

Analytical methods for cosmological correlators



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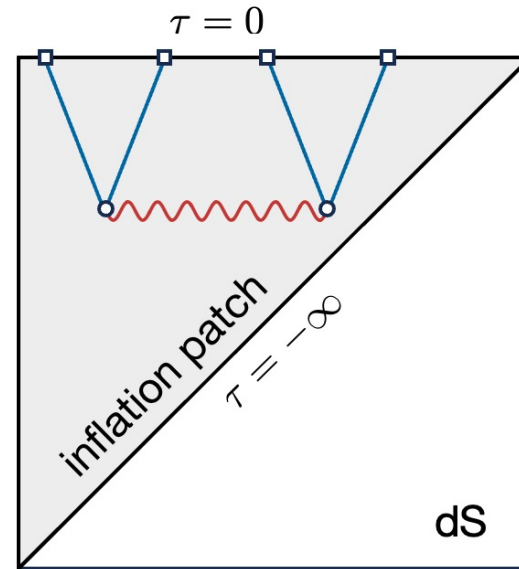
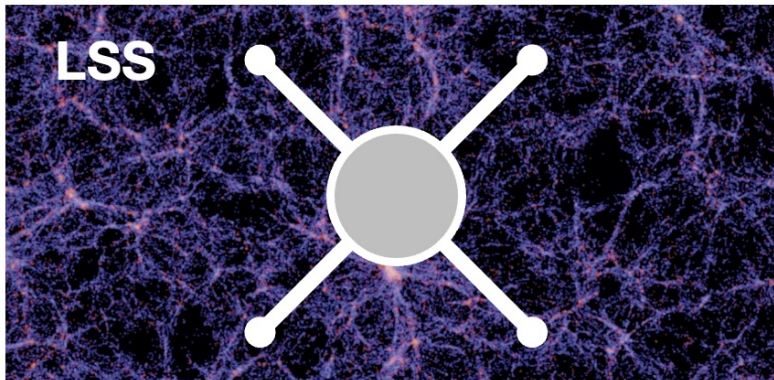
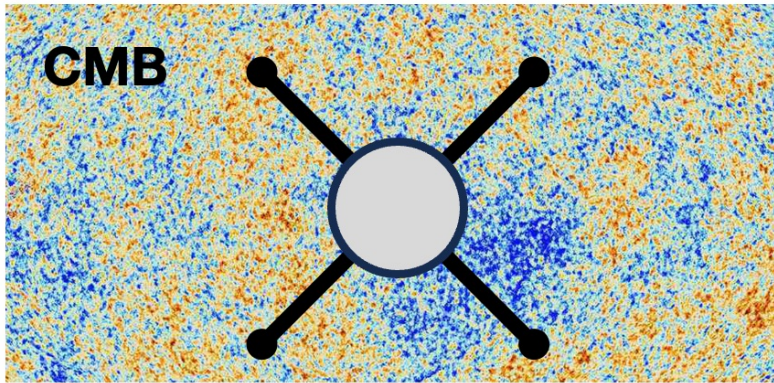
第四届高能物理理论与实验融合发展研讨会

辽宁师范大学 | September 20, 2025

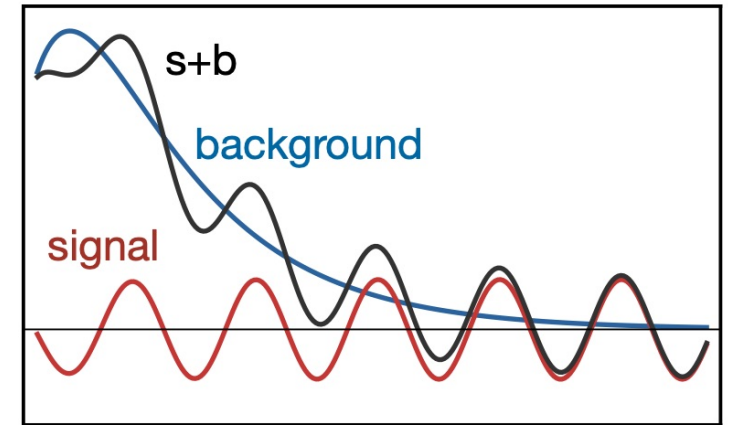
w/ Yunjia Bao, Xingang Chen, Yanou Cui, [Bingchu Fan](#), JiJi Fan, Soubhik Kumar, [Yuanzhao Li](#), [Haoyuan Liu](#), Tao Liu, Abraham Loeb, Qianshu Lu, Shiyun Lu, [Zhehan Qin](#), Matthew Reece, Xi Tong, Lian-Tao Wang, Yi Wang, [Jiayi Wu](#), [Jiaju Zang](#), [Hongyu Zhang](#), [Yisong Zhang](#), Yiming Zhong

A Cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]



Inflation \sim dS
particle production
mass $\sim 10^{14}$ GeV

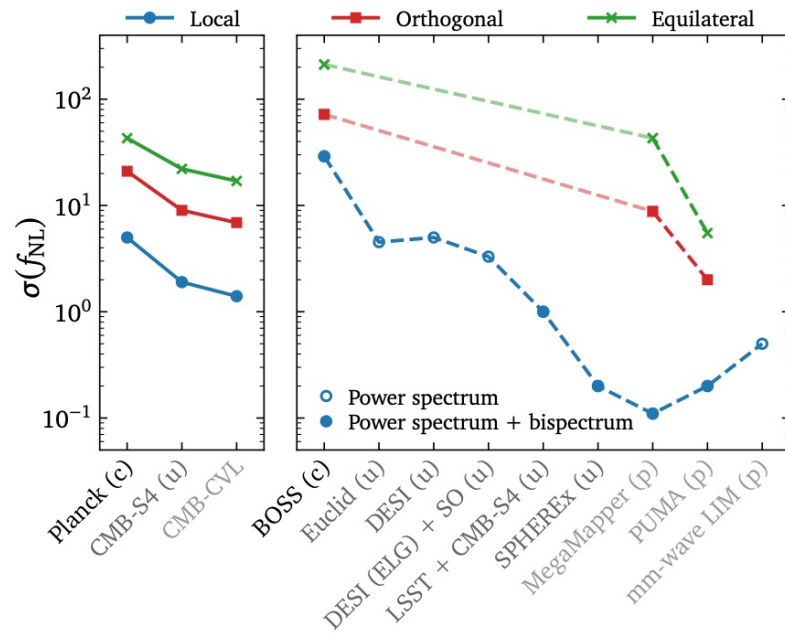


log of momentum ratio

superhorizon resonance
mass, spin, coupling, etc
amplitude nonanalyticity

Data are coming in!

