Quantum computing of chirality imbalance in SU(2) gauge theory

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Contents

Introduction

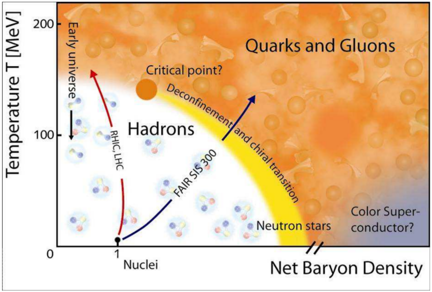
Non-perturbative nature of QCD

Quantum Chromodynamics:

- Spontaneous chiral symmetry breaking

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

- The phase diagram of quantum chromodynamics



Aarts, J. Phys. Conf. Ser. 706, 022004 (2016)

Classical methods VS Quantum computing

Lattice gauge theory:

- Gauge theories describe interactions between particles mediated by gauge fields.
- Lattice gauge theory is a successful discretization method that allows for non-perturbative calculations.

Classical MCMC methods:

- Action formulation of the theory
- Suffers from sign problem:
  - Real-time dynamics
  - High baryon density

Quantum simulation:

- Hamiltonian formulation
- Sign-problem free
- Parallel computation and global information processing
- Scalable to larger systems

SU(2) gauge theory

The Kogut-Susskind formulation

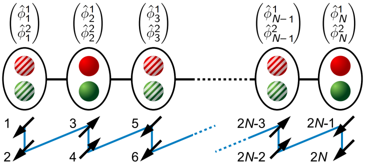
The Kogut-Susskind formulation of the Yang-Mills Hamiltonian is given by:

$$H = \frac{1}{2\Delta}\sum_{n=0}^{N-2}\left(\phi_n^\dagger U_n \phi_{n+1} + H.C.\right) + m\sum_{n=0}^{N-1}(-1)^{n+1}\phi_n^\dagger\phi_n + \frac{\Delta g^2}{2}\sum_{n=0}^{N-2}\mathrm{L}_n^2$$

As the conjugate momenta of the gauge field, the chromoelectric fields and the gauge link satisfy the canonical commutation relation:

$$\left[L_n^a, (U_n)_{\alpha\beta}\right] = (t^a U_n)_{\alpha\beta}\,,$$

$$\left[R_n^a, (U_n)_{\alpha\beta}\right] = (U_n t^a)_{\alpha\beta}\,.$$



Atas, Y. Y., et al., Nat. Commun. 12, 6499 (2021).

Purely fermionic formulation

For clarity in representation, we rewrite the Hamiltonian as:

$$H = \tilde{m} H_{\mathsf{m}} + \frac{\Delta^2 g^2}{2} H_{\mathsf{el}} + \frac{1}{2} H_{\mathsf{kin}}$$

To eliminate the gauge links, we consider the following local gauge transformation:

$$\Theta = \prod_{k} \exp\left(i\theta_k \cdot \sum_{j>k} \mathrm{Q}_j \right)$$

Under this transformation, the staggered fermion field transforms as:

$$\Theta \phi_n \Theta^\dagger = U_{n-1}^\dagger U_{n-2}^\dagger \cdots U_0^\dagger \phi_n$$

The mass term remains the same, and the hopping term becomes:

$$H'_{\mathsf{kin}} \to \Theta H_{\mathsf{kin}} \Theta^\dagger = \sum_{n=0}^{N-2} (\phi_n^\dagger \phi_{n+1} + \mathsf{H.C.})$$

Purely fermionic formulation

The Gauss's law:

$$G_n^a = L_n^a - R_{n-1}^a - Q_n^a = 0$$

The non-Abelian charge and the left and right color electric field transforms as:

$$\Theta Q_n^a \Theta^\dagger = (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_n^b$$

$$\Theta L_n^a \Theta^\dagger = L_n^a - \sum_{m>n} (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_m^b$$

$$\Theta R_{n-1}^a \Theta^\dagger = R_{n-1}^a - \sum_{m>n-1} (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_m^b$$

The Gauss's law therefore transforms as:

$$\Theta G_n^a \Theta^\dagger = \Theta L_n^a \Theta^\dagger - \Theta R_{n-1}^a \Theta^\dagger - \Theta Q_n^a \Theta^\dagger = L_n^a - R_{n-1}^a = 0, \qquad n > 0$$

$$\Theta G_0^a \Theta^\dagger = L_0^a - \sum_{m>0} Q_m^a - R_{-1}^a - Q_0^a = L_0^a - \sum_{m\geq 0} Q_m^a - R_{-1}^a = 0, \qquad n = 0$$

Purely fermionic formulation

Adopt the adjoint representation $R_n^a = \left(U_n^{adj}\right)^{ab} L_n^b$, Gauss's law can thus be solved recursively and gives:

$$L_n^a = (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m \geq 0} Q_m^b)$$

Finally, we obtain the form of the left color electric field under Θ-transformation:

$$\begin{aligned} \Theta L_n^a \Theta^\dagger &= (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m \geq 0} Q_m^b) \\ &\quad - \sum_{m > n} (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_m^b \\ &= (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m \leq n} Q_m^b) \end{aligned}$$

