Gauging non-invertible symmetries in 2D CFTs and von Neumann Subfactors

Based on 2504.05374 with Xingyang Yu

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Outline

Background: non-invertible symmetries in 2D

Review: gauging and group-theoretic fusion category

Motivation: beyond G and $\mathrm{Rep}(G)$, subfactors are helpful

Results: gauging non-invertible symmetry in $c = 1$ CFTs

Discussion and Outlook

Background: non-invertible symmetries in 2D

Non-Invertible Symmetries in 2D CFTs

In diagonal RCFTs, Verlinde lines are in 1-to-1 correspondence with conformal primaries [Verlinde '88].

They exhibit non-group-like fusion rules, and F-symbols as non-tivial isomorphism induced by an F-move.

See also [Fuchs, Ruhkel, Schweigert '02; Fuchs, Schweigert '03; Frohich, Fuch, Runkel, Schweigert '04]

Following the development of generalized symmetries [GKSW '14], such topological lines are identified and characterized in general 2D theories [Bhardwaj-Tachikawa '17], [Chang-Lin-Shao-Wang-Yin '17].

Example: Ising CFT

Ising CFT: the IR limit of critical Ising model on a square lattice.

Primary operators: $1_{(0,0)}$ the identity, $\epsilon_{(\frac{1}{2},\frac{1}{2})}$ the stress-energy tensor, and $\sigma_{(\frac{1}{16},\frac{1}{16})}$ the spin field, which are in 1:1 correspondence to topological lines $Id, \eta, \mathcal{N}$ with fusion rules

$$\eta^2 = 1, \quad \eta\mathcal{N} = \mathcal{N}\eta = \mathcal{N}, \quad \mathcal{N}^2 = 1 + \eta. \tag{1}$$

Action of topological lines on local operators:

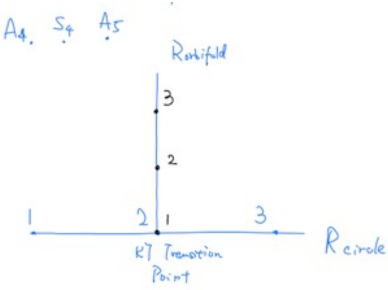
$$\eta 1 = 1, \quad \eta\epsilon = \epsilon, \quad \eta\sigma = -\sigma; \tag{2}$$

$$\mathcal{N}1 = \sqrt{2}1, \quad \mathcal{N}\epsilon = -\sqrt{2}\epsilon, \quad \mathcal{N}\sigma = 0. \tag{3}$$

F-symbol: this fusion rule allows two categories, Ising and $su(2)_2$, differing by a sign in $F^{\sigma}_{\sigma\sigma\sigma}$. Simplest examples of [Tambara-Yamagami '98].

Example: Rational CFT at $c = 1$

The moduli space of $c = 1$ CFT [Ginsparg '88] consists of the *circle branch*, *orbifold branch*, and three *expcetional points*:



Starting from $SU(2)_1$ WZW model at $R_{circle} = 1$, we can gauge discrete ADE subgroup of $SO(3)$ (which is a non-anomalous diagonal symmetry of the global symmetry $SO(4) = (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$) to get

- (A-type) $\mathbb{Z}_n$: other points on the circle branch
- (D-type) Dih_n of order 2n: points on the orbifold branch
- (E-type) A_4, S_4, A_5 (of order 12, 24, 60): exceptional points

Question: starting from a different point, possible gaugings?

Review: gauging non-invertible symmetries in group-theoretic fusion categories

Gauging discrete symmetries: development

Gauging a finite group symmetry: sum over partition functions with all possible charges, where a dual ("quantum") symmetry $\hat{\mathbb{Z}}_n = \mathrm{Hom}(\mathbb{Z}_n, U(1))$ appears [Vafa '88].

Gauging the $\hat{\mathbb{Z}}_n$ recovers the original theory with $\mathbb{Z}_n$ symmetry.

For non-Abelian G:

Gauging a non-Abelian G gives $\mathrm{Rep}(G)$.

[Bhardwaj - Tachikawa '17], in the dual theory, one should be able to "gauge $\mathrm{Rep}(G)$" to recover the original G theory. How to understand this formally?

Gauging discrete symmetry: current understandings

Rather than gauging G, we could gauge by a subgroup $K \subset G$.
We also have a choice of a discrete torsion $H^2(K, U(1))$.

In the category case, we gauge a symmetric separable Frobenius algebra object A in $\mathcal{C}$ In $Rep(G)$, gauging $A = \sum_i d_i \mathbf{d_i}$ recovers the original G symmetry.

