Constructing Bulk Topological Orders from Generalized Ising Models

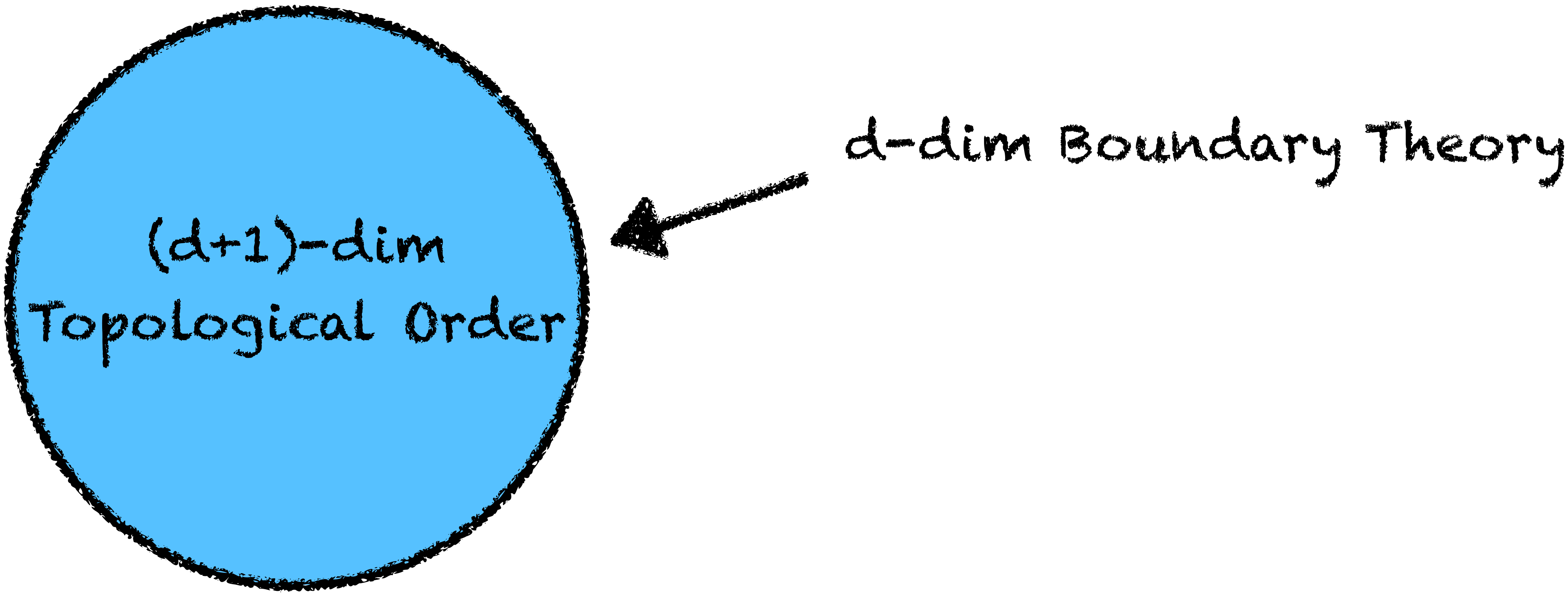
Shang Liu (IOP@CAS)

Workshop on Generalized Symmetries, 2025/07/28

SL, Wenjie Ji, SciPost Phys (2023).

物理研究所

Bulk-Boundary Correspondence of Topological Order



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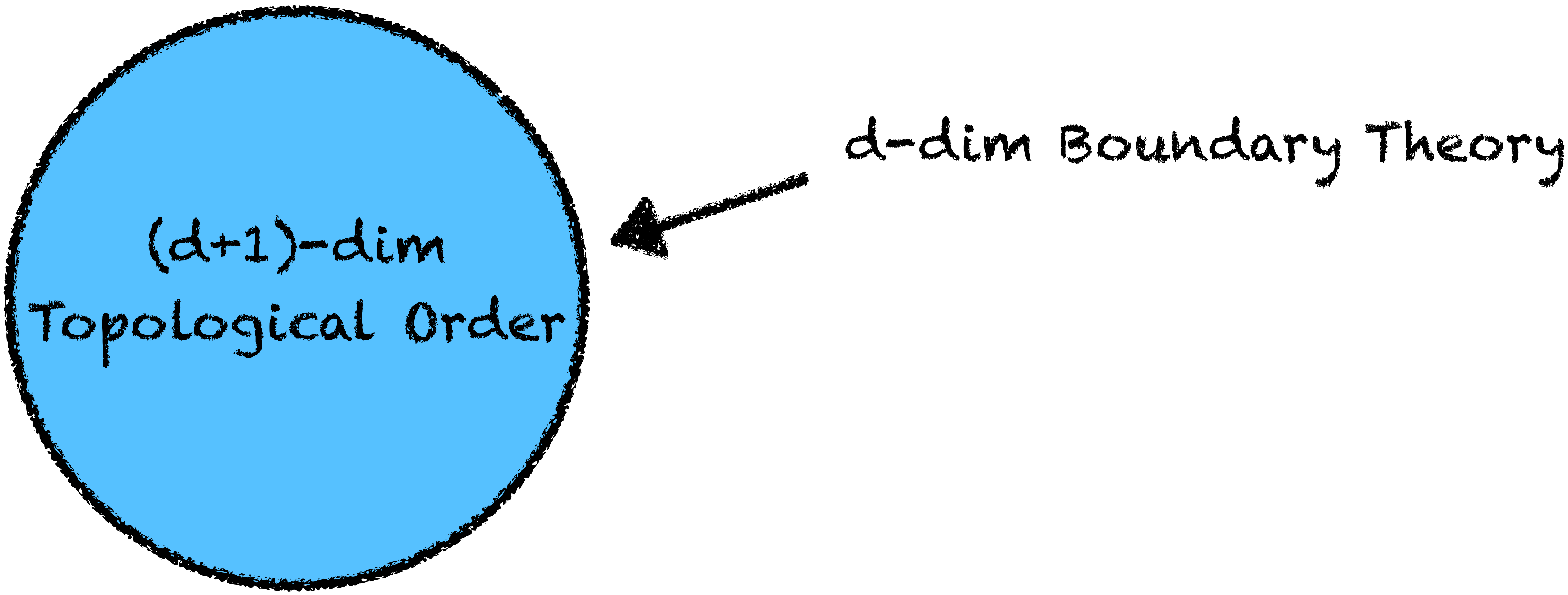

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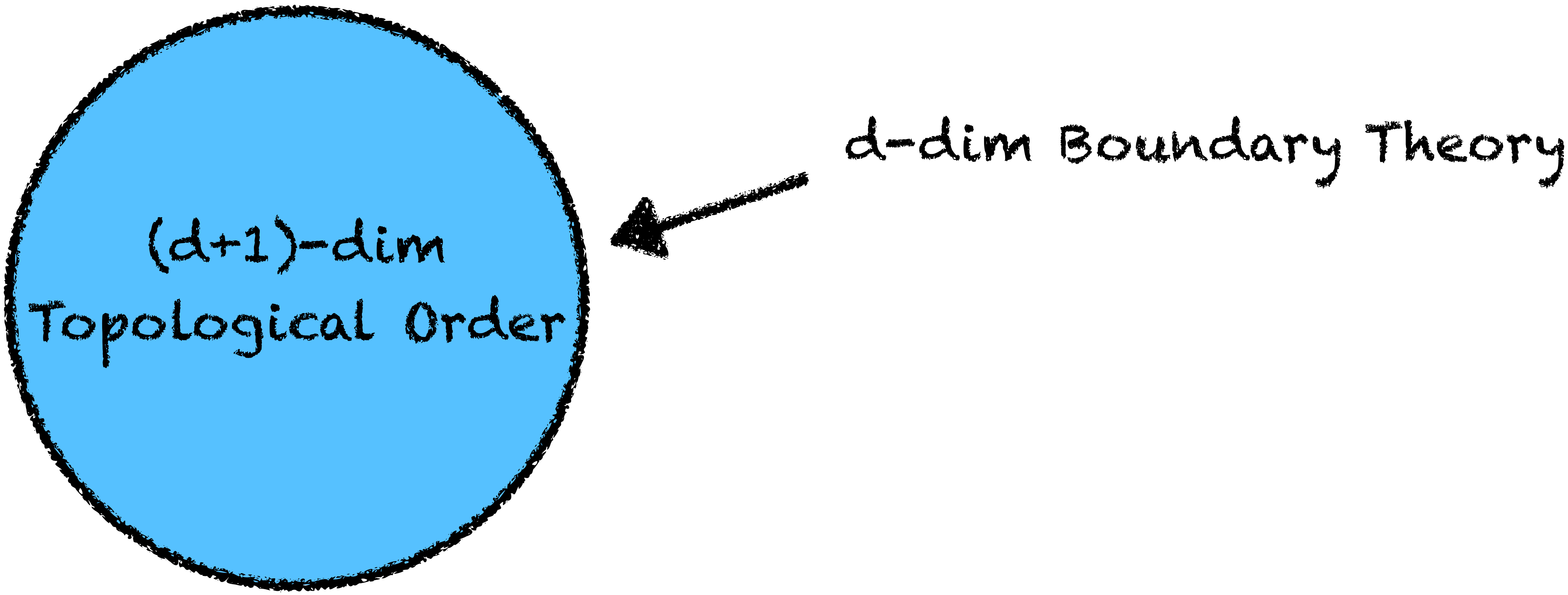

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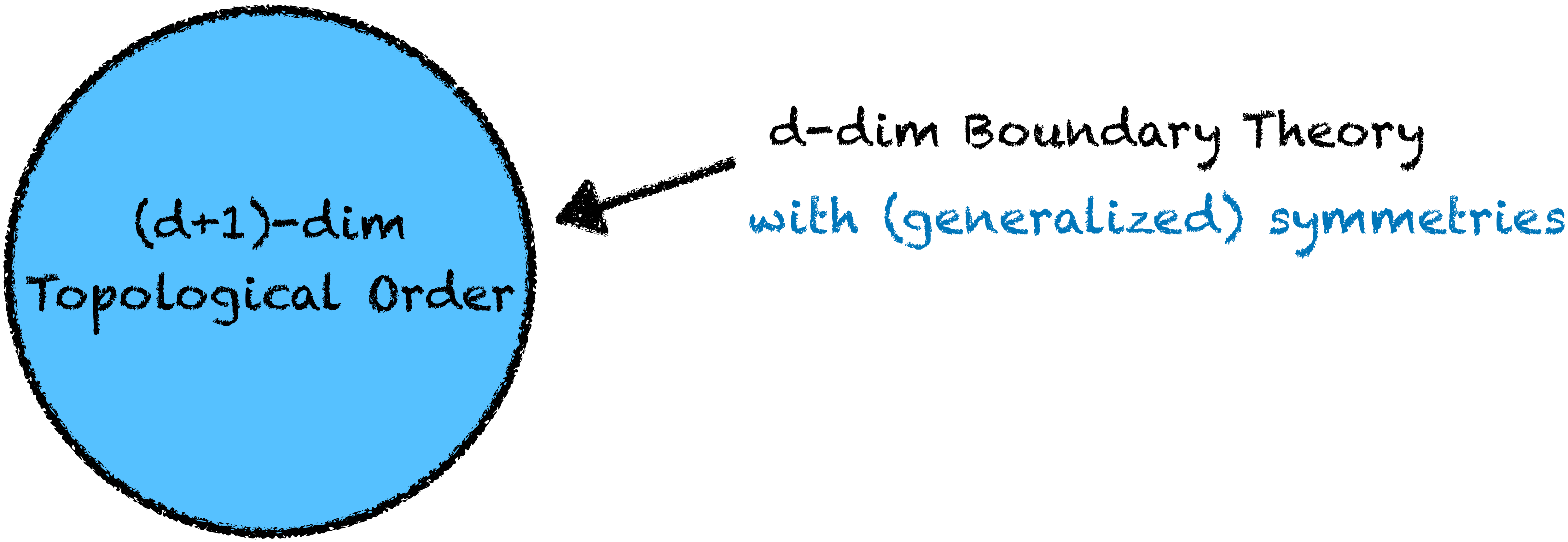