The chromoelectric energy term can be expressed in terms of the fermionic field:

$$H'_{\mathrm{el}} \to \Theta H_{\mathrm{el}} \Theta^\dagger = \sum_{n=0}^{N-2} \Theta \mathrm{L}_n^2 \Theta^\dagger = \sum_{n=0}^{N-2} \left(\sum_{k \leq n} \mathrm{Q}_k \right)^2$$

Qubit encoding

We employ the standard Jordan-Wigner transformation:

$$\varphi_n = \sigma_n^- \prod_{l=0}^{n-1} (-i\sigma_l^z)$$

Therefore, the lattice Hamiltonian can be mapped to the spin system as follows:

$$\begin{aligned} H \quad = \quad & \tilde{m} \sum_{n=0}^{N-1} \left[(-1)^{n+1} \frac{\sigma_{2n}^z + \sigma_{2n+1}^z}{2} + 1 \right] \\ & - \frac{1}{2} \sum_{n=0}^{N-2} \left(\sigma_{2n}^+ \sigma_{2n+1}^z \sigma_{2n+2}^- + \sigma_{2n+1}^+ \sigma_{2n+2}^z \sigma_{2n+3}^- + \text{H.C.} \right) \\ & + \frac{\Delta^2 g^2}{2} \sum_{n=0}^{N-2} \left(\sum_{k \leq n} \mathrm{Q}_k \right)^2 \end{aligned}$$

Qubit encoding

Where we replace the two components of the staggered fermion with:

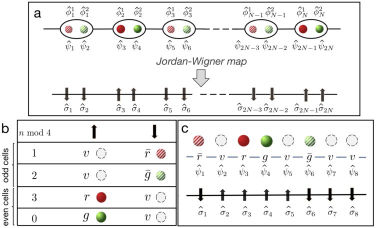
$$\phi_n = \begin{pmatrix} \phi_n^r \\ \phi_n^g \end{pmatrix} \longleftrightarrow \begin{pmatrix} \varphi_{2n} \\ \varphi_{2n+1} \end{pmatrix}$$

The three components of the non-Abelian charge are encoded as:

$$Q_n^x = \frac{1}{2}\left(\sigma_{2n+1}^+ \sigma_{2n}^- + \text{H.C.}\right)$$

$$Q_n^y = \frac{i}{2}\left(\sigma_{2n+1}^+ \sigma_{2n}^- - \text{H.C.}\right)$$

$$Q_n^z = \frac{1}{4}\left(\sigma_{2n}^z - \sigma_{2n+1}^z\right)$$



Atas, Y. Y., et al., Nat. Commun. 12, 6499 (2021).

Variational quantum algorithm

Motivation

The free energy of a system at finite temperature can be expressed as：

$$\begin{aligned} F(\beta) &= E(\beta) - TS(\beta) \\ &= \mathrm{Tr}\left[\rho(\beta)H\right] + T\mathrm{Tr}\left[\rho(\beta)\log\rho(\beta)\right] \end{aligned}$$

We propose a parametrized mixed state as:

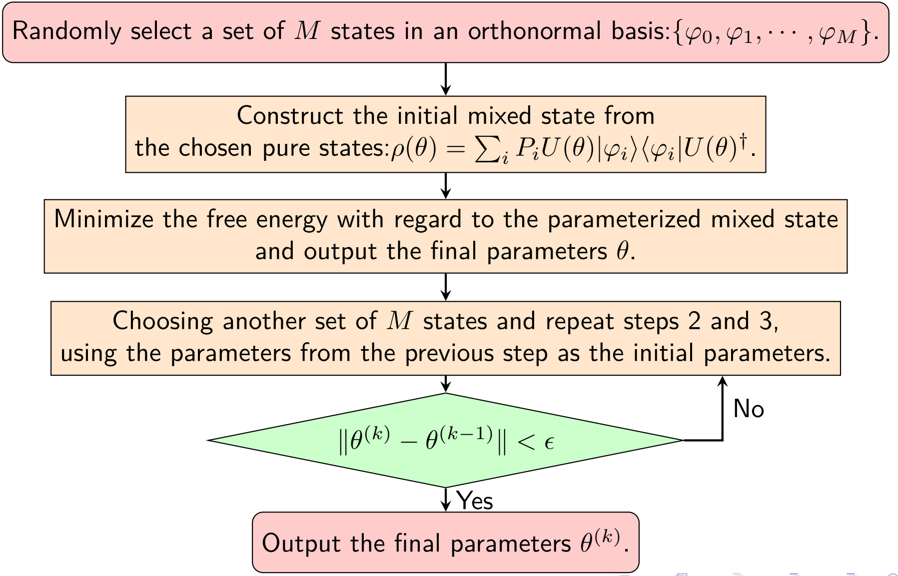
$$\rho(\theta) = \sum_i P_i U(\theta)|\varphi_i\rangle\langle\varphi_i|U(\theta)^\dagger$$

The parameterized unitary operator $U(\theta)$ is constructed by the QAOA:

