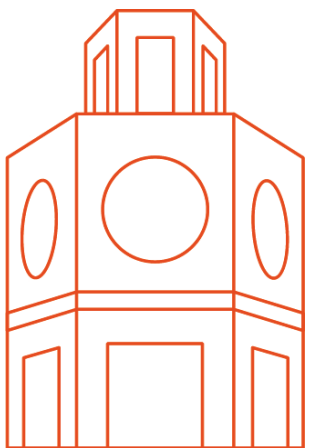


Constructing Bulk Topological Orders from Generalized Ising Models

Shang Liu (IOP@CAS)

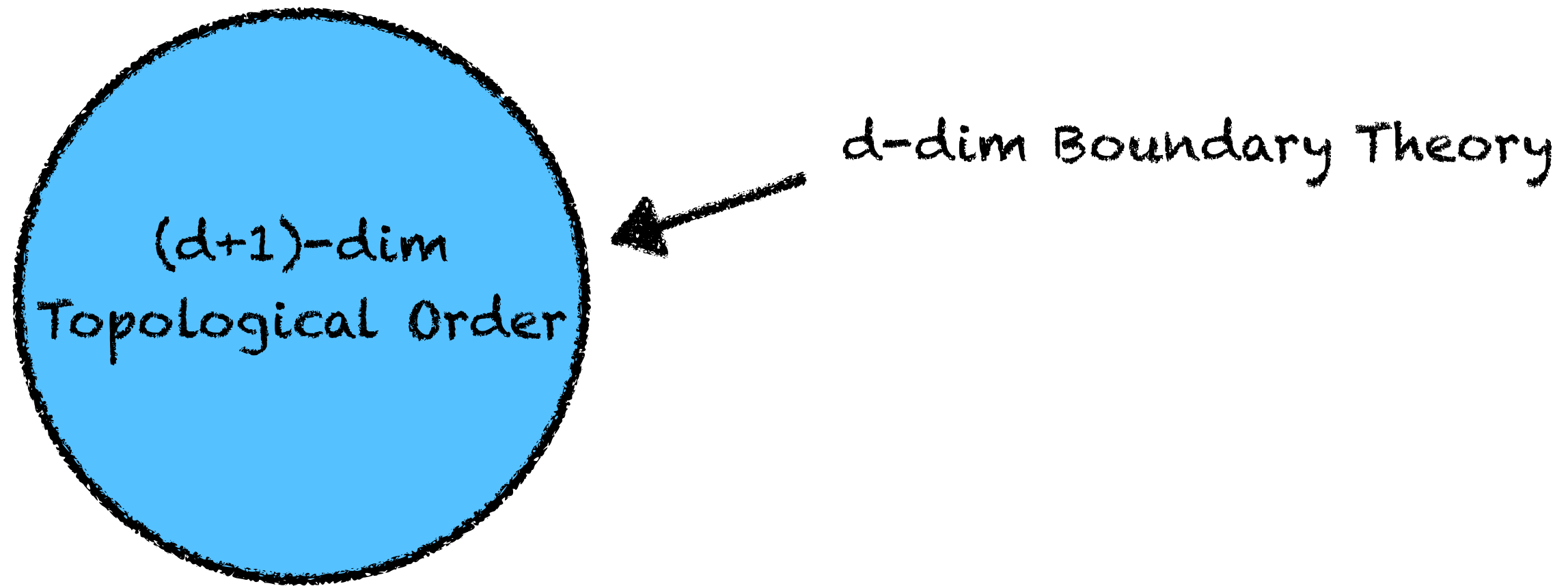
Workshop on Generalized Symmetries, 2025/07/28

SL, Wenjie Ji, SciPost Phys (2023).

