Selection Rules Revisited

Justin Kaidi

Generalized Symmetries in HEP and CMP
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[2402.00105] **JK**, Yuji Tachikawa, Hao Y. Zhang

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  - Subsystem symmetries [Lawler, Fradkin '04; Seiberg '19; Seiberg, Shao '20]
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- **The main focus of today's talk will actually *not* be symmetry itself, but something closely related.**

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- **This constraint on the Lagrangian leads to constraints on scattering amplitudes that hold *to all orders* in perturbation theory!**
- **Concretely, a process involving incoming fields $\phi_1, \dots, \phi_n$ and outgoing fields $\phi_{n+1}, \dots, \phi_N$ is allowed only if**

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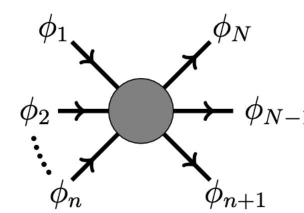

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- **Definition: A** *hypergroup* **is an algebra** $a \times b = \sum_c N_{ab}^c\, c$ **equipped with an involution** $a \mapsto \overline{a}$ **such that** $\overline{ab} = \overline{b}\,\overline{a}$ **and** $N_{ab}^e \neq 0$ **iff** $a = \overline{b}$.
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- **Definition: A** $fusion\ algebra$ **is a hypergroup such that all** $N_{ab}^c \in \mathbb{Z}$**.**

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- **Setup: Consider a QFT with fields ϕ_i labelled by elements a_i of a hypergroup A. We assume that the Lagrangian contains terms of the form $\phi_1 \ldots \phi_n \subset \mathcal{L}$ only if $e \prec a_1 \ldots a_n$.**

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- **When A is the hypergroup of representations of a group G, we saw that the constraint $e \prec a_1 \dots a_n$ on the Lagrangian extended to a constraint on scattering amplitudes at all loop orders. What about the more general case?**
- **Claim 1: Tree-level diagrams satisfy the same selection rules as the Lagrangian. In other words, a tree-level diagram involving incoming fields $\phi_1, \dots, \phi_n$ and outgoing fields $\phi_{n+1}, \dots, \phi_N$ is non-zero only if $e \prec \overline{a_1} \dots \overline{a_n} a_{n+1} \dots a_N$.**

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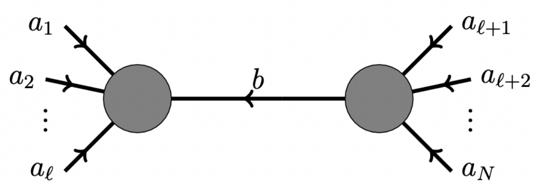

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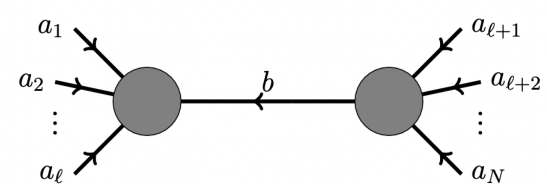

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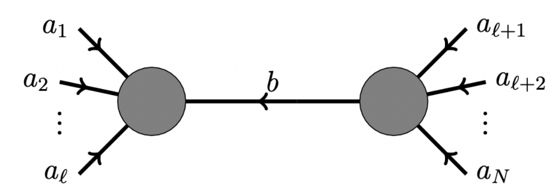

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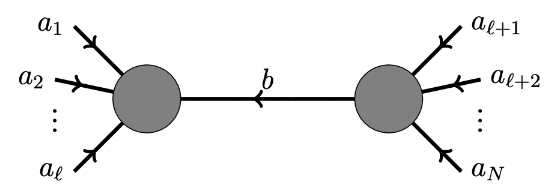

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  - But then we have that $e \prec b\overline{b} \prec a_1 \ldots a_N$ and hence the property is proven for $(k+1)$-vertices as well.

