Reaching the Ultimate Quantum Precision Limit at Colliders: Conditions and Case Studies

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Outlines

- Motivation and Background
- General Framework
- Case Studies
- Outlook and Discussions

The Central Question

- We have heard news like the following in AMO physics

REPORTS

Beating the Standard Quantum Limit with Four-Entangled Photons

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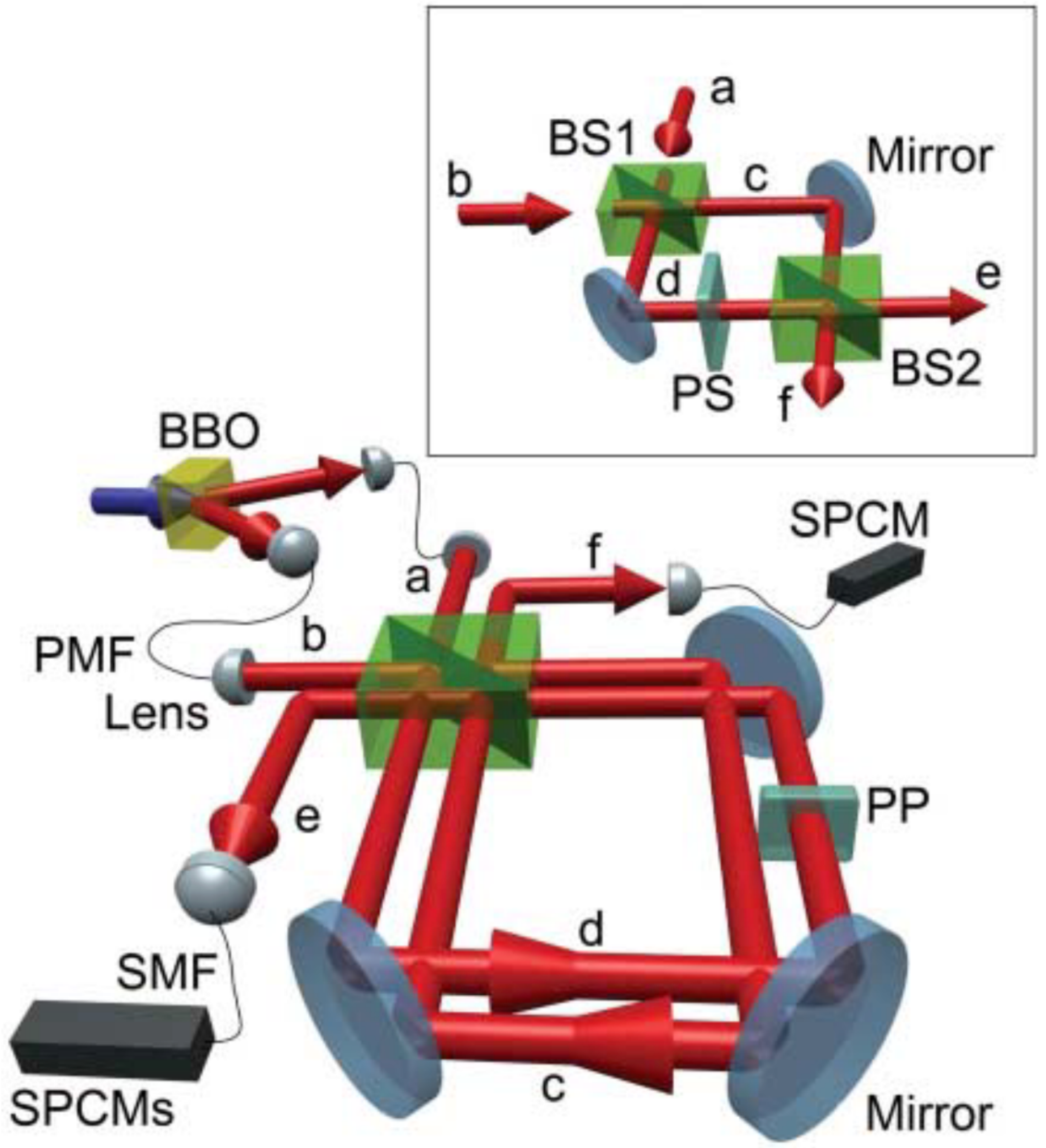

- Can collider experiments reach the Ultimate Quantum Limit?

- Challenge: colliders only access to classical observables (momenta)

Start with Classical Fisher Information (CFI)

- CFI: quantifies how much information an observable random variable carries about an unknown parameter d on which its probability distribution $f(x, d)$ depends
- $F_c(d) = \int dq\, f(q\,|\,d)\left[\frac{\partial}{\partial d}\log f(q\,|\,d)\right]^2$
- Cramér–Rao Bound: for any classical observable O_d for parameter d
  $\mathrm{Var}(O_d) \geq \left(\partial_d \langle O_d \rangle_\pi\right)^2 / F_c(d)$
- Only classical optimal observable O_d^{opt} reaches equality
- The larger CFI, the better the precision

Classical Fisher Information (CFI)

- Classical optimal observable reaching CFI
  - In perturbative sense: $f(q\,|\,d) = f_0 + d \cdot f_1 + \mathcal{O}(d^2)$
- Classical optimal observable $O_d^{\text{opt}} = f_1/f_0$
  - Saturating CR bound: $F_c(d) = \int dq\, f_0 \left(O_d^{\text{opt}}\right)^2$
- E.g. BELLE search of τ Electric Dipole Moments   PLB2003, JHEP04(2022)110
- E.g. machine learning techniques at colliders are optimizing sensitivities towards CFI

Quantum Fisher Information (QFI)

- QFI $F_q(d)$: quantum precision limit for estimating parameters d encoded in quantum density matrix ρ_d
  - Fundamental constraints imposed by Quantum Mechanics
    unbiased estimator of d: $(\Delta d)^2 \geq 1/[NF_q(d)]$
  - Non-correlated particles/photons with number n leads to
    Standard Quantum Limit (SQL): $F_q \sim n$
  - Entangled particles/photons (e.g. GHZ states) leads to
    Heisenberg Limit (HL): $F_q \sim n^2$
- **QFI** provides the **best achievable precision** for a given quantum state.