- ~ 2 orders in near future; ~ 4 ultimately with 21cm



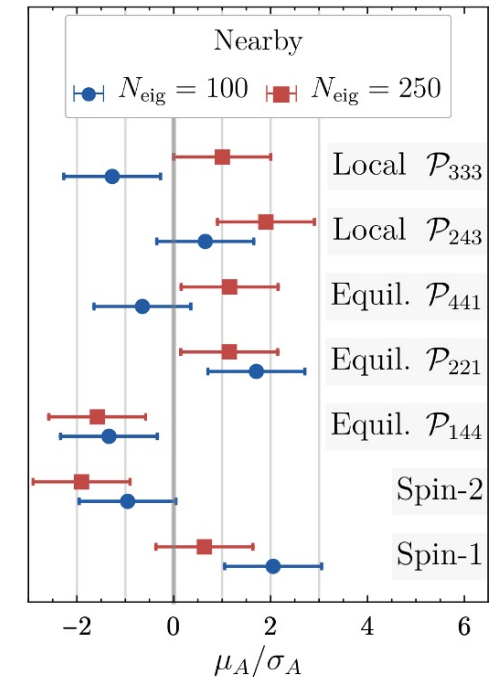
[Snowmass 2021: 2203.08128]

- Searches from CMB [Sohn et al. 2404.07203] and LSS data [Cabass et al. 2404.01894]

- Realistic particle models

- Parity violation [Bao, Wang, ZX, Zhong, 2504.02931]

- Quasi-single field inflation meets CMB [Kumar, Lu, ZX, Zhang, to appear]

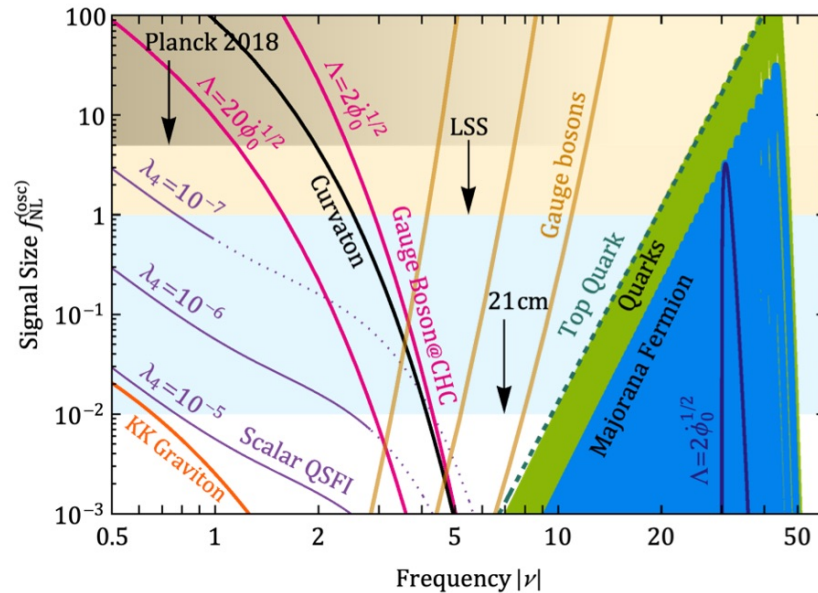


[Bao et al., 2504.02931]

Big questions

- Cosmological collider signals are cool, but:
 - Can they be true?
 - How to find them?

Can they be true? --- Particle Phenomenology



[Lian-Tao Wang, ZX, 1910.12876]

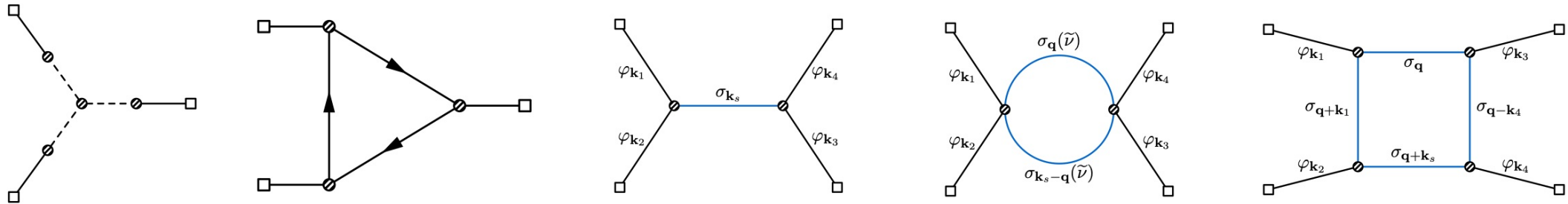
Over the years, many particle models identified in SM/BSM, with **naturally** large signals

Many fascinating stories which are still ongoing

The CC signals can be there, and deserve to be treated seriously

How to find them? --- Theory templates

- Behind the CC signals are “simple” Feynman graphs in the inflationary background:



- To look for CC signals in real data, we need a template bank
- Not a kinematic point, but the full shape; not for a parameter; but a multi-dim parameter grid
- We'd better compute them with **precision** and **efficiency**
- They may be hard, but let's not complain; Let's do it, **analytically**
- Developing fast! Many computations considered impossible a few years ago are now done

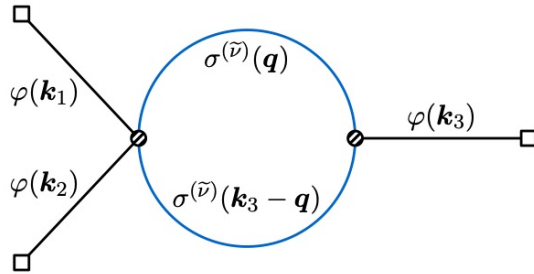
Why analytic?

- Data-wise: good analytical strategy speeds up numerical computation

Example: 3pt massive bubble: numerical $[O(10^5) \text{ CPU hrs}]$ vs. analytical $[O(10s) \text{ @ laptop}]$

[Wang, ZX, Zhong, 2109.14635]

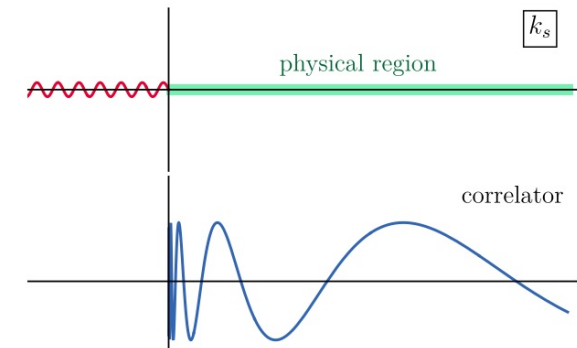
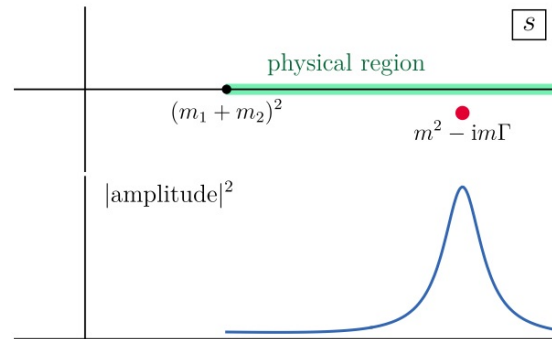
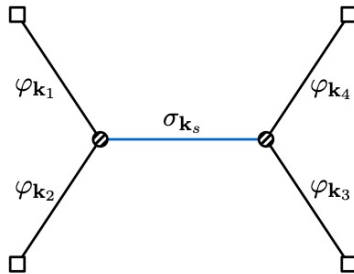
[Liu, Qin, ZX, 2407.12299]