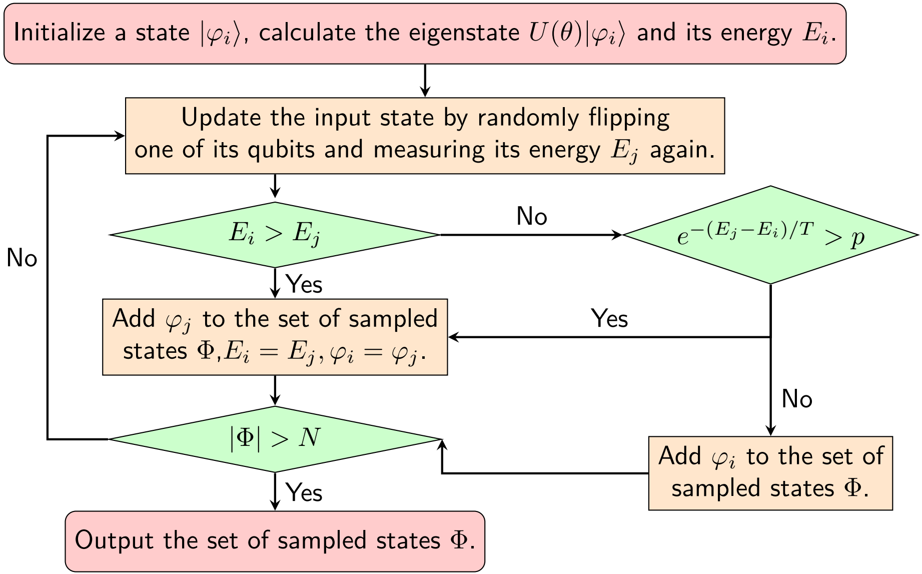
$$U(\theta) = \prod_{i=1}^{p}\prod_{j=1}^{d}\exp\left(i\theta_{ij}H_j\right)$$

Generally, the greater the values of p and d, the stronger the representation power of the ansatz.

Random sampling process



Monte Carlo sampling

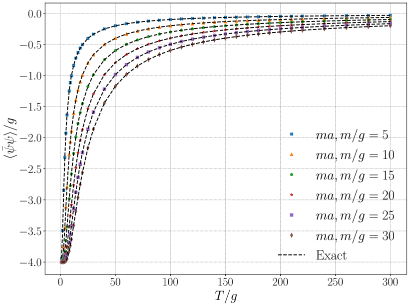


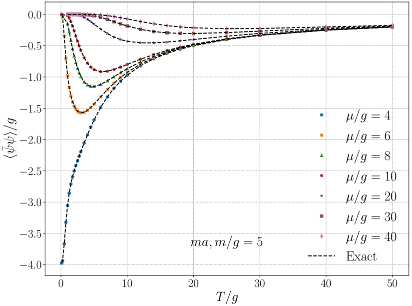
Result

The result from random sampling process

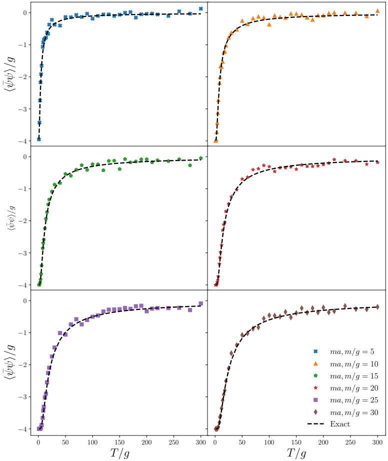
The thermal average of an observable O:

$$\langle O\rangle = \sum_i P_i \langle \varphi_i | U(\theta)^\dagger O U(\theta) | \varphi_i \rangle$$



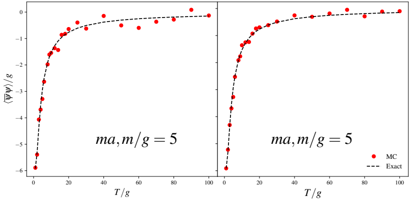


The result from Monte Carlo sampling



The thermal average of an observable:

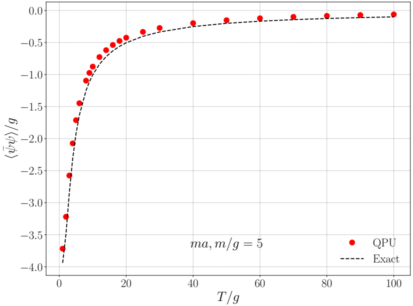
$$\langle O \rangle = \frac{1}{N}\sum_i \langle \varphi_i | U(\theta)^\dagger O U(\theta) | \varphi_i \rangle$$

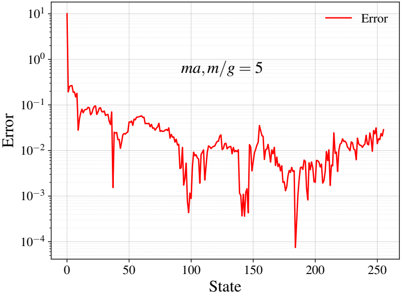


The result from IBM's quantum hardware

The thermal average of an observable O:

$$\langle O\rangle = \sum_i P_i \langle \varphi_i | U(\theta)^\dagger O U(\theta) | \varphi_i \rangle$$





Summary

Summary

In this work, we present the first direct simulation of chiral symmetry breaking using the 1+1D SU(2) non-Abelian gauge theory on a quantum computer:

1. We propose a variational quantum algorithm to prepare the Gibbs state.
2. We employ the VQA to calculate the chiral condensate of the SU(2) non-Abelian gauge theory with different m/g, temperatures, and chemical potentials.
3. We employ the Monte Carlo sampling to extend our quantum algorithm to large systems.

Our exploration highlights the potential of implementing current or near-term quantum computers to study QCD phase transition, especially at high chemical potential where the Taylor expansion used in classical lattice calculations fails.