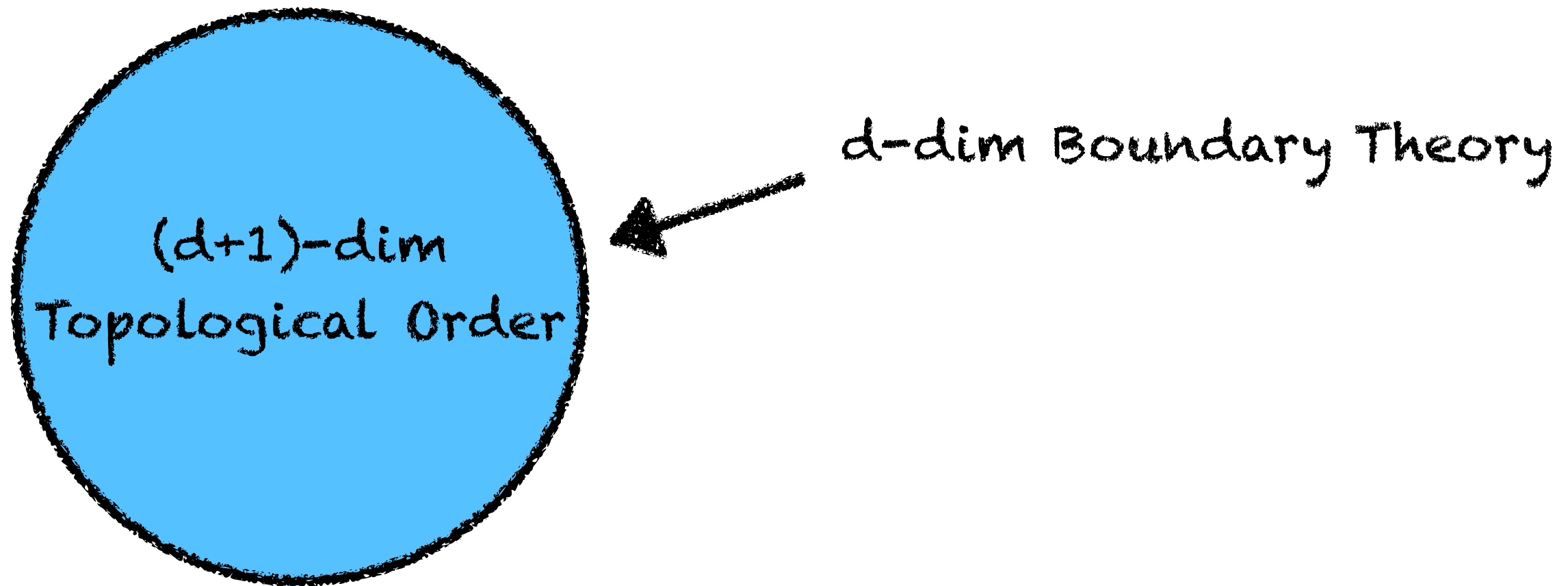


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Bulk-Boundary Correspondence of Topological Order

