Anomaly and fermionic unitary operators

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Generalized symmetries in HEP and CMP
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Outline

Introduction and Summary

Anomalies of fermionic and bosonic QFTs

There are recently significant improvements in our understanding of symmetries of quantum field theories (QFTs) and of their anomalies. (Cf. *Invertible phase, Anderson dual* $I_{\mathbb{Z}}(\Omega^{\mathrm{spin}}_{\bullet})$ ... [Freed, Hopkins 16'])

Among others, allowed anomalies of the **same** symmetry group in the same spacetime dimensions can differ between *bosonic* QFTs and *fermionic* QFTs.

- $U(1)$ in two dimensional spacetime, its anomalies are characterized by an integer "level" k, arbitary $k \in \mathbb{Z}$ is allowed in fermionic QFTs, while k has to be **even** in bosonic QFTs.
- Similarly, in the case of $\mathbb{Z}_2$ symmetry in 2d, its anomaly is classified by $\mathbb{Z}_2$ in bosonic QFTs but by $\mathbb{Z}_8$ in fermionic QFTs.

The boson/fermion anomaly difference can be understood in many different ways.

Hamiltonian point of view

Symmetry operations are implemented in terms of unitary operators acting on the Hilbert space $\mathcal{H}$ of the theory.

Our central observation

- d dimensional QFT with the Hilbert space $\mathcal{H}$ for a spatial slice M_{d-1}.
- Unitary operators U_1, U_2 representing symmetry operations localized respectively within disjoint regions $R_1, R_2 \subset M_{d-1}$, $R_1 \cap R_2 = \varnothing$.
- In a *bosonic* theory, the **locality** of the system dictates that they commute: $U_1 U_2 = U_2 U_1$.
- In a *fermionic* theory, they can either commute or anticommute: $U_1 U_2 = \pm U_2 U_1$.

$U(1)$ symmetry and current algebra in 2d

Current algebra and Zero-Winding-Number sector

Consider spacetime: $\mathbb{R}_{\text{time}} \times S^1_{\text{space}}$, the $U(1)_k$ current charge density operator $J^0(x,t)$ have the equal-time commutation relation

$$[J^0(x,t), J^0(y,t)] = k\frac{i}{2\pi}\frac{\partial}{\partial y}\delta(x-y)\,.$$

Picking functions $f_0, g_0 : S^1 \to \mathbb{R}$, with the assignment

$$LU(1)_0 \ni \exp(2\pi i f_0) \longmapsto U(f_0) := \exp 2\pi i \int_{S^1} f_0(x) J^0(x)\, dx\,,$$

describing **position dependent** $U(1)$ transformation.

- Loop group of $U(1)$: $LU(1) := \{f : S^1 \to U(1)\}$.
- $LU(1)_0$ is its zero-winding-number subgroup that is connected to the identity.

Using Baker-Campbell-Hausdorff formula to derive

$$\begin{aligned} U(f_0)U(g_0) &= \exp\left[2\pi i \beta_0(f_0, g_0)\right] U(f_0 + g_0) \\ &= \exp\left[2\pi i \gamma_0(f_0, g_0)\right] U(g_0)U(f_0) \end{aligned}$$

where the 2-cocycle map $\beta_0(f_0, g_0)$ and the commutator map $\gamma_0(f_0, g_0)$ on $LU(1)_0$ are given by

$$\beta_0(f_0, g_0) = \frac{k}{2} \int_{S^1} f_0\, g_0'\, dx, \qquad \gamma_0(f_0, g_0) = \beta_0(f_0, g_0) - \beta_0(g_0, f_0)\,.$$

Fact: current algebra determines a $U(1)$ central extension of $LU(1)_0$.

[Faddeev '84]: This is a manifestation of $U(1)$ anomaly.