Thank you!

Supplementary Materials

SU(2) gauge theory in the continuum

The Lagrangian density of SU(2) non-Abelian gauge theory:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + i\bar{\psi}\gamma^\mu(\partial_\mu + igA^a_\mu t^a)\psi - m\bar{\psi}\psi.$$

We adopt the temporal or Weyle guage $A^a_0 = 0$:

$$H = \int dx \left[\sum \pi(x)\dot{\phi}(x) - \mathcal{L}\right].$$

$$\Pi_1 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \bar{\psi}i\gamma^0, \quad \Pi_2 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^a_1)} = \frac{\partial(-\frac{1}{4}F^b_{\mu\nu}F^{\mu\nu,b})}{\partial(\partial_0 A^a_1)} = -F^{01,a}.$$

$$H = \int dx \left[-i\bar{\psi}\gamma^1(\partial_1 + igA^a_1 t^a)\psi + m\bar{\psi}\psi + \frac{1}{2}(L^a)^2\right].$$

SU(2) gauge theory on lattice

Dimensionless lattice variables:

$$M \to \frac{1}{a}\hat{M},$$

$$\psi_\alpha(x) \to \frac{1}{a^{1/2}}\hat{\psi}_\alpha(n),$$

$$\bar{\psi}_\alpha(x) \to \frac{1}{a^{1/2}}\hat{\bar{\psi}}_\alpha(n),$$

$$\partial_\mu \psi_\alpha(x) \to \frac{1}{a^{3/2}}\hat{\partial}_\mu \hat{\psi}_\alpha(n),$$

where $\hat{\partial}_\mu$ is the anti-Hermitian lattice derivative defined by

$$\hat{\partial}_\mu \hat{\psi}_\alpha(n) = \frac{1}{2}\left[\hat{\psi}_\alpha(n+\hat{\mu}) - \hat{\psi}_\alpha(n-\hat{\mu})\right].$$

SU(2) gauge theory on lattice

The quantum theory of the free Dirac field:

$$H_F = \int dx \ \left[-i\bar{\psi}\gamma^1\partial_1\psi + m\bar{\psi}\psi\right]$$

$$= \sum_n \left(-\frac{i}{2a}\right)\psi^\dagger(n)\gamma^0\gamma^1\left(\psi(n+1)-\psi(n-1)\right) + \frac{m}{a}\psi^\dagger(n)\gamma^0\psi(n)$$

The Canonical equation:

$$\dot{\psi}(n) = \frac{\partial H}{\partial(i\psi^\dagger(n))} = \left(-\frac{1}{2a}\right)\gamma^0\gamma^1\left[\psi(n+1)-\psi(n-1)\right] - i\frac{m}{a}\gamma^0\chi(n)$$

The plane wave solution: $\psi(n) = u(E,p)e^{-iEt+ipna}$

$$\psi(n+1)-\psi(n-1) = \psi(n)\left(e^{ipa}-e^{-ipa}\right) = 2i\sin(pa)\,\psi(n)$$

The matrix γ:

$$\gamma^0 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma^y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

SU(2) gauge theory on lattice

Thus:

$$\begin{pmatrix} E - \frac{m}{a} & -\frac{1}{a}\sin(pa) \\ -\frac{1}{a}\sin(pa) & E + \frac{m}{a} \end{pmatrix} \psi(n) = 0$$

We get the dispersion relation of ψ :

$$E^2 = \frac{m^2}{a^2} + \frac{1}{a^2}\sin^2(pa)$$

For negative energy solutions: $\psi(n) = v(E,p)e^{iEt - ipna}$

$$\psi(n+1) - \psi(n-1) = \psi(n)\left(e^{-ipa} - e^{ipa}\right) = -2i\sin(pa)\,\psi(n)$$

Thus:

$$\begin{pmatrix} E - \frac{m}{a} & \frac{1}{a}\sin(pa) \\ \frac{1}{a}\sin(pa) & E + \frac{m}{a} \end{pmatrix} \psi(n) = 0$$

We get the dispersion relation of ψ :

$$E^2 = \frac{m^2}{a^2} + \frac{1}{a^2}\sin^2(pa)$$

SU(2) gauge theory on lattice

$$H_K = \int dx\ \left(-i\bar{\psi}\gamma^1\partial_1\psi\right) = \sum_n \left(-\frac{i}{2a}\right)\psi^\dagger(n)\gamma^0\gamma^1\left(\psi(n+1)-\psi(n-1)\right)$$

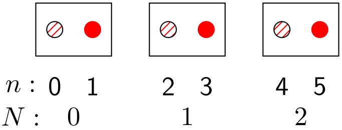
Transformation:

$$\psi(n) = (\gamma^0\gamma^1)^n\chi(n), \quad \psi^\dagger(n) = \chi^\dagger(n)(\gamma^0\gamma^1)^n$$

$$\begin{aligned} H_K = &\sum_n \left(-\frac{i}{2a}\right)\Big[\chi^\dagger(n)(\gamma^0\gamma^1)^n\gamma^0\gamma^1(\gamma^0\gamma^1)^{n+1}\chi(n+1) \\ &-\chi^\dagger(n)(\gamma^0\gamma^1)^n\gamma^0\gamma^1(\gamma^0\gamma^1)^{n-1}\chi(n-1)\Big] \\ =&\sum_n \left(-\frac{i}{2a}\right)\left[\chi^\dagger(n)\chi(n+1)-\chi^\dagger(n)\chi(n-1)\right] \\ =&\sum_{n,\alpha} \left(-\frac{i}{2a}\right)\left[\chi^\dagger_\alpha(n)\chi_\alpha(n+1)-\chi^\dagger_\alpha(n)\chi_\alpha(n-1)\right] \\ =&\sum_n \left(\frac{-i}{2a}\right)\left[\chi^\dagger(n)\chi(n+1)-\chi^\dagger(n)\chi(n-1)\right] \end{aligned}$$