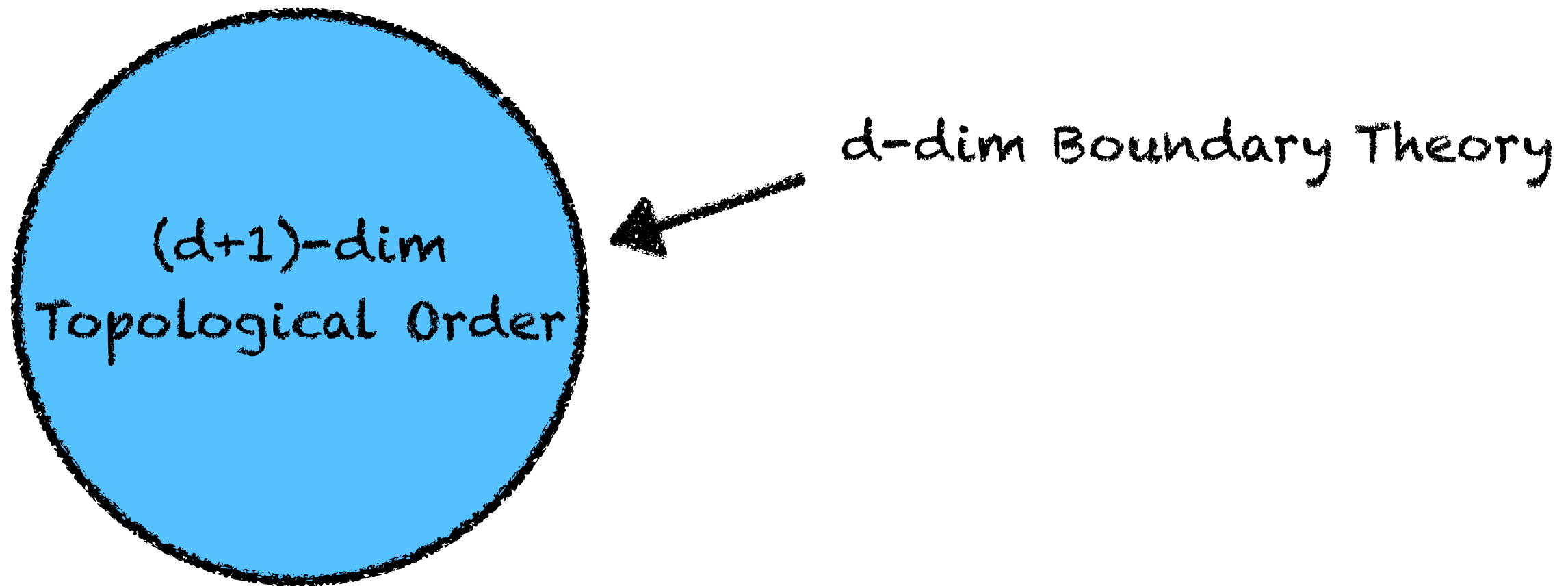


Bulk-Boundary Correspondence of Topological Order



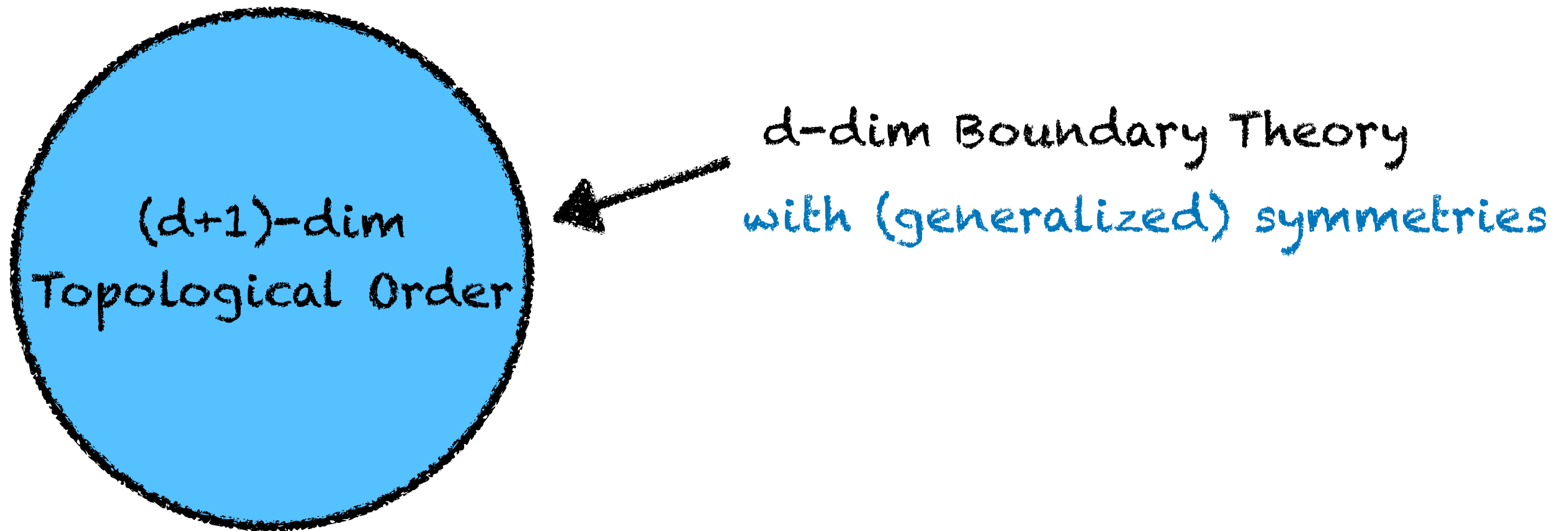
- A fundamental question in the theory of topological order (noninvertible anomaly).

