RIKEN Center for Interdisciplinary Theoretical and Mathematical Sciences

Quantum Simulations of Quantum Electrodynamics in Coulomb Gauge

QUANTUM COMPUTING AND MACHINE LEARNING WORKSHOP 2025

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Outline

I. Motivation

II. Hamiltonian formalism of quantum electrodynamics (QED) in Coulomb gauge

III. Map fermion fields and gauge fields to qubits.

IV. Complexity

V. Results

Steps of simulating quantum field theory (QFT) on quantum computers

Lagrangian of QFT
$\mathcal{L}(\psi, \partial_\mu \psi, A, \partial_\mu A)$

Steps of simulating QFT on quantum computers

Lagrangian of QFT
$\mathcal{L}(\psi, \partial_\mu \psi, A, \partial_\mu A)$

↓

Hamiltonian formalism
$\mathcal{H}(\psi, A, \Pi_\psi, \Pi_A)$

Gauge choice $A^0 = 0$.
(Does not fix all residual degrees of freedom)

$U(1)$ gauge theory as an example:

- Under the gauge transformation:

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

- If $A_0(x) = 0$, there exist time independent function $\alpha(\boldsymbol{x})$:

$$A'_0(x) = A_0(x) = 0$$

$$A'_i(x) = A_i(x) - \frac{1}{e}\partial_i \alpha(\boldsymbol{x})$$

Steps of simulating QFT on quantum computers

Lagrangian of QFT
$\mathcal{L}(\psi, \partial_\mu \psi, A, \partial_\mu A)$

Gauge choice $A^0 = 0$.

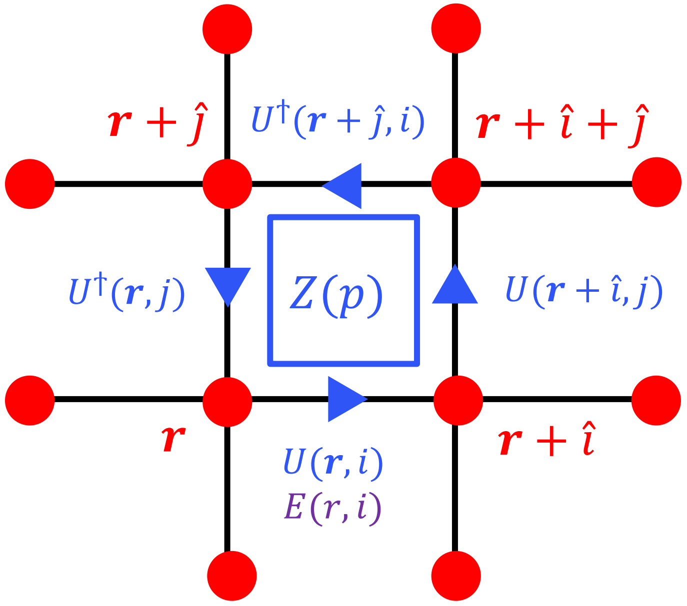
Hamiltonian formalism
$\mathcal{H}(\psi, A, \Pi_\psi, \Pi_A)$

Discretization of fields

Kogut-Susskind formalism
$H_{KS}(\hat{\psi}, U, \hat{\Pi}_\psi, \hat{E})$

◆ How to digitize the gauge fields while preserving Gauss's law?

J. B. Kogut, and L. Susskind, PRD, (1975)



- Gauss's law operator G generates residual gauge transformations.

$$[H_{\mathrm{KS}}, G] = 0,\ G|\mathrm{Phys}\rangle = 0.$$

- Quantization conditions (only holds in infinite-dimensional Hilbert space):

$$\left[E^a, U^\rho_{ij}\right] = -\sum_k \left(T^a_\rho\right)_{ik} U^\rho_{kj}, \ldots$$

Steps of simulating QFT on quantum computers

Lagrangian of QFT
$\mathcal{L}(\psi, \partial_\mu \psi, A, \partial_\mu A)$

↓ Gauge choice $A^0 = 0$

Hamiltonian formalism
$\mathcal{H}(\psi, A, \Pi_\psi, \Pi_\mathrm{A})$

↓ Discretization of fields

Kogut-Susskind formalism
$H_{KS}(\hat{\psi}, U, \hat{\Pi}_\psi, \hat{E})$

↓ Digitization of gauge fields

Choosing which digitization scheme depends on the problem.