SU(2) gauge theory on lattice

The staggered fermion:



We relabel n as: $n_\mu = 2N_\mu + \rho_\mu$

$$H_K = \sum_{N,\rho} \left(\frac{-i}{2a}\right) \left[\chi^\dagger(2N+\rho)\chi(2N+\rho+1) - \chi^\dagger(2N+\rho)\chi(2N+\rho-1)\right]$$

$$= \sum_{N,\rho,\rho'} \left(-\frac{i}{2a}\right) \chi^\dagger_\rho(N) \Big\{ \delta_{\rho,\rho'-1} \left[\chi_{\rho'}(N) - \chi_{\rho'}(N-1)\right] + \delta_{\rho,\rho'+1} \left[\chi_{\rho'}(N+1) - \chi_{\rho'}(N)\right] \Big\}$$

SU(2) gauge theory on lattice

The γ matrix:

$$\Gamma^1_{\rho\rho'} = i\sigma^y_{\rho\rho'} = (\delta_{\rho,\rho'-1} - \delta_{\rho,\rho'+1})\,, \qquad \Gamma^5_{\rho\rho'} = \sigma^x_{\rho\rho'} = (\delta_{\rho,\rho'-1} + \delta_{\rho,\rho'+1})$$

so:

$$H_K = \sum_{N,\rho,\rho'} \left(-\frac{i}{4a}\right) \chi^\dagger_\rho(N) \Big\{ \Gamma^5_{\rho\rho'} \left[\chi_{\rho'}(N+1) - \chi_{\rho'}(N-1)\right]$$

$$-\Gamma^1_{\rho\rho'} \left[\chi_{\rho'}(N+1) - 2\chi_{\rho'}(N) + \chi_{\rho'}(N-1)\right] \Big\}$$

$$= \sum_{N,\rho,\rho'} (-i)\, \chi^\dagger_\rho(N) \Big[\Gamma^5_{\rho\rho'} \partial_1 \chi_{\rho'}(N) - a\,\Gamma^1_{\rho\rho'}\, \Box_1\, \chi_{\rho'}(N) \Big]$$

where:

$$\partial_1 \chi_\rho(N) = \frac{1}{4a} \left(\chi_\rho(N+1) - \chi_\rho(N-1)\right)$$

$$\Box_1 \chi_\rho(N) = \frac{1}{4a^2} \left(\chi_\rho(N+1) - 2\chi_\rho(N) + \chi_\rho(N-1)\right)$$

SU(2) gauge theory on lattice

The mass term:

$$H_m = \int dx\, m\bar{\psi}\psi = \sum_n \frac{m}{a}\psi^\dagger(n)\gamma^0\psi(n)$$

$$= \sum_n \frac{m}{a}\chi^\dagger(n)(\gamma^0\gamma^1)^n\gamma^0(\gamma^0\gamma^1)^n\chi(n) = \sum_n (-1)^n\frac{m}{a}\chi^\dagger(n)\gamma^0\chi(n)$$

The quantum theory of the free Dirac field:

$$H_F = \sum_{N,\rho,\rho'} \Big\{ (-i)\,\chi^\dagger_\rho(N)\Big[\Gamma^5_{\rho\rho'}\partial_1\chi_{\rho'}(N) - a\,\Gamma^1_{\rho\rho'}\,\Box_1\,\chi_{\rho'}(N)$$

$$+i(-1)^n\frac{m}{a}\Gamma^0_{\rho\rho'}\chi_{\rho'}(N)\Big]\Big\}$$

The Canonical equation:

$$\dot{\chi}_\rho(N) = \frac{\partial H}{\partial(i\chi^\dagger_\rho(N))} = \sum_{\rho'}\Big\{ -\Big[\Gamma^5_{\rho\rho'}\partial_1 - a\,\Gamma^1_{\rho\rho'}\,\Box_1 + i(-1)^n\frac{m}{a}\Gamma^0_{\rho\rho'}\Big]\chi_{\rho'}(N)\Big\}$$

SU(2) gauge theory on lattice

We get:

$$\sum_{\rho'} \left\{ i\Gamma^0_{\rho\rho'}\dot{\chi}_{\rho'}(N) + \left[i\Gamma^1_{\rho\rho'}\partial_1 + ia\Gamma^5_{\rho\rho'}\Box_1 + (-1)^n \frac{m}{a}\delta_{\rho\rho'} \right] \chi_{\rho'}(N) \right\} = 0$$

The plane wave solution: $\chi_\rho(N) = u(E,p)e^{-iEt+ipN2a}$

The first-order derivative:

$$\partial_1\chi_\rho(N) = \frac{\chi_\rho(N+1) - \chi_\rho(N-1)}{4a}$$

$$= \frac{1}{4a}\chi_\rho(N)\left(e^{ip2a} - e^{-ip2a}\right) = \frac{i}{2a}\sin(p2a)\chi_\rho(N)$$