$$\mathcal{J}^{0,-2}(u) = Cu^3 - \frac{u^4}{128\pi \sin(2\pi i\tilde{\nu})} \sum_{n=0}^{\infty} \frac{(3 + 4i\tilde{\nu} + 4n)(1 + n)_{\frac{1}{2}}(1 + 2i\tilde{\nu} + n)_{\frac{1}{2}}}{(\frac{1}{2} + i\tilde{\nu} + n)_{\frac{1}{2}}(\frac{3}{2} + i\tilde{\nu} + n)_{\frac{1}{2}}} \\ \times \left\{ {}_2\mathcal{F}_1 \left[\begin{matrix} 2 + 2i\tilde{\nu} + 2n, 4 + 2i\tilde{\nu} + 2n \\ 4 + 4i\tilde{\nu} + 4n \end{matrix} \middle| u \right] u^{2n+2i\tilde{\nu}} - {}_3\mathcal{F}_2 \left[\begin{matrix} 1, 2, 4 \\ 1 - 2n - 2i\tilde{\nu}, 4 + 2n + 2i\tilde{\nu} \end{matrix} \middle| u \right] \right\} \\ + (\tilde{\nu} \rightarrow -\tilde{\nu})$$

- Theory-wise: good lessons about QFT in dS from analytical structures of correlators

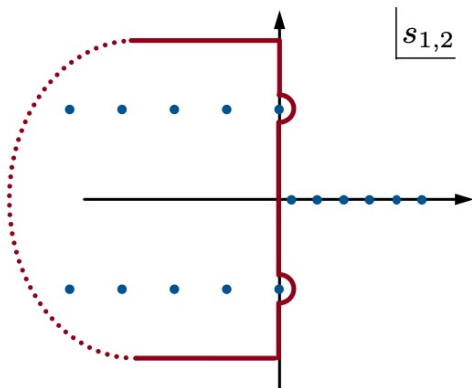
Whenever a correlator becomes singular, there is a physical reason



[Qin, ZX, 2308.14802]

Analytical methods

Partial Mellin-Barnes representation [Resolve!]



• Differential equations [Pinch!]

$$\mathcal{D}_{K_\alpha/E_i} \left(\begin{array}{c} \text{Diagram with vertices } p_i, p_j \text{ and edges } E_i, E_j \\ \text{with a red line connecting } p_i \text{ and } p_j \text{ labeled } K_\alpha, \tilde{\nu}_\alpha \end{array} \right) = \begin{array}{c} \text{Diagram with a single vertex } p_{ij+4} \text{ and edges } E_{ij} \end{array}$$

Family tree decomposition [Flip!]

$$\begin{array}{c} \bullet \xrightarrow{\quad} \tau_2 \\ \tau_1 \end{array} + \begin{array}{c} \tau_1 \xleftarrow{\quad} \bullet \\ \tau_2 \end{array} = \begin{array}{c} \bullet \text{---} \bullet \\ \tau_1 \quad \tau_2 \end{array}$$

• Dispersion relations [Glue!]

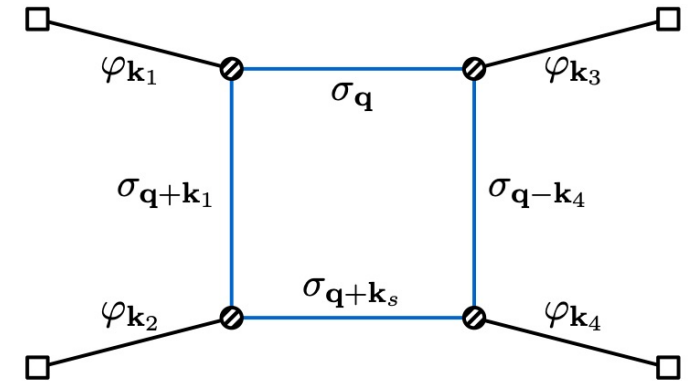
$$\begin{array}{c} \square \quad \square \\ \diagdown \quad \diagup \\ \circ \text{---} r \text{---} \circ \\ \diagup \quad \diagdown \\ \square \quad \square \end{array} = \int \frac{dr'}{2\pi i} \frac{1}{r' - r} \times \left(\begin{array}{c} \square \quad \square \\ \diagdown \quad \diagup \\ \circ \text{---} r' \text{---} \circ \\ \diagup \quad \diagdown \\ \square \quad \square \end{array} \times \begin{array}{c} \square \quad \square \\ \diagdown \quad \diagup \\ \circ \text{---} r' \text{---} \circ \\ \diagup \quad \diagdown \\ \square \quad \square \end{array} \right)$$

Massive inflation correlators

[See Chen, Wang, ZX, 1703.10166 for a review]

$$\mathcal{T}(\{\mathbf{k}\}) \sim \underbrace{\int d\tau}_{\text{vertex int}} \underbrace{\int d^d \mathbf{q}}_{\text{loop int}} \times (-\tau)^p \times e^{iE\tau} \times \underbrace{H_{i\tilde{\nu}} \left[-K(\mathbf{q}, \mathbf{k})\tau \right]}_{\text{ext line}} \times \underbrace{\theta(\tau_i - \tau_j)}_{\text{bulk line}}$$

- Massless / conformal external lines + (principal) massive internal lines
- Challenges:
 - Mode functions (Hankel, Whittaker, ...)
 - Loop momentum integrals
 - Nested time integrals
- Complexity increases with # of loops and # of vertices



Partial Mellin-Barnes representation

[Qin, ZX, 2205.01692, 2208.13790]

MB rep for all **bulk lines**; Resolving special functions into power functions

For example: Massive scalar propagator [Hankel function]

