Coherent Background and Multiple Solutions

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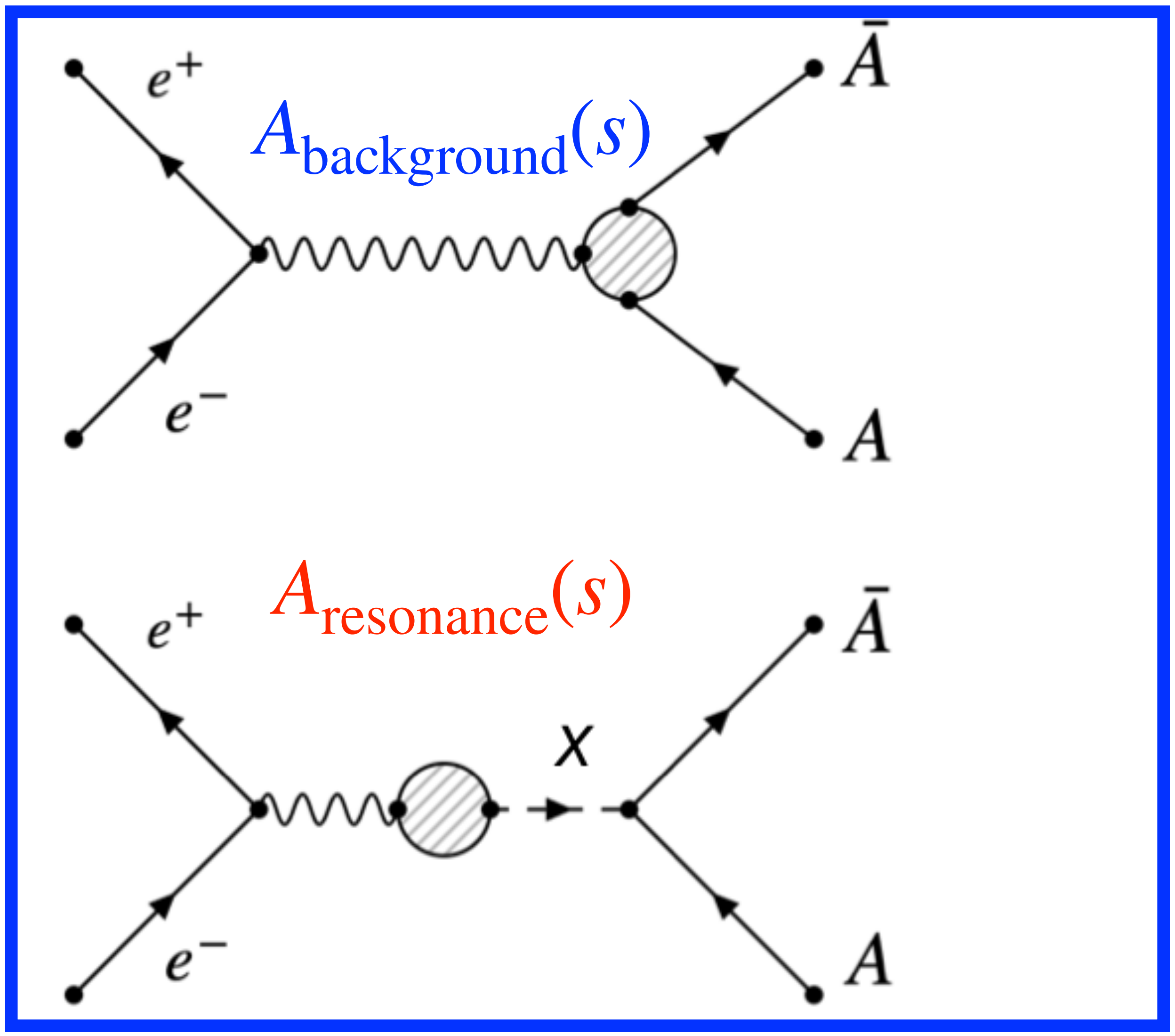
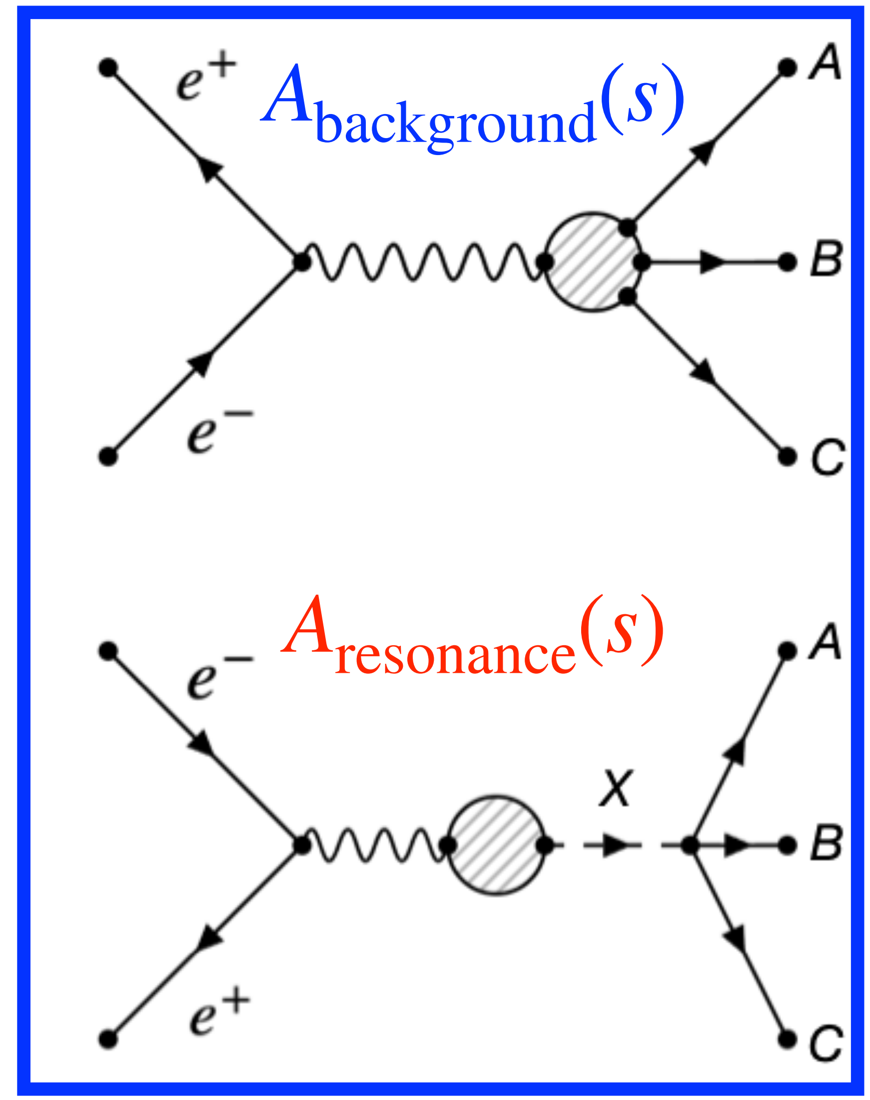
2025-09-21