Preserve Gauss's law strictly after digitization. (Including dynamical quark)

Impossible triangle

Unitary gauge link

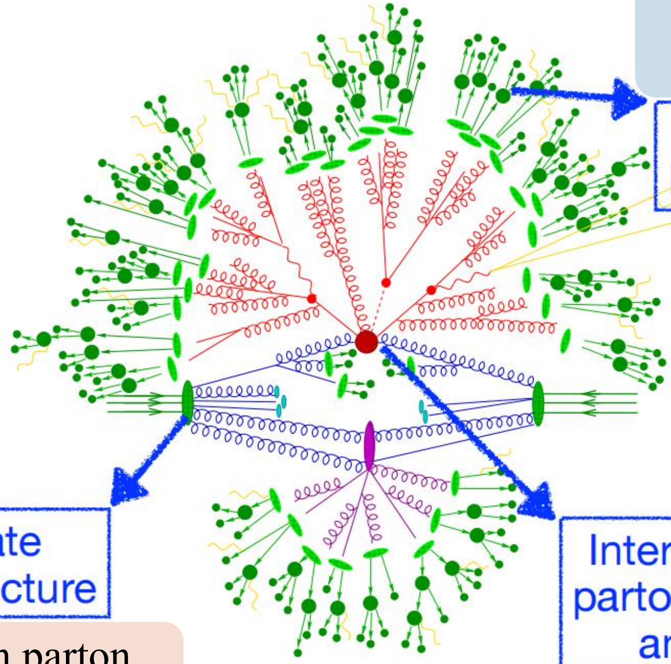
Has a controllable digitization error

- Truncation of E basis: T. Byrnes and Y. Yamamoto, PRA, (2006). E. Zohar, J. I. Cirac, and B. Reznik, PRL, (2012)…
- Discrete subgroup method: Y. Ji, H. Lamm, and S. Zhu, PRD, (2020)…
- Fixing to the maximum tree gauge: I. D'Andrea et al., PRD, (2024)…
- Quantum link model: D. Luo et al., PRA, (2020)…

…

Important observable in QFT: cross section

Gauge invariance
“out” states



Final state
hadronization

Initial state
hadron structure

Gauge link in parton
distribution functions

Intermediate state
partonic scatterings
and showers

Measuring the cross section of various processes are primary goals of RHIC, LHC, and other similar experiments.

Preserve Gauss’s law strictly after
digitization. (Including dynamical quark)

Unitary gauge link

Has a controllable
digitization error

We can try the Coulomb gauge!

Coulomb gauge QED as an attempt

- All residual degrees of freedom are eliminated, gauge field can be discretized directly:

$$\hat{\partial}_i^R \hat{A}^j(\boldsymbol{n}) \equiv \left[\hat{A}^j(\boldsymbol{n}+a\hat{\imath}) - \hat{A}^j(\boldsymbol{n})\right]/a,\ \hat{\partial}_i^R \hat{\Pi}^j(\boldsymbol{n}) \equiv \left[\hat{\Pi}^j(\boldsymbol{n}+a\hat{\imath}) - \hat{\Pi}^j(\boldsymbol{n})\right]/a$$

$$\hat{\Delta} A^i(\boldsymbol{n}) = \sum_j \left[\hat{A}^i(\boldsymbol{n}+\hat{\jmath}) - 2\hat{A}^i(\boldsymbol{n}) + \hat{A}^i(\boldsymbol{n}-\hat{\jmath})\right]/a^2$$

- Discretized version of four constraints ($\hat{\chi}_2$: Gauss's law, $\hat{\chi}_3$: Coulomb gauge condition):

$$\hat{\chi}_1(\boldsymbol{n}) = \hat{\Pi}^0 = 0$$

$$\hat{\chi}_2(\boldsymbol{n}) = \hat{\partial}_i^R \hat{\Pi}^i - J^0 = 0$$

$$\hat{\chi}_3(\boldsymbol{n}) = \hat{\partial}_i^R \hat{A}^i = 0$$

$$\hat{\chi}_4(\boldsymbol{n}) = \hat{\partial}_i^R \hat{\Pi}^i - \hat{\Delta}\hat{A}^0 = 0$$

$\Rightarrow$

$$\hat{A}^0(\boldsymbol{n}) = \sum_{\boldsymbol{m}} \sum_{\boldsymbol{p}\neq 0} \frac{J^0(\boldsymbol{m})}{\hat{E}_{\boldsymbol{p}}^2} e^{-i\boldsymbol{p}\cdot(\boldsymbol{n}-\boldsymbol{m})}$$

$$\hat{E}_{\boldsymbol{p}} = \frac{2}{a}\sqrt{\sum_i \sin^2\left(\frac{p^i a}{2}\right)}$$

- Quantization:

How to make the commutation relations simpler?