Bulk-Boundary Correspondence of Topological Order



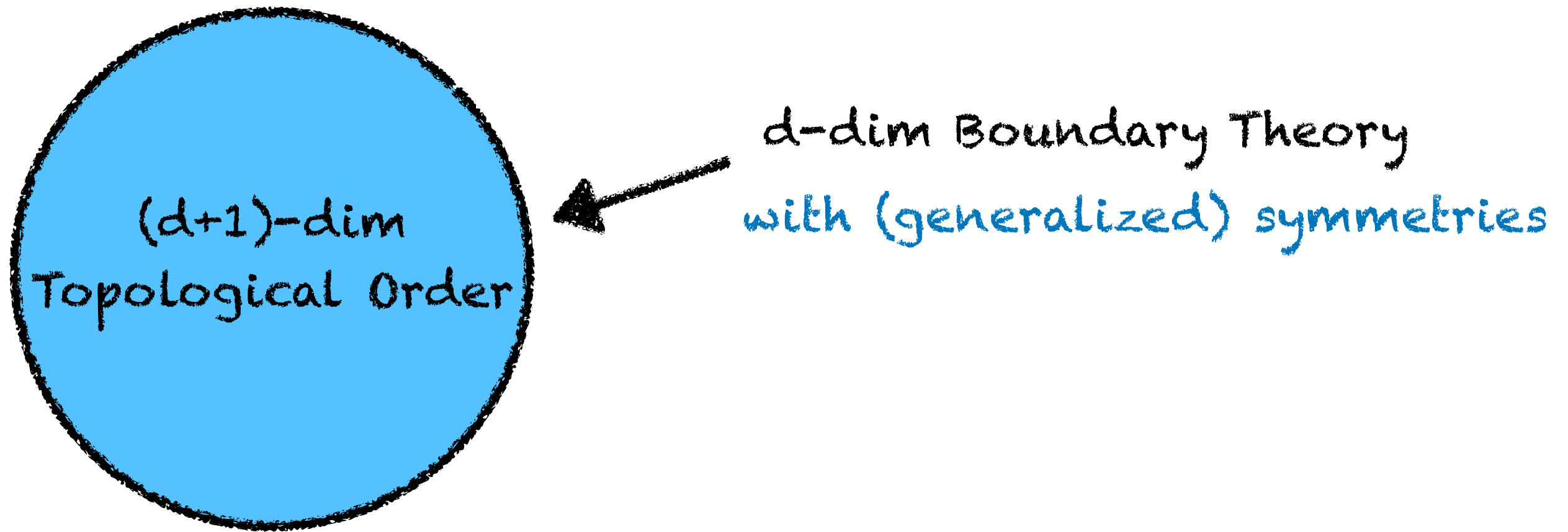
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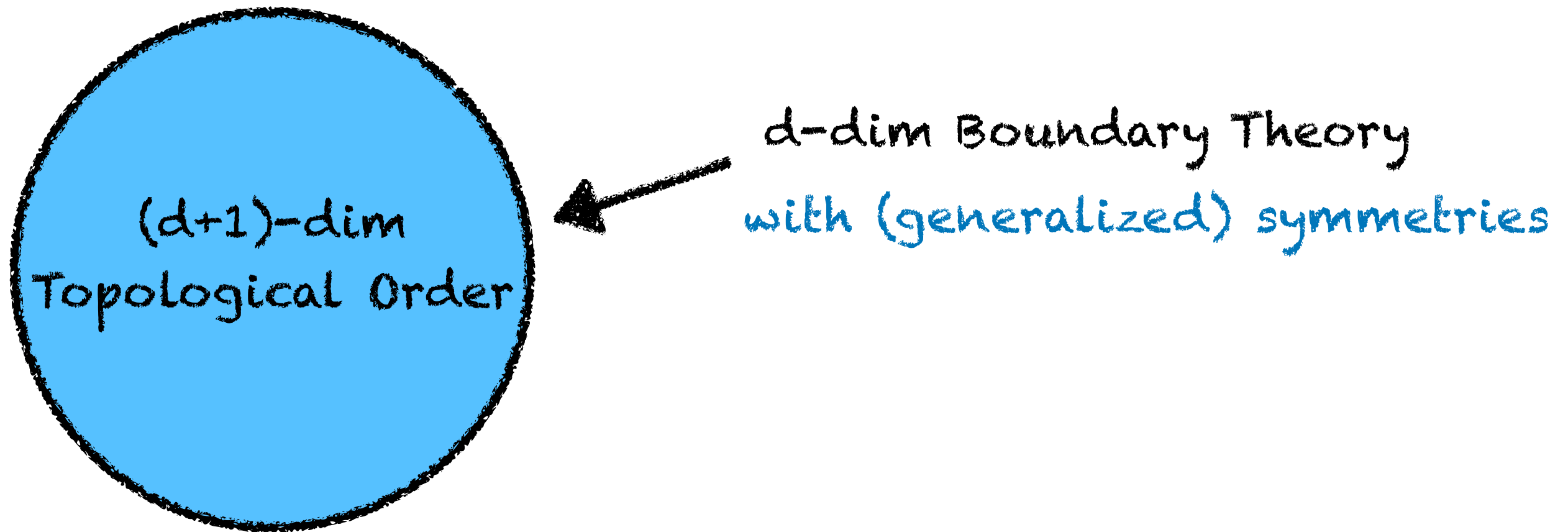
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Example 1: In (1+1)D, only one gapped phase with Z_2 KW symmetry.