Coherent Background





$$\sigma(s) \propto |A_{\text{bkg}}(s) + A_{\text{resonances}}(s)|^2$$

$$= |A_{\text{bkg}}(s)|^2 + |A_{\text{resonances}}(s)|^2 + A_{\text{bkg}}(s)A^*_{\text{resonances}}(s) + A_{\text{resonances}}(s)A^*_{\text{bkg}}(s)$$

Interference from Background

Coherent Background

$$\sigma = |A_{\rm bkg}(s) + A_{\rm resonances}(s)|^2$$
$$= |A_{\rm bkg}(s)|^2 + |A_{\rm resonances}(s)|^2 + \underbrace{A_{\rm bkg}(s)A^*_{\rm resonances}(s) + A_{\rm resonances}(s)A^*_{\rm bkg}(s)}_{\text{Interference from Background}}$$

- $A_{\rm resonances}$ usually takes the form of Breit-Wigner (B-W) function： such as
$$\frac{\sqrt{12\pi\Gamma\Gamma_{ee}\times\mathscr{B}}e^{i\phi}}{s-M^2+iM\Gamma}\frac{\sqrt{\Phi(\sqrt{s})}}{\sqrt{\Phi(M)}}\frac{M}{\sqrt{s}}$$
- $A_{\rm bkg}(s)$ usually takes the form of a smooth function, no pole structure
- Generally $A_{\rm bkg}(s)$ is difficult to determined from first principle， usually empirical(constant/phase space， polynomial, exponential, power-law etc.)

Example of Coherent Background in Data Analysis

$$\sigma = |A_{\rm bkg}(s) + A_{\rm resonances}(s)|^2$$

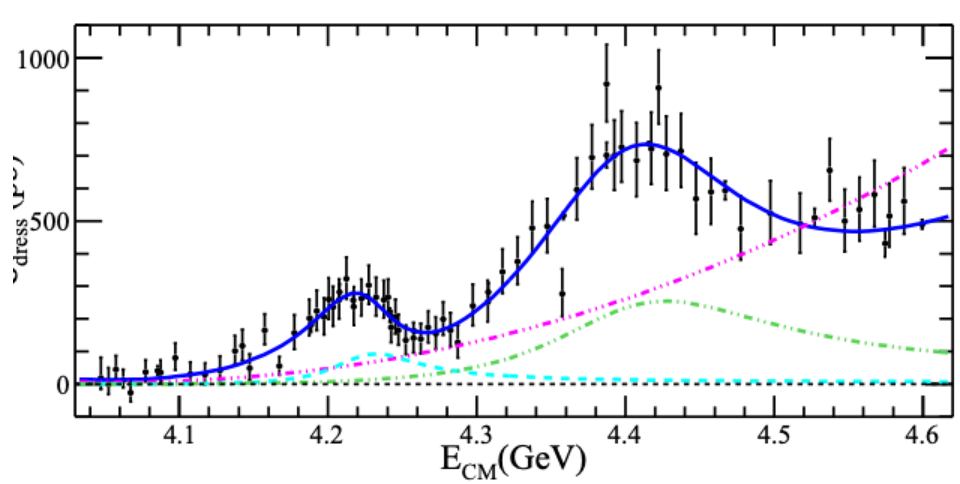
$$= |A_{\rm bkg}(s)|^2 + |A_{\rm resonances}(s)|^2 + \underbrace{A_{\rm bkg}(s)A^*_{\rm resonances}(s) + A_{\rm resonances}(s)A^*_{\rm bkg}(s)}_{\text{Interference from Background}}$$

Interference from Background

Phys. Rev. Lett, 122, 102002 (2019)

$$e^+e^- \to \pi^+ D^0 D^{*-}$$

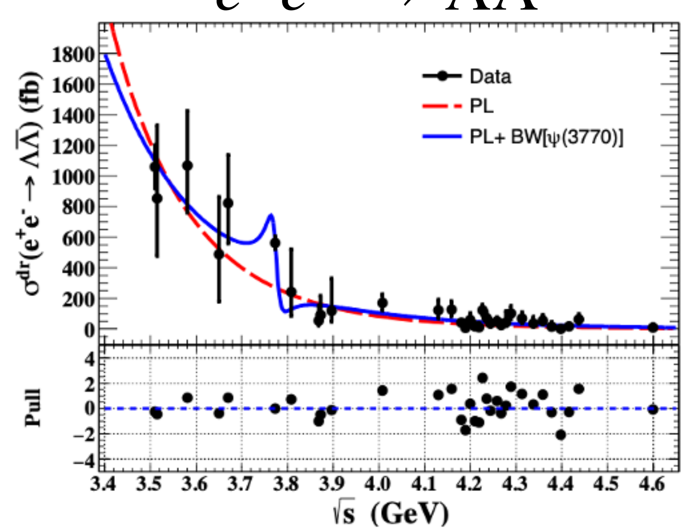
$$\sigma_{\rm dress}(m) = \left|c\sqrt{P(m)} + e^{i\phi_1}B_1(m)\sqrt{P(m)/P(M_1)} + e^{i\phi_2}B_2(m)\sqrt{P(m)/P(M_2)}\right|^2$$



Phys. Rev. D, 104, L091104 (2021)

$$\sigma^{\rm dr}(s) = \left|\sqrt{\sigma_0}\left(\frac{M}{\sqrt{s}}\right)^n + e^{i\phi}{\rm BW}(s)\right|^2$$

$$e^+e^- \to \Lambda\bar{\Lambda}$$



Phys. Rev. D, 104, 052012 (2021)

$$e^+e^- \to \pi^+\pi^-\Psi(2S)$$

$$\sigma^{\rm dressed}(\sqrt{s}) = \left|\sum_k e^{i\phi_k}\cdot BW_k(s) + e^{i\phi_{\rm cont}}\cdot\psi_{\rm cont}\right|^2$$

$$\psi_{\rm cont} = \frac{a}{(\sqrt{s})^n}\sqrt{\Phi(\sqrt{s})}.$$

$$\sqrt{\sigma_{NY}} = \sqrt{\Phi(\sqrt{s})e^{p_0 u}p_1},$$

$$u = \sqrt{s} - M_{thresh}$$

Phase space (const) background

power law background

exponential background

Coherent Background and Multiple Solutions

Phys. Rev. Lett, 122, 102002 (2019)

$$\sigma_{\text{dress}}(m) = \left|c\sqrt{P(m)} + e^{i\phi_1}B_1(m)\sqrt{P(m)/P(M_1)} + e^{i\phi_2}B_2(m)\sqrt{P(m)/P(M_2)}\right|^2$$