$$\left[\hat{A}_i(\boldsymbol{x}), \hat{\Pi}_\perp^j(\boldsymbol{y})\right] = i\delta_i^j \delta_{\boldsymbol{x},\boldsymbol{y}} + \sum_{\boldsymbol{p}\neq \boldsymbol{0}} \frac{\mathrm{i}}{M^3 a^2 \hat{E}_{\boldsymbol{p}}^2} \left(e^{-ip^i}a - 1\right)\left(e^{ip^j a} - 1\right)$$

Gauge fields in the momentum space

- Expansions of gauge fields

$$\hat{A}_i(\boldsymbol{x}) = \sum_{\boldsymbol{p}\neq\boldsymbol{0}} \frac{1}{\sqrt{2\hat{E}_p M^3}} \sum_{r=\pm} \left[\hat{\epsilon}_i^r(\boldsymbol{p}) a_{\boldsymbol{p}}^r e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \mathrm{H.c.}\right]$$

$$\hat{\Pi}_{\perp i}(\boldsymbol{x}) = \sum_{\boldsymbol{p}\neq\boldsymbol{0}} \sqrt{\frac{\hat{E}_P}{2M^3}} \sum_{r=\pm} \left[-\mathrm{i}\hat{\epsilon}_i^r(\boldsymbol{p}) a_{\boldsymbol{p}}^r e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \mathrm{H.c.}\right]$$

Unitary gauge link:
$U_i(\boldsymbol{x}) = \exp\left(-ig\hat{A}_i(\boldsymbol{x})\right)$

- A simple commutation relation:

$$\left[a_p^r, a_q^{s\dagger}\right] = \delta_{p,q}\delta_{r,s}$$

- Preserve Coulomb gauge condition and Gauss's law by solving:

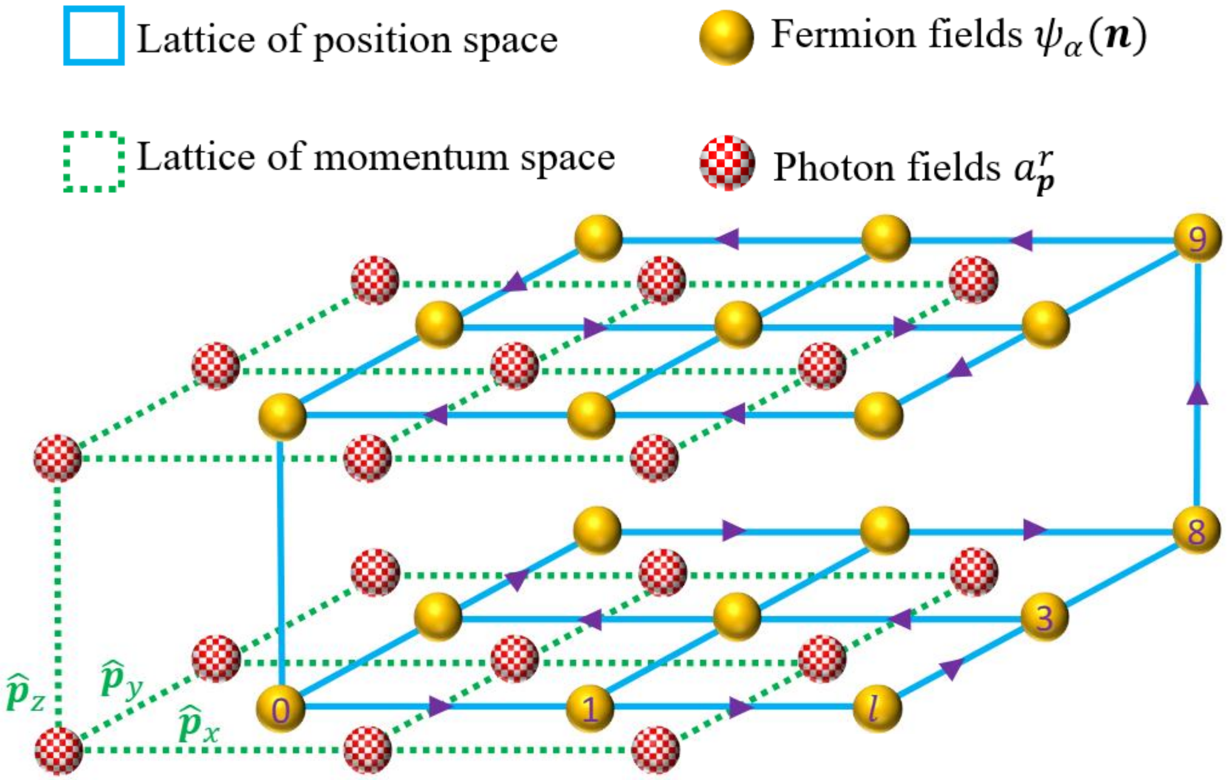
$$\sum_i \left(e^{ip^i a} - 1\right) \hat{\epsilon}_i^r(p) = 0$$