Physically, gauging A = insert a mesh of A-topological lines
Then, topological lines the dual theory $\mathcal{T}/A$ are described by $A - A$ bimodules, elements of ${}_A\mathcal{C}_A$.

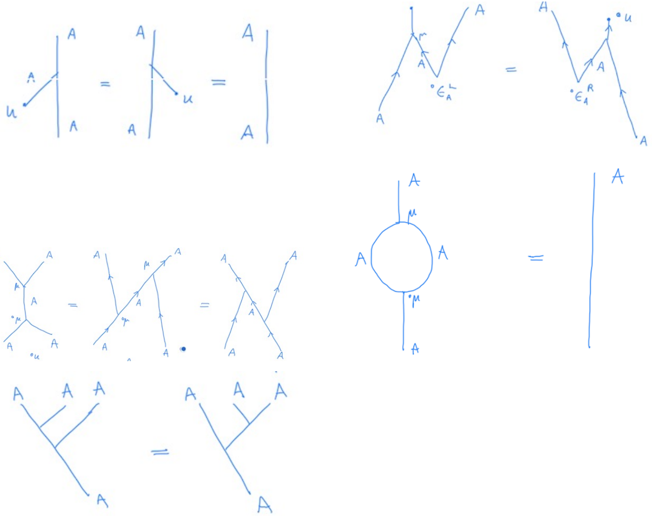
For the Ising example, there is an algebra object $A = 1 + \eta$ implementing the Krammer-Wannier duality.

The dual theory is isomorphic to the original one in a non-trivial way: spin σ is mapped to an non-genuine operator μ

Consistency conditions for an algebra object

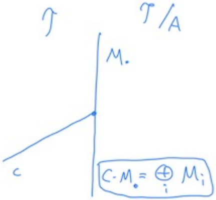
Unital, symmetric, Frobenius, separable, associative

To summarize: A should be "sufficiently deformable"



Gauging Interfaces of 2D CFTs: intuition

Gauging interfaces are right A-modules.
They further admit a $\mathcal{C}$-action, forming a module category $\mathcal{C}_A$.



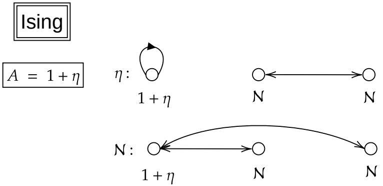
For a given gauging, there can be multiple gauging interfaces exchanged by a junction with $\mathcal{C}$ lines.

Gauging interfaces in the Ising CFT

Pick a basis M_i of gauging interfaces, $\mathcal{C}$ acts on M_i as matrices $R(\mathcal{C})$, Non-negative Integer Matrix (NIM) representation of $\mathcal{C}$.

Ising: $R(\eta)^2 = Id, \;\; R(\mathcal{N})^2 = Id + R(\eta), \;\; R(\eta)R(\mathcal{N}) = R(\mathcal{N})R(\eta) = R(\mathcal{N})$.

$$\text{Solution:} \quad R(\eta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad R(\mathcal{N}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (4)$$



Adjacency graphs of $R(\eta), R(\mathcal{N})$. Node: gauging interfaces; arrows: η (resp., $\mathcal{N}$) action on the gauging interfaces.

Gauging and Group-theoretic fusion category

A large family of fusion category can be obtained by starting from a group G, and gauging a subgroup K.

The resulting group-theoretic fusion category is denoted as $\mathcal{C}(G, \omega = 0; K, \psi)$:

- $\psi \in H^2(K, U(1))$ is the discrete torsion,
- and we work with trivial anomaly: $\omega = 0 \in H^3(G, U(1))$

Elements in $C(G, 0; K, \psi = 0)$ are $K - K$ bimodules.

When one add non-trivial ψ, we get twisted $K - K$ bimodules (i.e., projective representation for each K)

Dual fusion category for gauging in G

The fusion coefficient of such $K-K$ bimodules are given explicitly in [Kosaki-Munemasa-Yamagami '97].

$K-K$ bimodules in G can be parametrized by

($K-K$ double coset U in G,
representation of a centralizer $\Gamma(x,y)\subset K$ of an element in U),

which is reminiscent of Drinfeld Center of a finite group.