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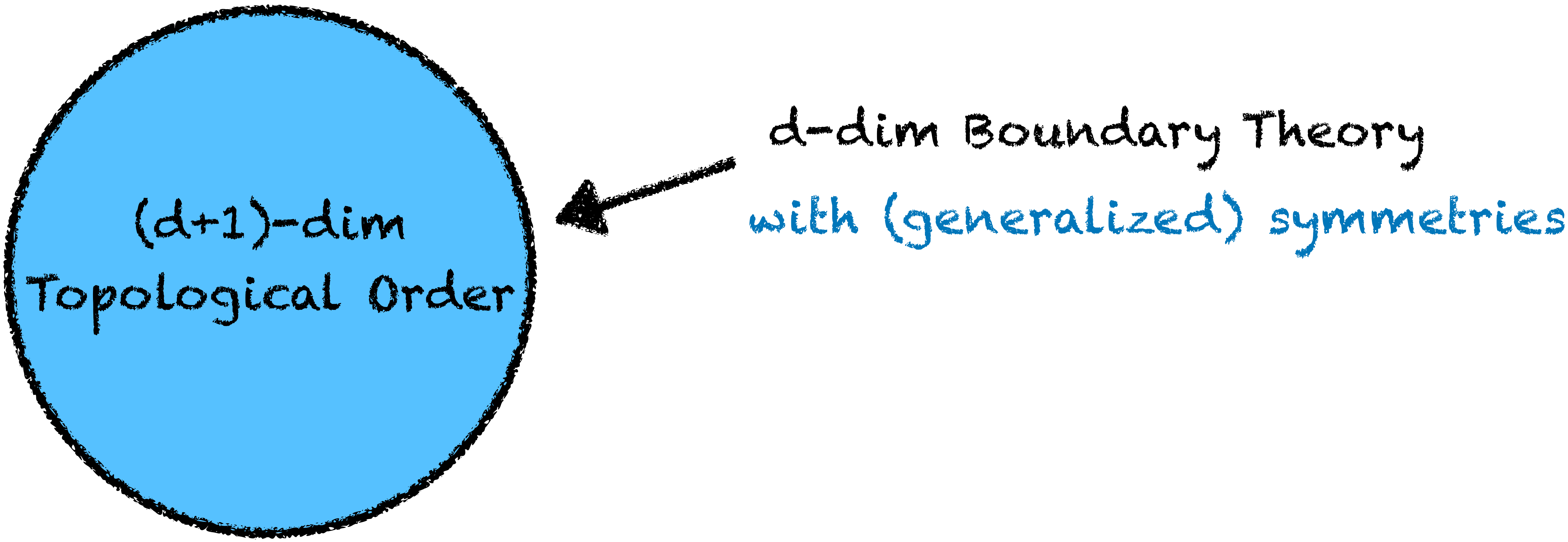

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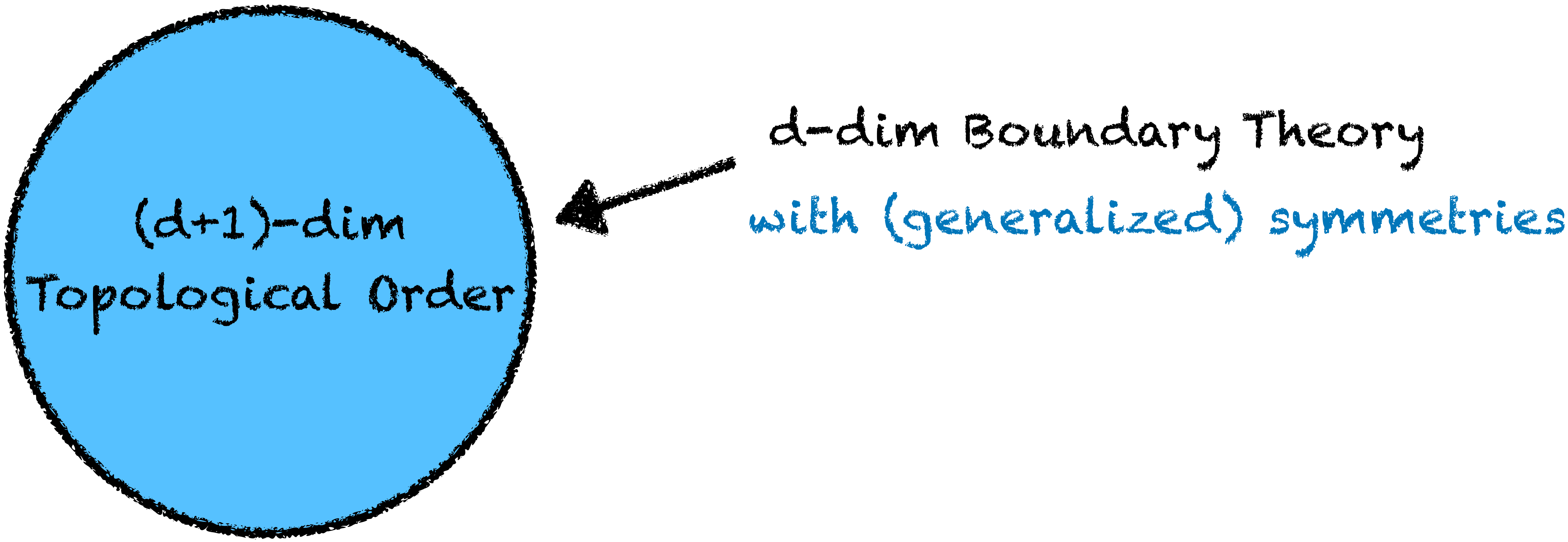

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Example 1: In (1+1)D, only one gapped phase with Z_2 KW symmetry.

Example 2: In (1+1)D, unitary minimal models are all anomaly free.

Cheng, Williamson, PRR (2020).

Constructing Bulk Topological Order

- Given a d-dim generalized symmetry, how to construct the corresponding (d+1)-dim topological order?

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0-form symmetry G	Dijkgraaf-Witten
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d-dim generalized Ising models *anomaly-free subsystem or higher-form Z_2 symmetries*	**(d+1)-dim stabilizer models** *topological or fracton* SL, Ji, SciPost Phys (2023).
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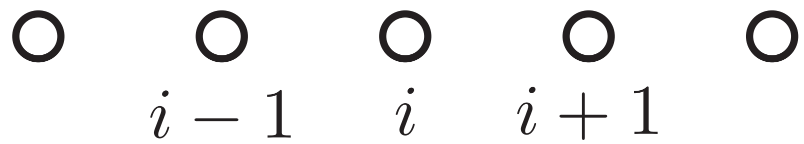
Subsystem Symmetry?

Bulk Construction: Example 1

- Standard (1+1)D transverse-field Ising model:



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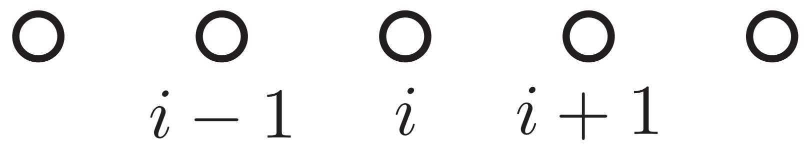

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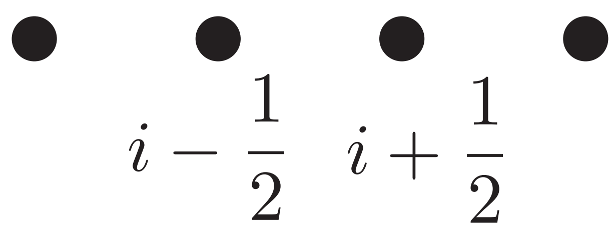

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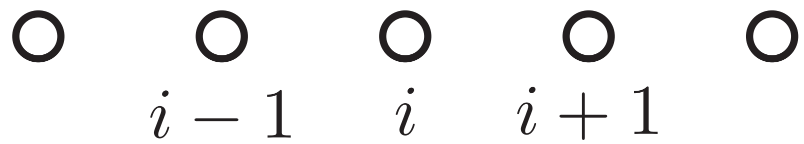

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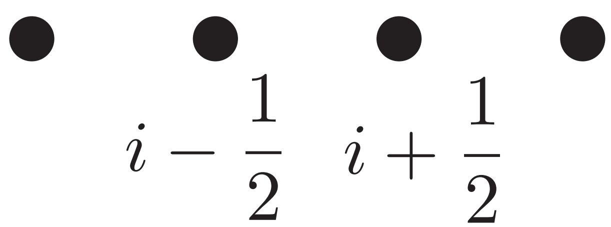

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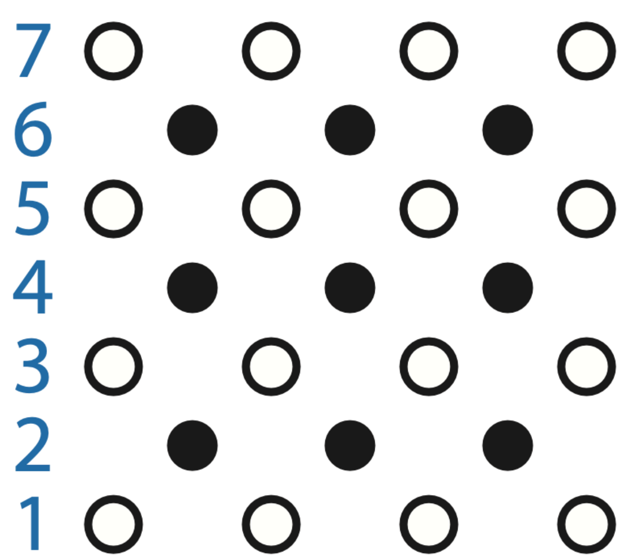

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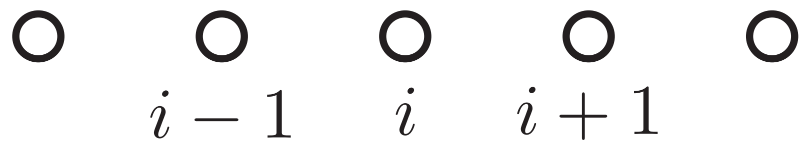

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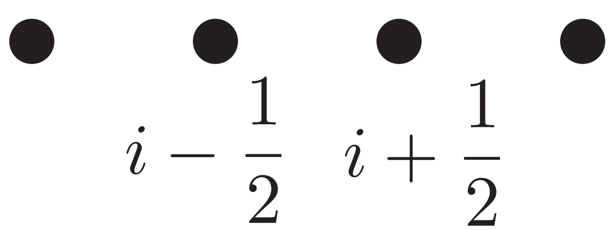

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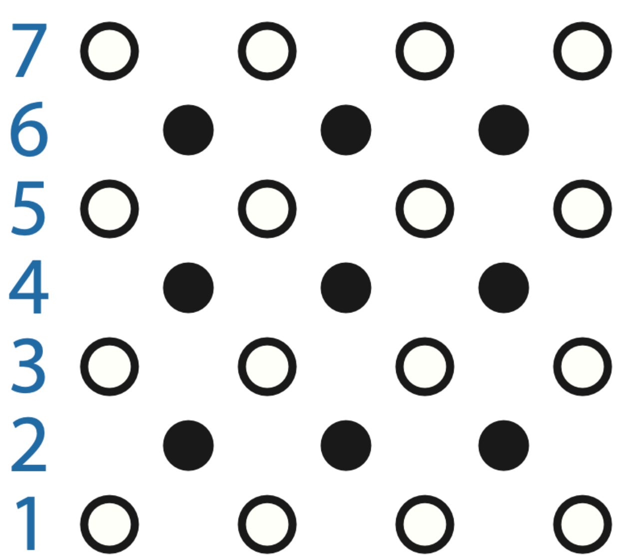
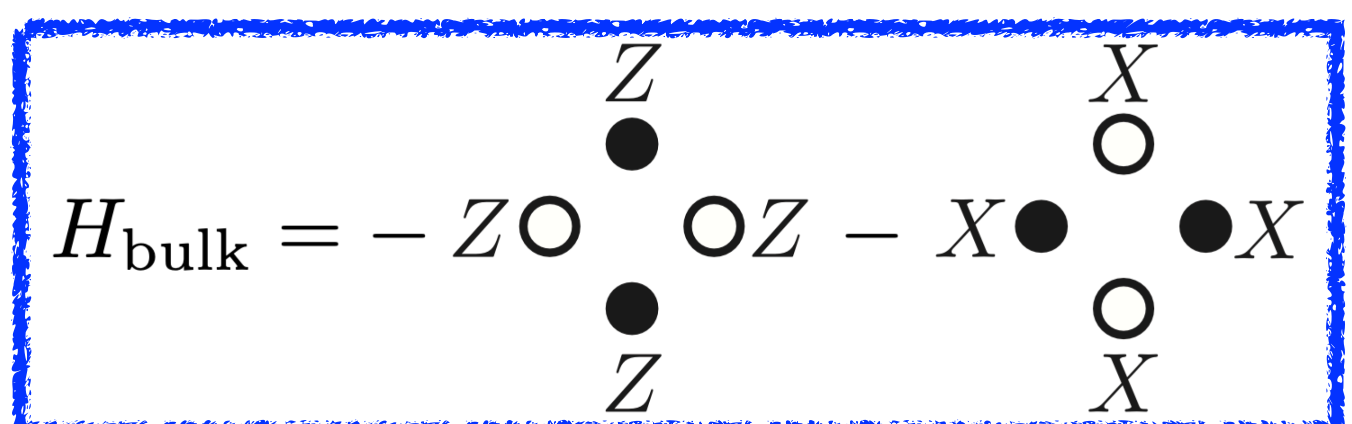
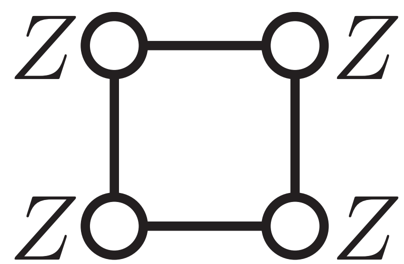