Parameter	Solution I	Solution II	Solution III	Solution IV
c (MeV$^{-3/2}$)	$(6.2 \pm 0.5) \times 10^{-4}$			
M_1 (MeV/c^2)	4228.6 ± 4.1			
Γ_1 (MeV)	77.0 ± 6.8			
M_2 (MeV/c^2)	4404.7 ± 7.4			
Γ_2 (MeV)	191.9 ± 13.0			
Γ_1^{el} (eV)	77.4 ± 10.1	8.6 ± 1.6	99.5 ± 14.6	11.1 ± 2.3
Γ_2^{el} (eV)	100.4 ± 13.3	64.2 ± 8.0	664.2 ± 80.0	423.0 ± 47.0
ϕ_1 (rad)	-2.0 ± 0.1	3.0 ± 0.2	-0.9 ± 0.1	-2.2 ± 0.1
ϕ_2 (rad)	2.1 ± 0.2	2.5 ± 0.2	-2.3 ± 0.1	-1.9 ± 0.1

Phys. Rev. D, 104, L091104 (2021)

$$\sigma^{\text{dr}}(s) = \left|\sqrt{\sigma_0}\left(\frac{M}{\sqrt{s}}\right)^n + e^{i\phi}\text{BW}(s)\right|^2$$

	Fit I	Fit II	
σ_0 (fb)	379 ± 22	320^{+750}_{-340}	
n	8.8 ± 0.4	8.2 ± 0.6	
ϕ (°)	–	183^{+57}_{-40}	240^{+17}_{-115}
σ_ψ (fb)	0(fixed)	240^{+1470}_{-190}	1440^{+270}_{-1390}
$\chi^2/ndof$	62.0/31	34.6/29	
$\mathcal{B}$ ($\times 10^{-5}$)	–	$2.4^{+15.0}_{-1.9}$	$14.4^{+2.7}_{-14.0}$

Phys. Rev. D, 104, 052012 (2021)

$$\sigma^{\text{dressed}}(\sqrt{s}) = \left|\sum_k e^{i\phi_k} \cdot BW_k(s) + e^{i\phi_{\text{cont}}} \cdot \psi_{\text{cont}}\right|^2$$

Parameters	Solution I	Solution II	Solution III	Solution IV
M($Y4220$) (MeV/c^2)	4234.2 ± 3.5			
Γ^{tot}($Y4220$) (MeV)	18.0 ± 8.8			
$B\Gamma^{ee}$($Y4220$) (eV)	1.63 ± 0.82	1.64 ± 0.83	0.02 ± 0.01	0.02 ± 0.01
M($Y4390$) (MeV/c^2)	4390.9 ± 7.4			
Γ^{tot}($Y4390$) (MeV)	143.6 ± 11.3			
$B\Gamma^{ee}$($Y4390$) (eV)	10.64 ± 4.36	19.73 ± 5.57	9.79 ± 3.46	19.12 ± 2.01
M($Y4660$) (MeV/c^2)	4652.5 ± 41.0			
Γ^{tot}($Y4660$) (MeV)	154.9 ± 25.3			
$B\Gamma^{ee}$($Y4660$) (eV)	4.75 ± 4.28	10.21 ± 5.21	4.69 ± 3.59	10.58 ± 3.78
$\phi_{Y(4220)}$ (rad)	1.68 ± 0.04	1.44 ± 0.05	6.27 ± 0.05	6.03 ± 0.04
$\phi_{Y(4660)}$ (rad)	6.07 ± 0.04	4.65 ± 0.04	6.03 ± 0.04	4.71 ± 0.04
$\phi_{\sigma_{\text{NY}}}$ (rad)	3.14 ± 0.83	2.70 ± 0.55	2.99 ± 0.67	2.45 ± 0.13
p_0($\times 10^5$)	3.80 ± 2.49	4.04 ± 2.66	3.79 ± 2.44	3.70 ± 1.92
p_1	9.47 ± 5.46	9.79 ± 5.60	9.50 ± 5.37	9.09 ± 3.82

QUESTION

- How interference cause the problem of multiple solutions
- Is there connection between the parameters of different solutions
- Can the solutions be found by a reliable and efficient way

Interfering B-W functions and Multiple Solutions

- Interference cause multiples solution, even without background (interference between B-W functions)

- $\sigma(s) \propto |A(s)|^2 = |\sum_{k=1}^{k=n} \frac{z_i}{s - p_k}|, p_k = M_k^2 - iM_k\Gamma_k$

- Multiplicity of solutions： 2^{n-1}, in which n is the number of resonances.

- Multiple solutions share **the same mass, width** parameters, but with **different phase angle and** $Br \times \Gamma^{ee}$

Phys. Rev. Lett, 129, 102003, 2022

$$e^+e^- \to \pi^+\pi^-\psi(3823)$$

Parameters	Solution I	Solution II
$M[R_1]$	$4406.9 \pm 17.2 \pm 4.5$	
$\Gamma_{\rm tot}[R_1]$	$128.1 \pm 37.2 \pm 2.3$	
$\Gamma_{e^+e^-}\mathcal{B}_1^{R_1}\mathcal{B}_2$	$0.36 \pm 0.10 \pm 0.03$	$0.30 \pm 0.09 \pm 0.03$
$M[R_2]$	$4647.9 \pm 8.6 \pm 0.8$	
$\Gamma_{\rm tot}[R_2]$	$33.1 \pm 18.6 \pm 4.1$	
$\Gamma_{e^+e^-}\mathcal{B}_1^{R_2}\mathcal{B}_2$	$0.24 \pm 0.07 \pm 0.02$	$0.06 \pm 0.03 \pm 0.01$
ϕ	$267.1 \pm 16.2 \pm 3.2$	$-324.8 \pm 43.0 \pm 5.7$

Studied Throughly

Int. J. Mod. Phys A 26, 4511 (2011)

arXiv: 0710.05627

Chin. Phys. C 42, no. 4, 043001 (2018)

arXiv: 1505.01509

Phys. Rev. D 99, 072007 (2019)

Explanation of the multiple solutions of interfering B-W functions

- Multiple solutions and the zeros of amplitude（[***Phys. Rev. D, 99, 072007, 2019***]）：

  - Set a zero of the amplitude to its conjugate, get a new solution

- $$A(s) = \sum_{k=1}^{k=n} \frac{z_i}{s - p_k} = (\prod_{k=1}^{k=n} \frac{1}{s - p_k}) Poly(s, n-1) = \frac{\prod_{i=1}^{i=n-1} (s - q_i)}{\prod_{k=1}^{k=n} (s - p_k)}$$

- $$\sigma(s) = |A(s)|^2 = |\frac{\prod_{i=1}^{i=n-1} (s - q_i)}{\prod_{k=1}^{k=n} (s - p_k)}|^2 = |\frac{\prod_{i=1}^{i=n-1} (s - q_i')}{\prod_{k=1}^{k=n} (s - p_k)}|^2,$$ $q'_i = q_i$ or $q'_i = q^*_i$

Multiple solutions by B-W interference： Determined by the zeros of the amplitude

‘Exact Multiple Solution’ in case of Coherent Background

- **Generalize the conclusion of interfering B-W functions to the case with coherent background:**