Higher loop result

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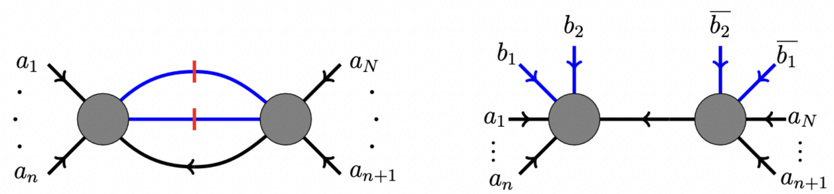

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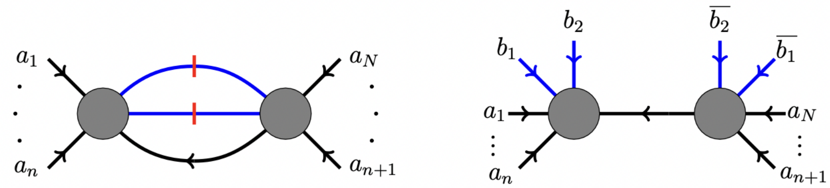

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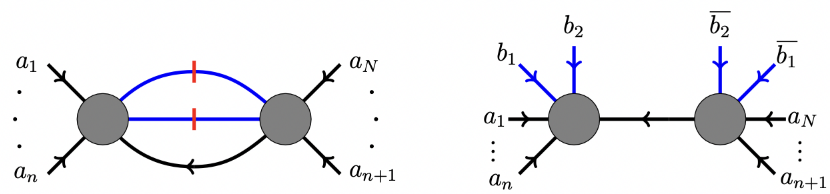

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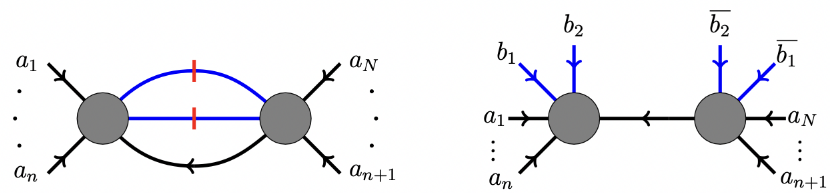

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  - Because A was a hypergroup, we can choose $d \prec c_L \ldots c_1$ such that $d \prec a_1 \ldots a_N$. This is what we wanted to prove, upon introducing the following definitions:

$$\mathrm{Com}(A) := \{c \mid c \prec b\overline{b} \text{ for some } b \in A\}$$

$$\mathrm{Com}(A)^L := \{d \mid d \prec c_1 \ldots c_L \text{ for some } c_1, \ldots, c_L \in \mathrm{Com}(A)\}$$

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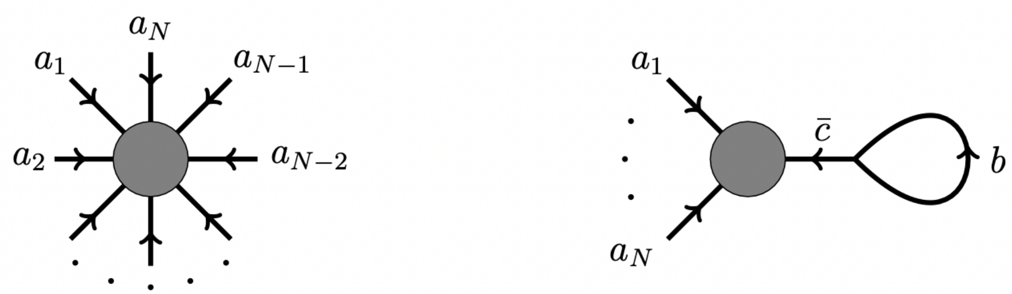

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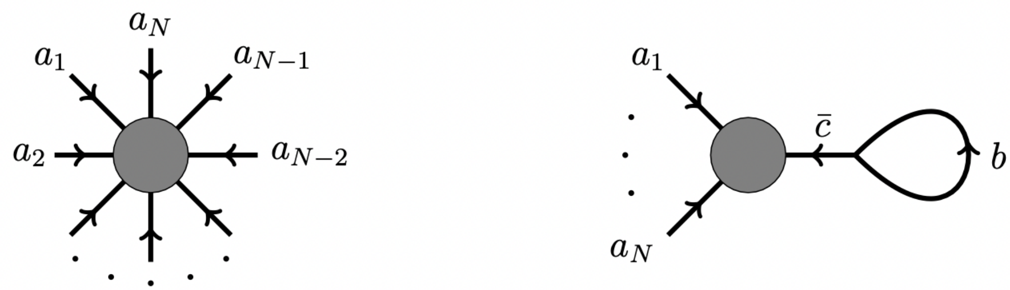