Quantum Fisher Information (QFI)

- Consider small parameter d (perturbative sense): $\rho_d = \rho_0 + d \cdot \rho_1 + \mathcal{O}(d^2)$

- There exists Symmetric Logarithmic Derivative (SLD) operator $\hat{Q}^{\text{opt}}$:

$$\rho_1 = \frac{1}{2}\left\{\rho_0, \hat{Q}^{\text{opt}}\right\}$$

- QFI in a perturbative form: $F_q(d) = \text{Tr}\left[\rho\left(\hat{Q}^{\text{opt}}\right)^2\right] \simeq \text{Tr}\left[\rho_0\left(\hat{Q}^{\text{opt}}\right)^2\right]$

Ultimate Quantum Limit: Quantum Cramér–Rao bound

- Quantum Cramér–Rao bound: for any quantum operator $\hat{Q}$ for measuring d

  - $\mathrm{Var}\left(\hat{Q}\right) \geq \left(\partial_d \langle \hat{Q} \rangle\right)^2 / F_q(d)$

  - Mean $\langle \hat{Q} \rangle$ and Variance $\mathrm{Var}\left(\hat{Q}\right)$

  - A higher QFI corresponds to a more precise measurement limit

- QFI saturation ⇔ projective measurements $\hat{\Pi}_i = |\psi_i\rangle\langle\psi_i|$

in SLD $\hat{Q}^{\mathrm{opt}} = \sum_i \lambda_i \hat{\Pi}_i$ eigenbasis

(S. L. Braunstein et al *Phys.Rev.Lett*. 1994)

(J. Liu, H. Yuan, X.-M. Lu, and X. Wang, *J. Phys.A* 2020)

The generalized quantum measurement at colliders

- Colliders perform classical measures on momenta of particles $|\vec{p}\rangle$
- But particle decay can serve as generalized quantum measurement
- Suppose a fermion decay $A \to BC$, with decay amplitude $M(A_\alpha \to BC)$
  - Its decay perform a measurement to spin density matrix: $\rho_A \to M\rho_A M^\dagger \propto \rho_A^{\text{post}}$
  - Generalized measurement : $\hat{D}_{\alpha\alpha'} \equiv N_D^{-1} M(A_\alpha \to BC) M^\dagger(A_{\alpha'} \to BC)$
    - Completeness condition: $\mathbf{I} = \int d\Omega_B \hat{D}(p_B)$
    - Normalized differential distribution and conservation:

$$1 = \text{Tr}[\rho_A] = \int d\Omega_B \text{Tr}[\rho_A \hat{D}(p_B)] = \int d\Omega_B \text{Tr}[\rho_A^{\text{post}}] = \int d\Omega_B f(p_B)$$

Q. Wang et al, 2402.16574, CPL
C.F. Qiao et al, 2002.04284, PRD
R. Ashby-Pickering et al, 2209.13990, JHEP

The relation between QFI and CFI

- A classical observable $O(p)$ corresponds to a quantum measurement $\hat{Q}_p$

$$\hat{Q}_O = \int d\Omega_p \hat{E}(p) O(p)$$

where $\hat{E}(p)$ is the generalized quantum measurement operator

- Same expectation value but different variance
  - $\langle \hat{Q}_O \rangle = \mathrm{Tr}\left[\rho \hat{Q}_O\right] = \langle O(p) \rangle$
  - Using Cauchy inequality: $\mathrm{Var}(O) \geq \mathrm{Var}(\hat{Q}_O)$ (J. Liu, H. Yuan, X.-M. Lu, and X. Wang, *J. Phys.A* 2020)

- Quantum optimal measurement precision is always better than classical optimal precision
  - $F_q(d) \geq \frac{\left(\partial_d \langle \hat{Q}_{O_d^{\mathrm{opt}}} \rangle\right)^2}{\mathrm{Var}(\hat{Q}_{O_d^{\mathrm{opt}}})} \geq \frac{\left(\partial_d \langle O_d^{\mathrm{opt}} \rangle\right)^2}{\mathrm{Var}(O_d^{\mathrm{opt}})} = F_c(d)$

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τ Decay as Quantum Measurement

- $\tau^+\tau^-$ production and spin density matrix:

$$\rho_{\alpha\alpha';\beta\beta'}(\hat{k}) = \frac{1}{|\overline{M}|^2} \overline{\sum_{\text{initial}}} M(\text{initial} \to \tau^+_\alpha(\hat{k})\tau^-_\beta(-\hat{k})) M^*(\text{initial} \to \tau^+_{\alpha'}(\hat{k})\tau^-_{\beta'}(-\hat{k}))$$

  - Introduce spin density matrix $\rho_d = \rho_0 + d \cdot \rho_1 + \mathcal{O}(d^2)$

- Two-body τ decay: $\tau^\pm \to \pi^\pm \overset{(-)}{\nu}$

  - Single τ spin measurement operator

$$\hat{D}^\pm_{\alpha\alpha'}(\hat{q}_\pm) = \frac{f_\pi^2}{2m_\tau^3|q|} M(\tau^\pm_{\alpha'} \to \pi^\pm(\hat{q}_\pm)X) M^*(\tau^\pm_\alpha \to \pi^\pm(\hat{q}_\pm)X)$$

