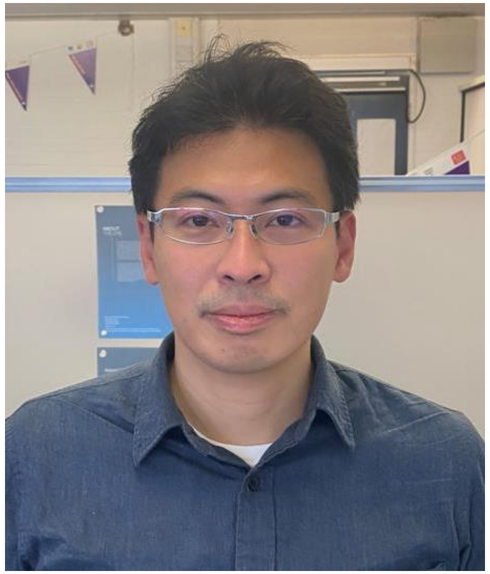
# Anomaly AND Generalized statistics on lattices

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Generalized Symmetries in HEP and CMP

July 28 , 2025

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arXiv:2412.01886
arXiv:2412.07653 (Hanyu single author)

# Statistics of particles and extended excitations

Statistics of quasiparticles (anyons): crucial properties of phases of matter; topological order, spin liquids

[Levin-Gu, Wen, Wang-Senthil,…]

Nontrivial statistics often implies nontrivial low-energy spectrum, as only bosons can condense.

Typically associated with ‘t Hooft anomalies of higher-form symmetries.

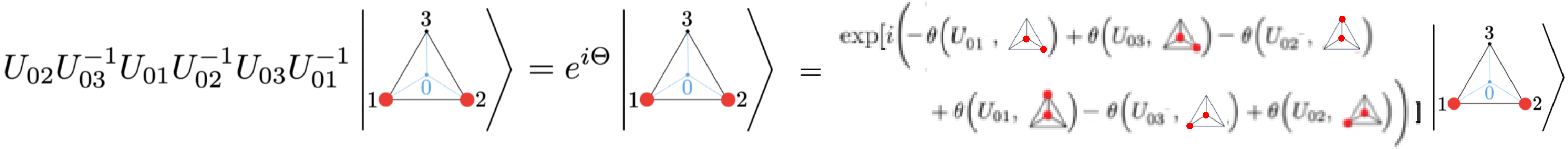
Dynamical constraints e.g. forbids confined phases

Anyons can be non-invertible, but in this talk we are mostly interested in invertible excitations and symmetries.

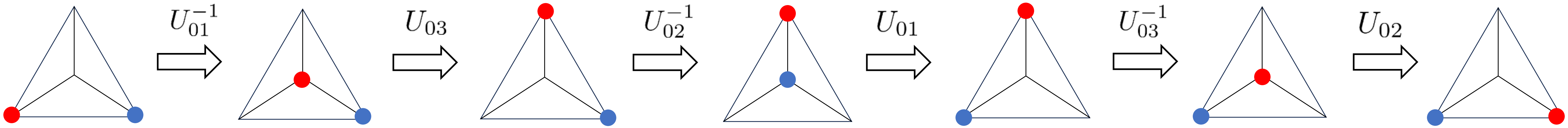
# Microscopic definition of statistics

Gapped local Hamiltonian system: How to define statistics of quasiparticles in microscopic lattice models?

T-junction: [Levin-Wen]




This process indeed does half-braiding of two identical particles:

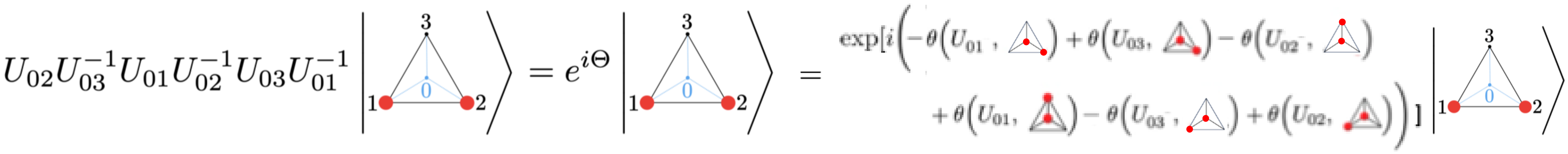



To be qualified as invariants, we further need to check the stability against perturbations.

Microscopic definition of statistics

Gapped local Hamiltonian system:  How to define statistics of quasiparticles in microscopic lattice models?

T-junction:    [Levin=Wen]




✓ Invariant under redefinitions of unitary by phases

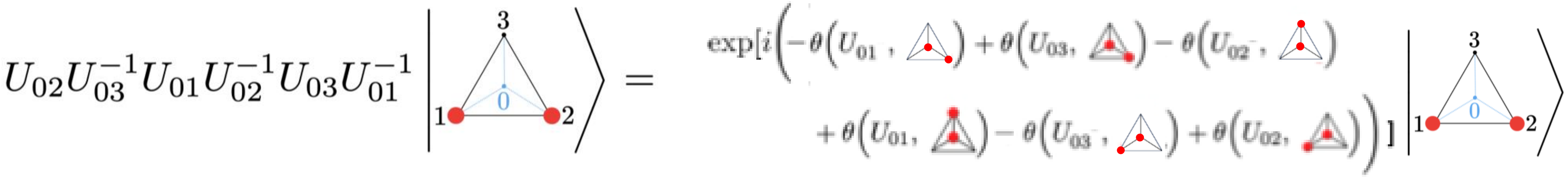
✓ Invariant under perturbations nearby the ends of unitaries

✓ Invariant under choices of initial excitation configurations

Question: Spins of Abelian anyons should be quantized.   Is this T junction a quantized invariant?