$$\sum_i \hat{\epsilon}_i^r(p)\hat{\epsilon}_i^s(p) = \delta_{rs}$$

$$\sum_r \hat{\epsilon}_i^r(p)\hat{\epsilon}_j^s(p) = \delta_{ij} - \frac{1}{\hat{E}_p^2}\left(e^{-ip^i}a - 1\right)\left(e^{ip^j a} - 1\right)$$

Those equations do not depend on the Fock state truncation of photons.

Coulomb gauge QED Hamiltonian on lattice



$$\hat{H}_E + \hat{H}_B = \sum_{\boldsymbol{p}\neq 0}\sum_r \hat{E}_{\boldsymbol{p}} a_{\boldsymbol{p}}^{r\dagger} a_{\boldsymbol{p}}^r$$

$$\hat{H}_I = \sum_{\boldsymbol{n},i}\sum_{\boldsymbol{p}}\sum_r \frac{J^i(\boldsymbol{n})}{\sqrt{2\hat{E}_p M^3}}\left[\hat{\epsilon}_i^r a_{\boldsymbol{p}}^r e^{i\boldsymbol{p}\cdot\boldsymbol{n}} + \mathrm{H.c.}\right]$$

$$\hat{H}_V = \frac{1}{2}\sum_{\boldsymbol{m},\boldsymbol{n}}\sum_{\boldsymbol{p}} \frac{J^0(\boldsymbol{m})J^0(\boldsymbol{n})}{\hat{E}_{\boldsymbol{p}}^2} e^{-i\boldsymbol{p}\cdot(\boldsymbol{m}-\boldsymbol{n})}$$

$$\hat{H}_M = \sum_{\boldsymbol{n}} \bar{\psi}(\boldsymbol{n})\left[-i\gamma^i \frac{\psi(\boldsymbol{n}+\hat{\imath}) - \psi(\boldsymbol{n}-\hat{\imath})}{2} + m\bar{\psi}(\boldsymbol{n})\psi(\boldsymbol{n})\right]$$

$$\hat{H}_W = \sum_{\boldsymbol{n}} -\frac{w}{2}\bar{\psi}(\boldsymbol{n})\hat{\Delta}\psi(\boldsymbol{n})$$

- We set $a = 1$ here.
- We have M lattice sites for each special dimension.
- Wilson term $\hat{H}_W$ comes from Wilson fermion.

Why and how map fields to qubits

$$\hat{H}_E + \hat{H}_B = \sum_{\boldsymbol{p}\neq 0}\sum_r \hat{E}_{\boldsymbol{p}} a_{\boldsymbol{p}}^{r\dagger} a_{\boldsymbol{p}}^r$$

$$\hat{H}_I = \sum_{\boldsymbol{n},i}\sum_{\boldsymbol{p}}\sum_r \frac{J^i(\boldsymbol{n})}{\sqrt{2\hat{E}_{\boldsymbol{p}} M^3}}\left[\hat{\epsilon}_i^r a_{\boldsymbol{p}}^r e^{i\boldsymbol{p}\cdot\boldsymbol{n}} + \mathrm{H.c.}\right]$$

$$\hat{H}_V = \frac{1}{2}\sum_{\boldsymbol{m},\boldsymbol{n}}\sum_{\boldsymbol{p}} \frac{J^0(\boldsymbol{m})J^0(\boldsymbol{n})}{\hat{E}_{\boldsymbol{p}}^2} e^{-i\boldsymbol{p}\cdot(\boldsymbol{m}-\boldsymbol{n})}$$

$$\hat{H}_M = \sum_{\boldsymbol{n}} \bar{\psi}(\boldsymbol{n})\left[-i\gamma^i \frac{\psi(\boldsymbol{n}+\hat{\imath}) - \psi(\boldsymbol{n}-\hat{\imath})}{2} + m\bar{\psi}(\boldsymbol{n})\psi(\boldsymbol{n})\right]$$

$$\hat{H}_W = \sum_{\boldsymbol{n}} -\frac{w}{2}\bar{\psi}(\boldsymbol{n})\hat{\Delta}\psi(\boldsymbol{n})$$

- Quantum computer can not understand $a_{\boldsymbol{p}}^r$, $J^i(\boldsymbol{n})$.
- We need to map those fields to qubits.