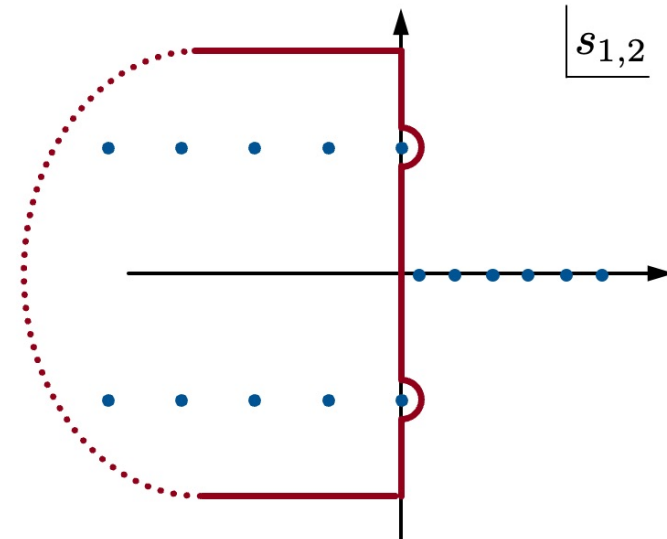
$$H_{\nu}^{(1)}(-k\tau) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \left(\frac{k}{2}\right)^{-2s} (-\tau)^{-2s} e^{(2s-\nu-1)\pi i/2} \Gamma\left[s - \frac{\nu}{2}, s + \frac{\nu}{2}\right]$$

Expanding in dilatation eigenmode, but no **dilatation or boost symmetry required**

Time and loop momentum integrals factorized, enabling separate treatments

$$\mathcal{T}(\{\mathbf{k}\}) \sim \int ds \times \mathcal{G}(s) \times \left[\int d^d \mathbf{q} K(\mathbf{q}, \mathbf{k})^\alpha \right] \quad \text{Loop int}$$

$$\times \left[\int d\tau e^{iE\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j) \right] \quad \text{Time int}$$



[See also Sleight 1907.01143 etc.]

Family tree decomposition

[ZX, Zang, 2309.10849; Fan, ZX, 2403.07050, 2509.02684]

Family tree decomposition: flip the directions such that all graphs are partially ordered

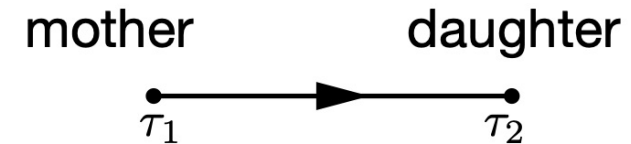
$$\theta(\tau_1 - \tau_2) + \theta(\tau_2 - \tau_1) = 1$$



Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



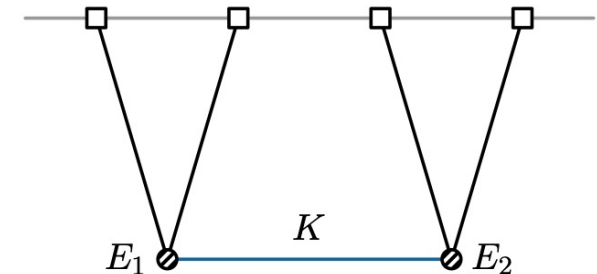
Every resulting nested graph can be interpreted as a **maternal family tree**

A notation for FTs: $\left[\overset{\text{sisters}}{\underset{\text{mother-daughter}}{12(34 \cdots)(5 \cdots)}} \right] = \int_{-\infty}^0 \prod_{i=1}^N \left[d\tau_i (-\tau_i)^{q_i-1} e^{i\omega_i \tau_i} \right] \theta_{21} \theta_{32} \theta_{52} \theta_{43} \cdots$

$$[\mathcal{P}(\hat{1}2 \cdots N)] = \frac{(-i)^N}{(i\omega_1)^{q_1 \cdots N}} \sum_{n_2, \dots, n_N=0}^{\infty} \Gamma(q_1 \cdots N + n_2 \cdots N) \prod_{j=2}^N \frac{(-\omega_j / \omega_1)^{n_j}}{(\tilde{q}_j + \tilde{n}_j) n_j!}$$

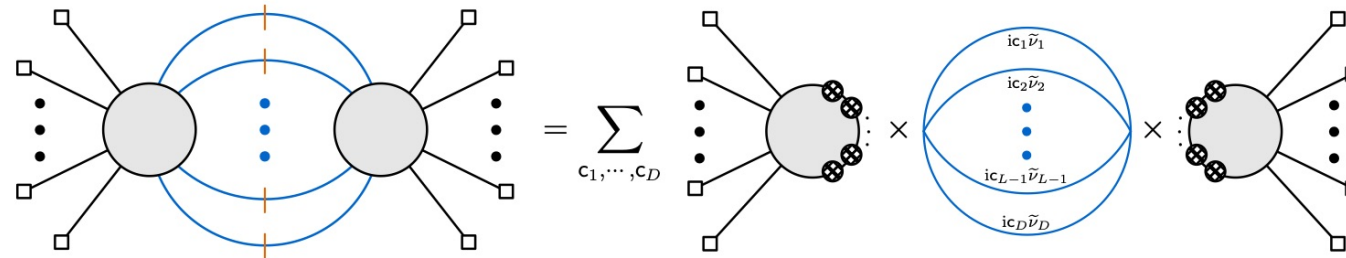
What can PMB + FTD do?

- Tree level: A trivial procedure to get analytical results for all trees [solved]
PMB => FTD => Collecting MB poles => Solutions in hypergeo series [ZX, Zang, 2309.10849]
- A byproduct: complete analytical answer for all conformal scalar tree amplitudes in power-law FRW space [automatically solve the kinematic flow diff eqs] [Fan, ZX, 2403.07050]
- Beyond tree level: Full computation remains challenging
However, very useful for studying analytical structure of arbitrary loop graphs
- Generally, a (tree or loop) correlator can exhibit singular behavior (branch point) at:
 - Nonlocal signal branch points (soft momentum limit)
 $K \rightarrow 0$
 - Local signal branch points (hard energy limit)
 $E_1 \rightarrow \infty$ or $E_2 \rightarrow \infty$
 - Partial energy branch points (zero energy sum limit)
 $E_1 + K \rightarrow 0$ or $E_2 + K \rightarrow 0$ or $E_1 + E_2 \rightarrow 0$



Singularity structure / factorization theorems / cutting rules

- Take the nonlocal signal as an example (very relevant to CC pheno)
- The nonlocal signal is factorized (and thus cut) and computable to the leading order in the soft momentum but to all loop orders [Qin, ZX, 2304.13295; 2308.14802]



$$\mathfrak{M}_{c_1 \dots c_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}} \Gamma \left[\begin{matrix} -\sum_{i=1}^D c_i i \tilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^D c_i i \tilde{\nu}_i \end{matrix} \right] \prod_{\ell=1}^D \left\{ \Gamma \left[\frac{3}{2} + c_\ell i \tilde{\nu}_\ell, -c_\ell i \tilde{\nu}_\ell \right] \left(\frac{P}{2} \right)^{2i c_\ell \tilde{\nu}_\ell} \right\}$$