# Quantization of T-junction

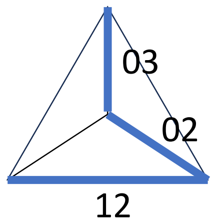
T junction is a quantized invariant.  Let's see this explicitly for Abelian anyons with Z2 fusion rule.

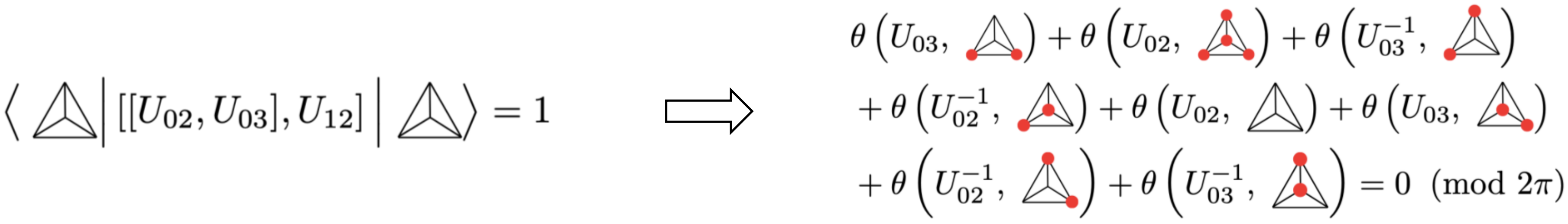


Key observation is that the triple commutator of operators with no common overlap must vanish:

(follows from assumption that unitaries are finite depth local circuit.)

For instance,



# Quantization of T-junction

( 4 x T junction ) for $\mathbb{Z}_2$ Abelian anyons is the combination of triple commutators:

$\exp[4i\Big(\theta(U_{01}^{-1},$  $) + \theta(U_{03},$  $) + \theta(U_{02}^{-1},$  $)$

$+\theta(U_{01},$  $) + \theta(U_{03}^{-1},$  $) + \theta(U_{02},$  $)\Big)]$

$$= \langle[[U_{02}, U_{03}], U_{12}]\rangle \times \langle[[U_{01}, U_{02}], U_{13}]\rangle \times \langle[[U_{03}, U_{01}], U_{23}]\rangle$$
$$\times \langle[[U_{02}^{-1}, U_{03}^{-1}], U_{12}]\rangle \times \langle[[U_{01}^{-1}, U_{02}^{-1}], U_{13}]\rangle \times \langle[[U_{03}^{-1}, U_{01}^{-1}], U_{23}]\rangle$$
$$\times \langle[[U_{03}, U_{02}], U_{23}]\rangle^2 \times \langle[[U_{02}, U_{01}], U_{12}]\rangle^2 \times \langle[[U_{01}, U_{03}], U_{13}]\rangle^2$$
$$= 1$$

This shows that the spin of $\mathbb{Z}_2$ Abelian anyons through T-junction must be quantized as 0, 1/4, 1/2, 3/4 ($\mathbb{Z}_4$ statistics).

$\mathbb{Z}_N$ Abelian anyons $\implies$ $\mathbb{Z}_{\gcd(2,N)\times N}$ statistics

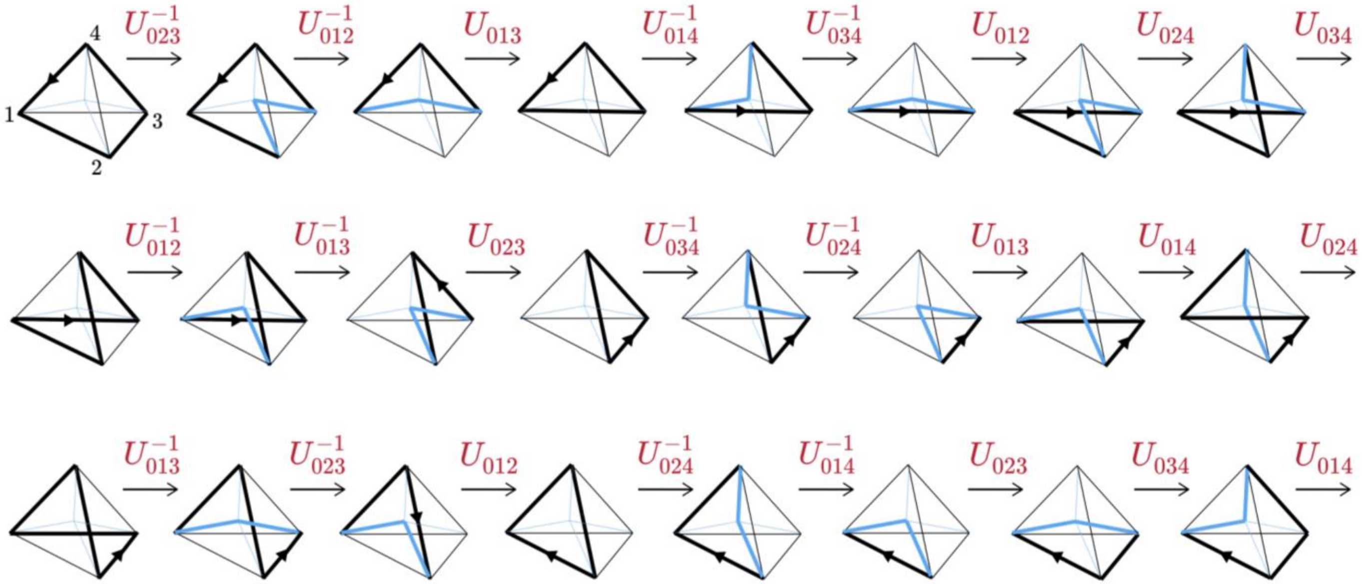
We will see that such mechanism for quantization is observed in a very general setup.