  Phys. Rev. D, 99, 072007, 2019

- $$\begin{cases} B(s) & \rightarrow \quad B(s)' = B(s) + \frac{(q_m - q_m^*)[B(s) - B(q_m)]}{s - q_m} \\ z_k & \rightarrow \quad z_k' = \frac{q_m^* - p_k}{q_m - p_k} z_k. \end{cases}$$

As the case of the B-W functions interference:

- Find the zeros of the amplitude
- Set one of some of the amplitude’s zero to its conjugate

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As the case of the B-W functions' interference:

- Find the zeros of the amplitude
- Set one of some of the amplitude's zero to its conjugate

Correct, but not very useful

Problem of 'Exact Multiple Solution' in Coherent Background

- **Except for some special case, the background function form will be altered**
    - Special cases like: constant background, polynomial background etc.
    - Generally the 'multiple solution' with coherent background is **approximate** if the background form unaltered
- **The widely used background function** (.e.g exponential, power law) **provide infinite multiplicity of zeros**
- **There are technical difficulties to reliably find zeros**
    - Usually no analytic solution nor closed form
    - Numeric method **depend on initial value heavily**

Multiple Solutions with Coherent Background, a closer look

Phys. Rev. Lett, 122, 102002 (2019)

Background contribute to new solutions

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Phys. Rev. D, 104, L091104 (2021)

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Phys. Rev. D, 104, 052012 (2021)

$$\sigma^{\text{dressed}}(\sqrt{s}) = \left|\sum_k e^{i\phi_k} \cdot BW_k(s) + e^{i\phi_{\text{cont}}} \cdot \psi_{\text{cont}}\right|^2$$

Background seems doesn't contribute to new solutions

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Factorization of Amplitudes

- **BW-only Case** （**Fundamental Theorem of Algebra**）：

$$A(s) = \sum_{k=1}^{k=n} \frac{z_i}{s - p_k} = (\prod_{k=1}^{k=n} \frac{1}{s - p_k}) Poly(s, n-1) = \frac{\prod_{i=1}^{i=n-1}(s - q_i)}{\prod_{k=1}^{k=n}(s - p_k)}$$

- **Coherent Background Case** （**Hadamard Factorization Theorem**，’Fundamental Theorem of Algebra’ for entire functions）：

- $$f(z) = \sum_{k=1}^{k=n} \frac{z_i}{s - p_k} + BKG(s) = Cz^m e^{h(z)} \prod (1 - \frac{z}{r_k}) e^{z/r_k + z^2/r_k^2 ... z^p/r_k^p}$$

    - r_k: zeros of $f(z)$
    - $p = [\alpha]$, in which α is the order of $f(z)$, i.e. $\log f(z) = O(|z|^{\alpha})$ when $|z|$ is sufficiently large
    - $h(z)$ polynomial of degree p

1 BW + Exponential Background, a case with closed form

- Consider the amplitude : $A(s) = \dfrac{z_1}{s - p_1} + z_0 e^{-s/\Lambda^2}$
  - zeros $r_k = p_1 - \Lambda^2 W_k(\dfrac{z_1}{z_0}\dfrac{e^{p_1/\Lambda^2}}{\Lambda^2})$,
  - $W_k(z)$: **Lambert W-function**, defined as the inverse function of $g(z) = ze^z$, **multiple value function, infinite values indexed by integer k**
  - $A(s)$ has **order 1**, thus $A(s) = \dfrac{1}{s - p_1} e^{As+B} \prod_k (1 - s/r_k) e^{s/r_k}$
  - Value of coefficient A和B can be determined by the value of $A(s)$ and its derivative some where
  - Moreover，numeric results implies $A(s) = \dfrac{1}{s - p_1} e^{-\frac{s}{2\Lambda^2}+B} \prod_k (1 - s/r_k)$ （who can prove?）

1 BW + Exponential Background, a case with closed form

- One Solution: $A(s) = \frac{1}{s-p_1} e^{-\frac{s}{2\Lambda^2}+B} \prod_k (1 - s/r_k)$,
- Another solution : $A'(s) = \frac{1}{s-p_1} e^{-\frac{s}{2\Lambda^2}+B'} \prod_k (1 - s/r'_k)$
- $A(s)/A'(s) \propto \frac{\prod_{j\in Z}(1-s/r_j)}{\prod_{j\in Z}(1-s/r'_j)}$

$$\begin{cases} B(s) \rightarrow B(s)' = B(s) + \frac{(q_m - q_m^*)[B(s)-B(q_m)]}{s-q_m} \\ z_k \rightarrow z'_k = \frac{q_m^* - p_k}{q_m - p_k} z_k. \end{cases}$$

Generally not possible to find another solution $A'(s)$ such that $A(s)/A'(s)$ is a constant

Infinite multiplicity of zeros, but **not all of them are important**

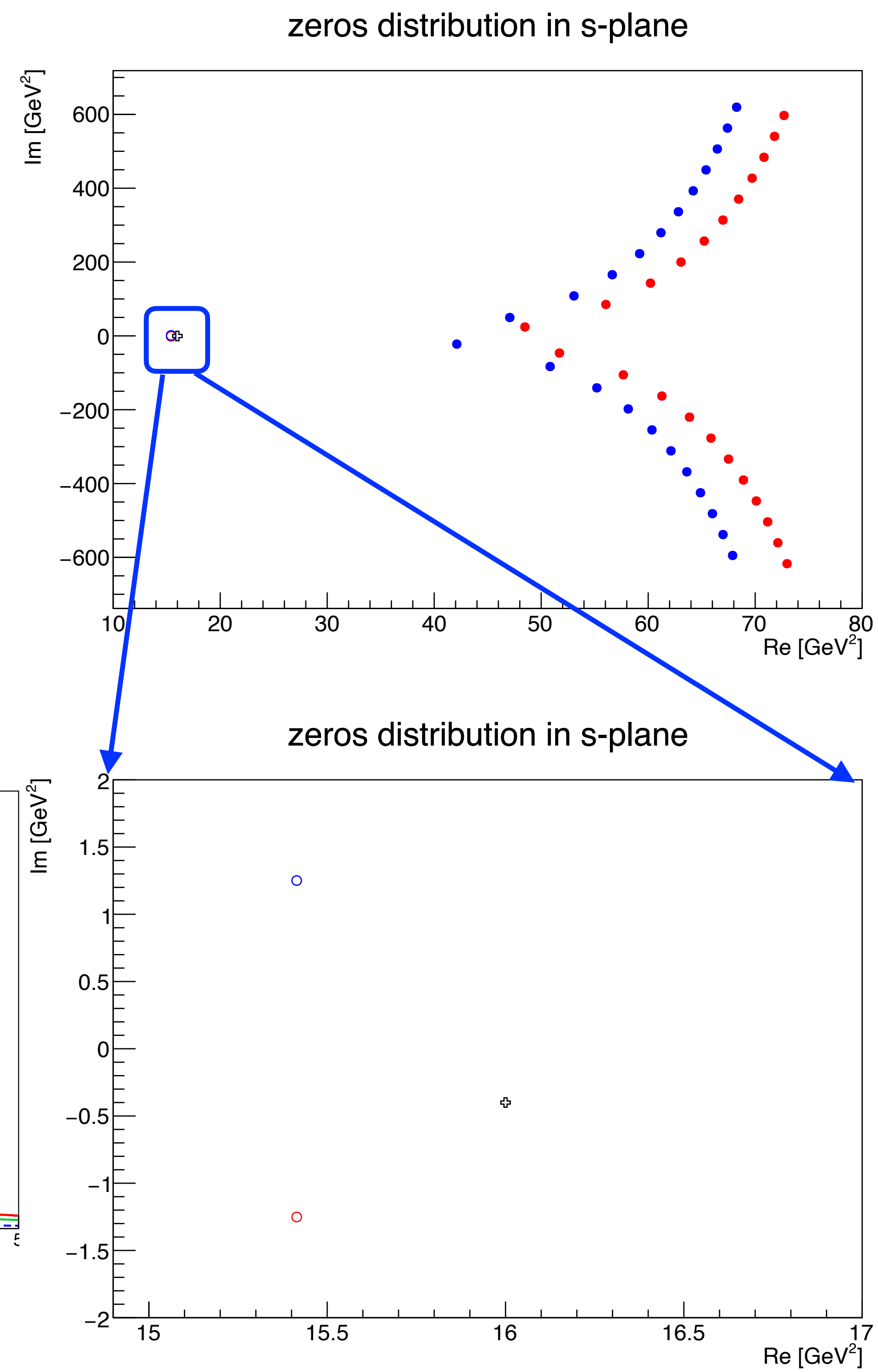
unimportant zeros: far from poles, very large or small imagery part