We get fusion coefficients N_{UVW} of U,V fusion into $\bar{W}$ via

$$N_{UVW}=\frac{1}{|G|}\sum_{x,y,z\in X}\sum_{\gamma\in\Gamma(x,y,z)}\chi_U^{x,y}(\gamma)\chi_V^{y,z}(\gamma)\chi_w^{(x,z)}(\gamma) \tag{5}$$

$x,y,z\in X$ are left cosets $K\backslash G$, $\Gamma(x,y,z)=\Gamma(x,y)\cap\Gamma(x,z)\cap\Gamma(y,z)$. $\chi_U^{(x,y)}(\gamma)$ is the character of a representation of the centralizer $\Gamma(x,y)$, evaluated at an element $\gamma\in\Gamma(x,y,z)\subset\Gamma(x,y)$ [KMY '97].

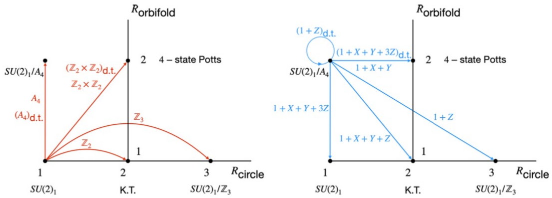
We wrote a code implementing this. It could run for $\mathcal{C}(G,1;K,1)$ for $G=A_4,S_4,A_5$, but it gets slower for larget groups. It takes a few minutes to run to find all fusion rules for $\mathcal{C}(A_5,0;\mathbb{Z}_2,0)$.

Dual fusion category for gaugin in Rep(G)

For each gauging in G by K (to get $\mathcal{C}(G, 0; K, 0)$), we know the corresponding gauging in $Rep(G)$ leading to the same $\mathcal{C}(G, 0; K, 0)$ via:

- Decompose G into K cosets G/K
- Write down the G rep. by examining G-action on G/K
- Decompose such a rep. into irreps of G.

This way, we know all possible gaugings in Rep(G) and the dual fusion category. Cases with $\psi \neq 0$ can be treated using [Putrov-Radhakrishnan '24] via $Z(G)$.



(a) $H \subset A_4$ Gaugings of $SU(2)_1$

(b) Rep(A_4) Gaugings of $SU(2)_1/A_4$

E.g., all possible gauging in A_4 and Rep(A_4),
taken from [Perez-Lona, Robbins, Sharpe, Vandermeulen, Yu '24]

Motivation: beyond G and $\mathrm{Rep}(G)$, subfactors are computationally helpful

Gauging in C^T and the need for subfactor theory

The branch point in the $c = 1$ moduli space is called Kosterlitz-Thouless (KT) point. It has a $\mathcal{C}^T$ triality symmetry, which is $\mathbb{Z}_2 \times \mathbb{Z}_2$ enhanced by $\mathcal{T}, \overline{\mathcal{T}}$:

$$\mathcal{T}\mathcal{T} = 2\overline{\mathcal{T}}, \quad \mathcal{T}\overline{\mathcal{T}} = 1 + \eta + \eta' + \eta\eta'. \tag{6}$$

There were no systematic approach to understand gaugings of this symmetry.

We realized subfactor theory to be a useful tool here:

Fusion category $\mathcal{C}$ and a gaugeable algebra object A is equivalent to "a pair of von Neumann subfactors $N \subset M$".

So we decided to delve into von Neumann subfactor theory.

von Neumann Subfactors

We won't need rigorous definition of von Neumann algebra, except:

- They are algebra of complex operators acting on a $\mathcal{H}$
- A factor is a building block of a von Neumann algebra (just like any integer decomposes into prime numbers).

See [Jia-Tian '25] and [Shao-Sorce-Srivastava '25] for recent works on von Neumann algebra and generalized symmetries.

There is a type classification of factors. We will be using *factors of type III*, ones that "does not admit any pure states or mixed states".

All the factor we cares about are isomorphic to each other, but the embedding $\iota : N \hookrightarrow M$ could contain non-trivial information.

Such a pair is called (von Neumann) **subfactors**.

Gaugeable algebra object via subfactors

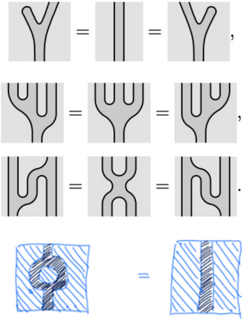
Using diagrams, we now intuitively explain how a subfactor $N \subset M$ (with embedding $\iota : N \hookrightarrow M$) knows about gauging.