- In QFT, we have commutators: $[A,\Pi]$ or $\{A,\Pi\}$.
- Quantum computer understand Pauli operators:

$$\sigma(A) = \sum_{\mu_0,\dots,\mu_{N_q-1}=0}^{3} A_{\mu_{N_q-1},\dots,\mu_0}\sigma_{N_q-1}^{\mu_{N_q-1}} \otimes \cdots \otimes \sigma_0^{\mu_0}$$

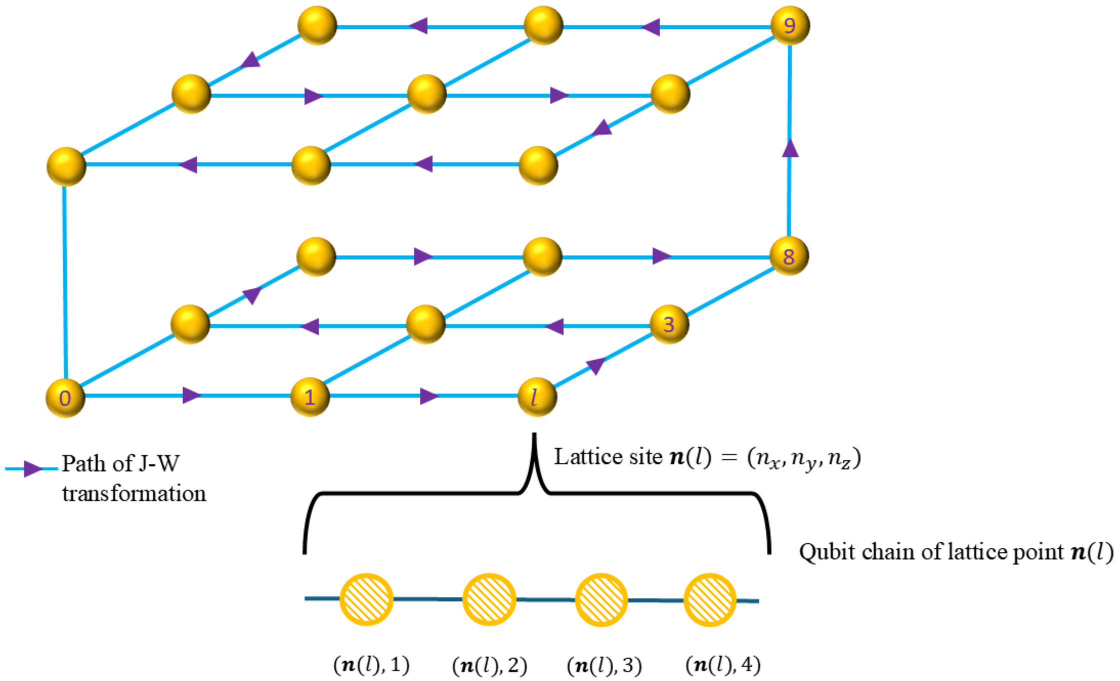
$$\sigma(\Pi) = \sum_{\mu_0,\dots,\mu_{N_q-1}} \Pi_{\mu_{N_q-1},\dots,\mu_0}\sigma_{N_q-1}^{\mu_{N_q-1}} \otimes \cdots \otimes \sigma_0^{\mu_0}$$

- Solve $A_{\mu_{N_q-1},\dots,\mu_0}$ and $\Pi_{\mu_{N_q-1},\dots,\mu_0}$:

$$[\sigma(A),\sigma(\Pi)] = (\approx)\sigma([A,\Pi]),$$

$$\{\sigma(A),\sigma(\Pi)\} = \sigma(\{A,\Pi\}).$$

Map fermion fields to qubits



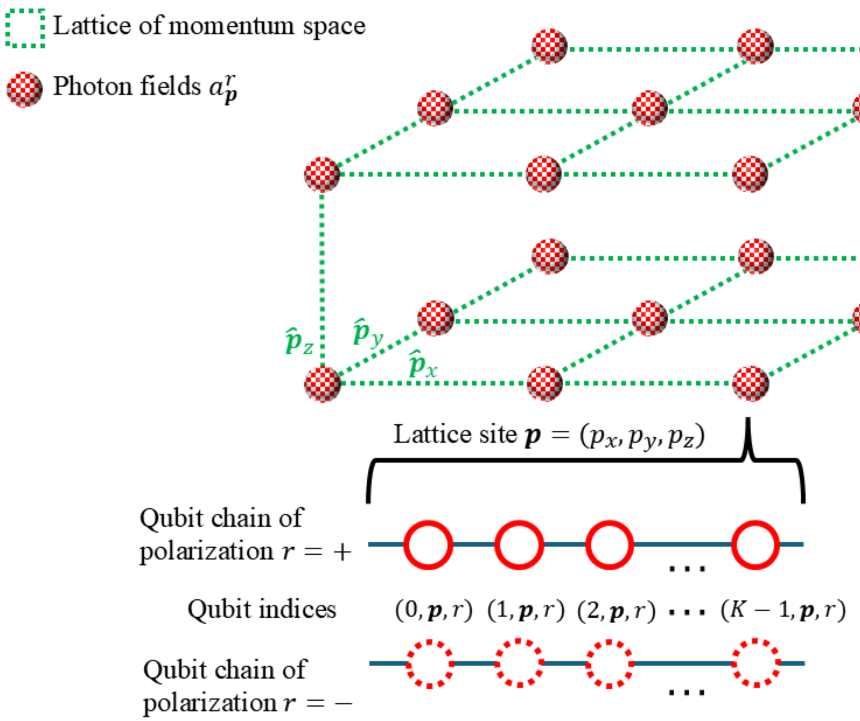
- Jordan-Wigner (J-W) transformation:

$$\psi_\alpha(\boldsymbol{n}) = \left[\prod_{l'<l}\left(\prod_{\beta=1}^{4}\sigma^3_{\boldsymbol{m}(l'),\beta}\right)\right]\left(\prod_{\beta=1}^{\alpha-1}\sigma^3_{\boldsymbol{n}(l),\beta}\right)\sigma^+_{\boldsymbol{n}(l),\alpha}$$

- J-W transformation preserves the anti-commutator:

$$\left\{\psi_\alpha(\boldsymbol{n}), \psi^\dagger_\beta(\boldsymbol{m})\right\} = \delta_{\alpha,\beta}\delta_{\boldsymbol{n},\boldsymbol{m}}$$

Map gauge fields to qubits



- Truncation of the Fock states:

$$a_{\boldsymbol{p}}^{r\dagger}\left|\Lambda = 2^{K} - 1\right\rangle_{\boldsymbol{p},r} = 0$$

- Given truncation Λ, the error ε_s of arbitrary state $|\psi\rangle$ scales as:

$$\varepsilon_s \sim O\left(\frac{g^2 M^{3d+3} + E M^{d+1}}{\Lambda}\right)$$

Controllable truncation error.

- Mapping of states: Fock states of photons $\leftrightarrow$ the computational basis of a quantum computer:

$$|\mathcal{N}(i_{K-1}, \dots, i_0)\rangle_{\boldsymbol{p},r} \leftrightarrow |i_{K-1}, \dots, i_0\rangle_{\boldsymbol{p},r}$$

- Mapping of operators:

$$a_{\boldsymbol{p}}^{r} \leftrightarrow \left\{\sum_{J=0}^{K-1}\left[\sigma_{J,\boldsymbol{p},r}^{+}\left(\prod_{L=0}^{J-1}\sigma_{L,\boldsymbol{p},r}^{-}\right)\right]\right\}\sqrt{\hat{\mathcal{N}}_{\boldsymbol{p},r}}$$

Map the Hamiltonian to qubits

$$\hat{H}_E + \hat{H}_B = \sum_{\mathbf{p}\neq 0}\sum_{r} \hat{E}_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} a^{r}_{\mathbf{p}} = \sum_{\mathbf{p}\neq 0}\sum_{r}\sum_{J=0}^{K-1} 2^J \left[\frac{1}{2}(I - \sigma^3_{J,\mathbf{p},r})\right]$$

$$\hat{H}_I = \hat{H}_I^S + \hat{H}_I^A$$

$$\hat{H}_I^S = \sum_{\mathbf{n},i}\sum_{\mathbf{p}\neq 0}\sum_{r}\sum_{L=0}^{K-1} \sum_{\substack{\mu_0,\ldots,\mu_L=1,2\\ \mu_0+\ldots+\mu_L-L-1=\text{even}}} \sum_{\mu_{L+1},\ldots,\mu_{K-1}=0,3} \frac{2^{-L} J^i(\mathbf{n})}{M^{\frac{3}{2}}\sqrt{2\hat{E}_{\mathbf{p}}}}$$
$$\times \left[\mathrm{Re}(\epsilon^r_i(\mathbf{p}))\cos(\mathbf{p}\cdot\mathbf{n}) - \mathrm{Im}(\epsilon^r_i(\mathbf{p}))\sin(\mathbf{p}\cdot\mathbf{n})\right] \mathcal{F}_{\mu_{K-1},\ldots,\mu_{L+1}} \mathcal{G}^S_{\mu_L,\ldots,\mu_0} \sigma^{\mu_{K-1}}_{K-1,\mathbf{p},r}\cdots\sigma^{\mu_{L+1}}_{L+1,\mathbf{p},r}\sigma^{\mu_L}_{L,\mathbf{p},r}\cdots\sigma^{\mu_0}_{0,\mathbf{p},r}$$