τ Decay with maximal spin analyzing power

- Single τ spin measurement operator for general decay

$$\hat{D}^{\pm}_{\alpha\alpha'}(\hat{q}_{\pm}) = \frac{1}{2}(\mathbf{I}_2 \mp a\hat{q}_{\pm} \cdot \vec{\sigma})_{\alpha\alpha'}$$

  - With $0 \leq |a| \leq 1$, $|a| = 1$ maximal spin analyzing power; $|a| = 0$ no spin analyzing power

- Two-body τ decay: $\tau^{\pm} \to \pi^{\pm} \overset{(-)}{\nu}$
  - Maximal spin analyzing power $\to \mathrm{Rank}[\hat{D}] = 1$
  - Projector: $\hat{D} = |\hat{q}\rangle\langle\hat{q}|$, $|\hat{q}\rangle = \{\cos(\theta/2), \sin(\theta/2)e^{i\phi}\}$
  - Properties: $\mathrm{Tr}[\hat{D}^{\pm}] = 1, \quad (\hat{D}^{\pm})^2 = \hat{D}^{\pm}, \quad \int d\Omega_{\pm}\hat{D}^{\pm}(\hat{q}_{\pm}) = 2\pi\mathbf{I}_2$

- We take projector measurement, and leave Positive Operator-Valued Measure (POVM) for future study

Collider Reality: Separable (local) Measurements

- $\tau^+\tau^-$, each decays independently
  $M^{\text{tot}}_{\alpha\beta} \propto M(\text{ini} \to \tau^+_\alpha(k_+)\tau^-_\beta(k_-)) \times M(\tau^+_\alpha(k_+) \to \pi^+\bar{\nu})M(\tau^-_\beta(k_-) \to \pi^-\nu)$

- Their decay forms separable quantum measurements

  - $\hat{E}_{\alpha\alpha';\beta\beta'}(\hat{q}_+, \hat{q}_-) = \hat{D}^+_{\alpha\alpha'}(\hat{q}_+) \otimes \hat{D}^-_{\beta\beta'}(\hat{q}_-)$

  - Properties:

$$\text{Tr}[\hat{E}] = 1, \quad \hat{E}^2(\hat{q}_+, \hat{q}_-) = \hat{E}(\hat{q}_+, \hat{q}_-), \quad \frac{1}{4\pi^2}\int d\Omega_+ d\Omega_- \hat{E}(\hat{q}_+, \hat{q}_-) = \mathbf{I}_4 \,.$$

- General collider measurements *cannot* access entangled states

Reaching QFI at collider is very difficult

- Recall to saturate QFI, one should measure with all four optimal projectors $\hat{\Pi}_i = |\psi_i\rangle\langle\psi_i|$, with orthonormal eigenstates $|\Psi_i\rangle$ of SLD $\hat{Q}^{\rm opt}$

- Collider separable (non-entangled) measurements: $\hat{E} = |\hat{q}_+, \hat{q}_-\rangle\langle\hat{q}_+, \hat{q}_-|$

  - Define first measurement: $|E_1\rangle = |\hat{q}_+, \hat{q}_-\rangle \equiv |\hat{q}_+\rangle \otimes |\hat{q}_-\rangle$

  - The other orthonormal separable measurements are forced to be $|E_2\rangle = |-\hat{q}_+, \hat{q}_-\rangle, \;\; |E_3\rangle == |\hat{q}_+, -\hat{q}_-\rangle, \;\; |E_4\rangle == |-\hat{q}_+, -\hat{q}_-\rangle$

- Conditions for QFI saturation:
$$[\hat{Q}, \hat{E}_j] = 0 \quad \Leftrightarrow \quad \hat{Q}|E_j\rangle = \lambda_j|E_j\rangle$$

Reaching QFI at collider is very difficult

- Conditions for QFI saturation: $[\hat{Q}, \hat{E}_j] = 0 \quad \Leftrightarrow \quad \hat{Q}\,|\,E_j\rangle = \lambda_j\,|\,E_j\rangle$

- $\hat{Q}^{\text{opt}}$ from production is independent with the measurement $\hat{E}$

  - If eigenstate $|\,\psi_i\rangle$ is entangled → *cannot* reach quantum optimal at colliders

- Schmidt Condition to be separated:

  - $|\,\psi\rangle = a\,|\uparrow\uparrow\rangle + b\,|\uparrow\downarrow\rangle + c\,|\downarrow\uparrow\rangle + d\,|\downarrow\downarrow\rangle$
    $|\,\psi\rangle$ separable $\longleftrightarrow ad - cb = 0$

  - Zero measure, very difficult to satisfy

Rank-deficient ρ saves the Day

- If ρ_0 is not full-rank, SLD is not unique——Most general form of SLD:

$$\hat{Q} = \sum_{\substack{i,j \\ p_i + p_j \neq 0}} \frac{\langle p_i | \rho_1 | p_j \rangle}{p_i + p_j} |p_i\rangle\langle p_j| + \sum_{\substack{i,j \\ p_i = p_j = 0}} r_{ij} |p_i\rangle\langle p_j|$$

  - $|p_i\rangle$: the eigenstate of ρ_0 with eigenvalue (probability) p_i
  - r_{ij} : arbitrary complex numbers satisfying $r_{ij} = r_{ij}^*$