- A fusion category C is encoded as as the category of certain ("DHR") endomorphisms of N
  - In particular, the F-symbol data is (intriguingly) implicit
- A gaugeable algebra object $A = (\theta$ in figure $)$ as $\bar{\iota}\iota$, where $\bar{\iota} : M \hookrightarrow N$ is a dual embedding

- All possible gauging interfaces (i.e., A-modules) as mixed $(N - M)$ bimodules

Recovering all the conditions on A

Then one can "resolve" all the objects A as "a thin strip of M inside N" Taken from [Bischoff-Kawahigashi-Longo-Rehren] except the last row. The right side acts first



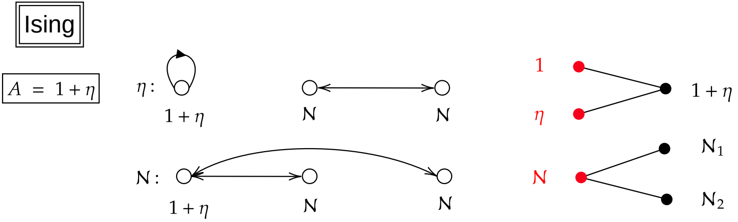
Light shadow: N; dark shadow: M.

Heuristically, we see all the conditions for the algebra object A to be naturally satisfied by deforming the M-N boundaries.

Principal graph: $\mathcal{A}$-action on modules

Recall that $\mathcal{C}$ elements are $N - N$ bimodules, and modules $\mathcal{C}_A$ are $N - M$ bimodules.

Mathematicians likes to organize these data into principal graph of a subfactor (*right*), which encodes the adjacency graph (*left*) of $\eta/\mathcal{N}$-action on the modules:

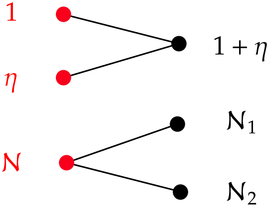


The odd part connecting to 1 is ι, the identity module.
Here we illustrate the case of Ising, where $\mathcal{A} = 1 + \eta$.

Finding Candidate Principal Graph from Fusion rules

For a fusion category $\mathcal{C}$, we look for its potential gaugings.

Here, we want to compute the adjacency matrix G_{rs} of the principal graph.



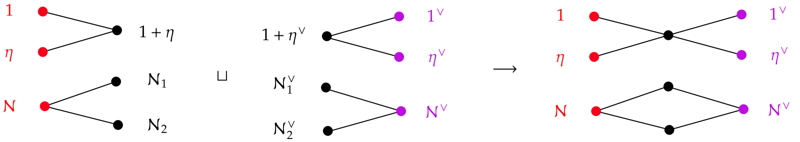
- Compute the fusion matrix $F^A_{r\times r}$ = triple fusion coefficient of (algebra obj. A, k_1^{th} topological line, k_2^{th} topological line).
- It can be shown that $F^A_{r\times r} = G_{r\times s}(G)^T_{s\times r}$, so we need to factorize F into GG^T.
- Then $G_{r\times s}$ gives the principal graph, s labeling the interfaces.

Dual and Glued Principal Graphs

$(\mathcal{C}, A)$	Physics theory $\mathcal{T}$	Subfactor $N \subset M$	principal graph
$\mathcal{C}$	topological lines in $\mathcal{T}$	$N - N$ bimod.	even part
$\mathcal{C}_A$	gauging interfaces	$N - M$ bimod.	odd part
${}_A\mathcal{C}_A$	dual topological lines $\mathcal{T}/A$	$M - M$ bimod.	dual even part
${}_A\mathcal{C}$	(dual) gauging interfaces	$M - N$ bimod.	(dual) odd part

$M - N$ bimodules are $N - M$ bimodules are in 1:1 correspondence,

So it is natural to glue a principal graph and a dual one along the odd parts.



Disclaimer Slide

Some feedback after previous talks.

The construction of M, N, despite mathematically valid, are very abstract and implicit.

We are only using M, N as a formal tool to extract key features from $\mathcal{C}$ (into the principal graph of $M \subset N$), and compare with a candidate dual $\mathcal{C}' =_{\mathcal{A}} \mathcal{C}_{\mathcal{A}}$

Suppose we know the group case, G, has N_G gaugings. If we have at most N_G candidate gaugings in ${}_{\mathcal{A}}\mathcal{C}_{\mathcal{A}}$, then these gaugings are all valid.

Results: gauging non-invertible symmetry in $c = 1$ CFTs

Results

We use the approach of subfactors to work out all possible gaugings and gauging interfaces for all $\mathcal{C}(G, 1; K, \psi)$ with $G = A_4, A_5$, following [Grossman-Snyder '11].

- We are able to explicitly determine all gaugings from each dual theories by focusing on the $A_4 \subset SO(3)$ symmetry.
- In particular, we understood all gaugings in the KT point with C^T symmetry - neither G nor $Rep(G)$.
- We identified a self-duality gauging in the A_5 exceptional point. This gauging allows us to construct a self-duality defect, which enhances the symmetry from Rep(A_5) to Rep($SL(2, \mathbb{F}_5)$), matching expectation from fusion rule of local operators.