$$\hat{H}_I^A = \sum_{\mathbf{n},i}\sum_{\mathbf{p}\neq 0}\sum_{r}\sum_{L=0}^{K-1} \sum_{\substack{\mu_0,\ldots,\mu_L=1,2\\ \mu_0+\ldots+\mu_L-L-1=\text{odd}}} \sum_{\mu_{L+1},\ldots,\mu_{K-1}=0,3} \frac{2^{-L} J^i(\mathbf{n})}{M^{\frac{3}{2}}\sqrt{2\hat{E}_{\mathbf{p}}}}$$
$$\times \left[\mathrm{Re}(\epsilon^r_i(\mathbf{p}))\sin(\mathbf{p}\cdot\mathbf{n}) + \mathrm{Im}(\epsilon^r_i(\mathbf{p}))\cos(\mathbf{p}\cdot\mathbf{n})\right] \mathcal{F}_{\mu_{K-1},\ldots,\mu_{L+1}} \mathcal{G}^A_{\mu_L,\ldots,\mu_0} \sigma^{\mu_{K-1}}_{K-1,\mathbf{p},r}\cdots\sigma^{\mu_{L+1}}_{L+1,\mathbf{p},r}\sigma^{\mu_L}_{L,\mathbf{p},r}\cdots\sigma^{\mu_0}_{0,\mathbf{p},r}$$

$$\mathcal{F}_{\mu_{K-1},\ldots,\mu_{L+1}} = \sum\nolimits_{\mu_0,\ldots,\mu_L=0,3}(-1)^{\mu_L} f_{\mu_{K-1},\ldots,\mu_0} \qquad f_{\mu_{K-1},\ldots,\mu_0} = \sum_{i_0,\ldots,i_{K-1}=0}^{1} \sqrt{\mathcal{N}(i_{K-1},\ldots,i_0)}(-1)^{i_{K-1}\mu_{K-1}} \times \ldots \times (-1)^{i_0\mu_0}$$

$$\mathcal{G}^S_{\mu_L,\ldots,\mu_0} = (-1)^{\mu_L-1}(-1)^{\left(-L-1+\sum_{J=0}^{L}\mu_J\right)/2} \qquad \mathcal{G}^A_{\mu_L,\ldots,\mu_0} = (-1)^{\mu_L-1}(-1)^{\left(-L-2+\sum_{J=0}^{L}\mu_J\right)/2}$$

Gate cost of simulating time evolution

Second-order Suzuki formula:

$$e^{-iHt} \approx \left(\prod_{\gamma=\Gamma}^{1} e^{-ih_\gamma \delta t/2} \prod_{\gamma=1}^{\Gamma} e^{-ih_\gamma \delta t/2}\right)^{N_T} + \mathcal{O}(\varepsilon = t^3/N_T^2)$$