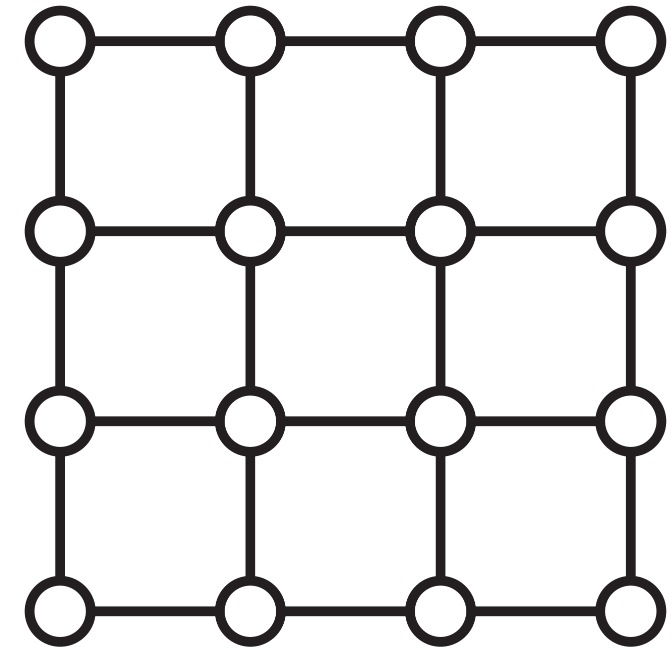


Bulk Construction: Example 2

- (2+1)D Plaquette Ising model:

$H = -J$ 

$- h$ 



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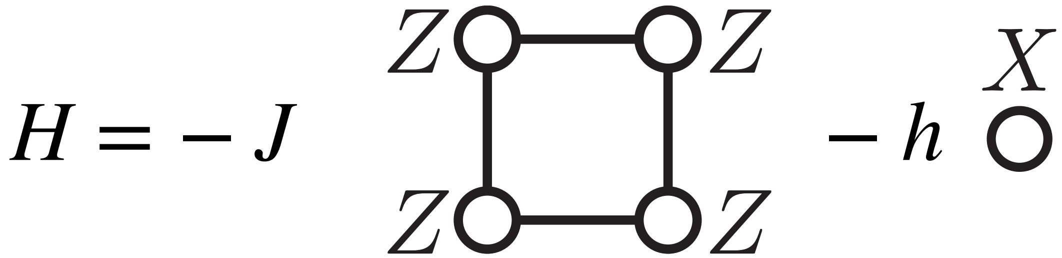

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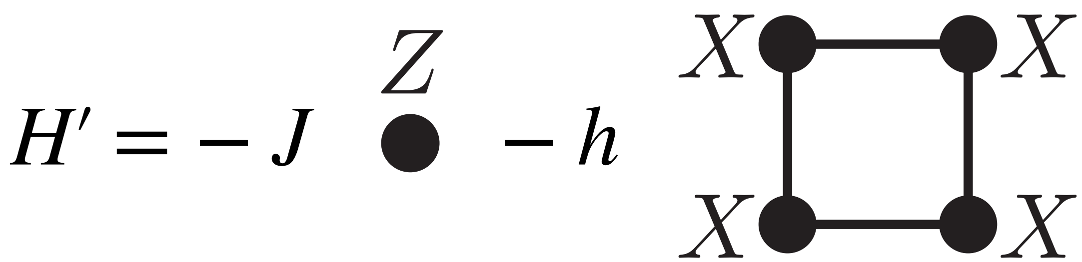

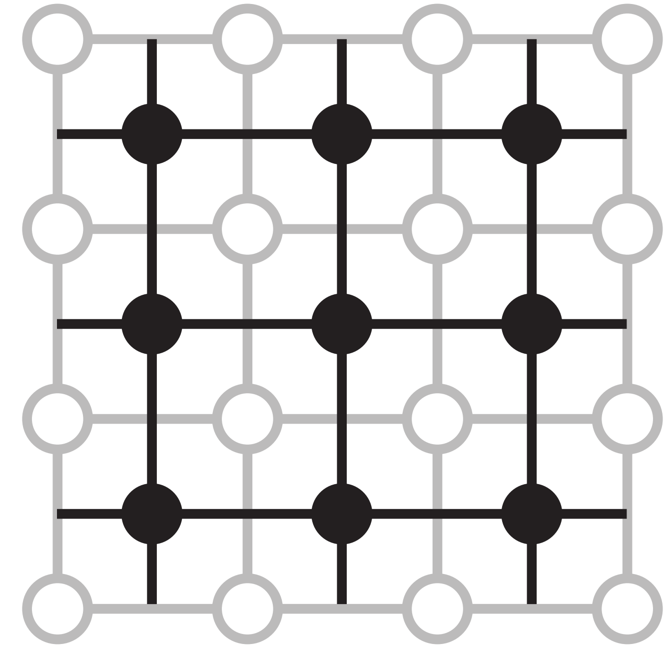

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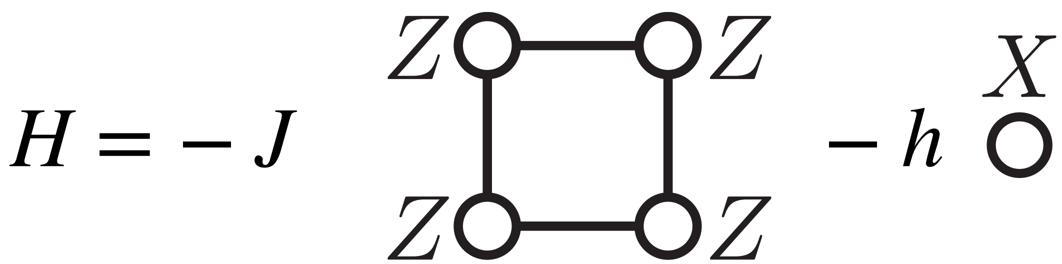

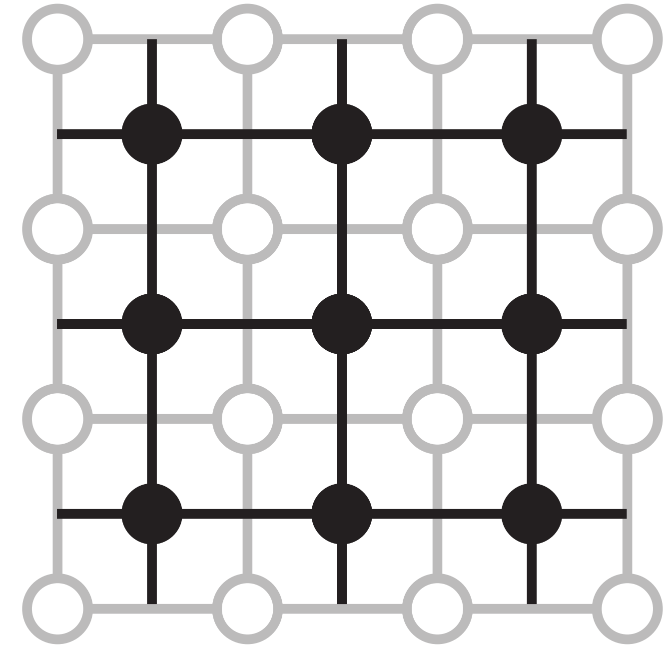

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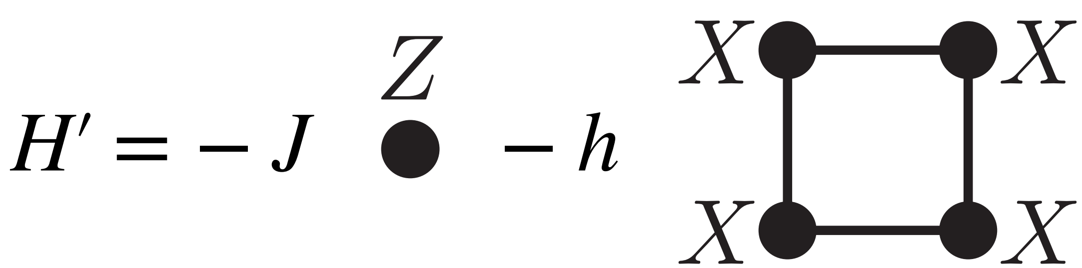
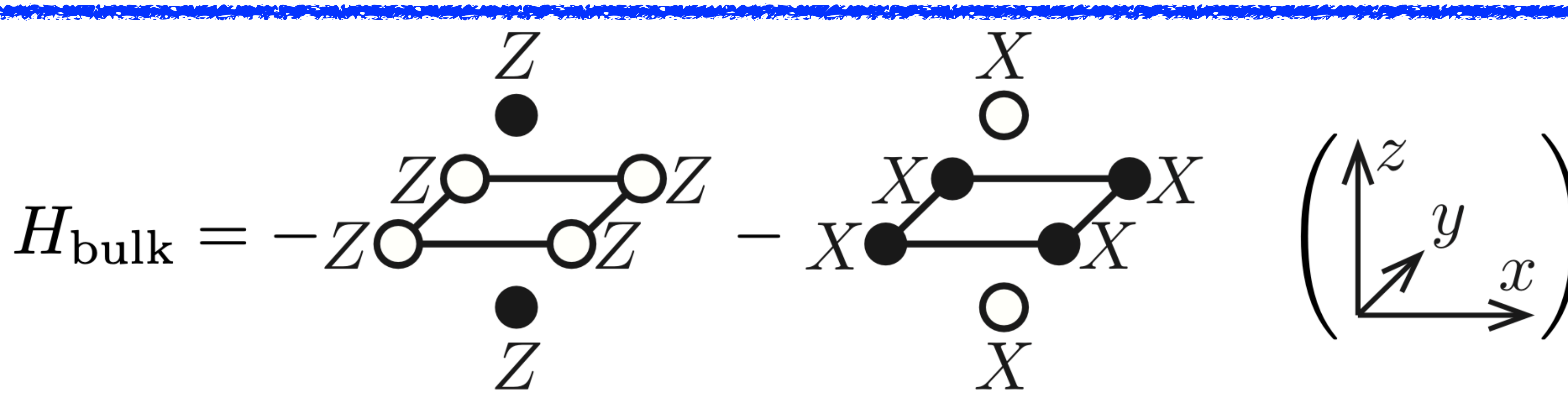


- The bulk model: Fuji, 1908.02257 (PRB, ’19).