Brauer-Picard Groupoid and Composing Gaugings

If there is a gauging that sends $\mathcal{T} \to \mathcal{T}'$, then all the gaugings in $\mathcal{T}$ are in 1:1 correspondence with gaugings in $\mathcal{T}'$, since we are allowed to compose gaugings.

$$\# \text{ of gaugings in } G = \# \text{ of } (K, \psi) \tag{7}$$

$$\stackrel{!}{=} \# \text{ of gaugings in } \mathcal{C}(G, 0; K, \psi). \tag{8}$$

[Choi, Lu, Sun '23], [Perez-Lona, Robbins, Sharpe, Vandermeulen, Yu '23 '24], [Diatlyk, Luo, Wang, Weller '23] used this to determine gaugings in Rep(G).

Mathematically, we are looking for the Brauer-Picard Groupoid of a fusion category

- where objects are fusion categories
- morphisms are "gauging" / Morita equivalences: map from $\mathcal{C}$ to its bimodule category ${}_A\mathcal{C}_A$.

Gaugings in the orbifold groupoid of A_4

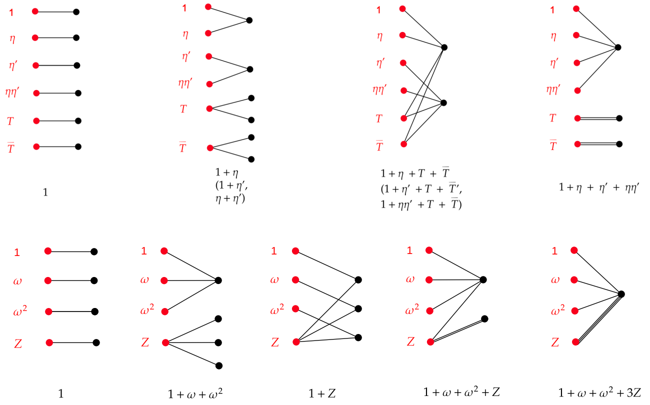
So can have the following table (two gaugings in the same line do not produce each other but they go to a common third theory):

Each row is a different theory, their symmetry may coincide since we just look at A_4.

Theory	${}_A\mathcal{C}_A$	$K \subset A_4$	A of $\mathrm{Rep}(A_4)$	A' of $\mathcal{C}_T$
$R_c = 1$	Vec_{A_4}	1	$(1 + X + Y + 3Z)_1$	???
K.T.	$\mathcal{C}^T$	$\mathbb{Z}_2$	$1 + X + Y + Z$	???
$R_c = 3$	$\mathrm{Rep}(A_4)$	$\mathbb{Z}_3$	$(1 + Z)_1$	???
$R_o = 2$	Vec_{A_4}	$(\mathbb{Z}_2)^2$	$1 + X + Y$	???
$R_o = 2$	Vec_{A_4}	$(\mathbb{Z}_2)^2_{\mathrm{d.t.}}$	$(1 + X + Y + 3Z)_2$	???
A_4 point	$\mathrm{Rep}(A_4)$	A_4	1	???
A_4 point	$\mathrm{Rep}(A_4)$	$(A_4)_{\mathrm{d.t.}}$	$(1 + Z)_2$	???

Where the $\mathrm{Rep}(A_4)$ column can be found by known techniques
But *the C_T column needs subfactors*.

Gaugings in the K.T. point via Gluing Principal Diagrams

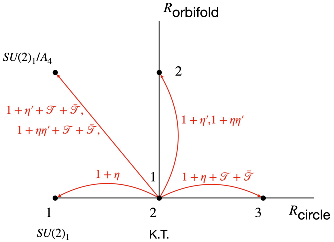


- Gauging 1 is trivial: remains C^T.
- Gauging $1+\eta$: dual to Gauging $\mathbb{Z}_2$ in G
- Gauging $1+\eta+\eta'+\eta\eta'$: dual to gauging $\mathbb{Z}_2\times\mathbb{Z}_2$ in G
- Gauging $1+\eta+\mathcal{T}+\overline{\mathcal{T}}$: dual to gauging $1+X+Y+Z$ in $\mathrm{Rep}(A_4)$. *Illustrated above*, second last elements in each row can be glued.