- Null space freedom provides the flexibility of $\hat{Q}$, *which provides hope to match* $\hat{E}_m(\hat{q}) = \hat{\Pi}_m(r)$

$$F_q(d) \xleftarrow{\hat{Q}^{\text{opt}}} \rho_d \xrightarrow{\hat{E}_m(\hat{q}_+, \hat{q}_-) = \hat{\Pi}_m(r)} \frac{df_m}{d\Omega_+ d\Omega_-} \xrightarrow{O_d^{\text{opt}}(\hat{q}_+, \hat{q}_-)} f_c(d)$$

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Three scenarios

- Higgs decay with CP-violating coupling: $h \to \tau^+\tau^-$

- $\mathscr{L}_{\text{CPV}} = -\dfrac{m_\tau}{v} h\,\bar{\tau}\left[\cos\delta_h + i\gamma_5 \sin\delta_h\right]\tau$

- τ Magnetic/Electric Dipole Moment: $e^+e^- \to \tau^+\tau^-$

- $\mathscr{L}_{\text{MDM/EDM}} = \dfrac{1}{2} F_{\mu\nu}\,\bar{\tau}\sigma^{\mu\nu}\left[-\dfrac{e}{2m_\tau} a_\tau - i\gamma_5 d_\tau\right]\tau$

- We use helicity basis for spin density matrix

Higgs CP-violating scenario

- Spin density matrix perturbatively: $\rho_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- Most simple and direct SLD: $Q_h = 2\rho_1$,

  - easy to check $\rho_1 = \frac{1}{2}\{\rho_0, Q_h\}$, and QFI: $F_q(\delta_h) = 4$

- Q_h is rank 2, with two entangled eigenstates:
$\hat{\Pi}_1 = \frac{1}{\sqrt{2}}(i|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \hat{\Pi}_2 = \frac{1}{\sqrt{2}}(-i|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

- $\hat{\Pi}_{1,2}$ cannot match to collider measurements

Higgs CP-violating scenario

- Most general SLD satisfying $\rho_1 = \frac{1}{2}\{\rho_0, \hat{Q}^{\text{gen}}\}$:

$$\hat{Q}^{\text{gen}}(r) = \begin{pmatrix} r_{44} & -\frac{r_{43}}{\sqrt{2}} & \frac{r_{43}}{\sqrt{2}} & r_{42} \\ -\frac{r_{34}}{\sqrt{2}} & \frac{r_{33}}{2} & -\frac{r_{33}}{2} - 2i & -\frac{r_{32}}{\sqrt{2}} \\ \frac{r_{34}}{\sqrt{2}} & -\frac{r_{33}}{2} + 2i & \frac{r_{33}}{2} & \frac{r_{32}}{\sqrt{2}} \\ r_{24} & -\frac{r_{23}}{\sqrt{2}} & \frac{r_{23}}{\sqrt{2}} & r_{22} \end{pmatrix}$$

- Scan all solutions across the auxiliary variables r's and particle momentum choices $\hat{q}_+$ and $\hat{q}_-$
    - to match $\hat{\Pi}_{1,2,3,4}(r) = \hat{E}_{1,2,3,4}(\hat{q}_+, \hat{q}_-)$

Two Construction Methods

- Matrix-based method
  - Brute force linear algebra: $[\hat{Q}, \hat{E}_j] = 0 \quad \Leftrightarrow \quad \hat{Q}\,|E_j\rangle = \lambda_j\,|E_j\rangle$
    - Direct, but computational challenge
  - Work in Pauli basis:
    $$\hat{Q} = a\mathbf{I}_4 + \sum_i b_i^+(\sigma_i \otimes \mathbf{I}_2) + \sum_j b_j^-(\mathbf{I}_2 \otimes \sigma_j) + \sum_{i,j} c_{ij}\,\sigma_i \otimes \sigma_j$$
    - If $\hat{Q}$ is separable and can match to $\hat{E}(\hat{q}_+, \hat{q}_-)$, if and only if,
      $$a = P_1\,, \qquad \vec{b}^\pm = P_{2/3}\,\hat{q}_\pm\,, \qquad c_{ij} = P_4\,\hat{q}^i_+\hat{q}^j_-$$
    - Requiring separable spin-spin correlation matrix: $\vec{\vec{c}} \propto \vec{\hat{q}}_+\vec{\hat{q}}_-$
- Amplitude-based method (focus on the transition amplitude M)
  - Closer to particle interactions and physics, see Appendix of arXiv:2506.10673

Higgs CP-violating scenario

- We found $\hat{\Pi}_{1,2,3,4}(r) = \hat{E}_{1,2,3,4}(\hat{q}_+, \hat{q}_-)$ for Higgs CPV scenario!
- Collider accessible SLD: $Q_h^{\rm opt} = \dfrac{2}{\sin\varphi}\begin{pmatrix} -\cos\varphi & 0 & 0 & e^{-i\Phi} \\ 0 & -\cos\varphi & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & -\cos\varphi & 0 \\ e^{i\Phi} & 0 & 0 & -\cos\varphi \end{pmatrix} \neq Q_h^{\rm old} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
- With $\varphi = \phi_1 - \phi_2$ and $\Phi = \phi_1 + \phi_2$; $\phi_{1,2}$ are free phase parameters