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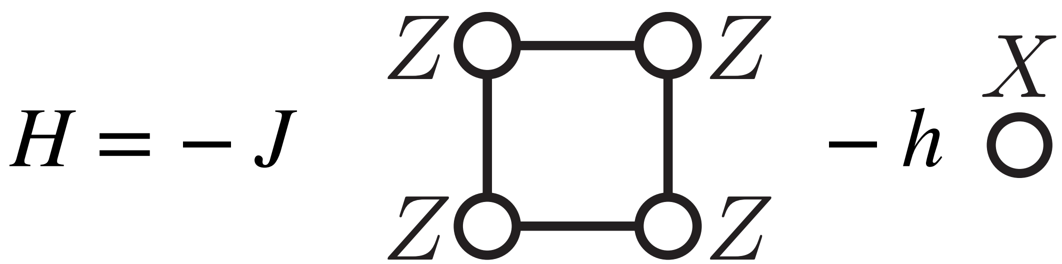

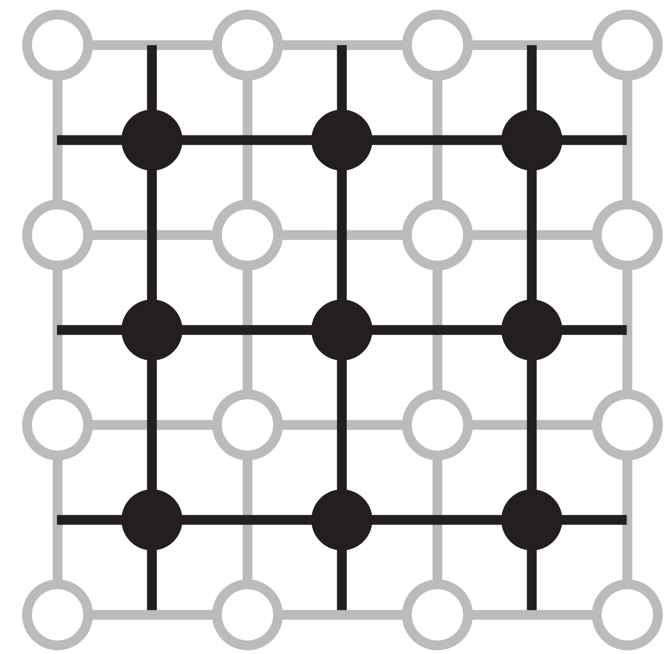

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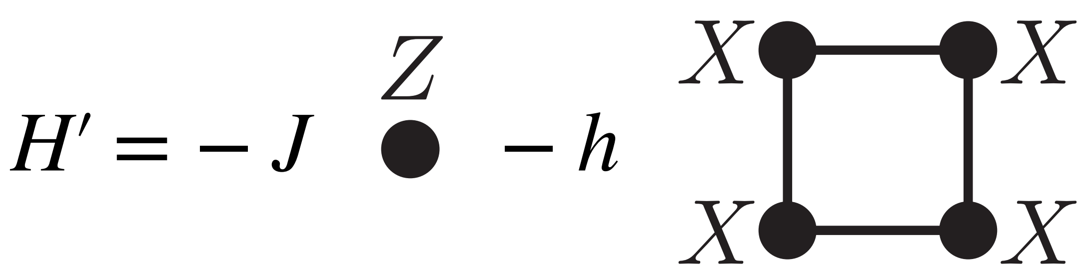

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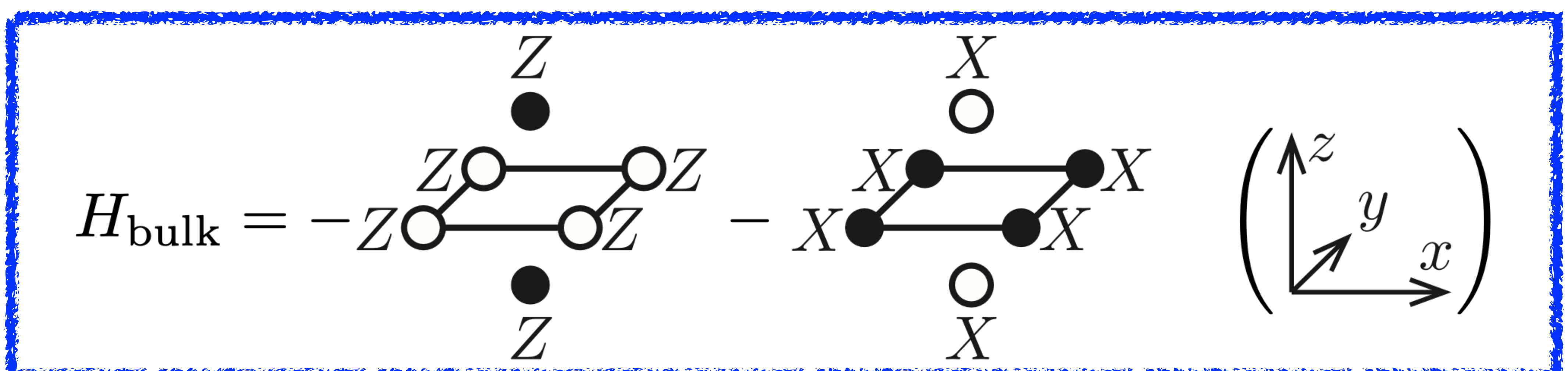
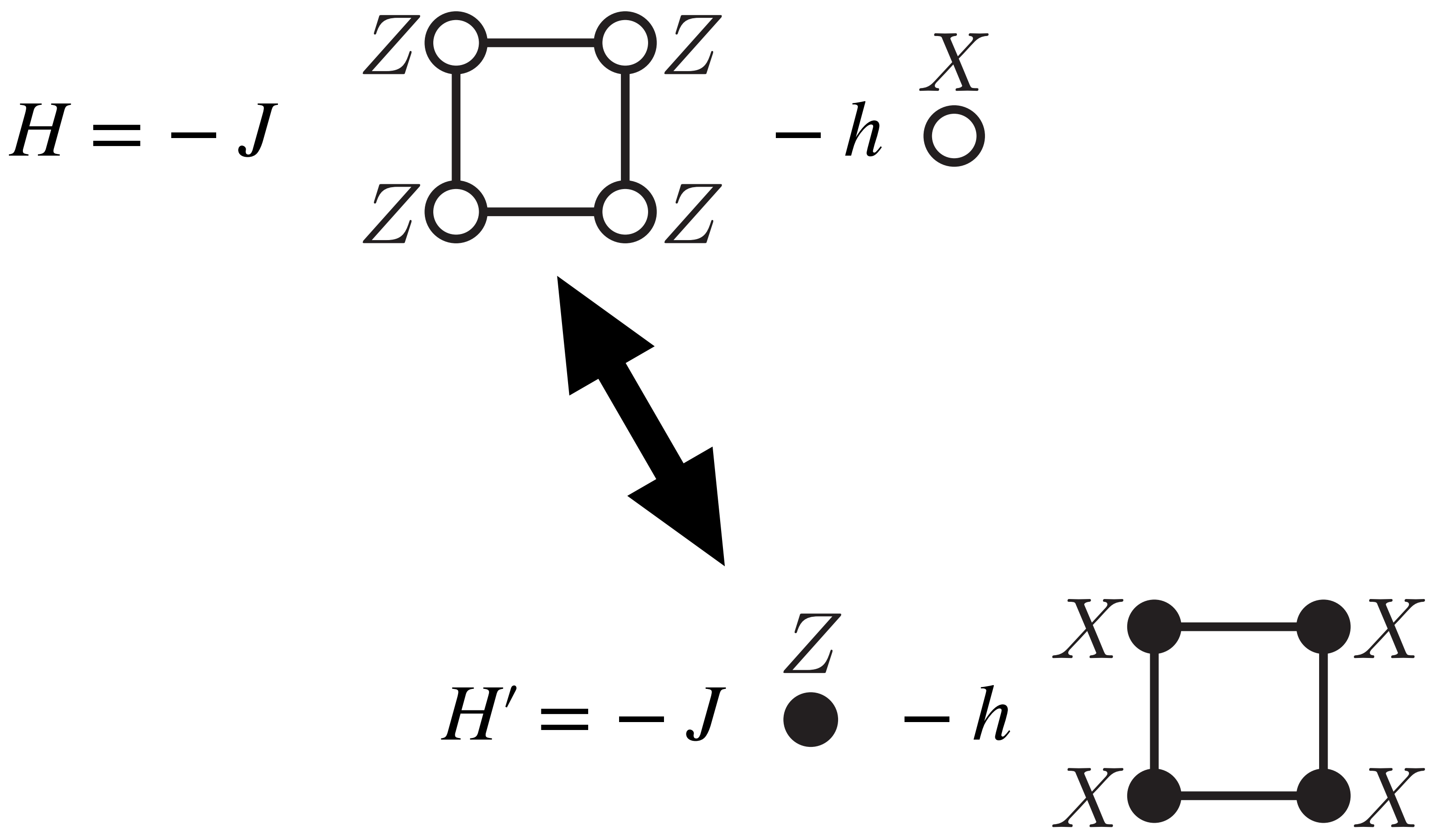


Restricted mobility along x and y directions.

Bulk Construction: Example 3



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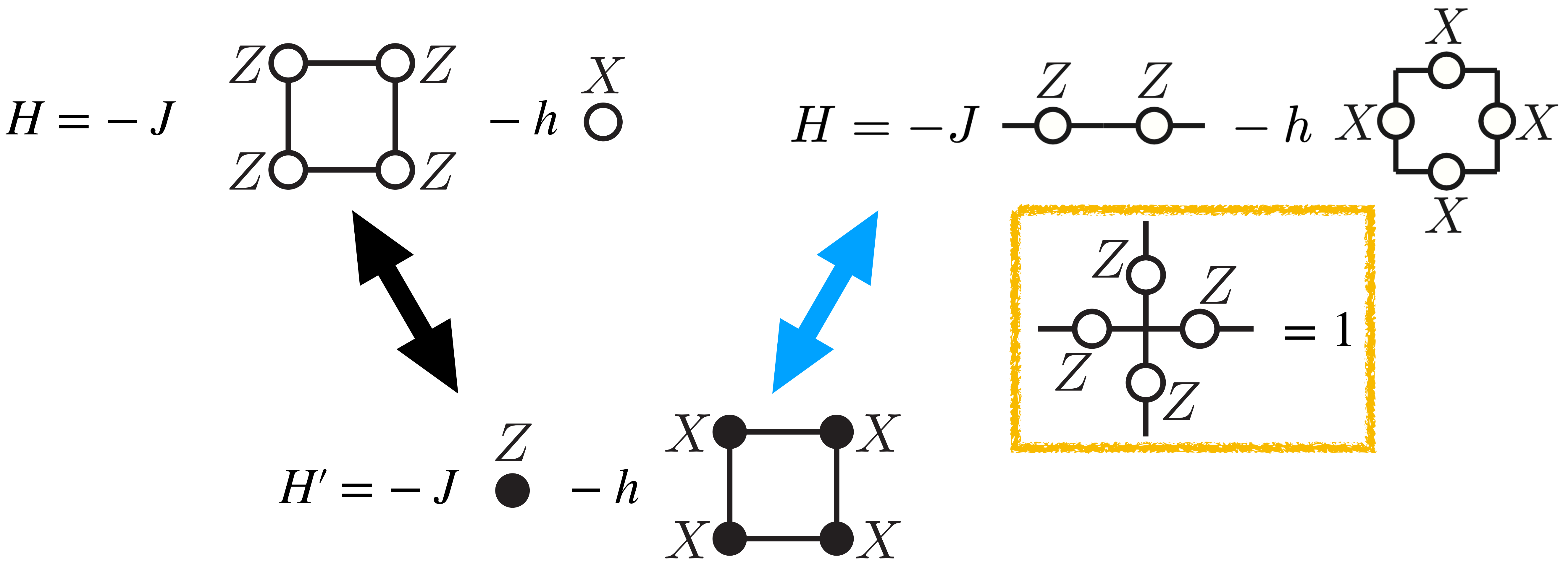