All possible gaugings in $C_T = C(A_4, 1; \mathbb{Z}_2, 1)$

Via the above subfactor analysis, we get

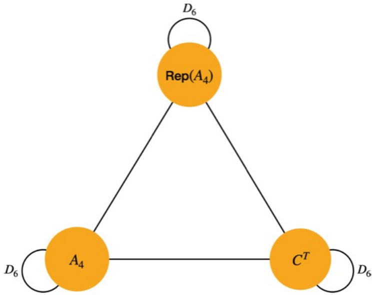
$\mathcal{C}(G,1;K,\psi)$	$K \subset A_4$	A of $\mathrm{Rep}(A_4)$	A' of $\mathcal{C}_T$
Vec_{A_4}	1	$(1+X+Y^2+3Z)_1$	$1+\eta$
$\mathcal{C}^T$	$\mathbb{Z}_2$	$1+X+Y^2+Z$	1
$\mathrm{Rep}(A_4)$	$\mathbb{Z}_3$	$(1+Z)_1$	$1+\eta+\mathcal{T}+\bar{\mathcal{T}}$
Vec_{A_4}	$(\mathbb{Z}_2)^2$	$1+X+Y^2$	$1+\eta'$
Vec_{A_4}	$(\mathbb{Z}_2)^2_{\text{d.t.}}$	$(1+X+Y^2+3Z)_2$	$1+\eta\eta'$
$\mathrm{Rep}(A_4)$	A_4	1	$1+\eta'+\mathcal{T}+\bar{\mathcal{T}}$
$\mathrm{Rep}(A_4)$	$(A_4)_{\text{d.t.}}$	$(1+Z)_2$	$1+\eta\eta'+\mathcal{T}+\bar{\mathcal{T}}$



$$\mathcal{T}\mathcal{T} = 2\bar{\mathcal{T}}, \quad \mathcal{T}\bar{\mathcal{T}} = 1+\eta+\eta'+\eta\eta'$$

Brauer-Picard Groupoid v.s. CFT Physics

Recall that objects are categories and morphisms are Morita equivalences / gaugings



Physical splitting: A_4 is seen both at $R_{circle} = 1$ and $R_{orbifold} = 2$, Rep(A_4) is seen both at $R_{circle} = 3$ and the A_4 point.

Discussions and outlook

Conclusions

The triplet of

$$(\text{original symmetry}, \text{dual symmetry}, \text{gauging interfaces})$$

can be effectively determined and conveniently packaged using the formalism of von Neumann subfactors.

Starting from the $SU(2)_1$ CFT, we gauged various subgroups of its $A_4 \subset SO(3)$ global symmetry. We further studied how the resulting theories are related to each other by gauging.

Future directions

We hope to better understand how the data of F-symbol / associator of a fusion category is encoded on the subfactor side.

Intermediate subfactors $N \subset P \subset M$ is well-studied on the subfactor side, but less on the fusion category side. We hope to use it to understand sequential gauging.

Subfactors are known to describe extension of local conformal nets, which appears to be a distinct physical setup than gauging non-invertible symmetries. Can we learn anything from comparing these two (apparently distinct) scenarios in physics that are both described by subfactors?


terima kasih
σας ευχαριστώ
આભાર
salamat
ačiū
dank u
Thank you
Danke
謝謝
شكرا
спасибо
ありがとう
감사합니다
merci
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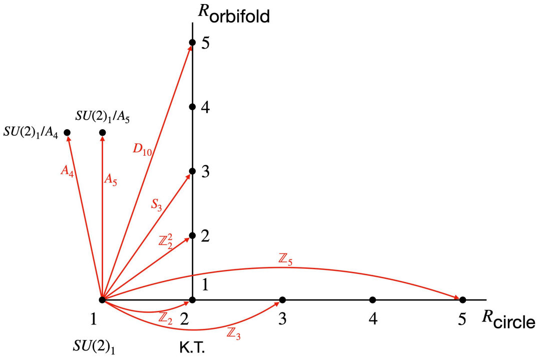
Bonus Content: Duality Enhancement

All possible gaugings in A_5

Our code explicitly computed the dual fusion rings



but further determine the gaugings among the dual categories $\mathcal{C}(A_5, 1; H, 1)$ besides $H \in \{1, A_5\}$ is difficult.

We leave this part for future work.

Duality enhancement and local operators

There is a self-dual gauging in $Rep(A_4)$ by $(1+Z)_{d.t.}$, which enhances the fusion category to $Rep(SL(2,\mathbb{F}_3))$.