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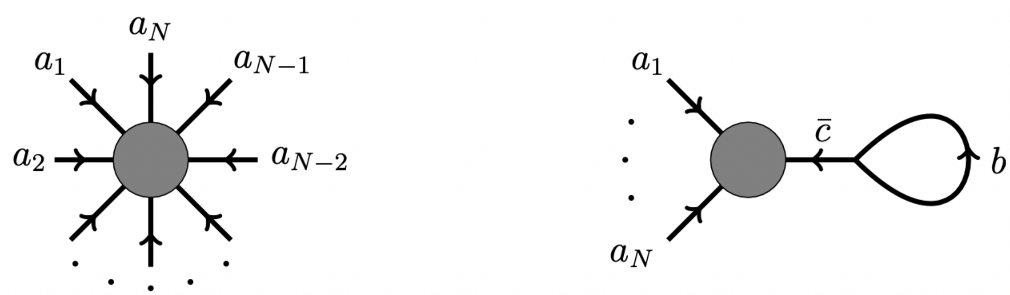

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- **Let's now give a concrete example.**

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- **Summary: at tree-level the amplitudes are constrained by the $\mathrm{Conj}(G)$ selection rules, but at arbitrary high loop order they obey only the selection rules for a standard $\mathrm{Ab}[G]$ symmetry.**

String Theory and Non-Invertible Symmetries

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- **So in the context of String Theory, the selection rules described above can be understood as coming from non-invertible *worldsheet* symmetries! (also studied in** [Heckman, McNamara, Montero, Sharon, Vafa, Valenzuela '24] **)**

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- **The worldsheet theory is known to have a non-invertible $\mathrm{Rep}(\Gamma)$ symmetry. The spacetime states are labelled by (a subset of) the elements of $\mathcal{Z}(\mathrm{Rep}(\Gamma))$. The associated hypergroup turns out to be that associated to $\mathrm{Conj}(\Gamma)$.**

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- Hence this is precisely the conjugacy class example studied before, and we conclude that in the presence of a non-Abelian orbifold, the tree-level scattering amplitudes are constrained by

$$[e] \prec [g_1^{-1}] \dots [g_n^{-1}][g_{n+1}] \dots [g_N]$$

In fact, this is a well-known fact about non-Abelian orbifolds! [Hamidi, Vafa '87]

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- **Furthermore, for $\Gamma \subset SU(2)$, one can show that at one-loop the selection rules already reduce to their final form, i.e. to those dictated by $\mathrm{Ab}[\Gamma]$.**

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$$U_{(\theta,\phi)}: \qquad \Phi_{m,w} \to e^{im\theta + iw\phi}\Phi_{m,w}$$

This is an invertible symmetry since $U_{(\theta,\phi)} \times U_{(\theta',\phi')} = U_{(\theta+\theta',\phi+\phi')}$.

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- **Now perform an orbifold by $X^9_{L,R} \to -X^9_{L,R}$. Since this $\mathbb{Z}_2$ symmetry acts as $\Phi_{m,w} \to \Phi_{-m,-w}$ and $U_{(\theta,\phi)} \to U_{(-\theta,-\phi)}$, the gauge-invariant operators in the orbifold theory are**

$$\begin{aligned}
\widehat{\Phi}_{m,w} &:= \frac{1}{\sqrt{2}}(\Phi_{m,w} + \Phi_{-m,-w}) \\
\widehat{U}_{(\theta,\phi)} &:= U_{(\theta,\phi)} + U_{(-\theta,-\phi)}
\end{aligned}$$

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- Similar results hold for more general toroidal orbifolds, e.g. $T^6/\mathbb{Z}_3$.

Conclusions

- **When a QFT has a group-like symmetry, the fields are labelled by representations, and terms in the Lagrangian of the form $\phi_1 \dots \phi_N \subset \mathcal{L}$ are allowed only if $\mathrm{id} \subset R_1 \otimes \cdots \otimes R_N$. This gives rise to selection rules on amplitudes that hold to all orders in perturbation theory.**

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- **We can instead imagine a QFT whose fields are labelled by elements of a hypergroup, and demand that all terms in the Lagrangian of the form $\phi_1 \dots \phi_N \subset \mathcal{L}$ satisfy $e \prec a_1 \dots a_N$. This gives rise to selection rules that hold at tree-level, but are increasingly broken at higher loop level, eventually reducing to selection rules coming from an Abelian group $\mathrm{Gr}[A]$.**

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- **The case of $T^6/\mathbb{Z}_3$ has some potentially interesting phenomenological applications, and will be explored in upcoming work** [**JK**, Shi, Shimamori] **.**

The End (for now)

Thank you!