# Generalized statistics: loop excitation in (3+1)D

Such invariants can be defined in generic space dimensions, with generic invertible extended excitations.

Example: Z2 1-form symmetry in (3+1)D. 24 step unitaries:

$$\begin{aligned}\mu_{24} := \ & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}\end{aligned}$$




"Fermionic loops"

Ryan Thorngren (2014):
$w_3$-obstruction

Theo Johnson-Freyd (2021):
Klein bottle invariant of fusion 2-categories

Chen-Hsin (2023):
Lattice $w_2 w_3$ (4+1)D TQFT

Fidkowski-Haah-Hastings (2023):
Loop-flipping process

We will give the framework for such invariants with generic dimensionality, and discuss physical consequences.

# Generalized statistics: particle fusion in (1+1)D

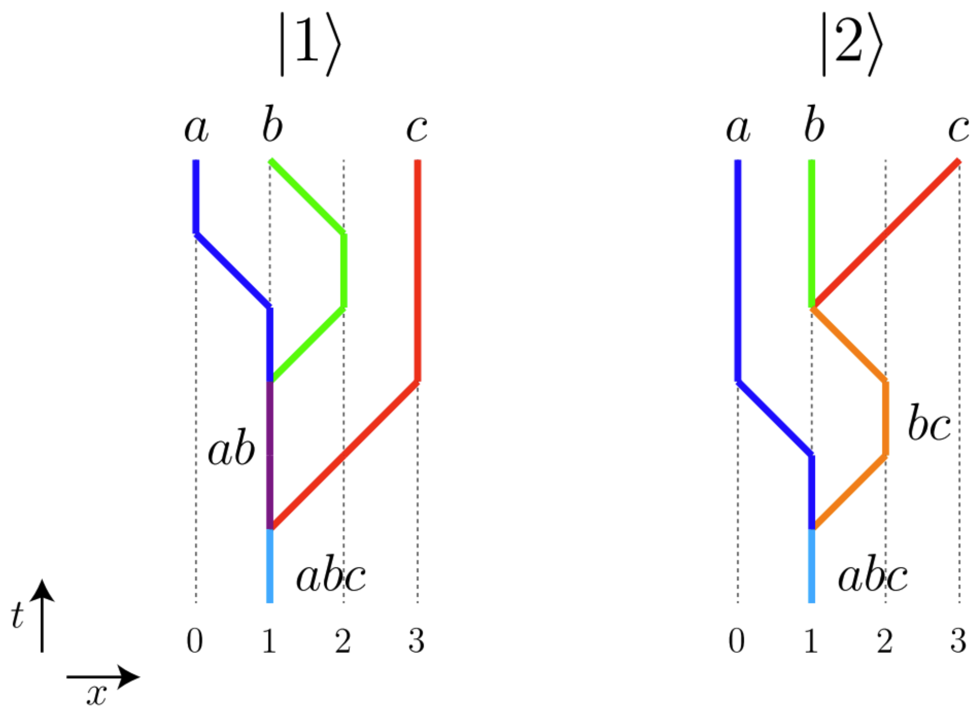



FIG. 4. The two processes that are compared in the microscopic definition of the $F$-symbol.

Kawagoe-Levin (2020)

For $\mathbb{Z}_2$ particle $s$, the "statistics" is equal to:

$$Z_3 = F(s,s,s)F(s,1,s).$$

We can further simplify this expression into hopping operators:

$$Z_3 = [U_{01}^2, U_{12}]$$

where $[A,B] \equiv ABA^{-1}B^{-1}$ and $U_{ij}$ is the hopping operator that moves particle from $i$ to $j$.

Therefore, in (1+1)D, bosons and fermions have trivial statistics, while semions exhibit nontrivial statistics. This is why higher gauging of fermions is possible within a (1+1)D subspace.

# Framework for Generalized statistics

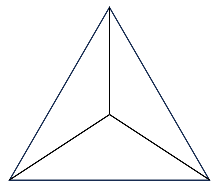
Setup:
- Local gapped lattice system, with tensor product Hilbert space
- Finite invertible p-form symmetry with fusion group G, generated by a finite depth unitary circuit
  (G can be non-abelian when p=0)

End of symmetry operators correspond to the extended excitations.