The second-order derivative:

$$\Box_1\chi_\rho(N) = \frac{1}{4a^2}\left(\chi_\rho(N+1) - 2\chi_\rho(N) + \chi_\rho(N-1)\right)$$

$$= \frac{1}{4a^2}\chi_\rho(N)\left(e^{ip2a} - 2 + e^{-ip2a}\right) = \frac{1}{2a^2}\chi_\rho(N)\left(\cos(p2a) - 1\right)$$

SU(2) gauge theory on lattice

Thus:

$$\sum_{\rho'}\left[E\Gamma^0_{\rho\rho'}-\frac{1}{2a}\sin(p2a)\Gamma^1_{\rho\rho'}+\frac{i}{2a}\left(\cos(p2a)-1\right)\Gamma^5_{\rho\rho'}+(-1)^n\frac{m}{a}\right]\chi_{\rho'}(N)=0$$

$$\begin{pmatrix} E+(-1)^n\frac{m}{a} & -\frac{1}{2a}\sin(p2a)+\frac{i}{2a}\left(\cos(p2a)-1\right) \\ \frac{1}{2a}\sin(p2a)+\frac{i}{2a}\left(\cos(p2a)-1\right) & -E+(-1)^n\frac{m}{a} \end{pmatrix}\chi(N)=0$$

We get the dispersion relation of $\chi(N)$:

$$E^2=\frac{m^2}{a^2}+\frac{1}{2a^2}\left(1-\cos(p2a)\right)=\frac{m^2}{a^2}+\frac{1}{a^2}\sin^2(pa)$$

For negative energy solutions: $\chi_\rho(N)=v(E,p)e^{iEt-ipN2a}$

$$\partial_1\chi_\rho(N)=\frac{\chi_\rho(N+1)-\chi_\rho(N-1)}{4a}$$

$$=\frac{1}{4a}\chi_\rho(N)\left(e^{-ip2a}-e^{ip2a}\right)=-\frac{i}{2a}\sin(p2a)\chi_\rho(N)$$

SU(2) gauge theory on lattice

The second-order derivative:

$$\Box_1 \chi_\rho(N) = \frac{1}{4a^2}\left(\chi_\rho(N+1) - 2\chi_\rho(N) + \chi_\rho(N-1)\right)$$

$$= \frac{1}{4a^2}\chi_\rho(N)\left(e^{-ip2a} - 2 + e^{ip2a}\right) = \frac{1}{2a^2}\chi_\rho(N)\left(\cos(p2a) - 1\right)$$

Thus:

$$\sum_{\rho'}\left[-E\Gamma^0_{\rho\rho'} + \frac{1}{2a}\sin(p2a)\Gamma^1_{\rho\rho'} + \frac{i}{2a}\left(\cos(p2a)-1\right)\Gamma^5_{\rho\rho'} + (-1)^n\frac{m}{a}\right]\chi_{\rho'}(N) = 0$$

$$\begin{pmatrix} -E + (-1)^n\frac{m}{a} & \frac{1}{2a}\sin(p2a) + \frac{i}{2a}\left(\cos(p2a)-1\right) \\ -\frac{1}{2a}\sin(p2a) + \frac{i}{2a}\left(\cos(p2a)-1\right) & E + (-1)^n\frac{m}{a} \end{pmatrix}\chi(N) = 0$$

We get the dispersion relation of $\chi(N)$:

$$E^2 = \frac{m^2}{a^2} + \frac{1}{2a^2}\left(1 - \cos(p2a)\right) = \frac{m^2}{a^2} + \frac{1}{a^2}\sin^2(pa)$$

Eliminate the gauge field

We consider the following local gauge transformation

$$\Theta = \prod_k \exp(i\theta_k \cdot \sum_{j>k} \mathrm{Q}_j)$$

Under this transformation, the staggered fermion field transforms as

$$\begin{aligned}
W_k \phi_n W_k^\dagger &= exp(i\theta_k^a \sum_{m>k} Q_m^a)\phi_n exp(-i\theta_k^a \sum_{m>k} Q_m^a) \\
&= exp\left(i\theta_k^a Q_n^a\right)\phi_n exp\left(-i\theta_k^a Q_n^a\right) \\
&= \phi_n + [i\theta_k^a Q_n^a, \phi_n] + \frac{1}{2!}[i\theta_k^a Q_n^a, [i\theta_k^a Q_n^a, \phi_n]] + \cdots \\
&= \phi_n - i\theta_k^a T^a \phi_n - \frac{1}{2!}\left(\theta_k^a T^a\right)^2 \phi_n + \cdots \\
&= (1 - i\theta_k^a T^a + \frac{1}{2!}\left(-i\theta_k^a T^a\right)^2 + \cdots)\phi_n \\
&= exp\left(-i\theta_k^a T^a\right)\phi_n, \qquad n > k, \\
W_k \phi_n W_k^\dagger &= \phi_n, \qquad \text{otherwise.}
\end{aligned}$$

Eliminate the gauge field

Under this transformation, the staggered fermion field transforms as

$$\Theta \phi_n \Theta^\dagger = U_{n-1}^\dagger U_{n-2}^\dagger \cdots U_0^\dagger \phi_n$$

The mass term remains the same, and the hopping term becomes

$$H'_{\text{kin}} \to \Theta H_{\text{kin}} \Theta^\dagger = \sum_{n=0}^{N-2} (\phi_n^\dagger \phi_{n+1} + \text{H.C.})$$

The Gauss's law

$$G_n^a = L_n^a - R_{n-1}^a - Q_n^a = 0$$

Although Gauss's law is formally gauge-invariant, its explicit representation may be modified under the Θ-transformation. In the following, we shall explicitly derive the form of Gauss's law in the transformed coordinate system.