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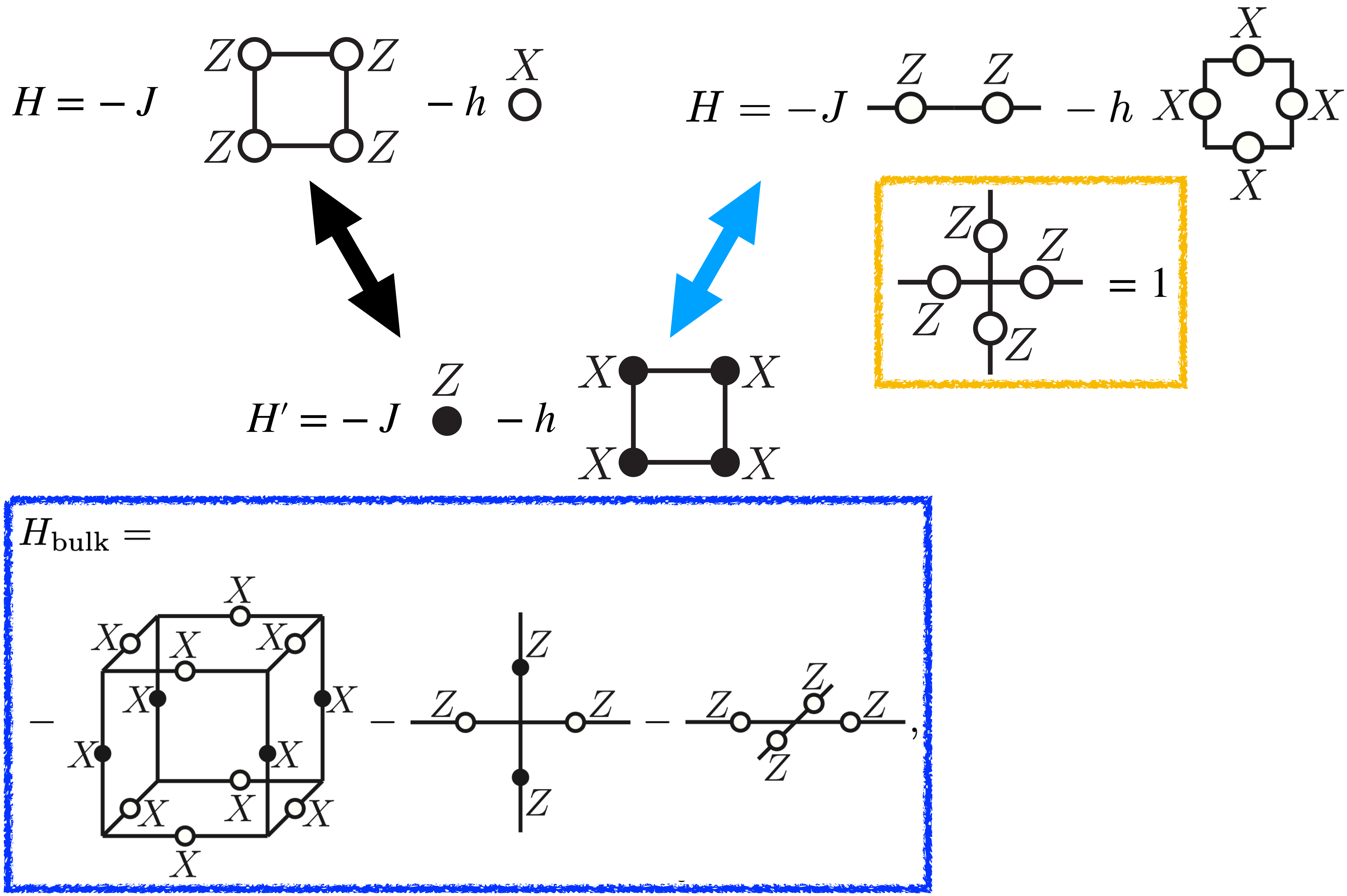

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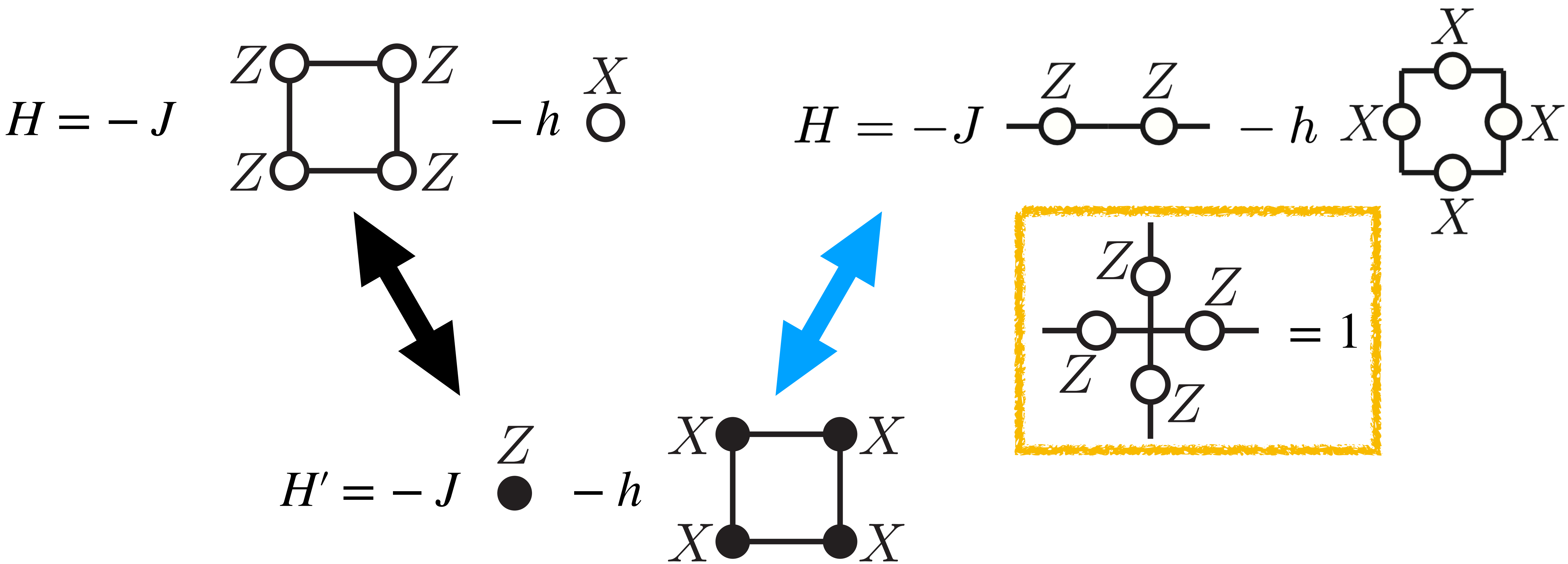
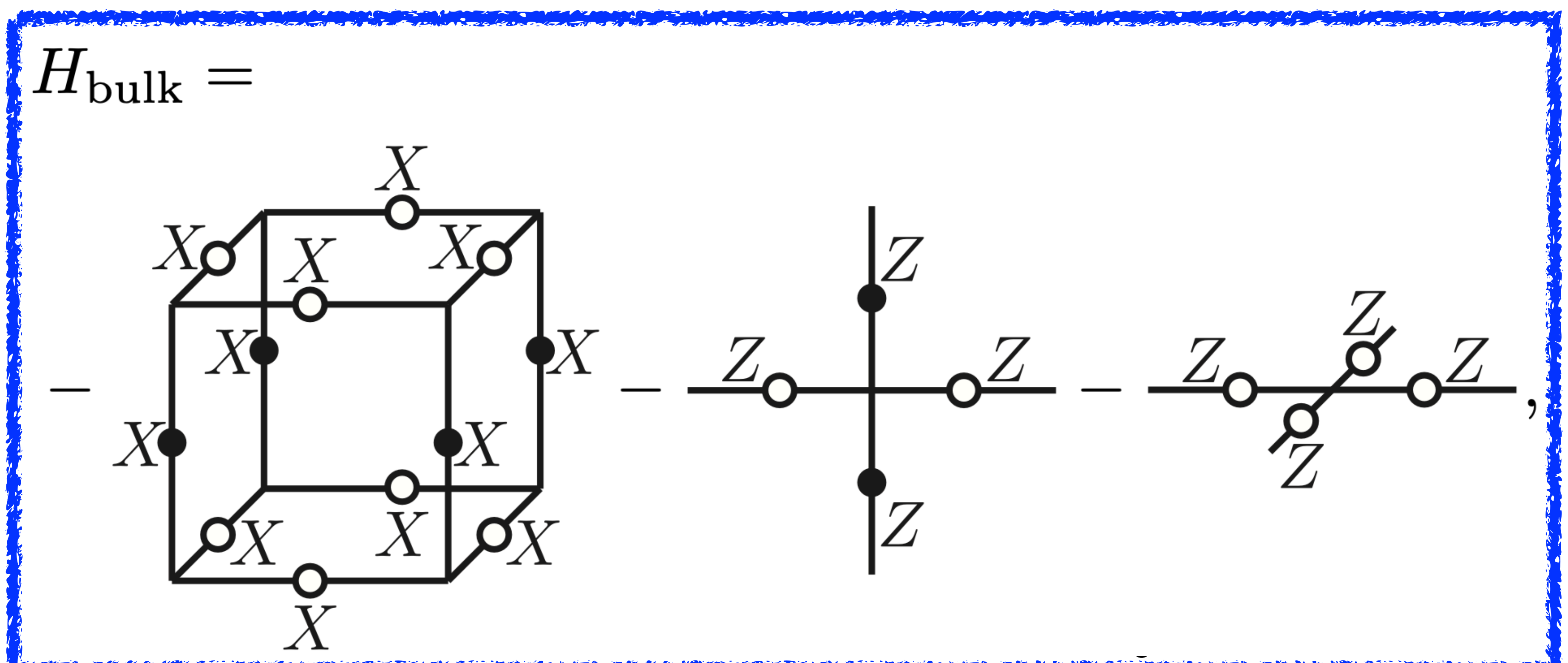
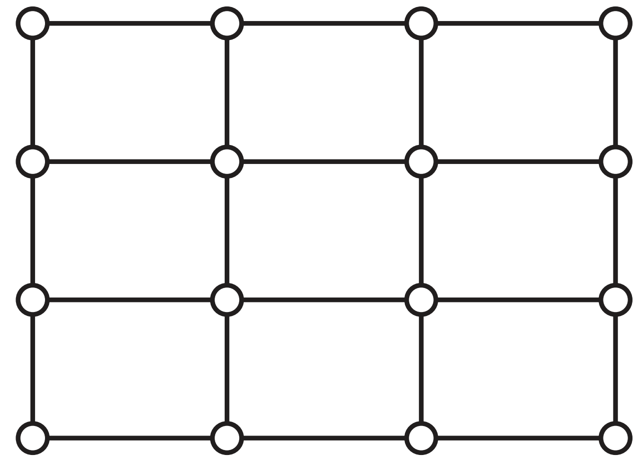
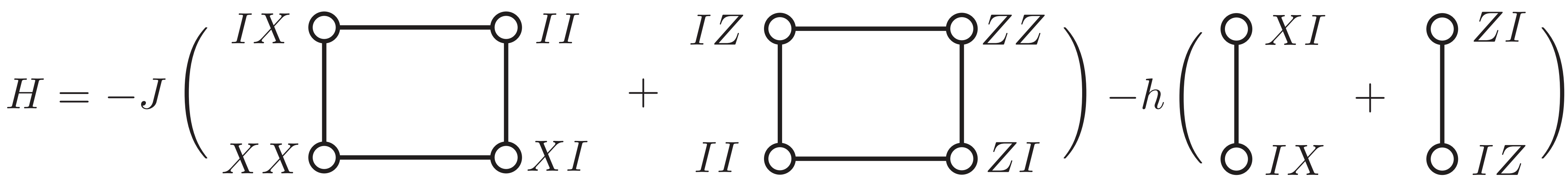




Same boundary theory for two distinct bulks

Bulk Construction: Example 4





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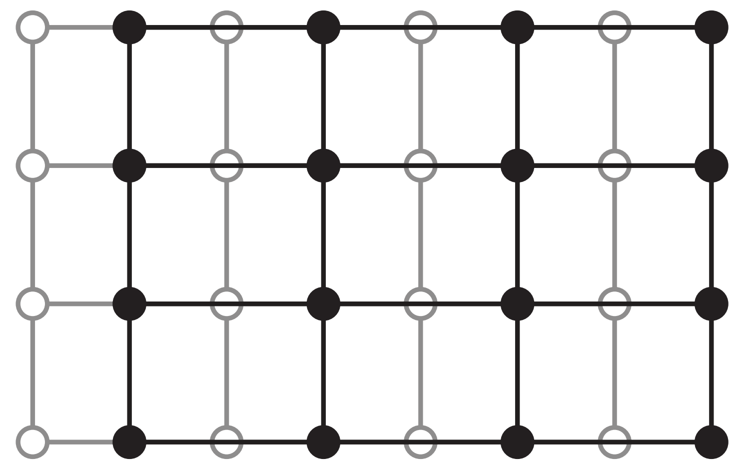