Input:
- Possible configurations of excitations $\mathcal{A}$ (on a simplicial complex embedded in space): finite group
- Set of patch symmetry operators $\mathcal{S}$ : symmetry generators creating excitation configurations

Example... T junction

- $\mathcal{A}$ : G (=ZN) anyon configurations on a triangulation of 2d sphere:



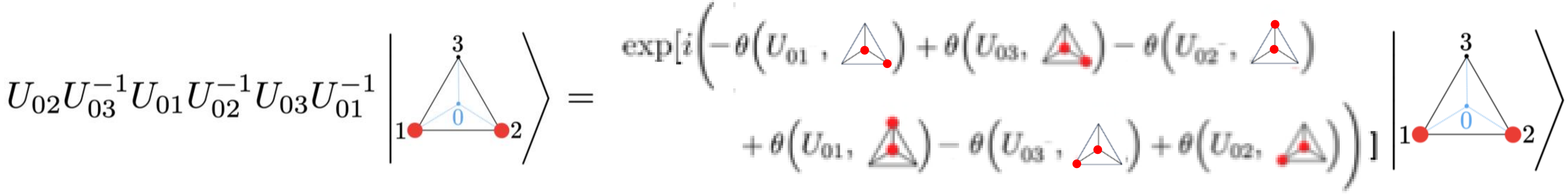
$\mathcal{A} = G^3$

(anyons on four vertices fuse to vacuum)

- $\mathcal{S}$ : set of anyon string operators on edges. Six generators of $G^6$ (# of edges) $\partial : \mathcal{S} \to \mathcal{A}$

# Framework for Generalized statistics

Invariant is a sequence of unitaries acting on a state, getting back to the original one




In general, it is sum of the phases $\theta(s,a)$ $\quad s \in \mathcal{S}, a \in \mathcal{A}$

$$U(s)\,|a\rangle = \exp(i\theta(s,a))\,|a+\partial s\rangle$$

It is convenient to introduce a formal sum of the objects $\quad E = \bigoplus_{s\in\mathcal{S},a\in\mathcal{A}} \mathbb{Z}\theta(s,a)$

The invariant is formulated as a specific subgroup $\quad E_{\text{inv}} \subset E$

( Let us restrict ourselves to the Abelian fusion group G in this talk. Can be safely generalized to non-Abelian groups. )

## Group of invariants: $E_{\mathrm{inv}} \subset E$

The condition for being an invariant: Linear constraints on integer coefficients $\epsilon(s,a)$ of

$$e = \bigoplus_{(s,a)} \epsilon(s,a)\theta(s,a) \in E = \bigoplus_{s\in\mathcal{S},a\in\mathcal{A}} \mathbb{Z}\theta(s,a)$$

1. The invariant corresponds to sequence of unitaries, with same initial and final state (Berry phase).

$$\sum_{s\in\mathcal{S}} \epsilon(s,a) - \sum_{s\in\mathcal{S}} \epsilon(s,a-\partial s) = 0, \quad \text{for any } a \in \mathcal{A} .$$

2. The invariant has to be stable against phase redefinitions of the unitary operators.

$$\sum_{a\in\mathcal{A}} \epsilon(s,a) = 0, \quad \text{for any } s \in \mathcal{S} .$$

3. The invariant has to be stable against perturbations nearby the boundaries of unitary operators.

$$\sum_{\substack{a\in\mathcal{A} \\ a|_{\sigma_j}=a_*^{(j)}}} \epsilon(s,a) = 0 , \qquad \sigma_j \in \mathrm{supp}(s)$$

( Stability against perturbations within a j-simplex $\sigma_j$ )

( uses exponentially decaying correlation length = gapped )

The three types of linear constraints together define $E_{\mathrm{inv}} \subset E$

# Trivial invariants from locality: $E_{\mathrm{id}} \subset E_{\mathrm{inv}}$

Some invariants $e \in E_{\mathrm{inv}}$ correspond to the trivial invariants (identity).

Trivial invariants originate from higher commutator:

$$\langle a| \, [[[U(s_1), U(s_2)], \cdots], U(s_n)] \, |a\rangle = 1 \qquad \mathrm{supp}(s_1) \cap \cdots \cap \mathrm{supp}(s_n) = \varnothing$$

Let $E_{\mathrm{id}} \subset E_{\mathrm{inv}}$ be the group of higher commutators. Then define generalized statistics as

$$T = E_{\mathrm{inv}} / E_{\mathrm{id}}$$

Though $E_{\mathrm{inv}}$ is an infinite group (direct sum of integers), the genuine invariant T is a finite Abelian group.

Invariants are torsions, and quantized.

# Quantization of Generalized statistics

Let's explicitly show that the invariant $T = E_{\text{inv}}/E_{\text{id}}$ is a finite group (torsion).

First, one can show that the equivalence class $[e] \in E_{\text{inv}}/E_{\text{id}}$ doesn't depend on initial state, i.e., the ratio

$$\frac{\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle}{\langle a_0' | \prod U(s_j)^{\pm} | a_0' \rangle} \in E_{\text{id}}$$

for any pair of initial states.