Schwinger-term and descent equation for $2d$ $U(1)$

Chain of descent equations

$$
\begin{aligned}
I_4(F) &= \frac{k}{2}\frac{F}{2\pi}\wedge\frac{F}{2\pi}\,, \quad I_4(F) = d\,I_3(A) \\
I_3(A) &= \frac{k}{2}\frac{A}{2\pi}\wedge\frac{F}{2\pi}\,, \quad \delta_{f_0}\,I_3(A) = d\,I_2(A, f_0) \\
I_2(A, f_0) &= \frac{k}{2} f_0 \wedge \frac{dA}{2\pi}\,, \quad \ldots \\
&\ldots
\end{aligned}
$$

One step further

$$
\delta_{g_0} I_2(A, f_0) = I_2(A^{g_0}, f_0) - I_2(A, g_0 + f_0) + I_2(A, g_0) = \frac{k}{2} df_0 \wedge dg_0 \equiv dI_1(f_0, g_0)\,,
$$

$$
\Longrightarrow I_1(f_0, g_0) = \frac{k}{2} f_0 \wedge dg_0\,.
$$

But, this is not the end of the story, cf. $LU(1)_0 \subset LU(1)$.

The loop group $LU(1)$

Consider

$$\mathcal{L} := \{f \in \mathcal{C}^\infty([0, 2\pi], \mathbb{R}) \mid f(0) = f(2\pi) \mod \mathbb{Z}\}$$

An arbitrary element in $LU(1)$ can be described as $\exp(2\pi i f)$ for some $f \in \mathcal{L}$ and its winding number is $w_f := f(2\pi) - f(0)$.
Lie algebra of $LU(1)$ is $L\mathbb{R} := \{f_0 : S^1 \to \mathbb{R}\}$ with the surjection

$$\exp 2\pi i(\cdot) : \mathcal{L}\mathbb{R} \longrightarrow LU(1)_0 \, .$$

- Naive definition

  $$LU(1) \ni \exp(2\pi i f) \longmapsto U(f) := \text{``}\exp\left[2\pi i \int_0^{2\pi} dx\, f(x) J^0(x,t)\right]\text{''} \, ,$$

  would fail and one needs careful regularization to proceed [OSTZ].

- Alternatively, extending the 2-cocycle β_0 and commutator map γ_0 from $LU(1)_0$ to $LU(1)$, e. g. β and γ.

The commutator map γ

Only aiming at the commutator map γ without considering the cocycle β is sufficient for $U_1 U_2 = e^{2\pi i\gamma} U_2 U_1 = \pm U_2 U_1$.

(γ-0) Recall $\mathcal{L} = \{f \in \mathcal{C}^\infty([0, 2\pi], \mathbb{R}) \mid f(0) = f(2\pi) \mod \mathbb{Z}\}$,
γ: $\mathcal{L} \times \mathcal{L} \to \mathbb{R}/\mathbb{Z}$ such that

$$\gamma(f + 1, g) = \gamma(f, g + 1) = \gamma(f, g) \mod 1.$$

(γ-1) γ is bi-additive

$$\gamma(f+h, g) = \gamma(f, g)+\gamma(h, g), \quad \gamma(f, g+h) = \gamma(f, g)+\gamma(f, h) \mod 1,$$

and alternating $\gamma(f, f) = 0 \mod 1$.

(γ-2) For $f_0, g_0 \in \mathcal{C}^\infty(S^1, \mathbb{R})$ with winding number zero, γ reduces to γ_0

$$\gamma(f_0, g_0) = \gamma_0(f_0, g_0) = \frac{k}{2} \int_{S^1} (f_0(x)g_0'(x) - g_0(x)f_0'(x))dx \mod 1.$$

(γ-3) γ satisfies the graded locality condition

$$\gamma(f, g) \in \frac{1}{2}\mathbb{Z} \quad \text{if} \quad \operatorname{supp} f \cap \operatorname{supp} g = \varnothing.$$

Here, $\operatorname{supp} f := \overline{\{x \in [0, 2\pi] \mid f(x) \neq 0 \mod 1\}}$ for $f \in \mathcal{L}$.

