quantum space is complex projective space, and the classical space is the cartesian product of the complex projective spaces, which can be defined in the quantum spacce via the discriminant loci. Thus, the quantum entanglement space is the difference of these two spaces and can be define exactly.
- We can reconstruct the discriminants from various measurements at high energy physics experiments, and probe the quantum entanglement spaces via a fundamental approach at exact level.

# Summary

- For the specific approach, we propose a generic method to calculate the quantum range and classical range for the expectation value of any physics observable, and can probe the quantum entanglement ranges which the previous ways cannot.
- We define the quantum non-locality tests as the tests for quantum entanglement space via the space-like separated measurements, which can be done at colliders as well.

# Thank You Very Much for Your Attention!

# No-Go Theorem for Collider Physics

- No-Go Theorem for collider physics [6].
- Choose a proper process, and rule out the Local Hidden Variable Theories (LHVTs).
- Choose another process, and measure the overall constant from QFT calculations. Point: avoid a circular argument.
- Munchhausen trilemma in philosophy: a circular argument, an infinite regression, and dogmatism.

[6] S. A. Abel, M. Dittmar and H. K. Dreiner, Phys. Lett. B **280**, 304-312 (1992); H. K. Dreiner, [arXiv:hep-ph/9211203 [hep-ph]]; S. Li, W. Shen and J. M. Yang, Eur. Phys. J. C **84**, no.11, 1195 (2024); M. Fabbrichesi, R. Floreanini and L. Marzola, [arXiv:2503.18535 [quant-ph]]; P. Bechtle, C. Breuning, H. K. Dreiner and C. Duhr, [arXiv:2507.15947 [hep-ph]]; S. A. Abel, H. K. Dreiner, R. Sengupta and L. Ubaldi, [arXiv:2507.15949 [hep-ph]].