In other words, it is equal to product of higher commutators, and actually $\frac{\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle}{\langle a_0' | \prod U(s_j)^{\pm} | a_0' \rangle}$ = 1

Then, sum up the phase over all choices of initial states:

$$|\mathcal{A}|[e] = \sum_{a_0 \in \mathcal{A}} \sum_{(s,a)} \epsilon(s,a)\theta(s, a + a_0)$$

$$= \sum_{a_0 \in \mathcal{A}} \sum_{(s,a)} \epsilon(s, a - a_0)\theta(s,a) = \sum_{(s,a)} \left( \sum_{a_0 \in \mathcal{A}} \epsilon(s, a_0) \right) \theta(s,a) = 0$$

$\Rightarrow$

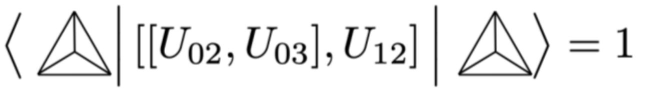
[e] has finite order,

Showing T is a finite group

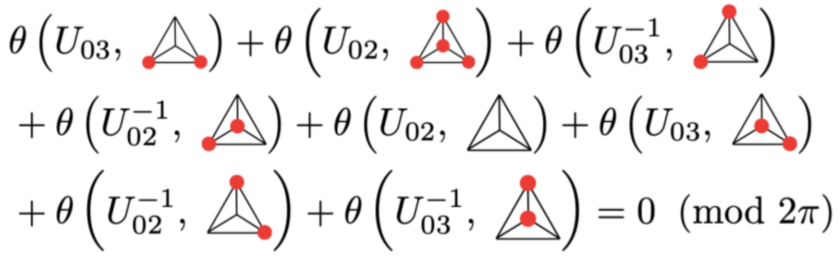
# Computation of invariants: Smith normal form

One can systematically compute $T = E_{\text{inv}}/E_{\text{id}}$ based on a simple algorithm.

The idea is to first list all possible higher commutators:




$\Longrightarrow$




and many other <[U,[U,U]]> =1 type equations

and many equations $\sum_{s,a} \epsilon(s,a)\theta(s,a) = 0$

Then, some linear combinations of higher commutators happen to have overall integer N factor:

<[U,[U,U]]> x <[U',[U',U']]> x ... $\Longrightarrow$ $N \sum_{s,a} \epsilon'(s,a)\theta(s,a) = 0$

This implies the existence of invariant quantized in $\mathbb{Z}_N$: $\sum_{s,a} \epsilon'(s,a)\theta(s,a) \in E_{\text{inv}}$

# Computation of invariants: Smith normal form

The combination of higher commutators with overall integer factor can be obtained by Smith normal form

Let's say we have higher commutators

$$\theta_1 + 2\theta_2 + 3\theta_3 = 0 \pmod{2\pi} ,$$
$$4\theta_1 + 5\theta_2 + 6\theta_3 = 0 \pmod{2\pi} ,$$
$$7\theta_1 + 8\theta_2 + 9\theta_3 = 0 \pmod{2\pi}$$

Then make an integer matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad (\text{columns } \theta_1, \theta_2, \theta_3)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{columns } \theta'_1, \theta'_2, \theta'_3)$$

Linear combination of eqs (row)

Redefiniton of theta by linear combinations (column)

SNF

$$\theta'_1 = \theta_1 + 2\theta_2 + 3\theta_3 ,$$
$$\theta'_2 = \theta_2 + 2\theta_3 ,$$
$$\theta'_3 = \theta_3 .$$

$1\theta'_1 = 0, \quad 3\theta'_2 = 0 , \quad 0\theta'_3 = 0$ correspond to quantized invariants (single nontrivial one is Z3)

# Computation of invariants: Smith normal form

Summarizing, the algorithm for computing the statistics is as follows:

1. First fix the simplicial complex and fusion group G, the configurations of excitations $\mathcal{A}$, and unitaries $\mathcal{S}$

2. Enumerate all possible higher commutators of unitaries which evaluates trivially

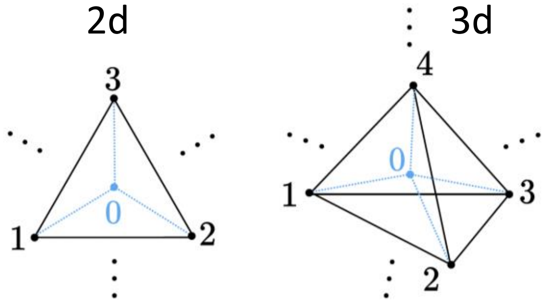
$$\begin{aligned} \theta_1 + 2\theta_2 + 3\theta_3 &= 0 \pmod{2\pi} , \\ 4\theta_1 + 5\theta_2 + 6\theta_3 &= 0 \pmod{2\pi} , \\ 7\theta_1 + 8\theta_2 + 9\theta_3 &= 0 \pmod{2\pi} \end{aligned}$$

3. Put the higher commutators into a matrix, and compute its Smith normal form

4. Invariants are classified by the entries of Smith normal form:

$$T = E_{\text{inv}}/E_{\text{id}} = \bigoplus_{a_{ii} \neq 0,1} \mathbb{Z}_{a_{ii}}$$

# Conjecture: Generalized Statistics = Group Cohomology




Take a triangulation on a sphere embedded in d dimensional space.

p-dimensional excitation ((d-p-1)-form symmetry) with fusion group G.