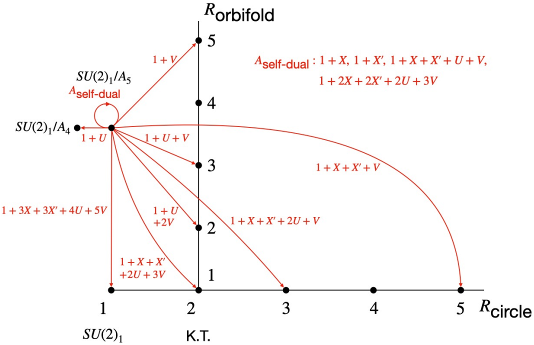
Concretely, the original $Rep(A_4)$ generated by $\mathbf{1}_0, \mathbf{1}_1, \mathbf{1}_2, \mathbf{3}$ is augmented by a defects D_0 and two more 2-dimensional elements D_1, D_2 where $D_i^2 = \mathbf{1}_i + \mathbf{3} \quad (i = 0, 1, 2)$.

We now take a pause and compare the above result with *the operator algebra of orbifold models* by [Dijkgraaf-Vafa-Verlinde-Verlinde '88] .

They already considered the $SU(2)_1/A_4$ orbifold model, and they found that the untwisted sector organizes into the representation ring of $SL(2,\mathbb{F}_3)$, whose order is 24: twice of that of A_4.

All possible gaugings in $Rep(A_5)$

We look again all the gaugings starting from $Rep(A_5)$



We know there is a loop in the upper-left, which is a self-dual gauging.

$Rep(A_5)$ self-dual gaugings via odd parts

Recall that $Rep(A_5)$ can be produced by gauging the whole A_5 with non-trivial algebraic structure

Under this correspondence, the $Rep(A_5)$ objects should be thought of as twisted $A_5 - A_5$ bimodules, and the odd parts have a twisted A_5-module structure (by the non-trivial element of $H^2(A_5, U(1)) \cong \mathbb{Z}_2$).

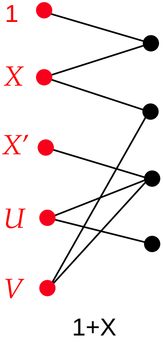
i.e., we want projective representations of A_5, and math literature tell us that they are of dimension 2, 2, 4, 6.

Recall: representations of $\mathcal{C}$ and $\mathcal{C}^\vee$ are given by odd parts of the principal diagram!

So we want to find algebra objects whose odd parts had ratio of dimension 1:1:2:3

Example of self-dual gauging in $Rep(A_5)$

The dimension of the odd part is defined by the (weighted) sum of all even parts that it connects to.



So, in the above diagram of gauging $1 + X$, we get the ratio of odd parts to be

$d(1 + X_3) : d(U_4) : d(X_3 + V_5) : d(X_3' + U_4 + V_5) = 1 : 1 : 2 : 3$,

matching the spectrum of twisted A_5-representations.

Duality Defect and Symmetry Enhancement

The self-dual gauging has multiple candidate algebra objects

$$1+X_3,\ 1+X_3',\ 1+X_3+X_3'+U_4+V_5',\ 1+2X_3+2X_3'+2U_4+3V_5 \quad (9)$$

of dimension $4, 4, 16, 36$ (as sums of A_5 reps of dimension 1, 3, 3, 4, 5).

We learn that we need to include four more duality defects of dimension

$$2, 2, 4, 6 \quad (10)$$

so that $\mathcal{N}_2\overline{\mathcal{N}}_2 = 1 + X_3$, etc.

These objects, together with $Rep(A_5)$, exactly reproduces the fusion rule for $Rep(SL(2,\mathbb{F}_5))$ (binary isocahedral group) of dimension 120, matching our expectation from the A_4 case.

Different looks at a same theory

Physical remark: the gauging of $\mathbb{Z}_2$ takes from $SU(2)_1$ to the K.T. point, irrespective of which symmetry we focus on to begin with (A_4, A_5, etc.)

Therefore, we should expect the $C(A_5, 1; \mathbb{Z}_2, 1)$ extends the "triality extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$", $\mathcal{C}(A_4, 1; \mathbb{Z}_2, 1)$.

We hope to see how this works explicitly in the future.

In a long term, one might expect both these symmetries fits into a $\mathbb{Z}_2$ gauging of the anomalous $SO(4)$, whose mathmatical description is not available yet.

Non-commutative non-invertible fusion rules when gauging A_5 subgroup

We take $\mathcal{C}(A_5, 1; \mathbb{Z}_2, 1)$ as an example.

There are 4 invertible elements forming $\mathbb{Z}_2 \times \mathbb{Z}_2$, and 14 elements each of quantum dimension 2.

We named them $X_2^a, X_2^b, \ldots\ldots$ For non-self-conjugate elements, we also use $X_2^a, X_2^{a'}$ to label conjugate pairs.