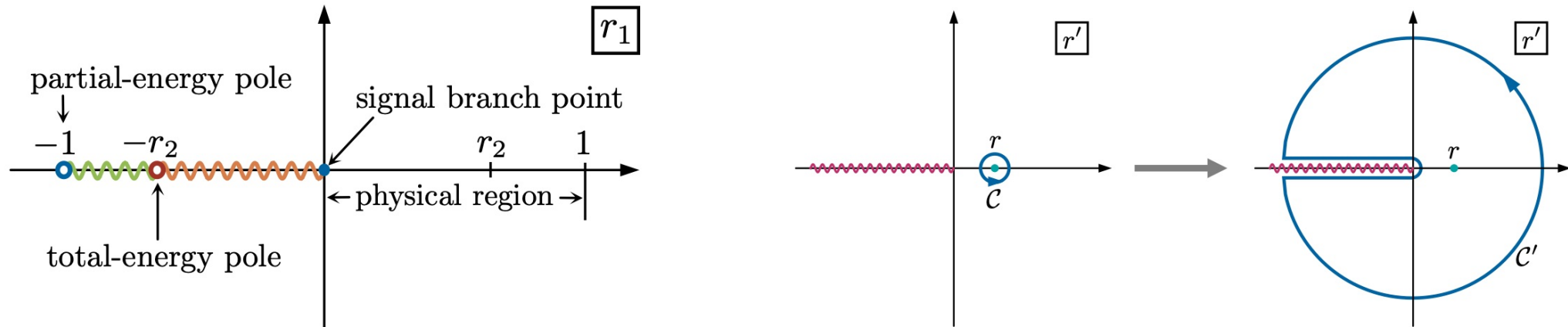
- Similar factorization and cutting rules hold for the local signal [Qin, ZX, to appear] and partial-energy limit [Wu, ZX, Zhang, to appear]
- In a sense, the nonanalytic part is always “simpler” than the analytic part

A dispersive bootstrap

[Liu, Qin, ZX, 2407.12299; Liu, Qin, Wu, ZX, Zhang, to appear]

The study of analyticity allows us to locate all singularities on the complex plane

=> Bootstrapping complex graphs by gluing simpler ones. The glue: dispersion integral



Dispersion integrals are insensitive to UV (local) physics

New and much simplified analytical expression for loops; UV and IR neatly separated

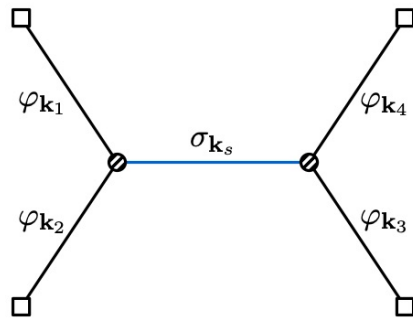
In particular: we identify an “irreducible background” demanded by analyticity

Lesson: UV div/regularization artificial and avoidable; but renormalization physical

[See also: Werth, 2409.02072]

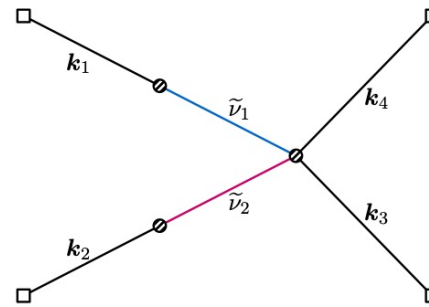
Differential equations

- “Old” technique, first used in cosmo correlators as a “bootstrap” equation
- However, much easier to derive and to generalize in the bulk



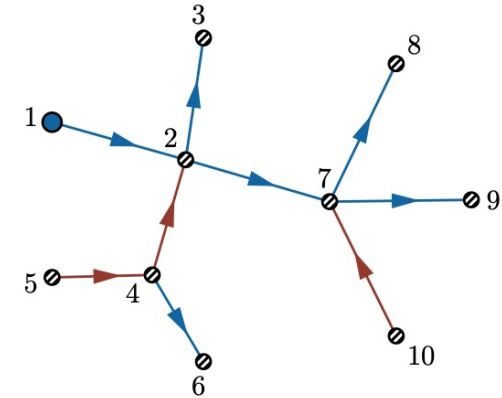
1 exchange (2018)

“Cosmological bootstrap”
[Arkani-Hamed, Baumann,
Lee, Pimentel, 1811.00024]



2 exchanges (2024)

[ZX, Zang, 2309.10849]
[Aoki, Pinol, Sano, Yamaguchi,
Zhu, 2404.09547]



Arbitrary exchanges (2023)

Partial Mellin-Barnes
[Qin, ZX, 2205.01692, 2208.13790]
Family tree [ZX, Zang, 2309.10849]

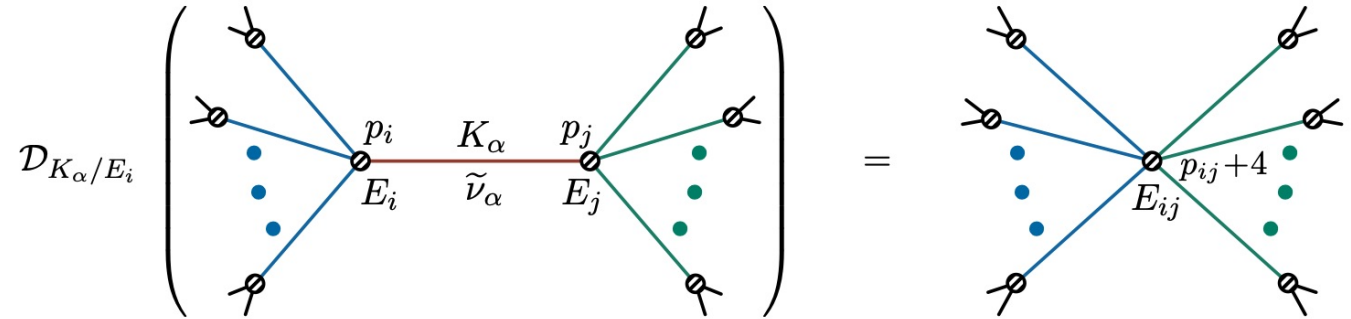
- **Partial Mellin-Barnes + family-tree decomposition** reduce all analytical computation to a trivial but tedious routine; The results involve too many layers of summations
- **It'd be good to have a rule to write down the results without doing any computation**

[See also Pimentel, Wang, 2205.00013, Qin, ZX, 2208.13790, 2301.07047, Jazayeri, Renaux-Petel, 2205.10340, Qin, Renaux-Petel, Tong, Werth, Zhu, 2506.01555, etc]

Differential equations for arbitrary massive trees

[Liu, ZX, 2412.07843]

- An internal line (bulk propagator) is collapsed to 0 or δ by a Klein-Gordon operator
- The KG operator can be pulled out of the integral with IBP at a given vertex
- We obtain a 2nd order diff eq for the graph by picking up a line + one of its two endpoint
- There are a total of $2I$ choices $\Rightarrow 2I$ diff eqs for $2I$ indep energy ratios. A complete set!