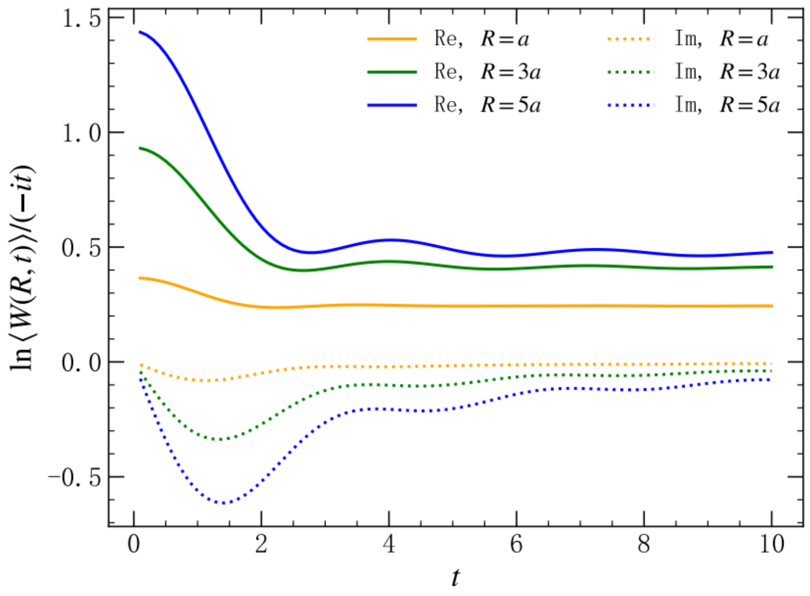
Evolution operators	**Cost of one-qubit gate**	**Cost of CNOT gate**
$\exp[-i(H_E + H_B)\delta t/2]$	$N_{G,1}^{EB} \sim M^d \log_2 \Lambda$	$N_{G,2}^{EB} = 0$
$\exp(-iH_I \delta t/2)$	$N_{G,1}^{I} \sim M^{2d} \Lambda\, (\log_2 \Lambda)^2$	$N_{G,2}^{I} \sim M^{2d} \Lambda\, (\log_2 \Lambda)^2$
$\exp(-iH_V \delta t/2)$	$N_{G,1}^{V} \sim M^{2d}$	$N_{G,2}^{V} \sim M^{2d}$
$\exp(-iH_M \delta t/2)$	$N_{G,1}^{M} \sim M^{d}$	$N_{G,2}^{M} \sim M^{2d-1}$
$\exp(-iH_W \delta t/2)$	$N_{G,1}^{W} \sim M^{d}$	$N_{G,2}^{W} \sim M^{2d-1}$

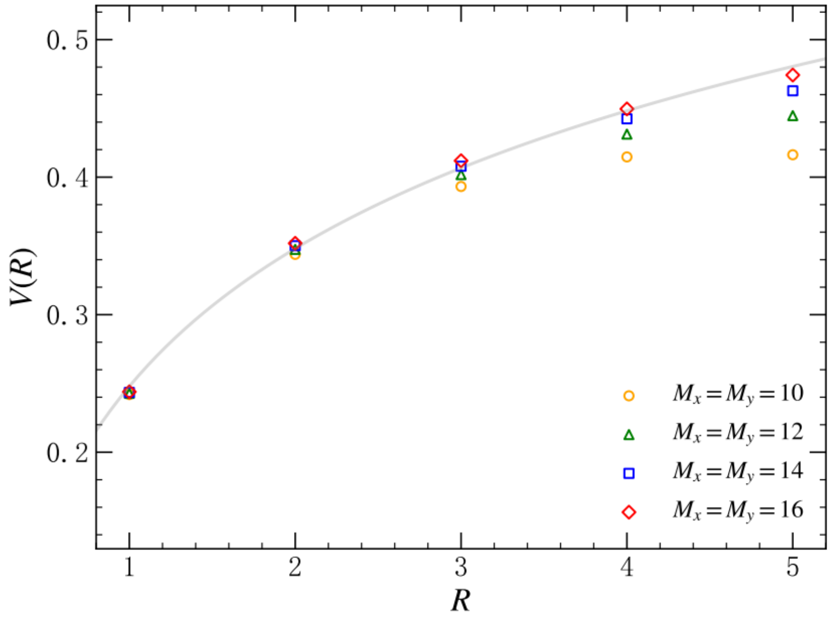
- Gate cost of our formalism: $O\left(\frac{t^{3/2}}{\varepsilon^{1/2}} M^{2d} \Lambda (\log_2 \Lambda)^2\right)$
- Gate cost for simulating e^{-iHt} in K-S formalism: $O\left(\frac{t^{3/2}}{\varepsilon^{1/2}} M^{3d/2} \Lambda_{KS} (\log_2 \Lambda_{KS})^2\right)$

Results of Wilson loop in (2+1)-D pure QED

Potential between electron and positron can be extracted from Wilson loop $W(R,t)$:

$$V(r) = \lim_{t\to\infty} \frac{1}{-it} \ln\langle\Omega|W(R,t)|\Omega\rangle$$





Grey curve: $\mathrm{V(R)} = a + b\log R$

Summary of Coulomb gauge formalism

- Our formalism overcomes the "Impossible triangle" of other formalisms:
  - Preserve Gauss's law strictly after digitization by solving polarization vector of photon.
  - Has unitary gauge link $U_i(\boldsymbol{x}) = \exp\left(-ig\hat{A}_i(\boldsymbol{x})\right)$.
  - Has controllable truncation error $\varepsilon_s \sim O\left(\frac{g^2 M^{3d+3} + EM^{d+1}}{\Lambda}\right)$.
- The cost of our formalism: more gate cost for simulating e^{-iHt}.
- Need more effort to go to the non-abelian case.