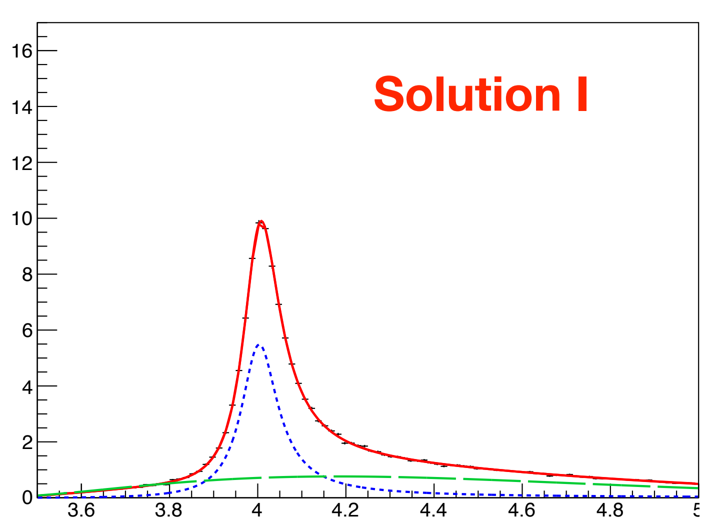
1 BW + Exponential Background, an Example

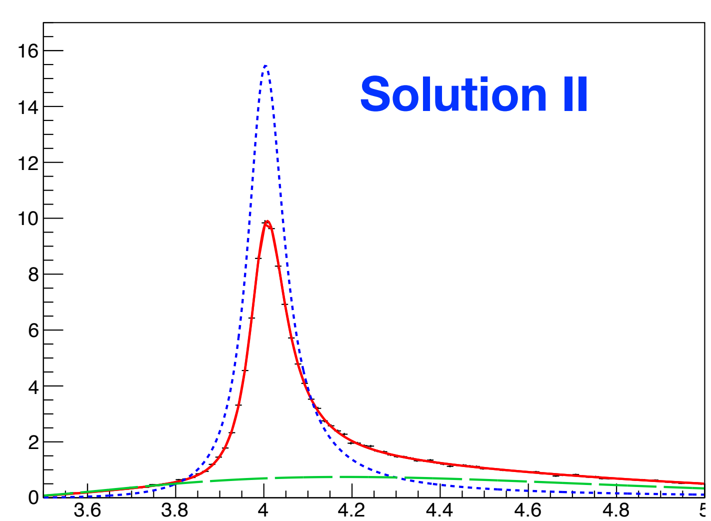
Example in *PRD, 99, 072007, 2019*

$$A(s) = \frac{z_1}{s - p_1} + z_0 e^{-s/\Lambda^2}$$

$$\begin{cases} z_0 &= 0.15 \\ z_1 &= (0.0125 + 0.025i)\ \text{GeV}^2 \\ p &= M^2 - iM\Gamma = (16.0 - 0.4i)\ \text{GeV}^2 \\ \Lambda &= 3\ \text{GeV} \end{cases}$$







1 BW + Exponential: Method

$$A(s)/A'(s) \propto \frac{\prod_{j\in Z}(1-s/r_j)}{\prod_{j\in Z}(1-s/r'_j)} = \frac{1-s/r_J}{1-s/r'_J}\frac{\prod_{j\in Z, j\neq J}(1-s/r_j)}{\prod_{j\in Z, j\neq J}(1-s/r'_j)},$$

- **All the zeros but one are outside the region of interest**
    - **regions close to poles, covered in center-of-mass energy (.e.g *s* in cross section scan)**
- $\frac{1-s/r_J}{1-s/r'_J}$, **significant impact on the resonance parameters**
- $\frac{\prod_{j\in Z, j\neq J}(1-s/r_j)}{\prod_{j\in Z, j\neq J}(1-s/r'_j)}$, **vary slowly in the region of interest, can be approximately treated as constant**

Set $r'_J = r^*_J$, get another solution

Multiple BW functions + exponential background

- **Need to evaluate generalized Lambert W function, no tools at hand**
- **Numerical method applied to find zeros. To avoid initial value sensitive problem, a mathematical trick is implemented**:
  - assuming we have function f(z) and h(z), with **known poles and zeros**
  - **Seeds of zeros**: consider function $F(z|\epsilon) = f(z) + \epsilon h(z)$, when $\epsilon \to 0$, zeros of $F(z|\epsilon)$ either localized **around the zeros of f(z), or poles of h(z). One can easily find the zeros of** $F(z|\epsilon)$ **when** ϵ **is sufficiently small. Denote these zeros as 'seeds' of zeros**
  - ϵ **evolves from a small value to 1**, zeros of $F(z|\epsilon)$ evolves **from 'seed' to the zeros of f(z)+h(z)**
- **In *n* BW functions cases, define** $f(s) = \frac{z_1}{s - p_1} + z_0 e^{-s/\Lambda^2}$, $h(s) = \sum_{i=2}^{i=n} \frac{z_i}{s - p_i}$
- **Amplitudes satisfy**: $A(s) = \prod_{i=1}^{i=n} \frac{1}{s - p_i} e^{-\frac{s}{2\Lambda^2} + B} \prod_k (1 - z/r_k)$