Bulk-Boundary Correspondence of Topological Order



- A fundamental question in the theory of topological order (noninvertible anomaly).
- A useful framework for understanding quantum many-body states with (generalized) symmetries.

Example 1: In $(1+1)D$, only one gapped phase with Z_2 KW symmetry.

Example 2: In $(1+1)D$, unitary minimal models are all anomaly free.

Cheng, Williamson, PRR (2020).

Constructing Bulk Topological Order

- Given a d -dim generalized symmetry, how to construct the corresponding $(d+1)$ -dim topological order?

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0-form symmetry G	Dijkgraaf-Witten
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Subsystem Symmetry?

Constructing Bulk Topological Order

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Constructing Bulk Topological Order

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0-form symmetry G	Dijkgraaf-Witten
d-dim generalized Ising models <i>anomaly-free subsystem or higher-form Z_2 symmetries</i>	(d+1)-dim stabilizer models <i>topological or fracton</i> SL, Ji, SciPost Phys (2023).
...	...

Subsystem Symmetry?

Bulk Construction: Example 1

- Standard (1+1)D transverse-field Ising model:



$$H = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i$$

Bulk Construction: Example 1

- Standard (1+1)D transverse-field Ising model:

$$\begin{array}{c} \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \\ i-1 \quad i \quad i+1 \end{array} \quad H = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i$$

- Kramers-Wannier dual model:

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ i - \frac{1}{2} \quad i + \frac{1}{2} \end{array} \quad H' = -J \sum_i Z_{i+1/2} - h \sum_i X_{i-1/2} X_{i+1/2}$$