Then, computation results imply the correspondence with the group cohomology: $T = H^{d+2}(B^{d-p}G, U(1))$

| | $G$-particles with $G=\prod_i \mathbb{Z}_{N_i}$ | $G$-loops with $G=\prod_i \mathbb{Z}_{N_i}$ | $G$-membranes with $G=\prod_i \mathbb{Z}_{N_i}$ |
|---|---|---|---|
| (1+1)D | $H^3(BG,U(1)) = \prod_i \mathbb{Z}_{N_i} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}_{(N_i,N_j,N_k)}$ | | |
| (2+1)D | $H^4(B^2G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)\times N_i} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)}$ | $H^4(BG,U(1)) = \prod_{i<j} \mathbb{Z}^2_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}^2_{(N_i,N_j,N_k)} \prod_{i<j<k<l} \mathbb{Z}_{(N_i,N_j,N_k,N_l)}$ | |
| (3+1)D | $H^5(B^3G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)}$ | $H^5(B^2G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)}$ | $H^5(BG,U(1)) = \prod_i \mathbb{Z}_{N_i} \prod_{i<j} \mathbb{Z}^2_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}^4_{(N_i,N_j,N_k)} \prod_{i<j<k<l} \mathbb{Z}^3_{(N_i,N_j,N_k,N_l)} \prod_{i<j<k<l<m} \mathbb{Z}_{(N_i,N_j,N_k,N_l,N_m)}$ |

⇒ Verified for small groups G.

For instance, $d=2, p=0$, $G = \mathbb{Z}_N$ (anyons):

$T = \mathbb{Z}_{2N}$ even $N$

$T = \mathbb{Z}_N$ odd $N$

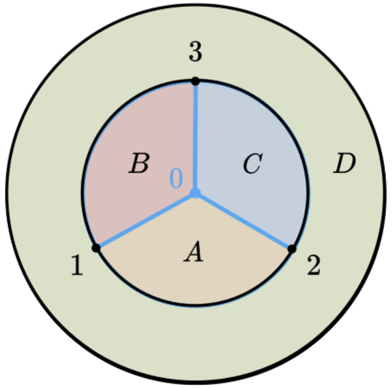
Spin quantization rule of anyons;
Checked up to N = 10 on laptop.

# Examples of invariants

- 1+1D: 0-form ZN symmetry $Z_3(g) := [U(g)_{01}^{|g|}, U(g)_{02}]$

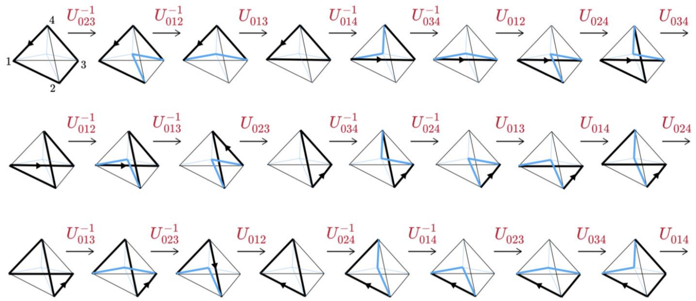

- 2+1D: 0-form ZN x ZN symmetry

$$Z_4^I(a,b) := (U(a)_{B+C})^{-N}\Big(U(a)_{B+C}\,[U(a)_B,[U(a)_A,U(b)_{A+B+C+D}]]\Big)^N,$$
$$Z_4^{II}(a,b) := (U(b)_{B+C})^{-N}\Big(U(b)_{B+C}\,[U(b)_B,[U(b)_A,U(a)_{A+B+C+D}]]\Big)^N.$$




- 3+1D: 1-form ZN symmetry

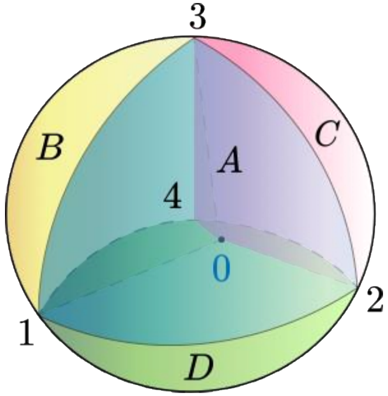
$$\mu_{24} := U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1}$$
$$\times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1}$$
$$\times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}$$



"Fermionic loops" for N = 2

0-form ZN symmetry

$$Z_5(g) := (U(g)_{0234}U(g)_{0124})^{-N}\,\Big(U(g)_{0234}[U(g)_{0134},U(g)_{0123}^N]^{-1}U(g)_{0124}[U(g)_{0134},U(g)_{0123}^N]\Big)^N$$



# Generalized statistics as anomalies: obstruction to gauging

The nontrivial invariant is directly regarded as obstruction to gauging the symmetry.

A take is that the product of unitaries $\langle a_0| U(s_{n-1})^{\pm} \dots U(s_j)^{\pm} \dots U(s_0)^{\pm} |a_0\rangle$ is the product of Gauss law operators.

$$G(\Delta) = 1, \quad U(s) = \prod_{\Delta \in s} G(\Delta)$$

Gauss law operator on local simplex Δ, and the unitary is product of Gauss laws

It means that the invariant obstructs commuting Gauss laws within the initial symmetric state.