Multiple BW functions + exponential background: Example

Phys. Rev. D, 104, 052012 (2021)

$$\sigma^{\text{dressed}}(\sqrt{s}) = \left|\sum_k e^{i\phi_k} \cdot BW_k(s) + e^{i\phi_{\text{cont}}} \cdot \psi_{\text{cont}}\right|^2$$

$$BW_k(s) = \frac{M_k}{\sqrt{s}} \frac{\sqrt{12\pi\Gamma_k^{\text{tot}}\Gamma_k^{ee}B_k}}{s - M_k^2 + iM_k\Gamma_k^{\text{tot}}} \sqrt{\frac{\Phi(\sqrt{s})}{\Phi(M_k)}},$$

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p_1	9.47 ± 5.46	9.79 ± 5.60	9.50 ± 5.37	9.09 ± 3.82

Zeros of Solution I ($\sqrt{s}$ as variable)

zeros evolved from Lambert W function (first 6 zeros)		zeros evolved from BW-functions' pole	
1	$-1.33\times10^{-4}-2.03\times10^{-5}i$	1	4.2360+0.0092i
2	5.04+0.43i	2	-4.2342-0.0090i
3	6.32+4.40i	3	4.5269-0.0598i
4	6.32-4.51i	4	-4.3914+0.0718i
5	6.87+8.01i	5	4.9280-0.5803i
6	6.87-8.13i	6	-4.653+0.0774i

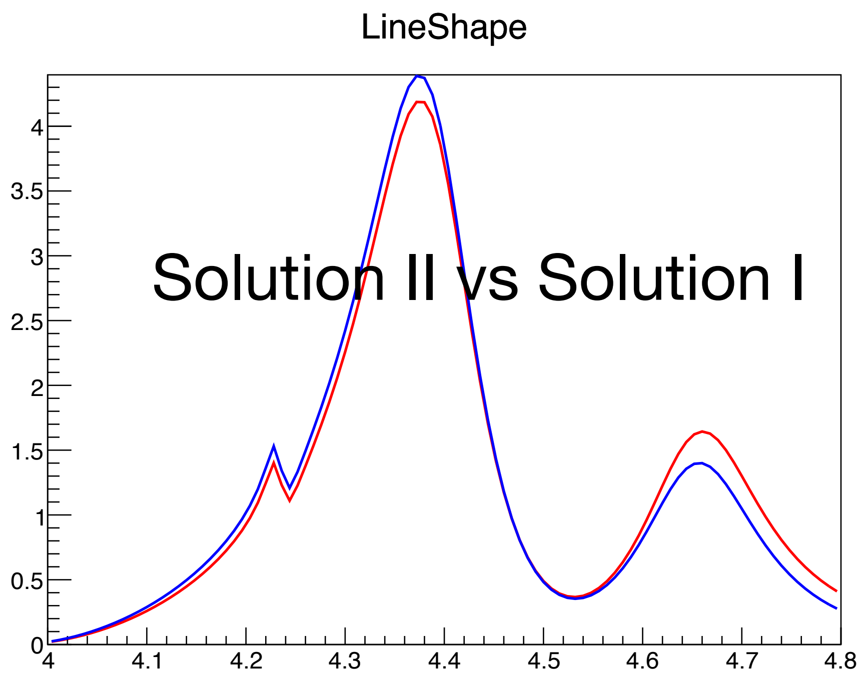
Multiple BW functions + exponential background: Example

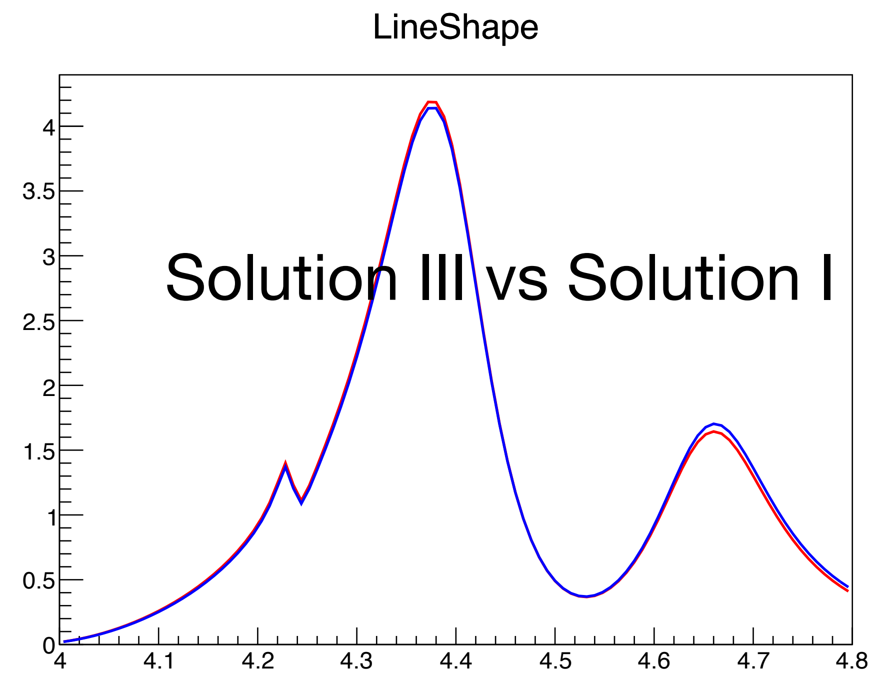
Only One zero flipped in the table

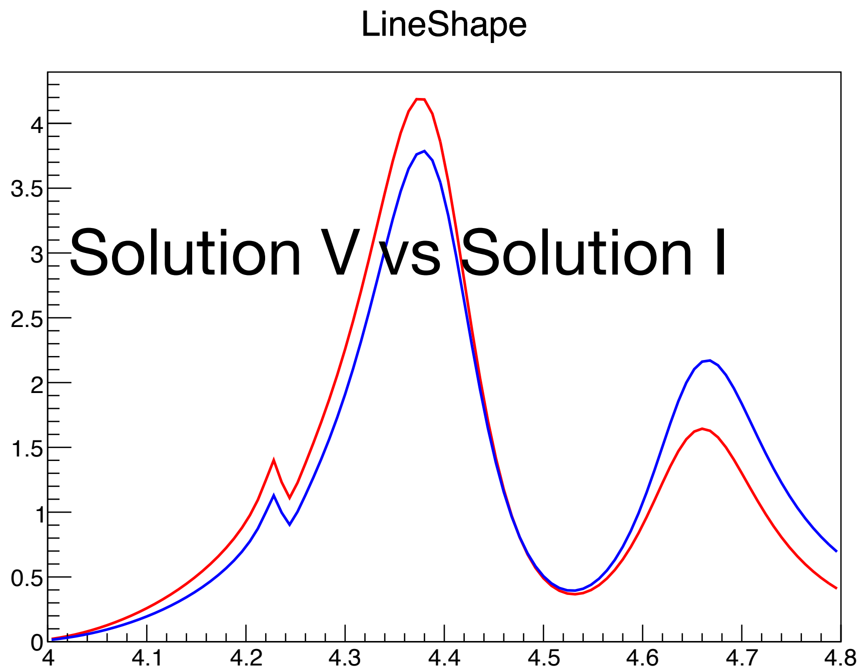
Parameters	Solution I	Solution II		Solution III		Solution IV	Solution V?
		By Scan	By Flip zero	By Scan	By Flip zero	By Scan	By Flip zero
M(Y(4220))/GeV/	4234.2						
Γ^{tot}(Y4220)/MeV	18.0						
$B\Gamma^{ee}$(Y4220)/eV	1.63	1.64	1.55	0.02	0.017	0.02	2.05
M(Y(4390))/GeV/	4390.9						
Γ^{tot}(Y4390)/MeV	143.6						
$B\Gamma^{ee}$(Y4390)/eV	10.64	19.73	17.54	9.79	10.10	19.12	15.36
M(Y(4660))/GeV/	4652.5						
Γ^{tot}(Y4660)/MeV	154.9						
$B\Gamma^{ee}$(Y4660)/eV	4.75	10.21	7.11	4.69	5.06	10.58	9.88
ϕ_{Y4220}(rad)	1.68	1.44	1.43	6.27	6.26	6.03	1.69
ϕ_{Y4660}(rad)	6.07	4.65	4.63	6.03	6.02	4.71	6.09
ϕ_{NY}(rad)	3.14	2.70	2.68	2.99	3.00	2.45	3.13
p_0	3.80	4.04	3.80	3.79	3.80	3.70	3.80
p_1	9.47	9.79	9.47	9.50	9.47	9.09	9.47