Theorem 1 [Okada-Shimamura-Tachikawa-Zhang '25]

There is a unique commutator map γ satisfying the consistency conditions $(\gamma\text{-}0)$–$(\gamma\text{-}3)$, and with the explicit formula given as

$$\gamma(f,g) = \frac{k}{2}\left(\int_0^{2\pi} (f(x)g'(x) - g(x)f'(x))\, dx + f(0)w_g - w_f g(0)\right) .$$

Comments:

- This formula first appeared in the work by Segal and collaborators in the 1980s, where the expression was derived as a unique solution generalizing the zero-winding-number result satisfying a **covariance** under the action of orientation preserving diffeomorphisms on S^1.
- Differential cohomology $\hat{H}^1(S^1) = \{f : S^1 \to S^1\} = LU(1)$ has the feature that the graded product [Cheeger, Simons '85]

$$\hat{H}^1(S^1) \times \hat{H}^1(S^1) \to \hat{H}^2(S^1)$$

naturally gives a function $\tilde{\gamma}(f,g) : LU(1) \times LU(1) \to \mathbb{R}/\mathbb{Z}$ that relates to the commutator map by [Freed, Moore and Segal '06]

$$\tilde{\gamma}(f,g) = \gamma(f,g) + \frac{1}{2} w_f w_g .$$

The 2-cocycle β

We then start with the following conditions.

(β-0) β is a map $\mathcal{L} \times \mathcal{L} \to \mathbb{R}/\mathbb{Z}$ such that

$$\beta(f+1, g) = \beta(f, g+1) = \beta(f, g) \mod 1.$$

(β-1) β satisfies the cocycle condition

$$\beta(g, h) - \beta(f+g, h) + \beta(f, g+h) - \beta(f, g) = 0 \mod 1.$$

(β-2) For $f_0, g_0 \in \mathcal{C}^\infty(S^1, \mathbb{R})$ with winding number zero, β reduces to β_0

$$\beta(f_0, g_0) = \frac{k}{2} \int_{S^1} f_0(x) g_0'(x) dx \mod 1.$$

(β-3) β is a 2-cocycle for the commutator map γ determined previously

$$\beta(f, g) - \beta(g, f) = \gamma(f, g) \mod 1.$$

Theorem 2 [Okada-Shimamura-Tachikawa-Zhang '25]

There is a unique 2-cocycle β up to coboundary satisfying the conditions $(\beta\text{-}0)$–$(\beta\text{-}3)$, and the explicit formula for a representative can be given as

$$\beta(f,g) = \frac{k}{2}\left(\int_0^{2\pi} f(x)g'(x)dx + w_g f(0)\right) .$$

Comments:

- In the choice of a representative β, there is a degree of freedom of adding a coboundary term. This leads to some variations of the 2-cocycles appearing in literature [Segal et. al; Cheeger and Simons; Bohm and Szlachanyi].
- In particular, when the level k is even, we can choose β so that it is $\mathrm{Diff}^+(S^1)$-invariant. For example, $\beta = \frac{1}{2}\gamma$ and $\beta(f,g) = \frac{k}{2}\left(\int_0^{2\pi} f(x)g'(x)dx - w_f g(0)\right)$ are $\mathrm{Diff}^+(S^1)$-invariant, but satisfy $(\beta\text{-}0)$ only when k is even.

$\mathbb{Z}_n \subset U(1)$ symmetry in 2d

Fermionic SPT for $(2+1)d$ bulk and $(1+1)d$ boundary

Anomaly consists of three data

$$(\mu, \nu, \alpha) \in C^1(BG; \mathbb{Z}_2) \times C^2(BG; \mathbb{Z}_2) \times C^3(BG; U(1)),$$

where $C^d(BG; A)$ is the set of A-valued cochains of degree d with the condition that

$$\delta\mu = 0, \quad \delta\nu = 0, \quad \delta\alpha = (-1)^{\nu^2}.$$

Gu-Wen ferminic SPT for $\mu = 0$.