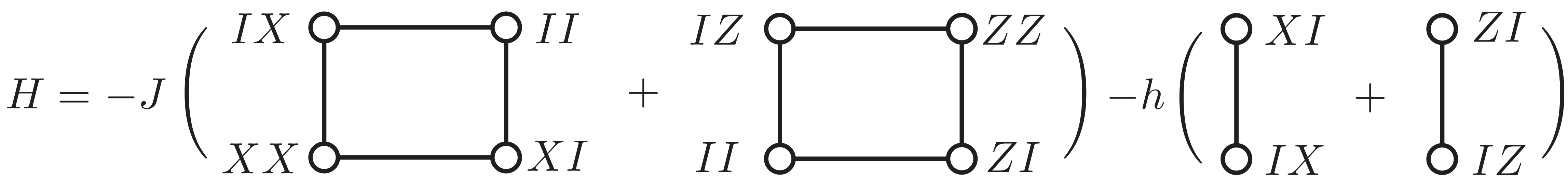

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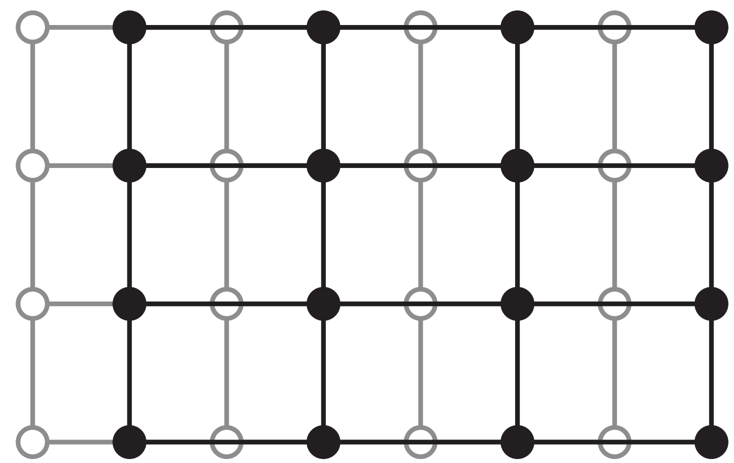

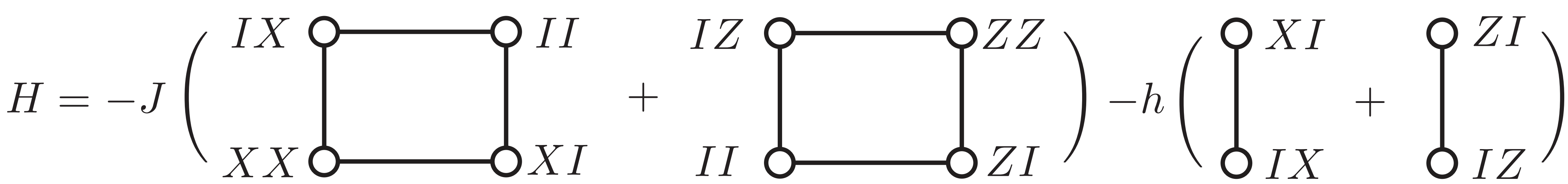

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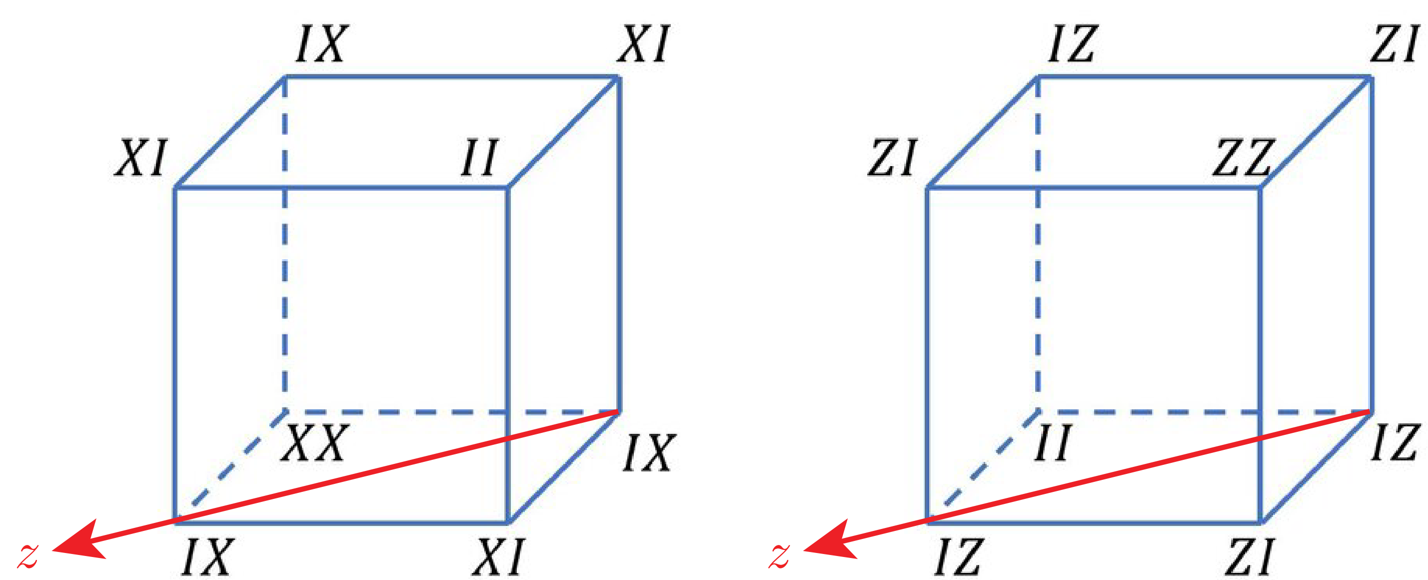
Bulk Stabilizers:



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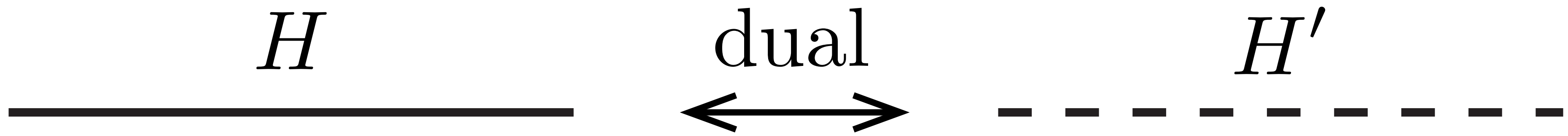

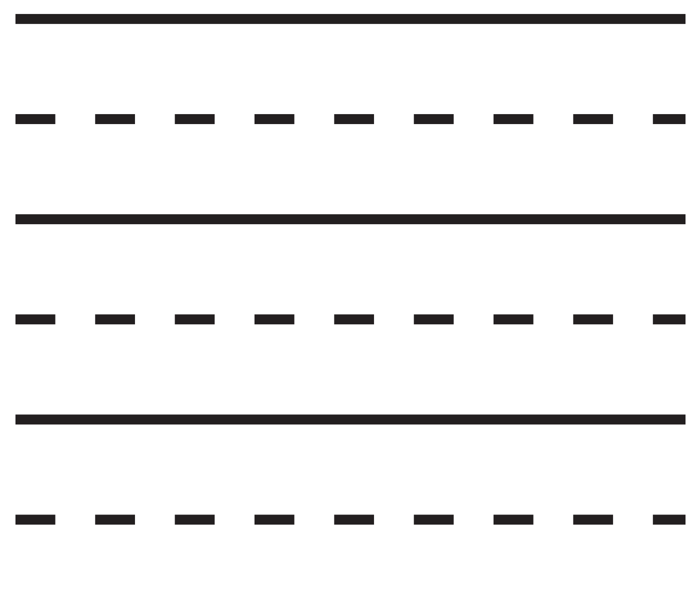

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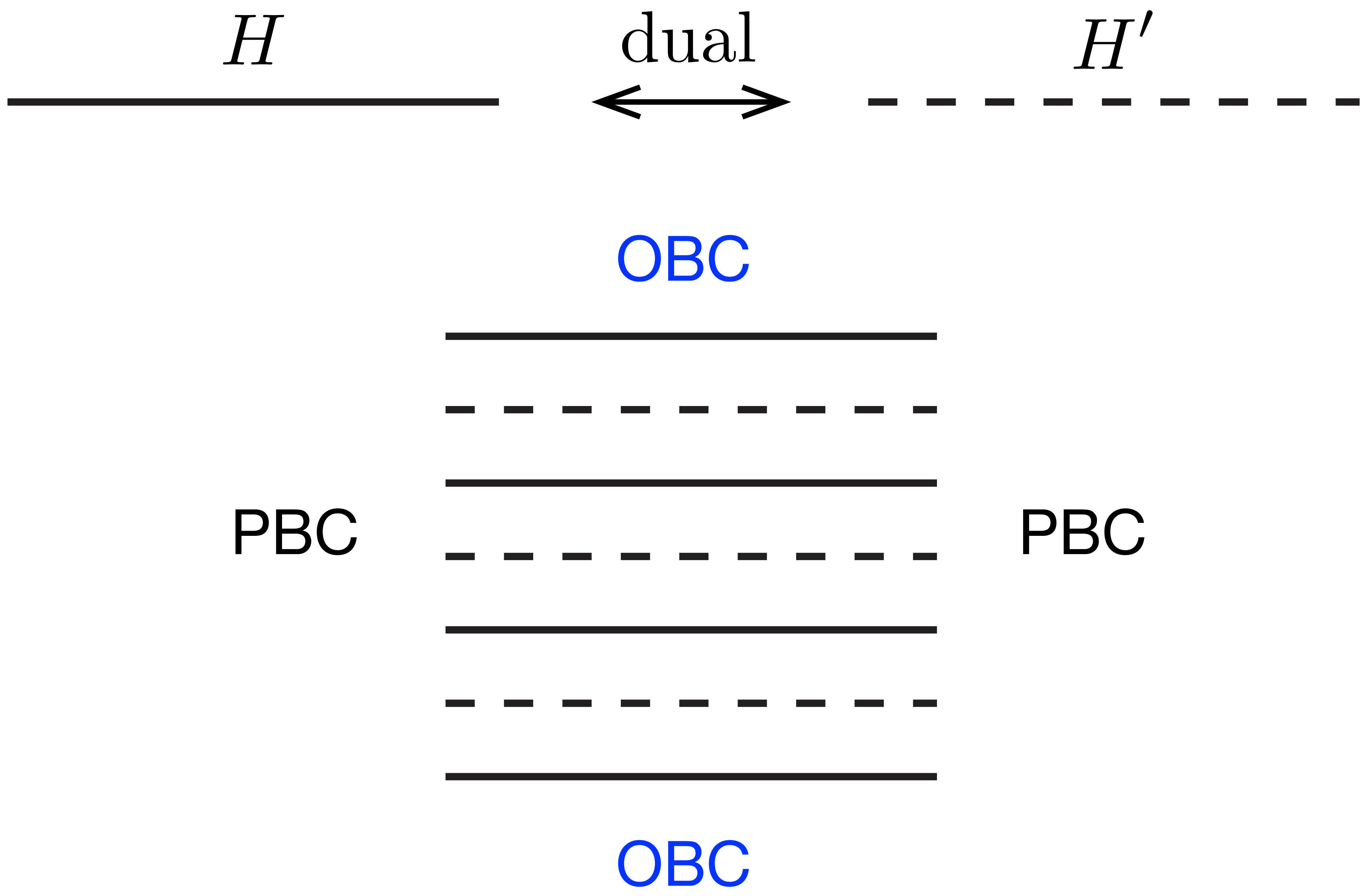