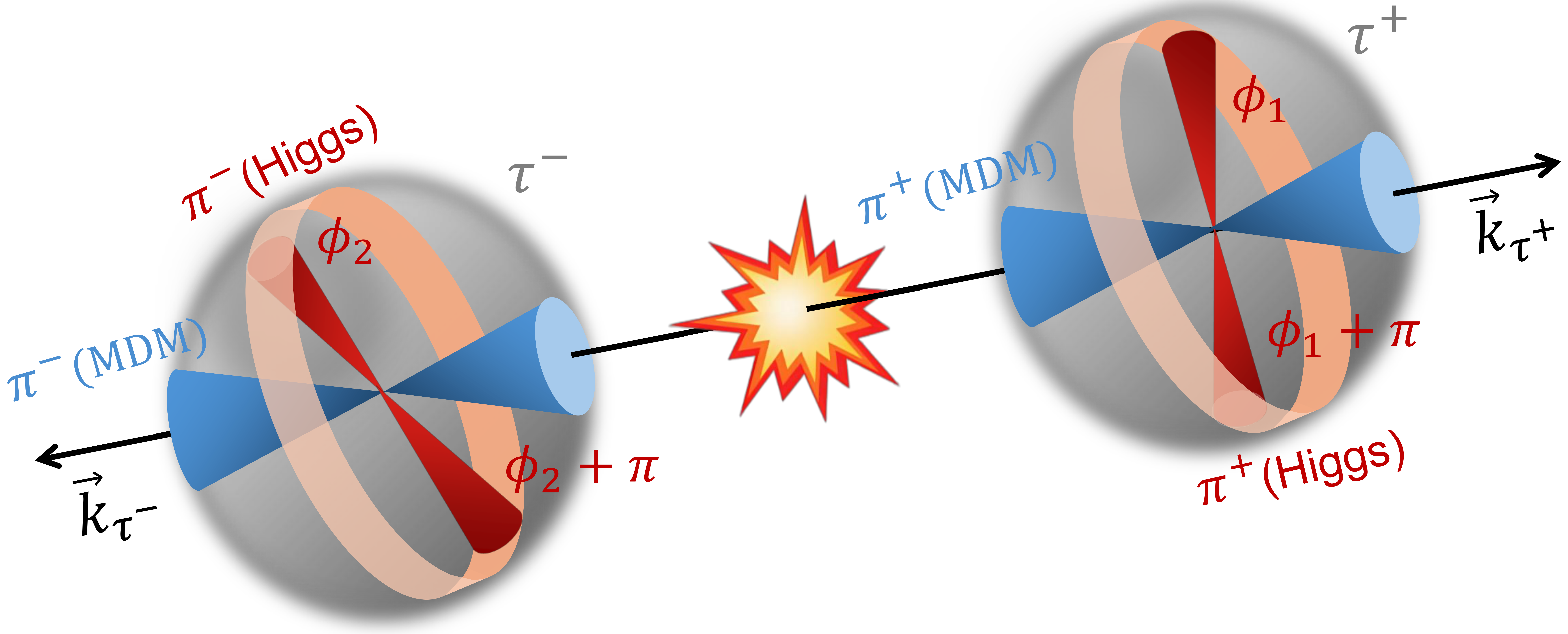
Quantum optimal collider measurements $\hat{E}_{1,2,3,4}$

CPV h	θ_+	θ_-	ϕ_+	ϕ_-
$\lvert E_1\rangle$	$\pi/2$	$\pi/2$	ϕ_1	ϕ_2
$\lvert E_2\rangle$	$\pi/2$	$\pi/2$	$\phi_1+\pi$	ϕ_2
$\lvert E_3\rangle$	$\pi/2$	$\pi/2$	ϕ_1	$\phi_2+\pi$
$\lvert E_4\rangle$	$\pi/2$	$\pi/2$	$\phi_1+\pi$	$\phi_2+\pi$

$\theta_\pm$: polar angle of $\hat{q}_\pm$

$\phi_\pm$: azimuthal angle of $\hat{q}_\pm$

Optimal collider measurements: four sets of directions



MDM scenario

- The spin density matrix for MDM:

$$\rho_0 = c_0 \begin{pmatrix} \frac{1}{2}(\cos 2\theta + 3) & -\frac{im}{\sqrt{s}} \sin 2\theta & -\frac{im}{\sqrt{s}} \sin 2\theta & -\sin^2\theta \\ \frac{im}{\sqrt{s}} \sin 2\theta & \frac{4m^2}{s} \sin^2\theta & \frac{4m^2}{s} \sin^2\theta & \frac{im}{\sqrt{s}} \sin 2\theta \\ \frac{im}{\sqrt{s}} \sin 2\theta & \frac{4m^2}{s} \sin^2\theta & \frac{4m^2}{s} \sin^2\theta & \frac{im}{\sqrt{s}} \sin 2\theta \\ -\sin^2\theta & -\frac{im}{\sqrt{s}} \sin 2\theta & -\frac{im}{\sqrt{s}} \sin 2\theta & \frac{1}{2}(\cos 2\theta + 3) \end{pmatrix}$$

$$\rho_1 = (1 - \frac{4m^2}{s})c_0^2 \begin{pmatrix} -2\sin^2\theta(\cos 2\theta + 3) & -\frac{ic_1\sqrt{s}}{4m} \sin 2\theta & -\frac{ic_1\sqrt{s}}{4m} \sin 2\theta & 4\sin^4\theta \\ \frac{ic_1\sqrt{s}}{4m} \sin 2\theta & 2\sin^2\theta(\cos 2\theta + 3) & 2\sin^2\theta(\cos 2\theta + 3) & \frac{ic_1\sqrt{s}}{4m} \sin 2\theta \\ \frac{ic_1\sqrt{s}}{4m} \sin 2\theta & 2\sin^2\theta(\cos 2\theta + 3) & 2\sin^2\theta(\cos 2\theta + 3) & \frac{ic_1\sqrt{s}}{4m} \sin 2\theta \\ 4\sin^4\theta & -\frac{ic_1\sqrt{s}}{4m} \sin 2\theta & -\frac{ic_1\sqrt{s}}{4m} \sin 2\theta & -2\sin^2\theta(\cos 2\theta + 3) \end{pmatrix}$$