$$\mathcal{D}_{(\alpha i)} \mathcal{G} = \frac{r_{(\alpha i)}^{p_j+4} r_{(\alpha j)}^{p_i+4}}{[r_{(\alpha i)} + r_{(\alpha j)}]^{p_{ij}+5}} \mathcal{C}_\alpha[\mathcal{G}],$$

$$\mathcal{D}_{(\alpha i)} \equiv \left(\vartheta_{(\alpha i)} - \frac{3}{2} \right)^2 + \tilde{\nu}_\alpha^2 - r_{(\alpha i)}^2 (\vartheta_{\{i\}} + p_i + 2) (\vartheta_{\{i\}} + p_i + 1)$$

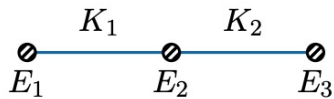
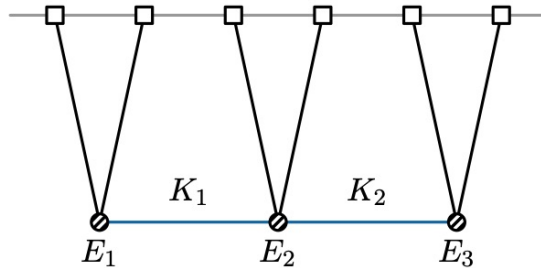
$$r_{(\alpha i)} = \frac{K_\alpha}{E_i} \quad \vartheta_{(\alpha i)} \equiv r_{(\alpha i)} \frac{\partial}{\partial r_{(\alpha i)}} \quad \vartheta_{\{i\}} \equiv \sum_{\beta \in \mathcal{N}(i)} \vartheta_{(\beta i)}$$

Complete solution

[Liu, ZX, 2412.07843]

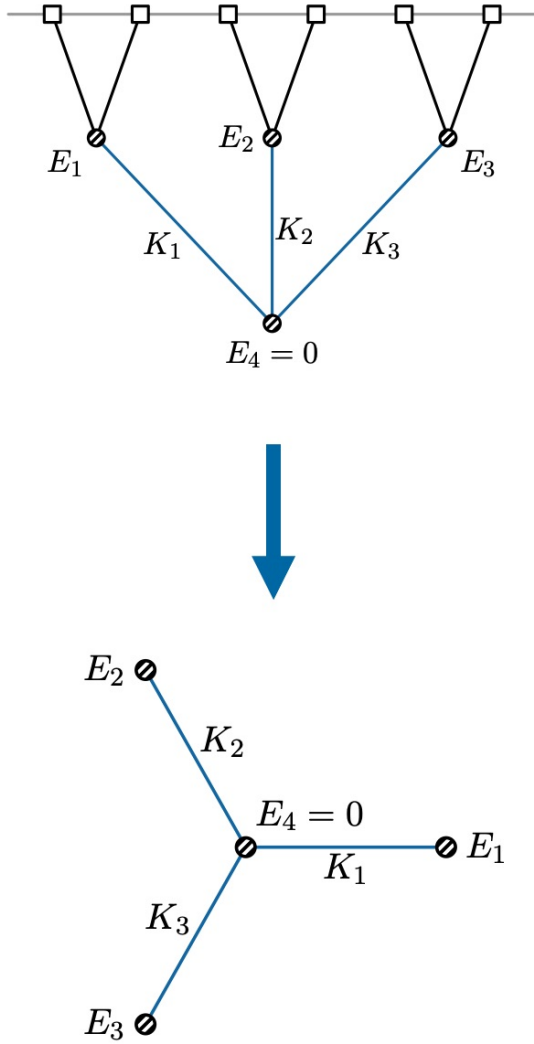
$$\mathcal{G} = \sum_{i \in 2^K} C_i^{\text{ut}} [\mathcal{G}]$$

- The complete solution to arbitrary massive tree is the sum of the CIS (completely inhom sol) and all of its cuts.
- CIS => massive family tree
- Cuts => “tuned” (# or ♭) massive family trees



$$\begin{array}{c} \begin{array}{c} \textcircled{\times} \xrightarrow{K_1} \textcircled{\times} \xrightarrow{K_2} \textcircled{\times} \\ E_1 \quad E_2 \quad E_3 \end{array} = \begin{array}{c} \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \\ \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \end{array} + \begin{array}{c} \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \\ \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \end{array} \\ + \begin{array}{c} \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \\ \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \end{array} + \begin{array}{c} \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \\ \textcircled{\times} \xrightarrow{\text{blue}} \textcircled{\times} \xrightarrow{\text{red}} \textcircled{\times} \end{array} \end{array}$$

$$\begin{aligned} \mathcal{G}_3 = & \llbracket 123 \rrbracket + \llbracket 1^{\sharp 1} \rrbracket \left(\llbracket 2^{\sharp 1} 3 \rrbracket + \llbracket 2^{\flat 1} 3 \rrbracket \right) + \llbracket 12^{\sharp 2} \rrbracket \left(\llbracket 3^{\sharp 2} \rrbracket + \llbracket 3^{\flat 2} \rrbracket \right) \\ & + \llbracket 1^{\sharp 1} \rrbracket \left(\llbracket 2^{\sharp 1 \sharp 2} \rrbracket + \llbracket 2^{\flat 1 \sharp 2} \rrbracket \right) \left(\llbracket 3^{\sharp 2} \rrbracket + \llbracket 3^{\flat 2} \rrbracket \right) + \text{shadows} \end{aligned}$$



$$\mathcal{G}'_4 = \text{CIS} [\mathcal{G}'_4] + \sum_{\alpha=1}^3 \text{Cut}_{K_\alpha} [\mathcal{G}'_4] + \sum_{\alpha \neq \beta} \text{Cut}_{K_\alpha, K_\beta} [\mathcal{G}'_4] + \text{Cut}_{K_1, K_2, K_3} [\mathcal{G}'_4]$$

$$\text{CIS} [\mathcal{G}'_4] = \llbracket 1\cancel{A}(2)(3) \rrbracket,$$

$$\begin{aligned} \sum_{\alpha=1}^3 \text{Cut}_{K_\alpha} [\mathcal{G}'_4] &= \llbracket 1^{\#1} \rrbracket \left(\llbracket 2\cancel{A}^{\#1} 3 \rrbracket + \llbracket 2\cancel{A}^{b1} 3 \rrbracket \right) + \llbracket 1\cancel{A}^{\#2} 3 \rrbracket \left(\llbracket 2^{\#2} \rrbracket + \llbracket 2^{b2} \rrbracket \right) \\ &\quad + \llbracket 1\cancel{A}^{\#3} 2 \rrbracket \left(\llbracket 3^{\#3} \rrbracket + \llbracket 3^{b3} \rrbracket \right) + \text{shadows}, \end{aligned}$$