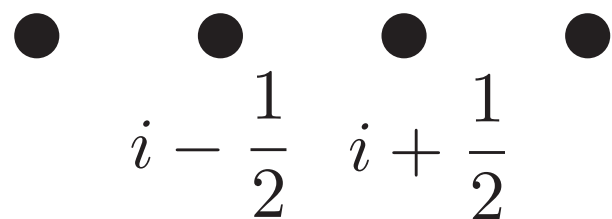
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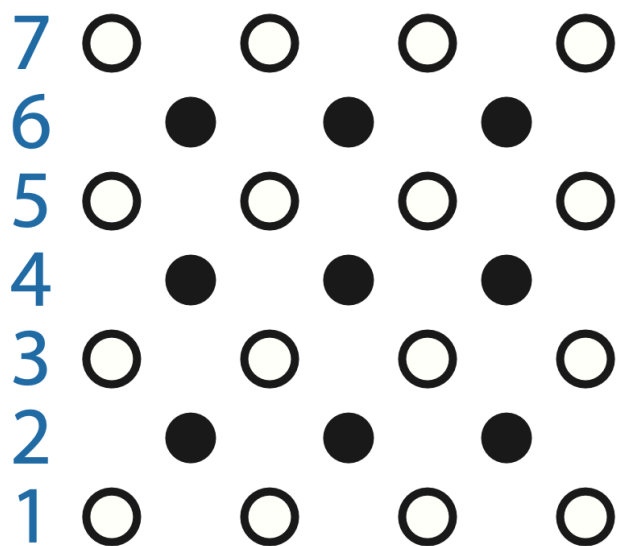
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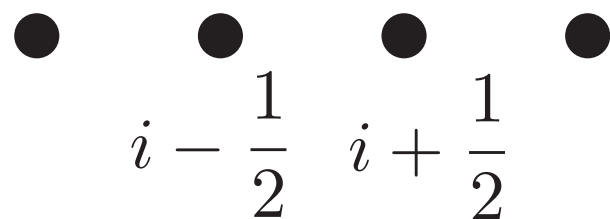
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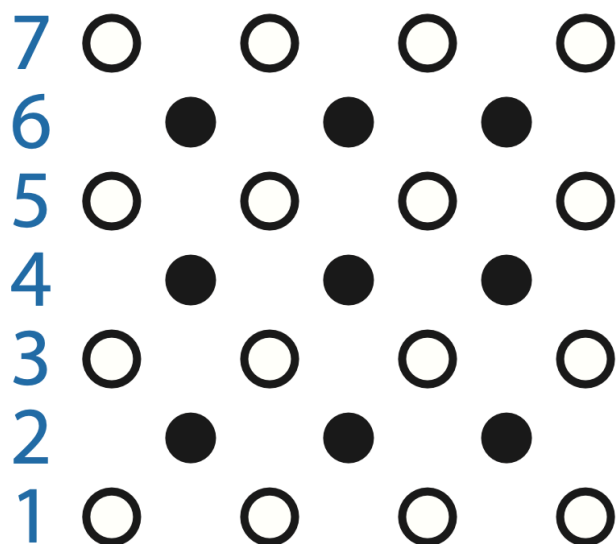
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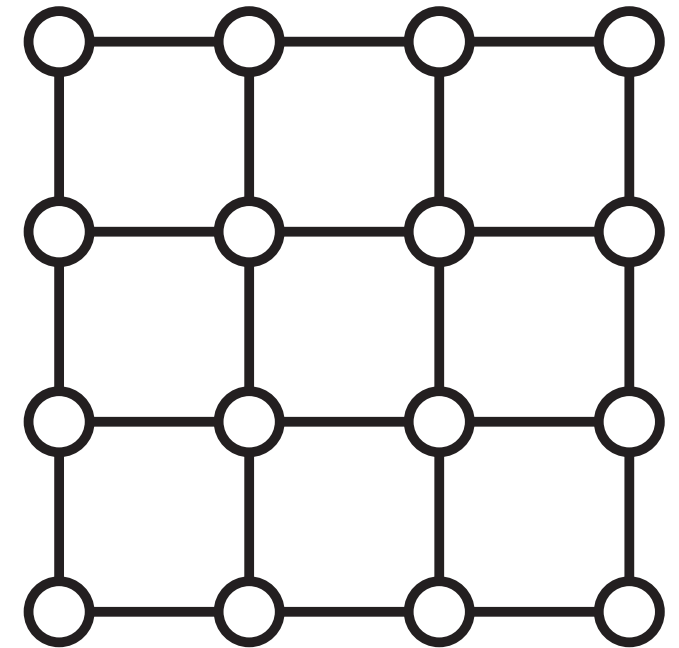


$$H_{\text{bulk}} = - \begin{array}{c} Z \\ \bullet \\ \circ Z \\ \bullet \\ Z \end{array} \begin{array}{c} X \\ \circ \\ \bullet X \end{array}$$

Bulk Construction: Example 2

- (2+1)D Plaquette Ising model:

$$H = -J \begin{array}{cc} Z \circ & \circ Z \\ | & | \\ Z \circ & \circ Z \end{array} - h \begin{array}{c} X \\ \circ \end{array}$$



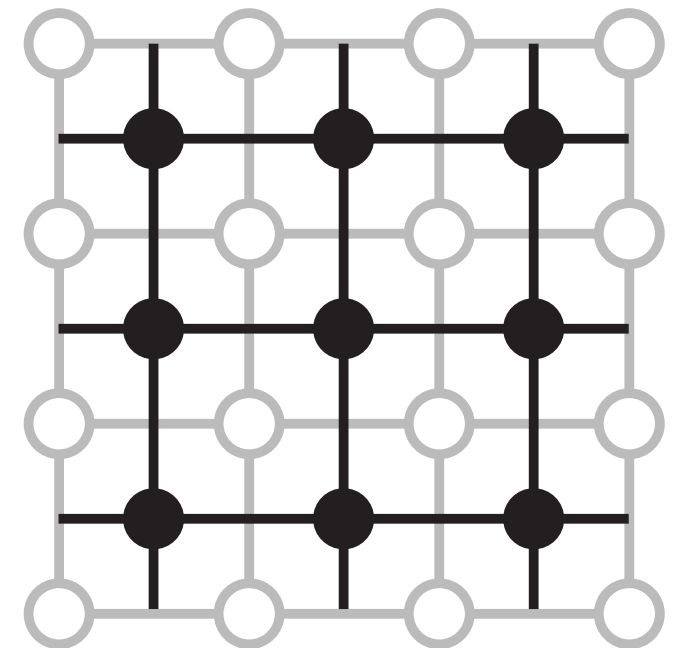
Bulk Construction: Example 2

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Bulk Construction: Example 2