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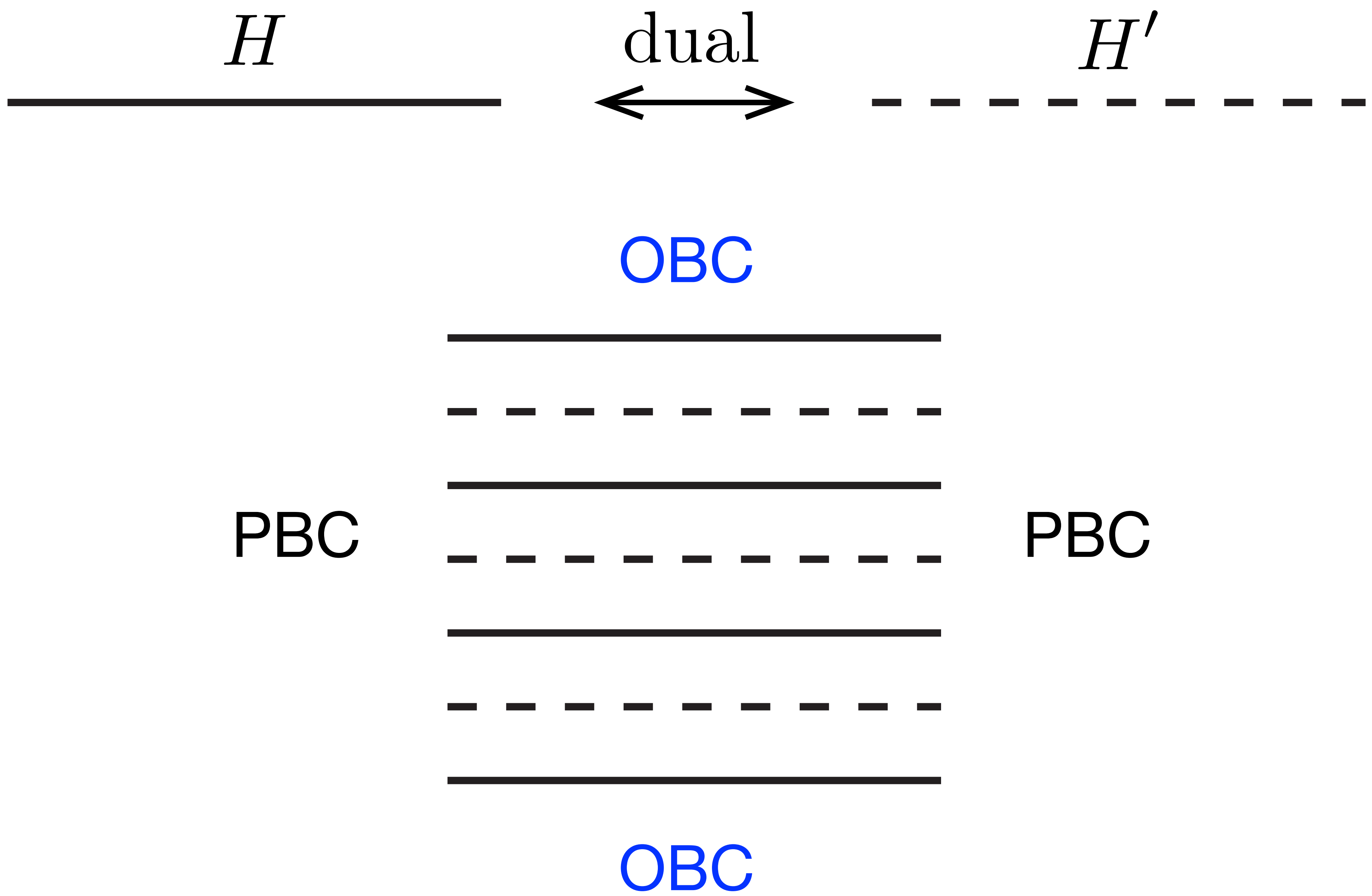

- The boundary theory is certain charge and boundary condition sectors of two copies of the model H.

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Ising — Dual Ising

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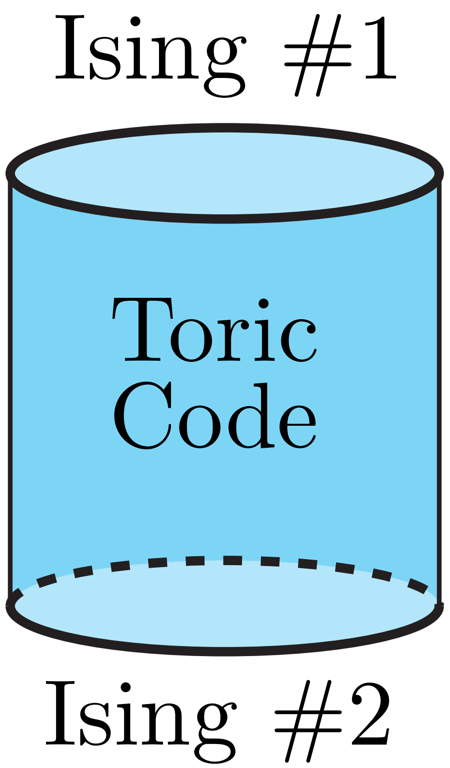

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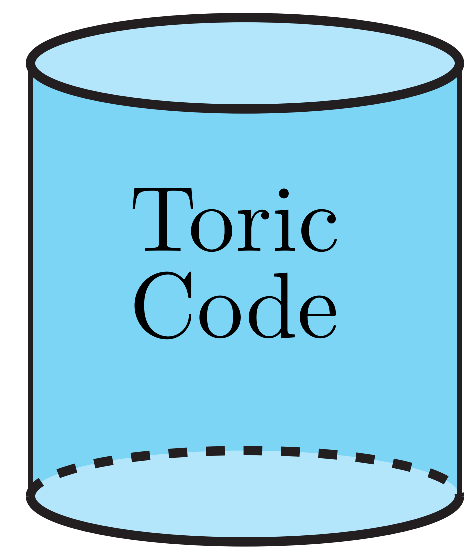
$$U^X \otimes U^X = 1$$



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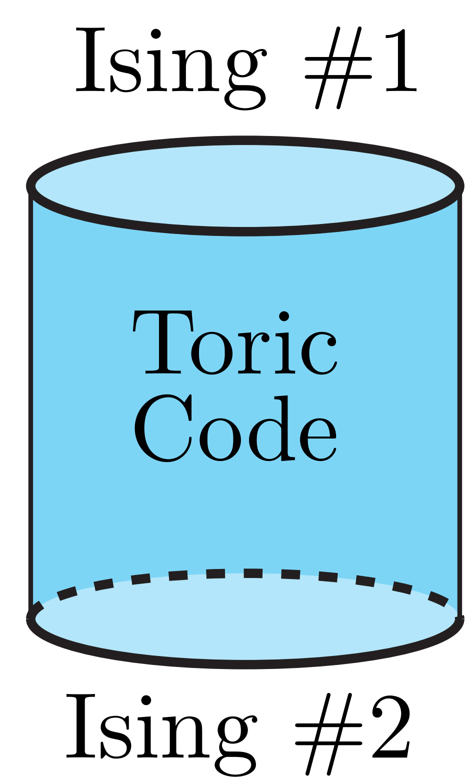

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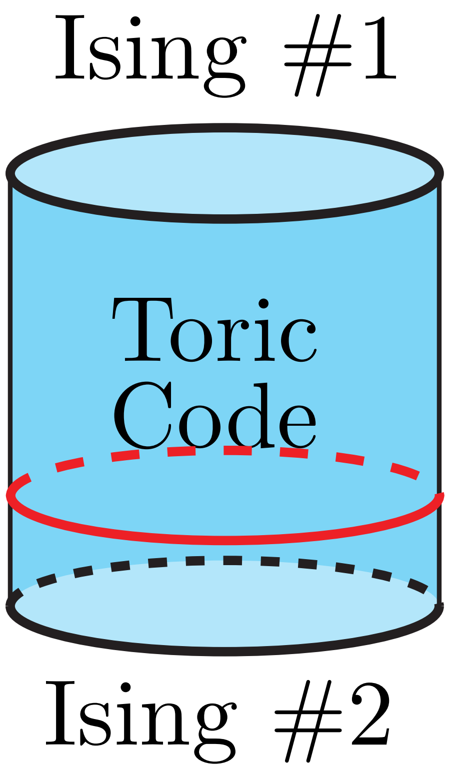

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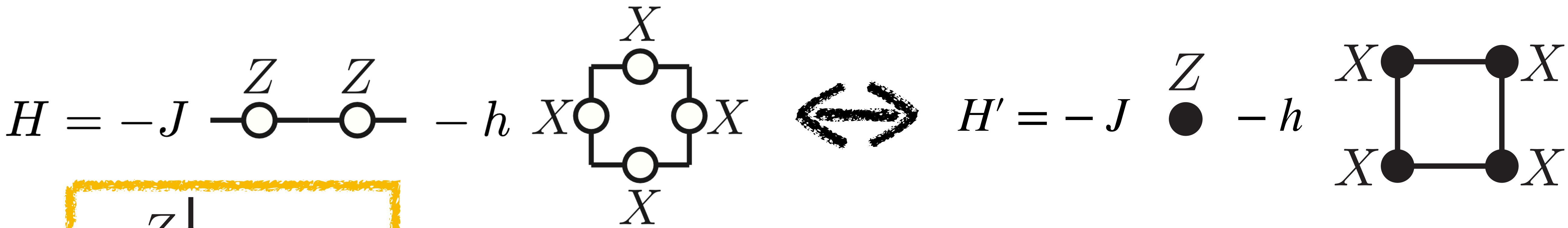
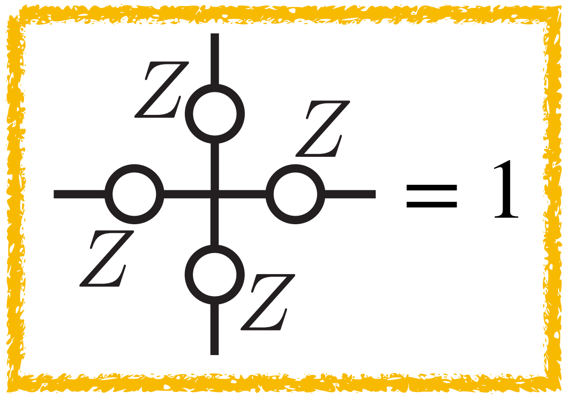
boundary condition labeled by the bulk string

Bulk-Boundary Correspondence: Example 2





Bulk = X-Cube Model

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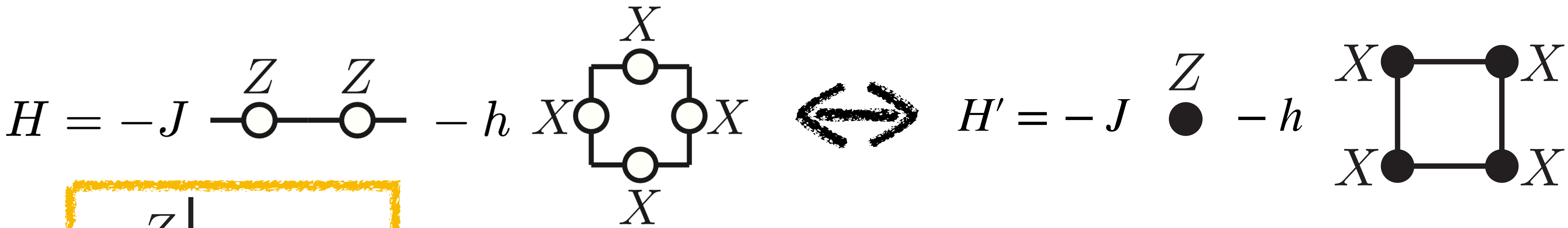