Eliminate the gauge field

We first consider the expression of the non-Abelian charge under the W-rotation

$$\begin{aligned}
W_k Q_m^a W_k^\dagger &= W_k \phi_m^\dagger T^a \phi_m W_k^\dagger \\
&= \phi_m^\dagger U_k T^a U_k^\dagger \phi_m \\
&= \phi_m^{\dagger i} \left(U_k\right)_{pl} \left(T^a\right)_{ln} (U_k^\dagger)_{nq} \phi_m^j \delta_{ip}\delta_{jq} \\
&= \phi_m^{\dagger i} \left(U_k\right)_{pl} \left(T^a\right)_{ln} (U_k^\dagger)_{nq} \phi_m^j [2(T^b)_{ij}(T^b)_{qp} + \frac{1}{2}\delta_{ij}\delta_{qp}] \\
&= (U_k^{adj})_{ab} Q_m^b, \quad m > k, \\
W_k Q_m^a W_k^\dagger &= Q_m^a, \quad \text{otherwise.}
\end{aligned}$$

Thus, we obtain the form of the non-Abelian charge under the Θ-transformation

$$\begin{aligned}
\Theta Q_n^a \Theta^\dagger &= W_0 W_1 \cdots W_{N-1} Q_n^a W_{N-1}^\dagger \cdots W_1^\dagger W_0^\dagger \\
&= W_0 W_1 \cdots W_{n-1} Q_n^a W_{n-1}^\dagger \cdots W_1^\dagger W_0^\dagger \\
&= (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_n^b.
\end{aligned}$$

Eliminate the gauge field

Given that the W-rotation and the gauge link U_n exhibit the same matrix structure

$$[L_n^a, W_n] = \left(\sum_{m>n} Q_m^a\right) W_n.$$

Thus, the left color electric field L_n^a takes the following form under the Θ-transformation

$$\begin{aligned}
\Theta L_n^a \Theta^\dagger &= W_0 W_1 \cdots W_n L_n^a W_n^\dagger \cdots W_1^\dagger W_0^\dagger \\
&= W_0 W_1 \cdots W_{n-1} (L_n^a - \sum_{m>n} Q_m^a) W_{n-1}^\dagger \cdots W_1^\dagger W_0^\dagger \\
&= L_n^a - \sum_{m>n} (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_m^b.
\end{aligned}$$

Eliminate the gauge field

Since the left and right color electric field are related by an adjoint representation, $R_n^a = \left(U_n^{\mathrm{adj}}\right)^{ab} L_n^b$, we can obtain the form of the right color electric field under the Θ-transformation as follows

$$\Theta R_n^a \Theta^\dagger = R_n^a - \sum_{m>n} (U_n^{adj} U_{n-1}^{adj} \cdots U_0^{adj})^{ab} Q_m^b.$$

The Gauss's law therefore transforms as

$$\Theta G_n^a \Theta^\dagger = \Theta L_n^a \Theta^\dagger - \Theta R_{n-1}^a \Theta^\dagger - \Theta Q_n^a \Theta^\dagger = L_n^a - R_{n-1}^a = 0, \qquad n > 0.$$

Adopt the adjoint representation $R_n^a = \left(U_n^{adj}\right)^{ab} L_n^b$, Gauss's law can thus be solved recursively and gives

$$L_n^a = (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} L_0^b$$

Eliminate the gauge field

For $n = 0$, Gauss's law gives

$$\Theta G_0^a \Theta^\dagger = L_0^a - \sum_{m>0} Q_m^a - R_{-1}^a - Q_0^a = L_0^a - \sum_{m\geq 0} Q_m^a - R_{-1}^a = 0$$

Thus we can represent the left color electric field by the background field and the non-abelian charge

$$L_n^a = (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m\geq 0} Q_m^b)$$

Finally, we obtain the form of the left color electric field under Θ-transformation:

$$\begin{aligned}\Theta L_n^a \Theta^\dagger &= (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m\geq 0} Q_m^b) \\ &\quad - \sum_{m>n} (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} Q_m^b \\ &= (U_{n-1}^{adj} U_{n-2}^{adj} \cdots U_0^{adj})^{ab} (R_{-1}^b + \sum_{m\leq n} Q_m^b)\end{aligned}$$

Eliminate the gauge field

The chromoelectric energy term can be expressed in terms of the fermionic field:

$$H'_{\mathsf{el}} \to \Theta H_{\mathsf{el}} \Theta^\dagger = \sum_{n=0}^{N-2} \Theta \mathrm{L}_n^2 \Theta^\dagger = \sum_{n=0}^{N-2} \left(\sum_{k \le n} \mathrm{Q}_k \right)^2$$

Consequently, the Hamiltonian can be reformulated as

$$\begin{aligned} H' \to \Theta H \Theta^\dagger = \frac{1}{2} \sum_{n=0}^{N-2} \left(\phi_n^\dagger \phi_{n+1} + \mathrm{H.C.} \right) \\ + \tilde{m} \sum_{n=0}^{N-1} (-1)^{n+1} \phi_n^\dagger \phi_n + \frac{\Delta^2 g^2}{2} \sum_{n=0}^{N-2} \left(\sum_{k \le n} \mathrm{Q}_k \right)^2 \end{aligned}$$

In the rotated frame,the Kogut-Susskind Hamiltonian is exclusively represented in terms of fermionic degrees of freedom.