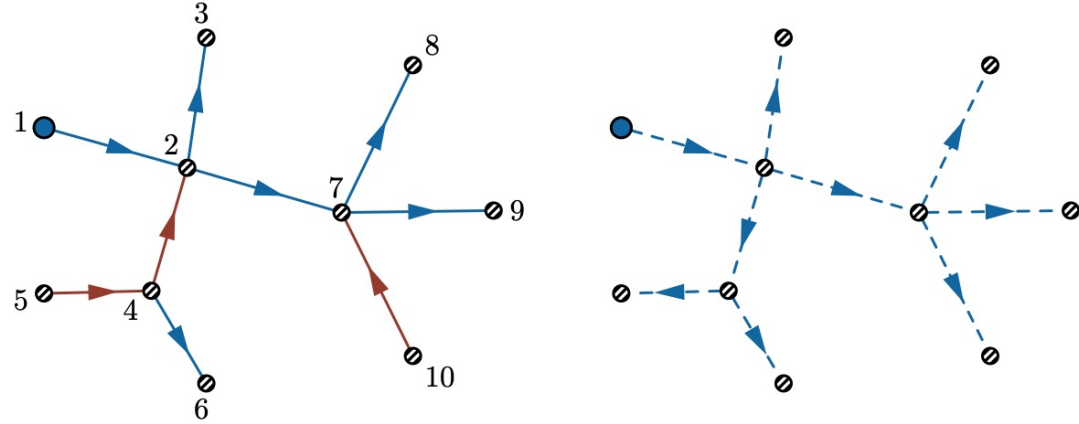
$$\begin{aligned} \sum_{\alpha \neq \beta} \text{Cut}_{K_\alpha, K_\beta} [\mathcal{G}'_4] &= \llbracket 1^{\#1} \rrbracket \llbracket 2^{\#2} \rrbracket \left(\llbracket 3\cancel{A}^{\#1\#2} \rrbracket + \llbracket 3\cancel{A}^{b1\#2} \rrbracket + \llbracket 3\cancel{A}^{\#1b2} \rrbracket + \llbracket 3\cancel{A}^{b1b2} \rrbracket \right) \\ &\quad + \llbracket 1^{\#1} \rrbracket \left(\llbracket 2\cancel{A}^{\#1\#3} \rrbracket + \llbracket 2\cancel{A}^{b1\#3} \rrbracket \right) \left(\llbracket 3^{\#3} \rrbracket + \llbracket 3^{b3} \rrbracket \right) \\ &\quad + \llbracket 1\cancel{A}^{\#2\#3} \rrbracket \left(\llbracket 2^{\#2} \rrbracket + \llbracket 2^{b2} \rrbracket \right) \left(\llbracket 3^{\#3} \rrbracket + \llbracket 3^{b3} \rrbracket \right) + \text{shadows}, \end{aligned}$$

$$\begin{aligned} \text{Cut}_{K_1, K_2, K_3} [\mathcal{G}'_4] &= \llbracket 1^{\#1} \rrbracket \llbracket 2^{\#2} \rrbracket \llbracket 3^{\#3} \rrbracket \left(\llbracket \cancel{A}^{\#1\#2\#3} \rrbracket + \llbracket \cancel{A}^{b1\#2\#3} \rrbracket + \llbracket \cancel{A}^{\#1b2\#3} \rrbracket + \llbracket \cancel{A}^{\#1\#2b3} \rrbracket \right. \\ &\quad \left. + \llbracket \cancel{A}^{b1b2\#3} \rrbracket + \llbracket \cancel{A}^{b1\#2b3} \rrbracket + \llbracket \cancel{A}^{\#1b2b3} \rrbracket + \llbracket \cancel{A}^{b1b2b3} \rrbracket \right) + \text{shadows} \end{aligned}$$

Completely inhomogeneous solution: massive family trees

- Quite remarkably, the CIS has a direct hypergeo rep: $\text{CIS} [\mathcal{G}_V] = \llbracket \mathcal{P}(1 \cdots V) \rrbracket$
- The solution expanded in the largest vertex energy ($1/E_1$), indep of the order of other energies
- Picking up a largest energy automatically generates a partial order: **massive family tree**
 q : a “family parameter” encoding the tree structure:

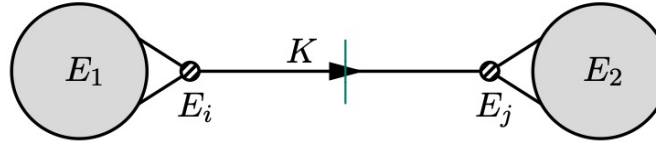
$$q_i \equiv \tilde{\ell}_i + 2\tilde{m}_i + \tilde{p}_i + 4N_i$$



$$\begin{aligned} \llbracket 1 \cdots V \rrbracket &= \sum_{\{\ell, m\}} 2^V \cos(\pi p_1 \cdots V / 2) \Gamma(q_1 + p_1 + 1) \\ &\times \prod_{i=2}^V \frac{(-1)^{\ell_i}}{\ell_i! \left(\frac{\ell_i + q_i + p_i}{2} + \frac{5}{4} \pm \frac{i\tilde{\nu}_i}{2} \right)_{m_i+1}} \left(\frac{K_i}{2E_1} \right)^{2m_i+3} \left(\frac{E_i}{E_1} \right)^{\ell_i + p_i + 1} \end{aligned}$$

Homogeneous solutions: cuts of massive family trees

- The homogeneous solutions are obtained by executing appropriate cuts:



$$\text{Cut}_{K_\alpha} [\tilde{\mathcal{G}}_V] = \llbracket \hat{1} \dots i^\# \dots V_1 \rrbracket \left\{ \llbracket (V_1 + 1) \dots j^\# \dots V \rrbracket + \llbracket (V_1 + 1) \dots j^\flat \dots V \rrbracket \right\} + \text{c.c.}$$

- The cut involves certain dressings of massive family trees: augmentation and flattening:

$$\llbracket \dots i^\# \dots \rrbracket \equiv \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\Gamma(-m - i\tilde{\nu}_\alpha)}{m!} \left(\frac{K_\alpha}{2E_i} \right)^{2m+i\tilde{\nu}_\alpha+3/2} \llbracket \dots i \dots \rrbracket_{p_i \rightarrow p_i+2m+i\tilde{\nu}_\alpha+3/2} ,$$

$$\llbracket \dots i^\flat \dots \rrbracket \equiv \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\Gamma(-m + i\tilde{\nu}_\alpha)}{m!} \left(\frac{K_\alpha}{2E_i} \right)^{2m-i\tilde{\nu}_\alpha+3/2} \left\{ \frac{\cos \left[\frac{\pi(p_{\text{tot}}+2i\tilde{\nu}_\alpha)}{2} \right]}{\cos \left(\frac{\pi p_{\text{tot}}}{2} \right)} \llbracket \dots i \dots \rrbracket \right\}_{p_i \rightarrow p_i+2m-i\tilde{\nu}_\alpha+3/2}$$

Where are we now?