[Gu, Wen '14]: anomaly data for $2d$ finite group $\mathbb{Z}_2$ symmetry

$$(\nu, \alpha) \in C^3(B\mathbb{Z}_2, U(1)) \times Z^2(B\mathbb{Z}_2, \mathbb{Z}_2)\,,$$

such that

$$\delta\alpha = (-1)^{\nu^2}\,.$$

Gaiotto-Kapustin Phase

[Gaiotto, Kapustin '14] explained how μ emerges and [Brumfiel, Morgan '16] proved $\{[(\mu, \nu, \alpha)]\} \cong \mathrm{Hom}(\Omega_3^{\mathrm{spin}}(BG), U(1))$, known from hep-th by eta-invariant.

‘Hamiltonian’ derivation of the anomaly 3-cocycle

What if we wish to extract the anomaly information from the Hamiltonian point of view?

- It is possible and the answer is known as Else-Nayak argument [Else, Nayak '14] from cond-mat.
- Known in the algebraic quantum field theory (**AQFT**) community already in the 80's or 90's, recent ref. see [Muger '05]
- Recently, S. Seifnashri had several articles discussing it on lattice models, e. g. [2308.05151].

Key observation

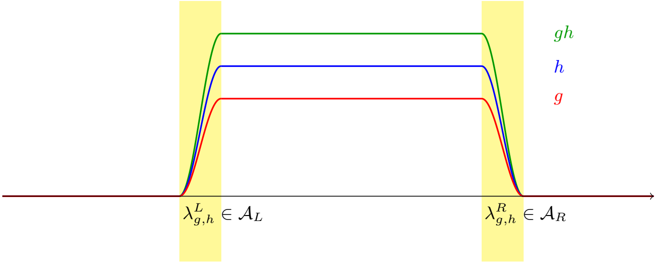
U_g can be “manipulated” such that

$$U_g U_h \propto u(g,h) U_{gh} \,,$$

where $u(g,h)$ is a **unitary** operator supported “somewhere”, instead of closes up to a phase. $u(g,h)$ itself is defined up to a phase.

Derive (α, ν) from Else-Nayak's argument

An application of $g \in G$ on a finite segment better defined than a semi-infinite segment, and abstractly using $\rho_g(O)$ as $U_g O U_g^{-1}$.



$$\rho_g \rho_h(O) = +\lambda^L_{g,h} \lambda^R_{g,h} \rho_{gh}(O) (\lambda^R_{g,h})^{-1} (\lambda^L_{g,h})^{-1}$$

In general, operators $O_L \in \mathcal{A}_L$ and $O_R \in \mathcal{A}_R$ should either commute or anti-commute,

$$O_L O_R = (-1)^{|O_L||O_R|} O_R O_L,$$

where $|O| = 0, 1$ is the fermion parity. (Impossible in a bosonic theory!)

Details

$$\rho_g \rho_h(O) = +\lambda^L_{g,h}\lambda^R_{g,h}\rho_{gh}(O)(\lambda^R_{g,h})^{-1}(\lambda^L_{g,h})^{-1} \tag{1}$$

Take O above to be $O_L \in \mathcal{A}_L$, as $\lambda^R_{g,h} \in \mathcal{A}_R$, we have

$$\lambda^R_{g,h} O_L = (-1)^{|O_L||\lambda^R_{g,h}|} O_L \lambda^R_{g,h}.$$

Plugging it in (1) we find

$$\rho_g \rho_h(O_L) = (-1)^{|O_L||\lambda^R_{g,h}|}\lambda^L_{g,h}\rho_{gh}(O_L)(\lambda^L_{g,h})^{-1}.$$

Also, by letting $O = (-1)^F$ in (1) and assuming ρ_g etc. preserve $(-1)^F$, we can show $|\lambda^R_{g,h}| = |\lambda^L_{g,h}| =: \nu(g,h)$. We then conclude

$$\rho_g \rho_h(O_L) = (-1)^{|O_L||\lambda^L_{g,h}|}\lambda^L_{g,h}\rho_{gh}(O_L)(\lambda^L_{g,h})^{-1}\,. \tag{2}$$

Assuming the existence of F-symbol (associator α)

$$\rho_g(\rho_h\rho_k)(O) = \alpha(g,h,k)(\rho_g\rho_h)\rho_k(O)\,,$$

one can derive that

- $\nu(g,h)$ is a 2-cocycle with value in $\mathbb{Z}_2$, cf.