And we get

$$X_2^a X_2^b = X_2^i + X_2^f, \quad X_2^b X_2^a = X_2^h + X_2^j \tag{11}$$

Together with many other pairs of non-commuting elements.

Backup Slides

[Backup] Another physical interpretation: Conformal Embedding

In Algebraic QFT, $M \subset N$ is sometimes interpreted as "conformal embedding", e.g. $SU(2)_{10} \subset SO(5)_1$.

These Lie algebra are "conformal nets". They describe Chiral CFTs by picking a single light ray, compactify it on S^1, and assigning a von Neumann algebra for each subset (interval) $I \subset S^1$.

But this "conformal embedding" appears to be physically unrelated to "gauging non-invertible symmetry", other than sharing the same mathematical structure.

So they will not play a role in our talk today.

K.T. point

Kosterlitz-Thouless CFT is the CFT that describes the Berezinskii-Kosterlitz-Thouless transition in 2D XY model.

I learned a bit of this from this slide
`https://oshikawa.issp.u-tokyo.ac.jp/Slides/BKT-SCGP-May202`
by prof. Oshikawa.

It can be seen in the XY model with Lagrangian

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \tag{12}$$

And it is also supposed to be found in the $S = 1/2$ XXZ chain:

$$\mathcal{H} = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^Z) \tag{13}$$

Numerical Works

See numerical work by [Huang-Lin-Ohmori-Tachikawa-Tezuya '21] (I heard these from slides of Yuji about this paper), [Vanhove-Lootens-Van Damme-Wolf-Osborne-Haegeman-Verstraete '21],

both conjectured the IR limit of an "anyon chain" gives a CFT with Haagerup symmetry.

An estimate of central charge $c \sim 2$ was obtained in both works.

After learning about these works, we appreciated the power of subfactor theory, and we decided to learn more about them.

von Neumann Algebra

They are C^* algebra, whose elements are bounded operators acting on a Hilbert $\mathcal{H}$. $*$ refers to a conjugation.

Given a Hilbert space $\mathcal{H}$ and the space of bounded operators on it $B(\mathcal{H})$, we can define the *commutant* of $A \subset B(H)$ to be $A' = \{T \in B(\mathcal{H}) | TS = ST, \forall S \in B(\mathcal{H})\}$.

Then, von Neumann algebra needs to satisfy $A = A''$ certain "completeness" condition w.r.t. double commutant. Textbook [Jones and Sander '97] (see also [Sorce '23])

A *factor* is a von Neumann algebra A that $A \cap A'$ only contains identity and its multiplets. They play a similar role as simple Lie algebras.

They are math objects that originate from physics of QM, so it is natural for physicists today to keep an eye on them. There are many conceptual discussions on von Neumann algebras in Holography.

How did subfactor theory became useful in non-invertible symmetries?

I first heard of subfactor theory from the fact that, there is a fusion category (called the Haagerup fusion category) that is first constructed using subfactor theory.

It has fusion algebra of

$$a^3 = 1, \quad aX = Xa = X, \quad X^2 = X + 1 + a + a^2. \tag{14}$$

At the same time, there are no construction of a CFT that admit Haagerup category as its categorical symmetries.

There were efforts of numerically constructing the Haagerup CFT.

[Huang-Lin-Ohmori-Tachikawa-Tezuya '21][Vanhove-Lootens-Van Damme-Wolf-Osborne-Haegeman-Verstraete '21]

Lesson for us: subfactor is a fruitful framework that produces physically interesting examples. There are probably more to be found.

Principal graph vs fusing ring and its representation

As we have seen, principal graphs keeps track of the fusion **ring** and part of its representation data.

However, given a principal graph, there could be 0, 1, or several underlying subfactors. So principal graph does not specify everything [**?**]ven mathematicians struggle in extracting explicit F-symbols from subfactors.

Nonetheless, the non-existence of the principal graph for a fusion rule would exclude it.

The glued principal graph builds in the condition that the $M - N$ bimodules and $N - M$ bimodules matches one-to-one.

Principal Graph vs Underlying Fusion Category

In fact, the first paper [Haagerup '93], only proved the existence of the principal graph of the Haagerup subfactor by analyzing its principal graph.

But he did not construct it explicitly.

Only a subsequenst paper by [Asaeda-Haagerup '97], the Haagerup subfactor is constructed (together with some generalizations by replacing $\mathbb{Z}_3$ with larger cyclic groups).

In [Grossman-Snyder '11], another fusion category with Haagerup fusing ring was identified, again using a lot of principal diagrams.

Still, they did not explicitly compute F-symbols, which was not done until [Osborn-Stiegemann-Wolf '19].