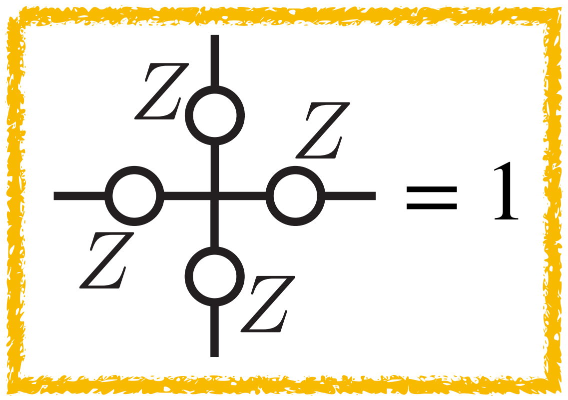

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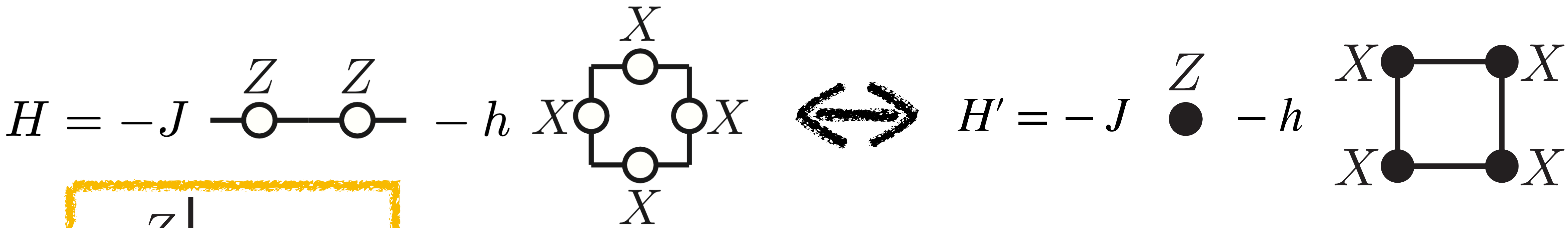
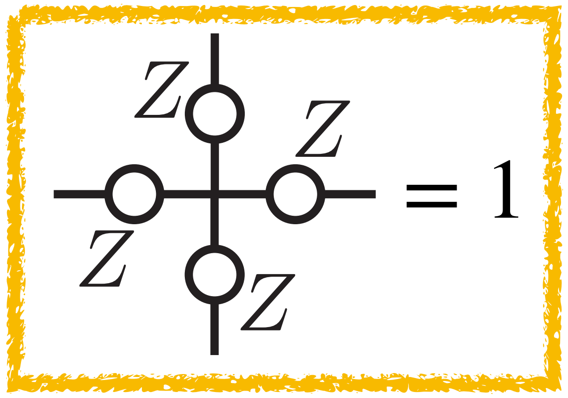
Boundary ≈ Two Copies of the Model H

Bulk-Boundary Correspondence: Example 2

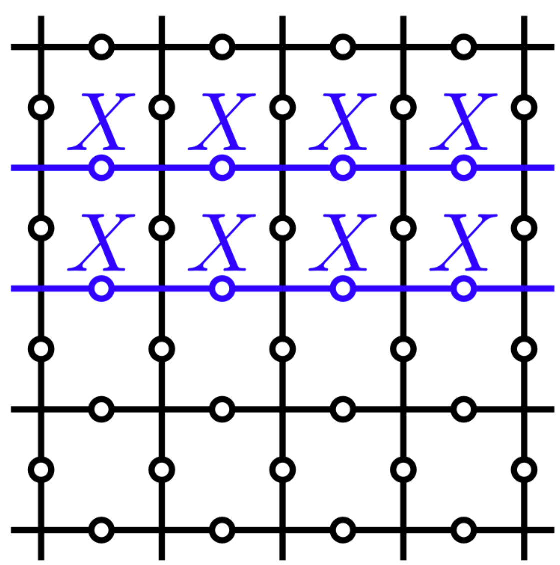




Bulk = X-Cube Model

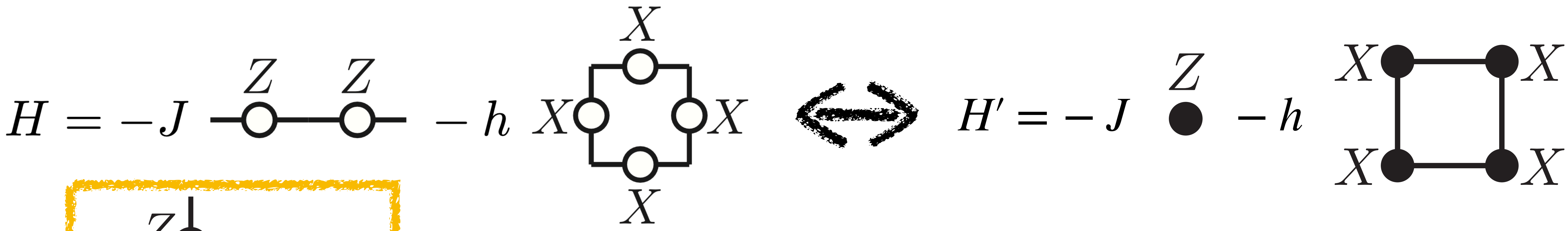
Boundary ≈ Two Copies of the Model H

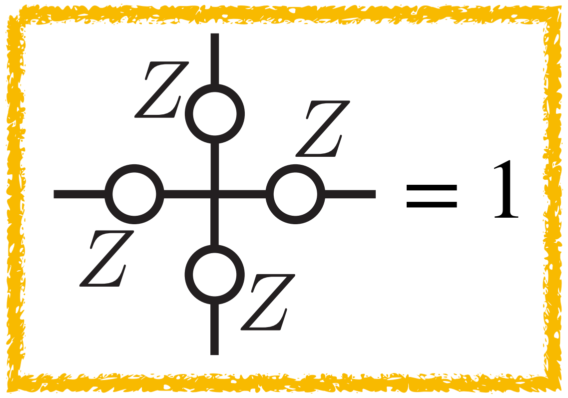
(1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1.$



(boundary layer)

Bulk-Boundary Correspondence: Example 2



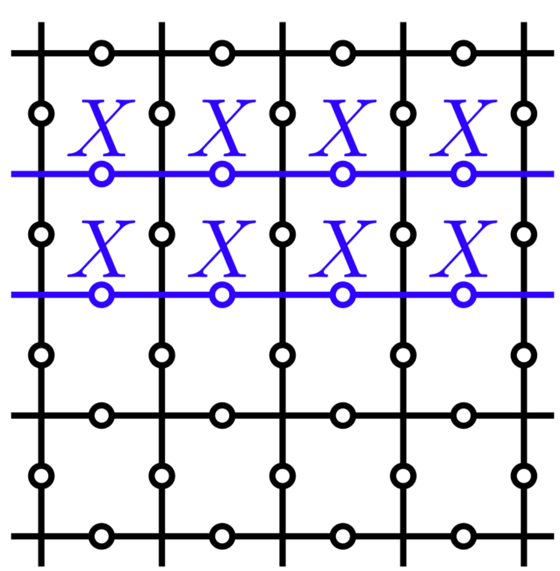


Bulk = X-Cube Model

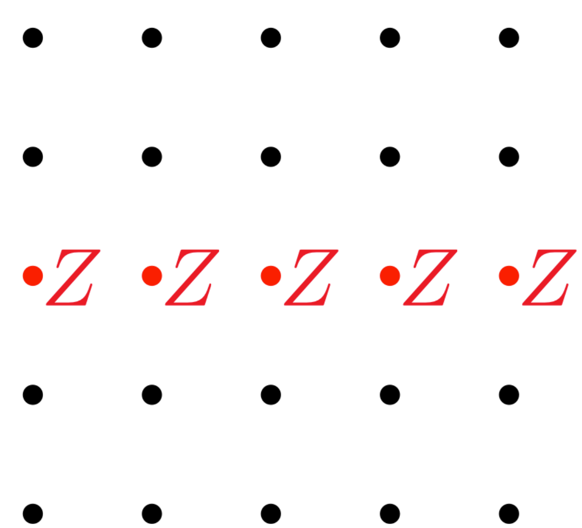
Boundary ≈ Two Copies of the Model H

(1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1.$

(2) Correlated boundary conditions.

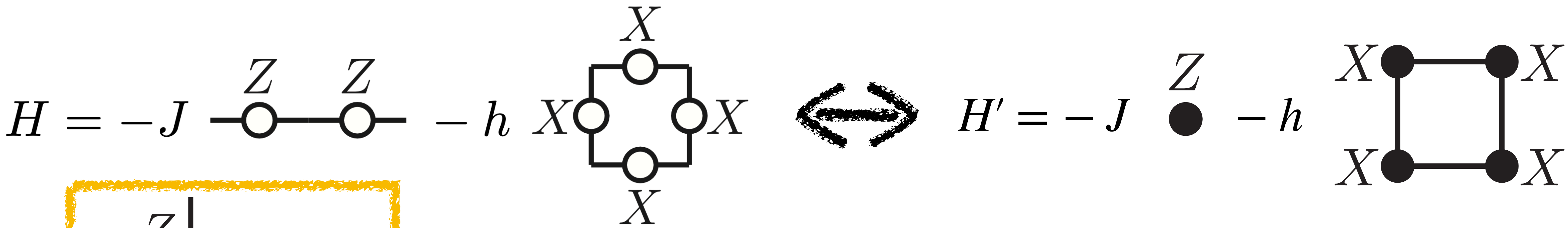


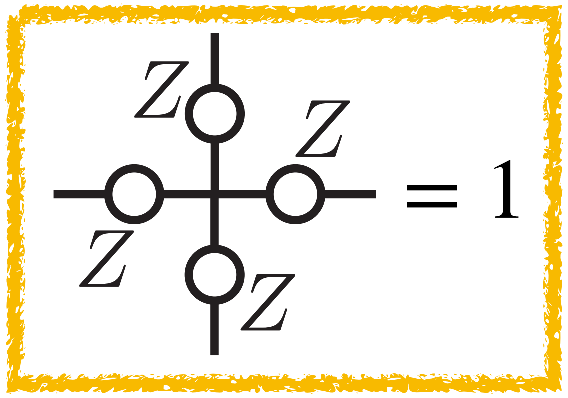
(boundary layer)



(some internal layer)

Bulk-Boundary Correspondence: Example 2





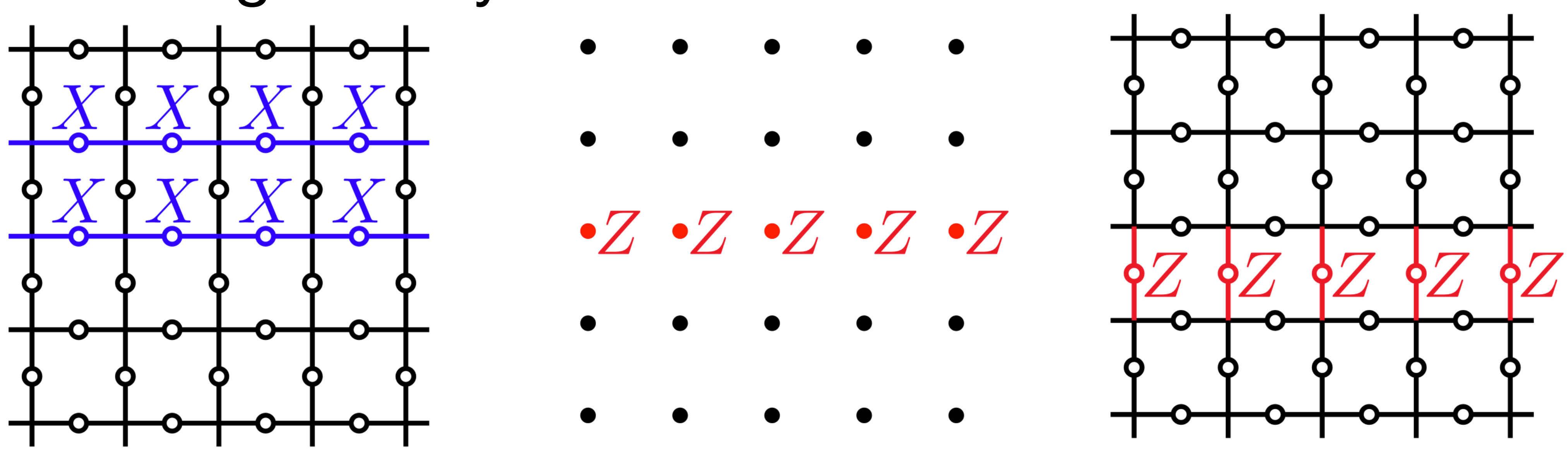
Bulk = X-Cube Model

Boundary ≈ Two Copies of the Model H

(1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1$.

(2) Correlated boundary conditions.

(3) Extra degeneracy.



(boundary layer) (some internal layer) (all internal layers)

Summary & Outlook

A *bulk construction approach* to the bulk-boundary correspondence of noninvertible topological phases.

- Subsystem symmetry / fracton.

SL, Ji, SciPost Phys (2023).

- Bulk excitations.
- Anomalous symmetries.
- New interesting topological/fracton orders.