- For models containing n BW function, we can **flip n zeros at most**

Exp BKG + BW: LineShape of the Multiple Solutions







- Fit Program can find Sol II and Sol III, but failed to find Sol V
    - Sol V close to Sol I
    - Poorer line shape agreement tp Solution I

B-W functions + Powerlaw Background

- Amplitude contains power law function： $A(s) = \sum_{i=1}^{i=n} \frac{z_i}{s-p_i} + z_0/s^p$

- **Cannot make Hadmard Factorization directly： not converge.**

- Let $w = \ln(s/\mu^2)$, Hadamard Factorization validated for variable w:

- $$A(s) = A(\mu^2 e^w) = \prod_{i=1}^{i=n} \frac{1}{s-p_i} \times e^{aw+b} \prod_{j \in Z} (1 - w/r_j)$$

- Zeros can also be achieved by the ‘seed evolution’ method:

- $A(s) = \sum_{i=1}^{i=n} \frac{z_i}{s-p_i} + \epsilon z_0/s^p$, zero of $\sum_{i=1}^{i=n} \frac{z_i}{s-p_i}$ and pole of z_0/s^p are known

- By flipping at most n zeros, one gets multiple solutions

BW functions+ Powerlaw Background: Example

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$$\sigma^{\text{dressed}}(\sqrt{s}) = \left| \sum_k e^{i\phi_k} \cdot BW_k(s) + e^{i\phi_{\text{cont}}} \cdot \psi_{\text{cont}} \right|^2$$

$$BW_k(s) = \frac{M_k}{\sqrt{s}} \frac{\sqrt{12\pi\Gamma_k^{\text{tot}}\Gamma_k^{ee}B_k}}{s - M_k^2 + iM_k\Gamma_k^{\text{tot}}} \sqrt{\frac{\Phi(\sqrt{s})}{\Phi(M_k)}},$$

$$\psi_{\text{cont}} = \frac{a}{(\sqrt{s})^n} \sqrt{\Phi(\sqrt{s})}.$$

TABLE III. Results of the fit to the $e^+e^- \to \pi^+\pi^-\psi(3686)$ cross section for the case when the continuous part is described by ψ_{cont} in Eq. (6). The uncertainties involve statistical and systematic ones propagated from the cross section measurement in Table I.

Parameters	Solution I	Solution II	Solution III	Solution IV
$M(Y4220)$ (MeV/c^2)		4234.4 ± 3.2		
$\Gamma^{\text{tot}}(Y4220)$ (MeV)		17.6 ± 8.1		
$B\Gamma^{ee}(Y4220)$ (eV)	1.59 ± 0.75	1.63 ± 0.78	0.02 ± 0.01	0.02 ± 0.01
$M(Y4390)$ (MeV/c^2)		4390.3 ± 6.0		
$\Gamma^{\text{tot}}(Y4390)$ (MeV)		143.3 ± 10.0		
$B\Gamma^{ee}(Y4390)$ (eV)	10.70 ± 4.13	20.72 ± 2.46	9.86 ± 4.11	19.44 ± 2.04
$M(Y4660)$ (MeV/c^2)		4651.0 ± 37.8		
$\Gamma^{\text{tot}}(Y4660)$ (MeV)		155.4 ± 24.8		
$B\Gamma^{ee}(Y4660)$ (eV)	4.72 ± 3.79	11.15 ± 3.23	4.66 ± 4.20	11.28 ± 3.25
$\phi_{Y(4220)}$ (rad)	1.68 ± 0.04	1.39 ± 0.06	6.24 ± 0.05	5.95 ± 0.04
$\phi_{Y(4660)}$ (rad)	6.07 ± 0.03	4.77 ± 0.04	6.03 ± 0.06	4.77 ± 0.03
ϕ_{cont} (rad)	3.14 ± 0.79	2.58 ± 0.13	2.99 ± 0.92	2.40 ± 0.08
$a(\times 10^5)$	4.81 ± 35.83	5.28 ± 33.23	5.13 ± 26.55	3.48 ± 24.16
n	8.65 ± 3.66	8.72 ± 3.40	8.69 ± 3.09	8.43 ± 3.53

Zeros of Solution I ($w = \log(s/\mu^2)$ as variable, μ= 1 GeV)

zeros evolved from origin		zeros evolved from interfering BW-functions' zeros	
1	3.2647+0.1953i	1	2.8874+0.0043i
2	3.2229-0.2721i	2	3.0208-0.0264i
3	3.7722+2.2786i	3	2.8812-6.2768i
4	3.7698-2.3479i	4	3.0359-6.2842i
5	3.7431+4.2888i	5	2.8915+6.2860i
6	3.7467-4.3578i	6	3.0374+6.2409i