$$(-1)^F\lambda^L_{g,h}(-1)^F = (-1)^{\nu(g,h)}\lambda^L_{g,h} \tag{3}$$

- And

$$\rho_g(\lambda^L_{h,k})\lambda^L_{g,hk} = \alpha(g,h,k)\lambda^L_{g,h}\lambda^L_{gh,k}\,, \tag{4}$$

  a lengthy but straightforward computation shows that

$$\delta\alpha = (-1)^{\nu^2}\,.$$

Anomaly of $\mathbb{Z}_n \subset U(1)$

Now define the carry for $b, c \in \{0, 1, 2, \ldots, n-1\}$

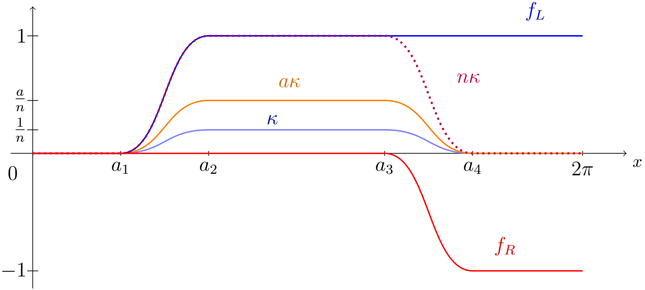
$$p(b,c) = \begin{cases} 0 & (b+c<n) \\ 1 & (b+c \geq n) \end{cases} ,$$

and the residue of an integer m modulo n is denoted by $\overline{m}$.
We specify the profile function $\kappa(x)$ of the generator $\exp 2\pi i \frac{1}{n} \in \mathbb{Z}_n \subset U(1)$

$$\kappa(x) = \begin{cases} 0 & (0 \leq x < a_1) \\ \text{interpolate} & (a_1 \leq x < a_2) \\ \frac{1}{n} & (a_2 \leq x \leq a_3) \\ \text{interpolate} & (a_3 < x \leq a_4) \\ 0 & (a_4 < x \leq 2\pi) \end{cases} ,$$

where $0 < a_1 < a_2 < a_3 < a_4 < 2\pi$.

$$f_L(x) = \begin{cases} 0 & (0 \le x < a_1) \\ n\kappa(x) & (a_1 \le x < a_2) \\ 1 & (a_2 \le x \le 2\pi) \end{cases} \quad , \quad f_R(x) = \begin{cases} 0 & (0 \le x < a_3) \\ n\kappa(x) - 1 & (a_3 \le x < a_4) \\ -1 & (a_4 \le x \le 2\pi) \end{cases}$$



When comparing $\rho_b\rho_c(O)$ and $\rho_{\overline{b+c}}(O)$, the fusion operators appear if and only if $b + c \ge n$ and we then conclude that

$$\lambda^L_{b,c} = U(f_L)^{p(b,c)} , \qquad \lambda^R_{b,c} = U(f_R)^{p(b,c)} .$$

We can compute

$$\rho_a(\lambda^L_{b,c}) = U(a\kappa)\left(U(f_L)^{p(b,c)}\right)U(a\kappa)^{-1} = (\exp 2\pi i\,\gamma(a\kappa, f_L))^{p(b,c)}\;U(f_L)^{p(b,c)}$$
$$= \left(\exp 2\pi i \frac{k}{2}\frac{a}{n}p(b,c)\right)\;U(f_L)^{p(b,c)}\,.$$

Put the result back to the identity (4), we get

$$\alpha(a,b,c) = \exp 2\pi i \frac{k}{2}\frac{a}{n}p(b,c)\,.$$

For the fermion parity we take $(-1)^F = U(-\frac{1}{2})$ with the constant map $g = -\frac{1}{2}$ and

$$(-1)^F\lambda^L_{b,c}(-1)^F = (-1)^{\nu(b,c)}\,\lambda^L_{b,c} \Longrightarrow (-1)^{\nu(b,c)} = \left(\exp 2\pi i\,\gamma(-\frac{1}{2}, f_L)\right)^{p(b,c)}$$
$$= \exp 2\pi i(-\frac{k}{2})\,p(b,c)\,.$$