- Massive tree graphs: solved; WYSIWYG solutions, in hypergeo series
- Loop level: simple 1-loop graphs (massive bubbles) computed, also in hypergeo series
- Analytical structures largely known for all trees and many loops: only poles / branch points of finite degrees
- **Conjecture:** Any graphic contribution to a renormalized massive cosmological correlator is a multivariate hypergeometric function with only power-law singularities (finite-deg poles or branch points)
- Most of these hypergeo functions are not yet named, and are like “black boxes”
- Then what does the analytical calculation mean other than giving correlators names?
- Why pF_q / Appell / Lauricella look like black boxes to us, but sine and cosine do not?

What is analytical computation?

- Using series solutions to define, identify, and represent family trees (hypergeo functions)
- Using the flexibility of FTD to link different reps of family trees => Analytical continuation!

$$\begin{array}{c} \bullet \\ \tau_1 \end{array} \longrightarrow \begin{array}{c} \bullet \\ \tau_2 \end{array} + \begin{array}{c} \bullet \\ \tau_1 \end{array} \longleftarrow \begin{array}{c} \bullet \\ \tau_2 \end{array} = \begin{array}{c} \bullet \\ \tau_1 \end{array} \text{-----} \begin{array}{c} \bullet \\ \tau_2 \end{array}$$

$$[12] = [12] \quad \frac{1}{\omega_1^{q_{12}}} {}_2\mathcal{F}_1 \left[\begin{matrix} q_2, q_{12} \\ q_2 + 1 \end{matrix} \middle| -\frac{\omega_2}{\omega_1} \right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}} {}_2\mathcal{F}_1 \left[\begin{matrix} 1, q_{12} \\ q_2 + 1 \end{matrix} \middle| \frac{\omega_2}{\omega_{12}} \right]$$

$$[12] + [21] = [1][2] \quad \frac{1}{\omega_1^{q_{12}}} {}_2\mathcal{F}_1 \left[\begin{matrix} q_2, q_{12} \\ q_2 + 1 \end{matrix} \middle| -\frac{\omega_2}{\omega_1} \right] + \frac{1}{\omega_2^{q_{12}}} {}_2\mathcal{F}_1 \left[\begin{matrix} q_1, q_{12} \\ q_1 + 1 \end{matrix} \middle| -\frac{\omega_1}{\omega_2} \right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1} \omega_2^{q_2}}$$

$$[123] + [2(1)(3)] = [1][23] \quad \frac{1}{\omega_1^{q_{123}}} {}^{2+1}\mathcal{F}_{1+1} \left[\begin{matrix} q_{123}, q_{23} \\ q_{23} + 1 \end{matrix} \middle| \begin{matrix} -, q_3 \\ -, q_3 + 1 \end{matrix} \middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1} \right] + \frac{1}{\omega_2^{q_{123}}} \mathcal{F}_2 \left[\begin{matrix} q_{123} \\ q_1 + 1, q_3 + 1 \end{matrix} \middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2} \right]$$

$$= \frac{\Gamma[q_1]}{\omega_1^{q_1} \omega_2^{q_{23}}} {}_2\mathcal{F}_1 \left[\begin{matrix} q_3, q_{32} \\ q_3 + 1 \end{matrix} \middle| -\frac{\omega_3}{\omega_2} \right]$$

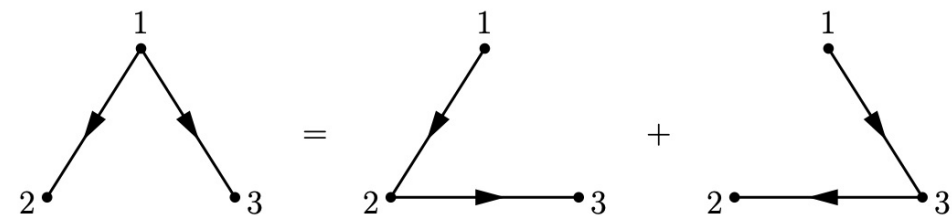
- We are currently able to find hypergeo series reps for any family trees at all of their singular points [Fan, ZX, 2509.02684]

Family trees are further decomposable
into chains [Fan, ZX, 2403.07050; 2509.]

$$\theta_{21}\theta_{31}(\theta_{32} + \theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$

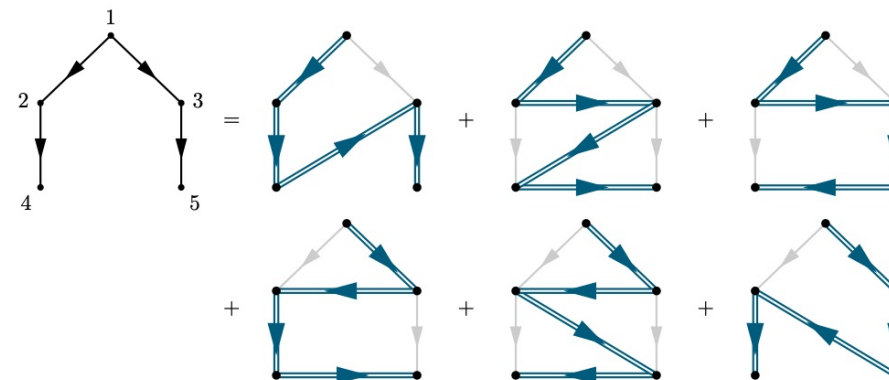
Shuffle product:

$$ab \sqcup cd = abcd + acbd + acdb \\ + cabd + cadb + cdab$$



Practically: taking shuffle product
recursively among all subfamilies

$$\begin{aligned} [1(24)(35)] &= \{1(24) \sqcup (35)\} \\ &= \{12435\} + \{12345\} + \{12354\} \\ &\quad + \{13245\} + \{13254\} + \{13524\} \end{aligned}$$



Family chain: standard iterated integrals; Hopf algebra; transcendental weight;
Higher weight functions cannot be fully reduced to lower weight functions

Final thoughts and outlooks

- We have found simple rules to identify & write down all hypergeo sols for any tree graphs
- Straightforward generalizations:
 - degenerate limits □ boost-breaking dispersion □ loop integrands [ongoing]
- However, we have to deal with unfamiliar hypergeo functions. Two bold programs:
 - Charting out all singularity structure of cosmological correlators
 - Obtaining hypergeo series reps at all singularities (at least for trees)
- In the meantime, many classic pheno examples remain challenging (triple exchange / strong mixing / chemical potential loops), even numerically. We should work harder
- Thinking pheno-wise: **all computations must be initiated analytically and finished numerically, the only question being where to execute the analytical-to-numerical transition**
- We hope that some of the analytical progress can provide new insights and better answers!

Thank you!