Multiple BW functions + power-law background: Example

Only One zero flipped in the table

Parameters	Solution I	Solution II		Solution III		Solution IV	Solution V?
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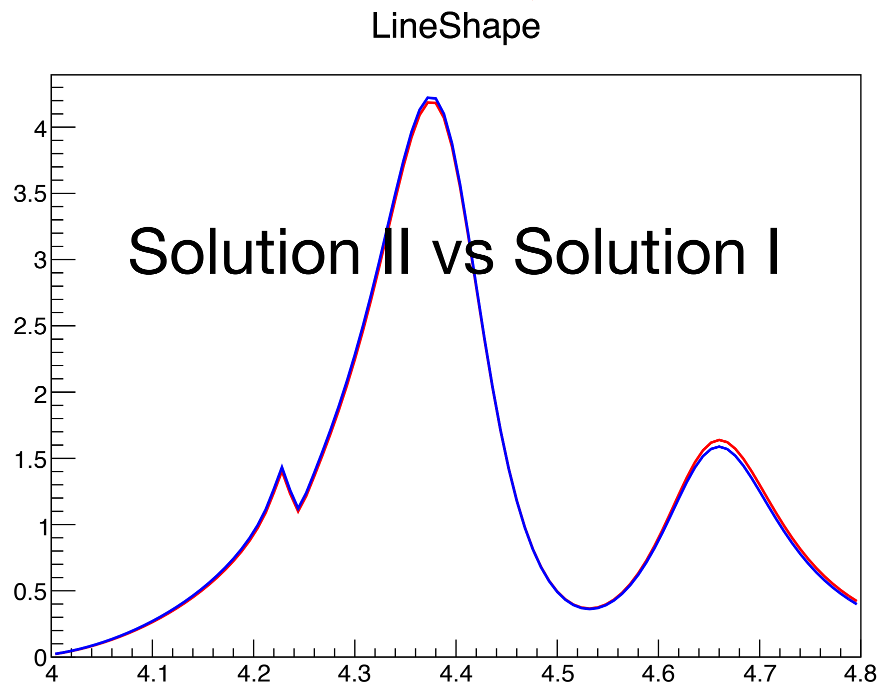
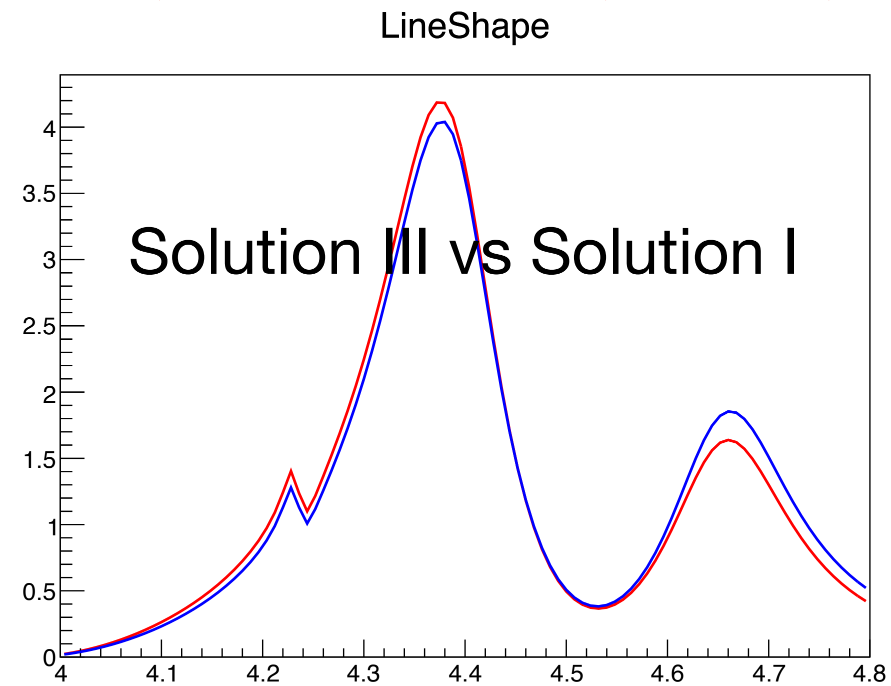
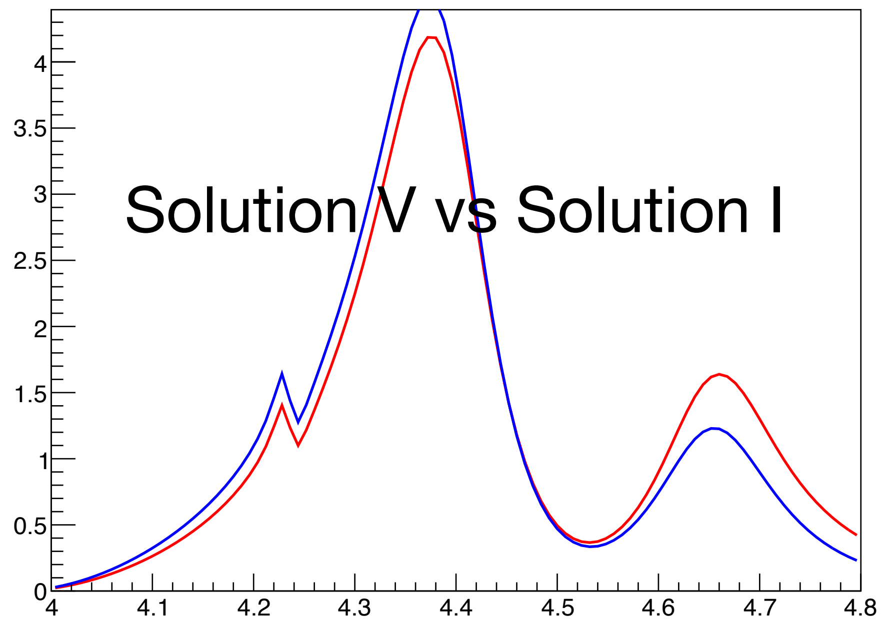
Γ^{tot}(Y4220)/MeV	17.6						
$B\Gamma^{ee}$(Y4220)/eV	1.59	1.63	1.83	0.02	0.016	0.02	1.36
M(Y(4390))/GeV/	4390.3						
Γ^{tot}(Y4390)/MeV	143.3						
$B\Gamma^{ee}$(Y4390)/eV	10.70	20.72	23.98	9.86	9.49	19.44	8.20
M(Y(4660))/GeV/	4651.0						
Γ^{tot}(Y4660)/MeV	155.4						
$B\Gamma^{ee}$(Y4660)/eV	4.72	11.15	14.46	4.66	4.32	11.28	2.55
ϕ_{Y4220}(rad)	1.68	1.39	1.44	6.24	6.26	5.95	1.66
ϕ_{Y4660}(rad)	6.07	4.77	4.68	6.03	6.03	4.77	6.07
ϕ_{NY}(rad)	3.14	2.58	3.36	2.99	3.68	2.40	3.81
a(10^5)	4.81	5.28	4.81	5.13	4.81	3.48	4.81
n	8.65	8.72	8.65	8.69	8.65	8.43	8.65

- For models containing n BW function, we can **flip n zeros at most**

Pow BKG + BW: LineShape of the Multiple Solutions







- Fit Program can find Sol II and Sol III, but failed to find Sol V
  - Sol V close to Sol I
  - Poorer line shape agreement tp Solution I

Summary and Prospective

- As that in BW-only case, the multiple solution is strongly connected to **the zeros of amplitude**
- The multiple solutions are **approximate**, rather than exact, if one keeps the background form
- We need only focus on the **few zeros in the region of interest**
  - Close to the poles, imagery part not too big or small
  - Determine a solution with zeros as much as the BW functions
- A reliable numerical method to find zeros is developed
- According to one solution, one can get others by **flipping the few zeros in the region of interest**, and **set it to initial value of parameters and redo the fit**
- In principle can be applied to more complex cases: more complex backgrounds, variable width, Flatté formula etc.

谢谢！