It is straightforward to verify that

$$(\delta\alpha)(a,b,c,d) = \exp 2\pi i(\frac{k}{2})p(a,b)p(c,d)\,.$$

$SU(2)$ in $4d$ via $2d$ invertible phase

$SU(2)$ symmetry transformation in $4d$

- Consider $4d$ theory with $SU(2)$ symmetry on $M_3 \times \mathbb{R}$ and consider the action of position-dependent symmetry operation specified by $f : M_3 \to SU(2)$.
- Take $f, g : M_3 \to SU(2)$ whose supports are distinct.
- Aim: show that

$$U(f)U(g) = (-1)^{w_f w_g} U(g)U(f) \,.$$

[Faddeev '84, Zumino 85']: Schwinger-term in spacetime dimensions higher than two **must** contain gauge field potential.

$$[G^a(x), G^b(y)] = if^{abc} G^c(x)\delta^3(x-y) - \frac{i}{12\pi^2} d^{abc}\epsilon_{ijk}\partial_i A_j^c(x)\partial_k\delta^3(x-y) \,.$$

$$G^a(x) := \partial_i E_i^a(x) + f^{abc} A_i^b E_i^c + i\bar{\psi}\gamma^0 T^a\psi.$$

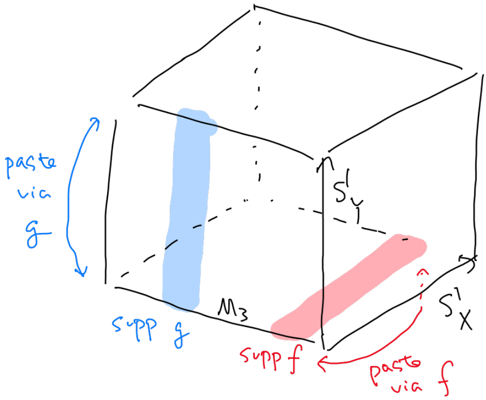
Effective gauge group $\mathrm{Map}(M_3, SU(2))$

1. Strategy: $M_5 = M_3 \times S^1_X \times S^1_Y$ construct an $SU(2)$ bundle on $M_3 \times S^1_X$ using the gauge transformation f, and an $SU(2)$ bundle on $M_3 \times S^1_Y$ using the gauge transformation g.

   (Just the mapping torus construction)

2. Now study the anomaly on $T^2_{X,Y} = S^1_X \times S^2_Y$, where the effective symmetry group on the torus $T^2_{X,Y}$ is the mapping space $\mathrm{Map}(M_3, SU(2)) := \{f : M_3 \to SU(2)\}$.

3. $\mathrm{Map}(M_3, SU(2))$-background on $T^2_{X,Y}$ have holonomy f on S^1_X and g on S^1_Y, respectively.



The light-red shaded region is where the $SU(2)$ field is nontrivial due to the gauge transformation by f. (This entire gauge configuration on $M_3 \times S^1_X$ is pulled back to S^1_Y, and therefore it might be better to fill by light-red along the S^1_Y direction too.)

Gluing property of eta-invariant

- The anomaly is detected by the so-called eta-invariant.

$$\eta(M_5) = \frac{1}{2}\left(\sum_{\lambda\neq 0} \mathrm{sign}\,\lambda + \mathsf{dim}\,\mathsf{ker}(D_5)\right)_{\mathrm{reg.}}$$

- $M_3 = (\mathsf{supp}(f)) \sqcup (\mathsf{supp}(g)) \sqcup M_3'$
- The anomaly $Z(M_5) = \exp(2\pi i \eta(M_5))$,

$$\eta(M_5) = \eta((\mathsf{supp}(f)) \times T^2_{X,Y})\eta((\mathsf{supp}(g) \times T^2_{X,Y})\eta(M_3' \times T^2_{X,Y}).$$

- This means, with a fixed spin structure on $T^2_{X,Y}$, we have

$$Z(M_5; f, g)Z(M_5; e, e) = Z(M_5; f, e)Z(M_5; e, g),$$

where now $Z(M_5; f, g)$ can be view as the $2d$ invertible phase on $T^2_{X,Y}$ with f and g holonomy on the 1-cycles.