⇨ Obstruction to gauging the symmetry = Microscopic definition of ‘t Hooft anomalies

# Generalized statistics as anomalies: dynamical consequences

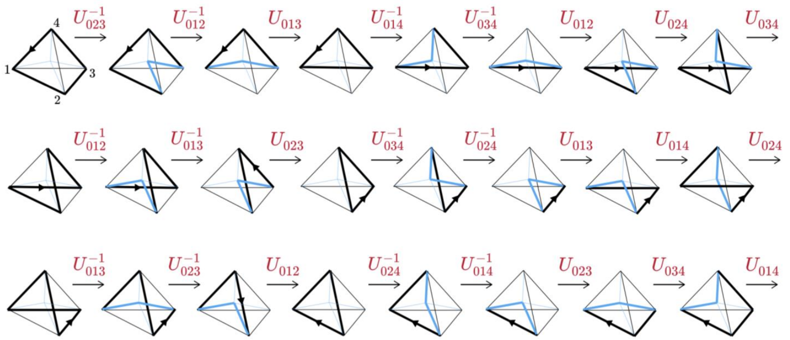
Generalized statistics is understood as the ’t Hooft anomaly.

Indeed, generalized statistics has a direct dynamical consequence (similar to Lieb-Schultz-Mattis):

Generalized statistics $T \neq 1$ on the symmetric state $|\psi\rangle$ implies that the state cannot be short-range entangled.

(i.e., cannot be connected to tensor product state by finite depth circuit)

For instance, Z2 1-form symmetry in (3+1)D:




$$
\begin{aligned}
\mu_{24} := \; & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\
& \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\
& \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}
\end{aligned}
= -1 \quad \Longrightarrow
$$

LRE state

Such result has been known for anyons in (2+1)D: T-junction must be trivial on SRE states [Bravyi-Hastings-Verstraete, Li-Lee-Yoshida]

## Warm-up (review): Anyons imply long-range entanglement

Let us see how the anyon T-junction forbids the short-range entanglement.

Suppose $|\psi\rangle$ is SRE state in 2d. i.e., $V|\psi\rangle = |0\rangle^n$ with a finite depth circuit V.

$$\langle\psi| U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1} |\psi\rangle = \langle 0|^n \tilde{U}_{02}\tilde{U}_{03}^{-1}\tilde{U}_{01}\tilde{U}_{02}^{-1}\tilde{U}_{03}\tilde{U}_{01}^{-1} |0\rangle^n \qquad \tilde{U} = V^\dagger U V$$

In this setup, each excited state $\tilde{U}|0\rangle^n$ is trivial product state away from the excitation:

$$|jk\rangle := \tilde{U}_{jk}|0\rangle^n = |j\rangle|k\rangle \otimes |0\rangle_{\overline{j,k}}$$

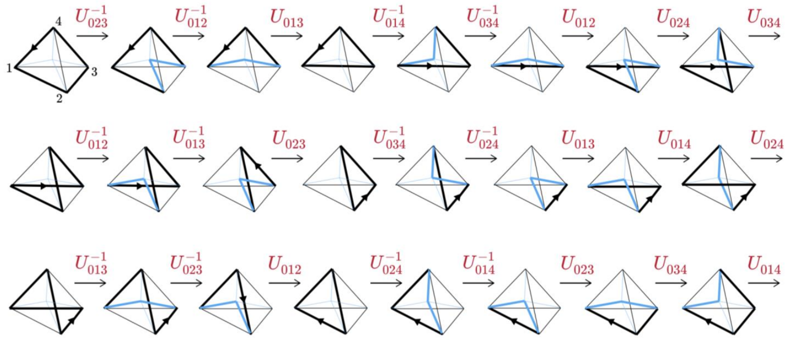
So excitation is just a disentangled 0d state. This greatly constrains the property of Berry phase.

The action of $\tilde{U}_{kl}$ is independent of the excitations away from k, l. This leads to cancellation of T-junction:

$$\begin{aligned}\Theta &= \theta\left(U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}, 12\right) \\ &= -\theta(U_{01},02) + \theta(U_{03},02) - \theta(U_{02},03) \\ &\quad + \theta(U_{01},03) - \theta(U_{03},01) + \theta(U_{02},01) = 0\ .\end{aligned}$$

# Fermionic loops imply long-range entanglement

Such argument can be generalized to extended excitations as well. Let's consider Z2 1-form symmetry in (3+1)D




$$\begin{aligned}\mu_{24} := \; & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}\end{aligned} = -1$$

$\Longrightarrow$ LRE state

Let's consider 3d SRE state $|\psi\rangle$ w/ Z2 1-form symmetry.

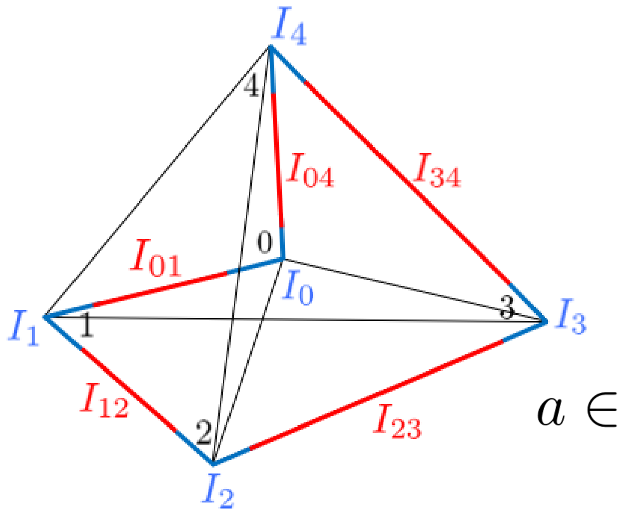
Then, each state $U\,|\psi\rangle$ can be taken to be a trivial product state away from excitations:

$$|\partial s\rangle := \quad U(s)\,|\psi\rangle = |a\rangle_{\partial s} \otimes |0\rangle_{\overline{\partial s}}$$

(up to finite depth circuit)

Then, each excited state is essentially a 1d gapped state, which can be described by matrix product state (MPS).