MDM and EDM scenarios

- Quantum optimal collider-accessible SLD for MDM:

$$\hat{Q} = 2\left(\frac{s}{4m^2} - 1\right) c_0 \begin{pmatrix} -\frac{8m^2}{s}\sin^2\theta & 0 & 0 & 0 \\ 0 & \cos 2\theta + 3 & 0 & 0 \\ 0 & 0 & \cos 2\theta + 3 & 0 \\ 0 & 0 & 0 & -\frac{8m^2}{s}\sin^2\theta \end{pmatrix}$$

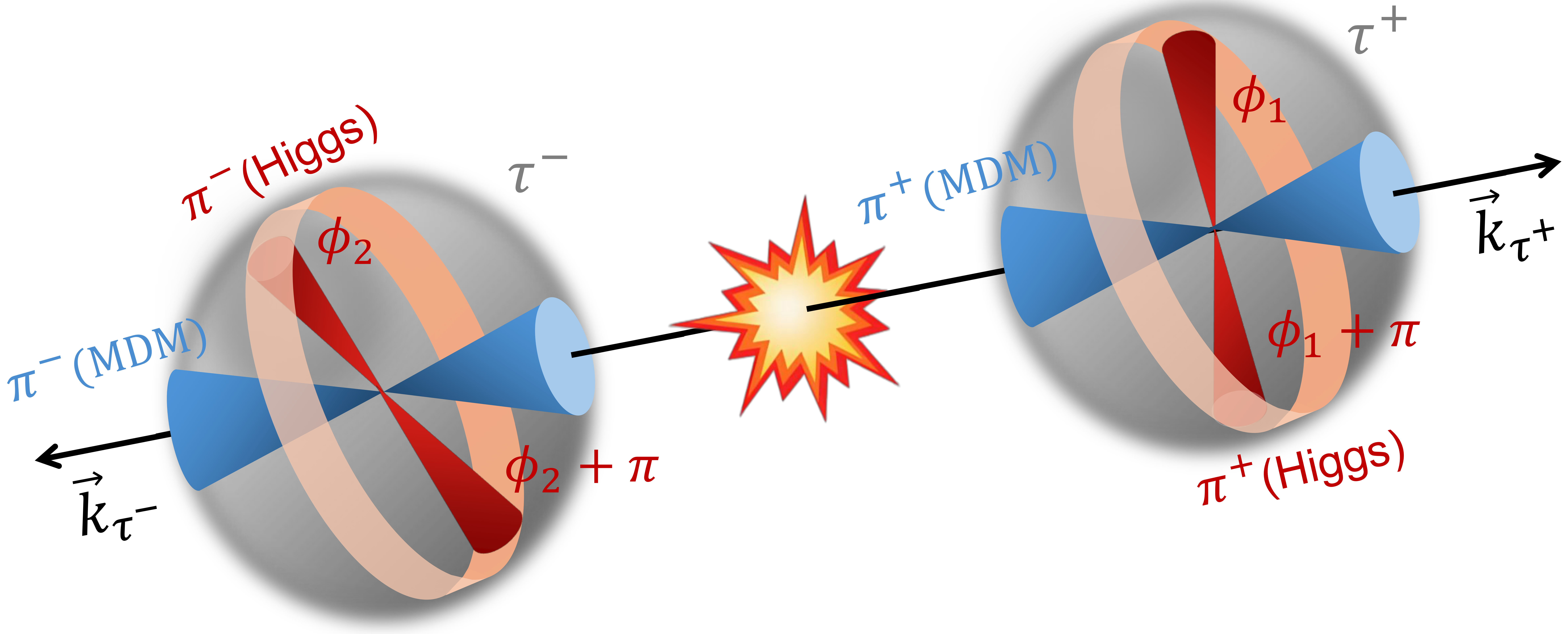
$$F_q^{\mathrm{MDM}}(\theta) = \frac{2s\sin^2\theta\,(\cos 2\theta + 3)\left(s - 4m^2\right)^2}{m^2\left(8m^2\sin^2\theta + s\cos 2\theta + 3s\right)^2}$$

- Corresponding classical optimal collider measurements:

MDM	θ_+	θ_-	ϕ_+	ϕ_-
$\lvert E_1\rangle = \lvert\downarrow\downarrow\rangle$	π	π	$-$	$-$
$\lvert E_2\rangle = \lvert\downarrow\uparrow\rangle$	π	0	$-$	$-$
$\lvert E_3\rangle = \lvert\uparrow\downarrow\rangle$	0	π	$-$	$-$
$\lvert E_4\rangle = \lvert\uparrow\uparrow\rangle$	0	0	$-$	$-$