Qubit encoding

As a second step, a Jordan-Wigner transformation is applied to map the fermionic matter degrees of freedom to Pauli spin operators

$$\begin{aligned}
\phi_n^\dagger \phi_{n+1} &= -\left(\sigma_{2n}^{+}\sigma_{2n+1}^{z}\sigma_{2n+2}^{-} + \sigma_{2n+1}^{+}\sigma_{2n+2}^{z}\sigma_{2n+3}^{-}\right), \\
\phi_n^\dagger \phi_n &= \frac{1}{2}\left(\sigma_{2n}^{z} + \sigma_{2n+1}^{z} + 2\right).
\end{aligned}$$

We can rewrite the chromoelectric energy term appearing in the Hamiltonian as

$$\begin{aligned}
H_{\text{el}} &= \sum_{n=0}^{N-2}\left(\sum_{m\leq n} \mathrm{Q}_m\right)^2 \\
&= \sum_{n=0}^{N-2}\left(\sum_{l=0}^{n} \mathrm{Q}_l^2 + 2\sum_{l=0}^{n-1} \mathrm{Q}_l \sum_{k=l+1}^{n} \mathrm{Q}_k\right) \\
&= \sum_{n=0}^{N-2} (N-n-1)\,\mathrm{Q}_n^2 + 2\sum_{n=0}^{N-3} \mathrm{Q}_n \sum_{k=n+1}^{N-2} (N-k-1)\,\mathrm{Q}_k.
\end{aligned}$$

Qubit encoding

We map the non-Abelian charge to Pauli spin operators

$$
\begin{aligned}
\left(Q_n^a\right)^2 &= \frac{1}{8}\left(1-\sigma_{2n}^z\sigma_{2n+1}^z\right), \quad a=x,y,z \\
Q_n^x Q_k^x &= \frac{1}{4}\left(\sigma_{2n+1}^+\sigma_{2n}^-\sigma_{2k+1}^+\sigma_{2k}^- + \sigma_{2n+1}^+\sigma_{2n}^-\sigma_{2k}^+\sigma_{2k+1}^- + \text{H.C.}\right), \\
Q_n^y Q_k^y &= -\frac{1}{4}\left(\sigma_{2n+1}^+\sigma_{2n}^-\sigma_{2k+1}^+\sigma_{2k}^- - \sigma_{2n+1}^+\sigma_{2n}^-\sigma_{2k}^+\sigma_{2k+1}^- + \text{H.C.}\right), \\
Q_n^z Q_k^z &= \frac{1}{16}\left(\sigma_{2n}^z-\sigma_{2n+1}^z\right)\left(\sigma_{2k}^z-\sigma_{2k+1}^z\right), \\
\mathrm{Q}_n^2 &= \frac{3}{8}\left(1-\sigma_{2n}^z\sigma_{2n+1}^z\right), \\
\mathrm{Q}_n\mathrm{Q}_k &= \frac{1}{2}\left(\sigma_{2n+1}^+\sigma_{2n}^-\sigma_{2k}^+\sigma_{2k+1}^- + \text{H.C.}\right) + \frac{1}{16}\left(\sigma_{2n}^z-\sigma_{2n+1}^z\right)\left(\sigma_{2k}^z-\sigma_{2k+1}^z\right).
\end{aligned}
$$

Qubit encoding

The explicit form of the kinetic term is given by

$$H_{\text{kin}} = -\frac{1}{2}\sum_{n=0}^{N-2}\left(\sigma^+_{2n}\sigma^z_{2n+1}\sigma^-_{2n+2} + \sigma^+_{2n+1}\sigma^z_{2n+2}\sigma^-_{2n+3} + \text{H.C.}\right).$$

The mass term reads

$$H_{\text{m}} = \tilde{m}\sum_{n=0}^{N-1}\left[(-1)^{n+1}\frac{\sigma^z_{2n}+\sigma^z_{2n+1}}{2} + 1\right].$$

Finally, the chromoelectric Hamiltonian is expressed as

$$\begin{aligned} H_{\text{el}} = {} & \frac{3}{8}\sum_{n=0}^{N-2}(N-n-1)(1-\sigma^z_{2n}\sigma^z_{2n+1}) \\ & + \frac{1}{8}\sum_{n=0}^{N-3}\sum_{k=n+1}^{N-2}(N-k-1)(\sigma^z_{2n}-\sigma^z_{2n+1})(\sigma^z_{2k}-\sigma^z_{2k+1}) \\ & + \sum_{n=0}^{N-3}\sum_{k=n+1}^{N-2}(N-k-1)(\sigma^+_{2n+1}\sigma^-_{2n}\sigma^+_{2k}\sigma^-_{2k+1} + \text{H.C.}) \end{aligned}$$