# Fermionic loops imply long-range entanglement




Each excited state in SRE is the 1d MPS state along excitations.

Let's consider a "patchwork" of MPS:

For instance, $|a\rangle = \mathrm{Tr}\left[V^0 E^{01} V^1 E^{12} V^2 E^{23} V^3 E^{34} V^4 E^{40}\right]$

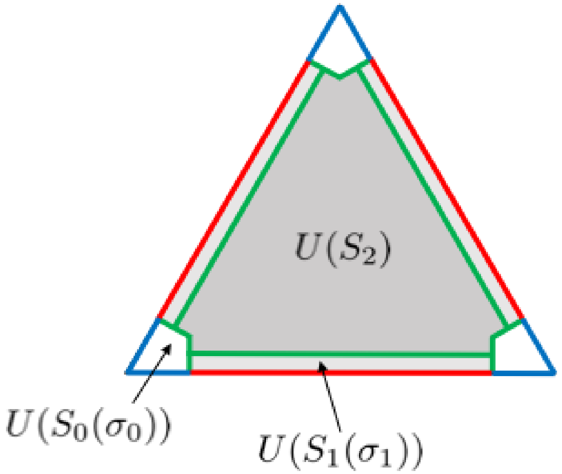
MPS V only depends on excitation configuration near a vertex, and E only depends on those near an edge.

This patchwork representation allows us to construct a canonical choice of excited state $|a\rangle$ for generic configuration.

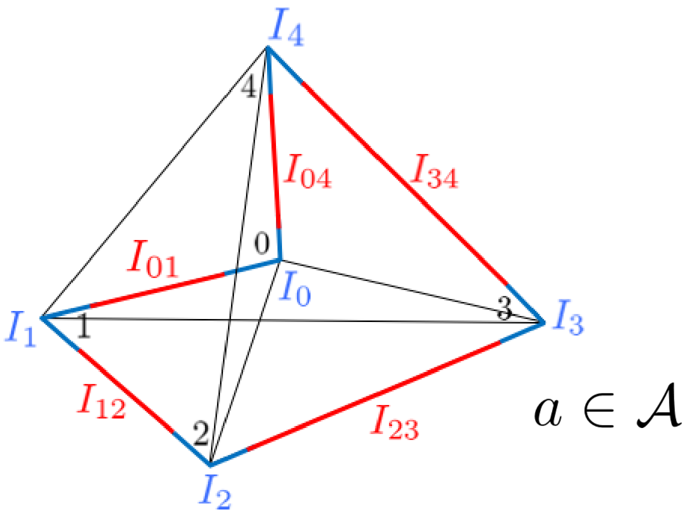
This specific structure of an excited state again greatly constrains the Berry phase $U(s)\,|a\rangle = \exp(i\theta(s,a))\,|a + \partial s\rangle$

# Fermionic loops imply long-range entanglement

The symmetry operator also decomposes into circuits near vertex, edge, bulk.




$$U_{jkl} = U_j^{(0)} U_k^{(0)} U_l^{(0)} U_{jk}^{(1)} U_{kl}^{(1)} U_{jl}^{(1)} U_{jkl}^{(2)}$$




Berry phase decomposes into smaller part, and each phase only depends on MPS on specific j-simplex:

$$\theta(U_{jkl}, a) = \theta(U_{j;jkl}^{(0)}, a) + \theta(U_{k;jkl}^{(0)}, a) + \theta(U_{l;jkl}^{(0)}, a) + \theta(U_{jk}^{(1)}, a) + \theta(U_{kl}^{(1)}, a) + \theta(U_{jl}^{(1)}, a) + \theta(U_{jkl}^{(2)}, a)$$

Then, invariance under local perturbations at j-simplex enforces the Berry phase on each j-simplex to cancel out.

One can then show $e \in E_{\text{inv}}$ has trivial invariant on SRE.

# Generalized statistics imply long-range entanglement

Such argument can be extended to generic setup: Need to assume tensor network representation of excited states.

Let's consider SRE state $|\psi\rangle$ w/ G p-form symmetry in generic dimensions.

Then, each state $U|\psi\rangle$ can be taken to be a trivial product state away from excitations:

$$|\partial s\rangle := \quad U(s)|\psi\rangle = |a\rangle_{\partial s} \otimes |0\rangle_{\overline{\partial s}}$$

(up to finite depth circuit)

Tensor network at the excitations

Then decompose the tensor network and operators into the ones localized nearby j-simplices.

We can use the conditions of $E_{\text{inv}}$ for the stability against perturbations at j-simplex, leads to cancellation of phases.

One can then show $e \in E_{\text{inv}}$ has trivial invariant on SRE.

## Summary

- Universal microscopic descriptions for statistics of invertible deconfined excitations

- Generalized statistics is quantized, and systematically computed using Smith normal form

- Generalized statistics gives microscopic definition of anomalies, and constrains low-energy spectrum

## Future directions

- Gapless systems? If the perturbation is always symmetric, the definition should also work for gapless systems.

- Non-invertible symmetries / non-Abelian anyons? Is there analogue of higher commutators of unitaries?

- Proof for the correspondence between generalized statistics and group cohomology?