- EDM has no solution

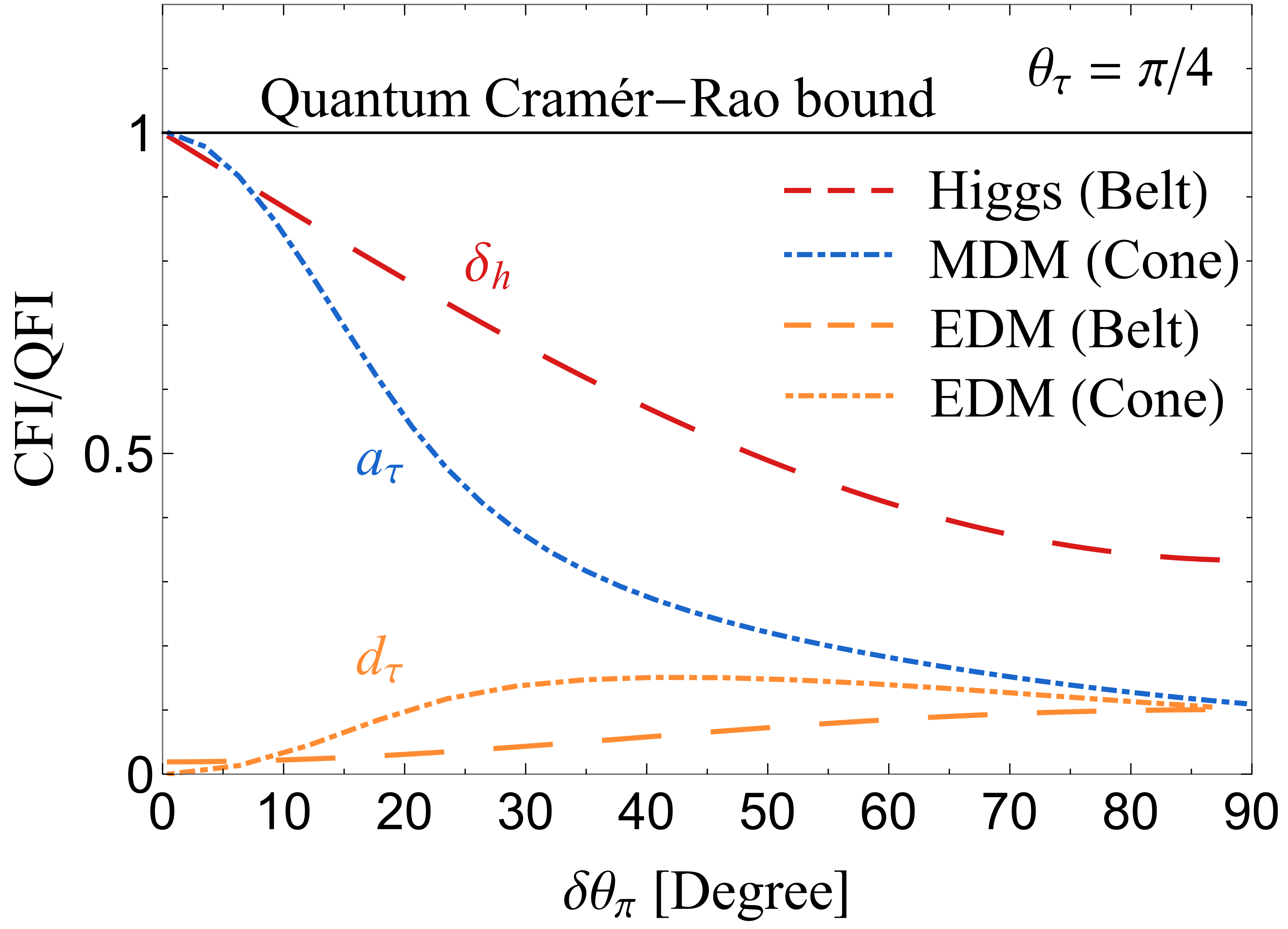
Optimal collider measurements: four sets of directions