Making use of $2d$ invertible phase (Anomaly)

- Recall, fermionic $2d$ invertible phase is the pairing between

$$(\alpha, \nu) \in H^2(BG, U(1)) \times Z^1(BG, \mathbb{Z}_2)$$

and $2d$ Spin bordism class $[(\Sigma, \varphi)]$ with a G-bundle.

- Brumfiel-Morgan pairing is

$$(\alpha, \nu) \times (\Sigma, \varphi) \longmapsto Z(\Sigma, \varphi) := \exp 2\pi i \left(\left(\int_\Sigma \varphi^*(\alpha) \right) + q_\Sigma(\varphi^*(\nu)) \right)$$

where $q_\Sigma : H^1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$ is the **quadratic refinement** associated to the spin structure σ on Σ.

- Practically, given the Arf invariant, $q_\Sigma(x) = \mathrm{Arf}_\Sigma(\sigma + x) - \mathrm{Arf}_\Sigma(\sigma)$, $\sigma + x$ is the spin structure obtained by adding $x \in H^1(\Sigma; \mathbb{Z}_2)$.

$2d$ invertible phase evaluation on two torus

- The pullback $\int_{T^2} \varphi^*(\alpha)$ evaluates to $\alpha(f,g) - \alpha(g,f) = \gamma(f,g)$ on T^2 with holonomy (f,g).
- On T^2, Arf invariant is nontrivial $1 \in \mathbb{Z}_2$ if and only if the fermion is periodic on all directions, e. g. (R,R).
- Specifying the spin structure (NS, R) and holonomy $(f, g = e)$, we get

$$Z(T^2_{NS,R}; f, e) = \begin{cases} +1 & (\nu(f) = 0), \\ -1 & (\nu(f) = 1). \end{cases}$$

- Next we consider $T^2_{NS,NS}$ with holonomy (f,g). Then the value is

$$Z(T^2_{NS,NS}; f, g) = \exp(2\pi i(\gamma(f,g) + \frac{1}{2}\nu(f)\nu(g))).$$

Compare the 5d eta with 2d invertible phase

- Back to 5d, $Z(M_5; f, e)$ corresponds to $M_5 = M_4 \times S^1_Y$ with f holonomy on S^1_X and holonomy e on S^1_Y.
- On $M_4 = M_3 \times S^1_X$, we have the gauge field potential $A_f = f^{-1}df$ on M_3, where d is the exterior derivative on M_3.
- The eta invariant we want is

$$\eta(M_4 \times S^1_Y) = \left(\int_{M_4} \frac{1}{2}\mathrm{tr}\,\frac{F_f}{2\pi} \wedge \frac{F_f}{2\pi}\right) \times \eta(S^1_Y) = \left(\int_{M_3} \frac{1}{2}\mathrm{CS}\left(\frac{A_f}{2\pi}\right)\right) \times \eta(S^1_Y)\,,$$

We exponentiate the eta invariant with $2\pi i$, taking into account the spin structure on S^1_Y and we will get the result

$$Z(M_5; f, e) = (-1)^{w_f a_Y}\,,$$

$a_Y = 0$ or 1 depending on spin structure of S^1_Y being NS or R.

Anti-commutation of Unitaries in QFT

From the previous computation, we find the fermion number $\nu(f)$ of the unitary operator $U(f)$ to be given by $\nu(f) = w_f$.
Then taking $a_X = a_Y = 0$, we see $Z(M_5; f, g) = 1$, and from the torus invertible phase

$$Z(T^2_{NS,NS}, f, g) = \exp(2\pi i(\gamma(f, g) + \frac{1}{2}\nu(f)\nu(g))),$$

we find

$$\gamma(f, g) = -\frac{1}{2}\nu(f)\nu(g).$$

Conclusion

If the support of f and g are distinct on M^3

$$U(f)U(g) = (-1)^{w_f w_g} U(g)U(f)\,.$$