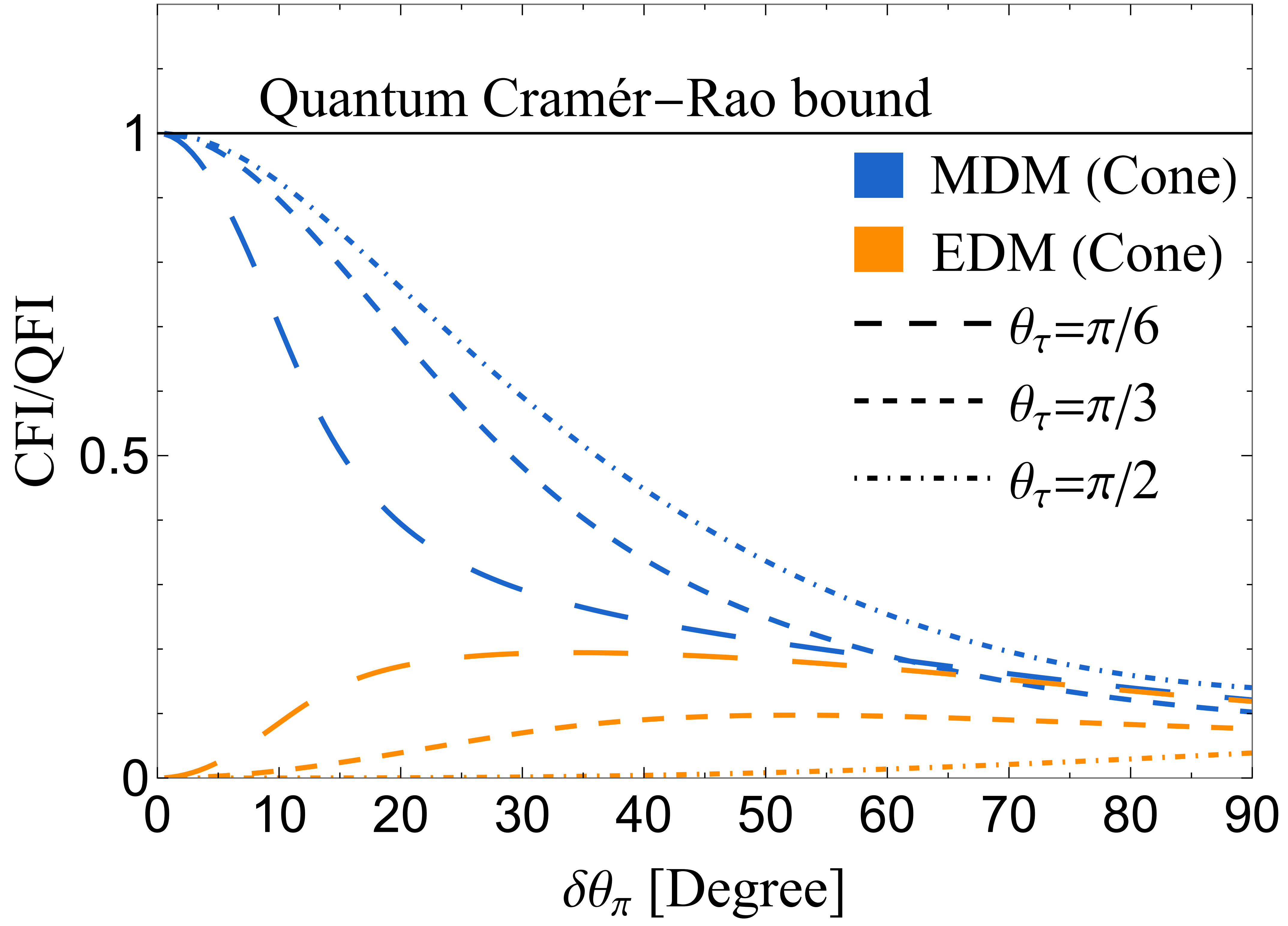


Collider Strategy: Phase-Space Restricted PS

- Expand each optimal direction into a cone of angular size $\delta\theta_\pi$
  - Allowing non-zero events
  - CFI asymptotic saturates QFI

$$F_c(d) = \int d\mathrm{PS}_{\mathrm{Belt/Cone}} \times \frac{\Sigma_1^2}{\Sigma_0}$$





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Non-maximal spin analyzing power

- Non-maximal case: $|\alpha_\pm| < 1$

$$\hat{D}^\pm(\hat{q}_\pm) = \frac{1}{2}\left(\mathbf{I}_2 + \alpha_\pm \times \hat{q}_\pm \cdot \vec{\sigma}\right)$$

- It reduces CFI comparing with maximal case
  - Differential σ: $\Sigma_0 = \frac{1}{4S}(1 + \alpha_+\alpha_- q_+^i q_-^j C_0^{ij}), \quad \Sigma_1 = \frac{1}{4S}(\alpha_+\alpha_- q_+^i q_-^j \partial_d C_d^{ij})$
  - CFI integrand: $\frac{\Sigma_1^2}{\Sigma_0} = \frac{1}{4S} \times \frac{(\alpha_+\alpha_- q_+^i q_-^j \partial_d C_d^{ij})^2}{1 + \alpha_+\alpha_- q_+^i q_-^j C_0^{ij}}$
- Inequality: $\frac{(\alpha_+\alpha_- q_+^i q_-^j \partial_d C_d^{ij})^2}{1 + \alpha_+\alpha_- q_+^i q_-^j C_0^{ij}} = |\alpha_+\alpha_-| \frac{(q_+^i q_-^j \partial_d C_d^{ij})^2}{\frac{1}{|\alpha_+\alpha_-|} + \frac{\alpha_+\alpha_-}{|\alpha_+\alpha_-|} q_+^i q_-^j C_0^{ij}} \leq \frac{(q_+^i q_-^j \partial_d C_d^{ij})^2}{1 + \text{Sign}\left[\alpha_+\alpha_-\right] q_+^i q_-^j C_0^{ij}}$
- $|\alpha_\pm| < 1$ CFI is smaller than $|\alpha_\pm| = 1$ CFI, thus cannot reach QFI

A comparison with AMO measurements

Aspect	Colliders	AMO systems
Type of Quantum Measurement	Generalized quantum measurements arising naturally from particle decays	Controlled projective or generalized quantum measurements
Measurement Structure	Forced to be separable measurements — no entanglement between unstable particle decays	Prefer local (separable) measurements to conserve quantum resources; typically do not exploit entanglement unless intentionally designed
Control over Measurement	Quantum measurements occur "spontaneously" via decay dynamics but lack experimental control	Quantum measurements are deliberately prepared and precisely controlled
QFI Saturation	Achievable for certain interactions; occurs passively at specific regions of phase space	Achieved by preparing specific quantum states (e.g., squeezed or entangled) tailored to maximize sensitivity

Summary and Outlook

- QFI represents the ultimate precision limit in parameter estimating, QFI ≥ CFI
- Quantum measurement connect quantum states to classical distribution
  Particle decay serves as generalized quantum measurements
- Collider measurements can only access separable projectors $\hat{E} = \hat{D}(q_+)\hat{D}(q_-)$
- Condition for QFI saturation: (specific directions)

$$F_q(d) \xleftarrow{\hat{Q}^{\rm opt}} \rho_d \xrightarrow{\hat{E}_m(\hat{q}_+,\hat{q}_-)=\hat{\Pi}_m(r)} \frac{df_m}{d\Omega_+ d\Omega_-} \xrightarrow{O_d^{\rm opt}(\hat{q}_+,\hat{q}_-)} f_c(d)$$

Thank you!

- Rank deficiency of $\rho \longrightarrow$ Flexibility of SLD
- QFI saturation is achievable for Higgs CPV decay and MDM cases, but not for EDM
- Framework easily applicable to other entangled biparticle systems:
  $t\bar{t}$ from ee, $t\bar{t}$ pseudo-scalar resonance, baryon pairs $(\Delta, \Lambda, \Lambda_b)$, gauge boson pairs