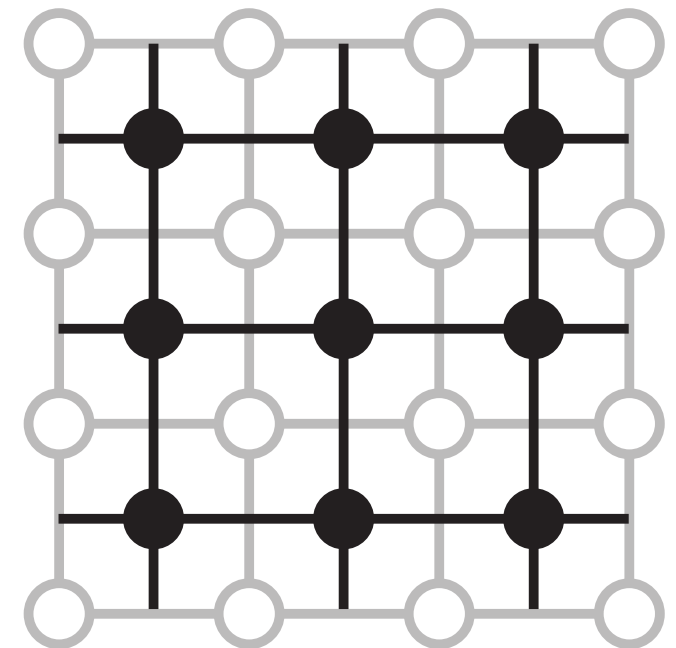
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- The bulk model: [Fuji, 1908.02257 \(PRB, '19\)](#).



$$H_{\text{bulk}} = - \begin{array}{cc} & Z \\ & \bullet \\ Z \circ & \circ Z \\ | & | \\ Z \circ & \circ Z \\ & \bullet \\ & Z \end{array} - \begin{array}{cc} & X \\ & \circ \\ X \bullet & \bullet X \\ | & | \\ X \bullet & \bullet X \\ & \circ \\ & X \end{array} \quad \left(\begin{array}{c} z \\ y \\ x \end{array} \right)$$

Bulk Construction: Example 2

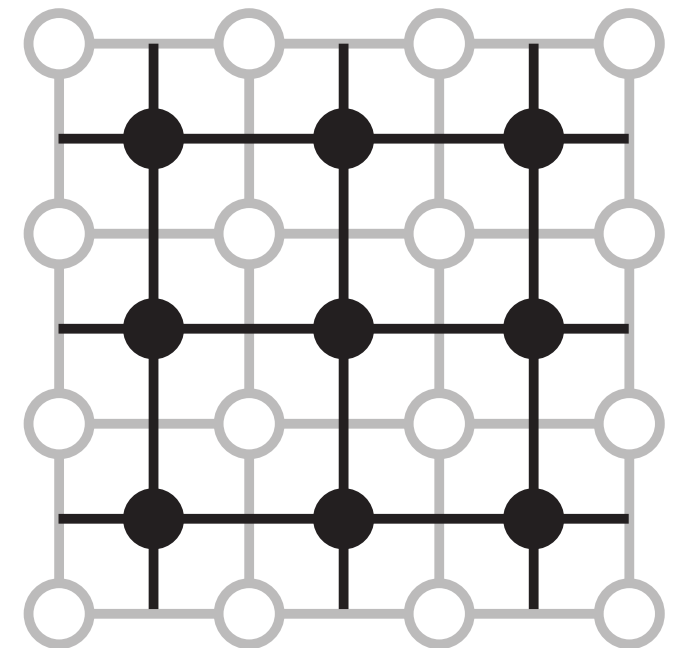
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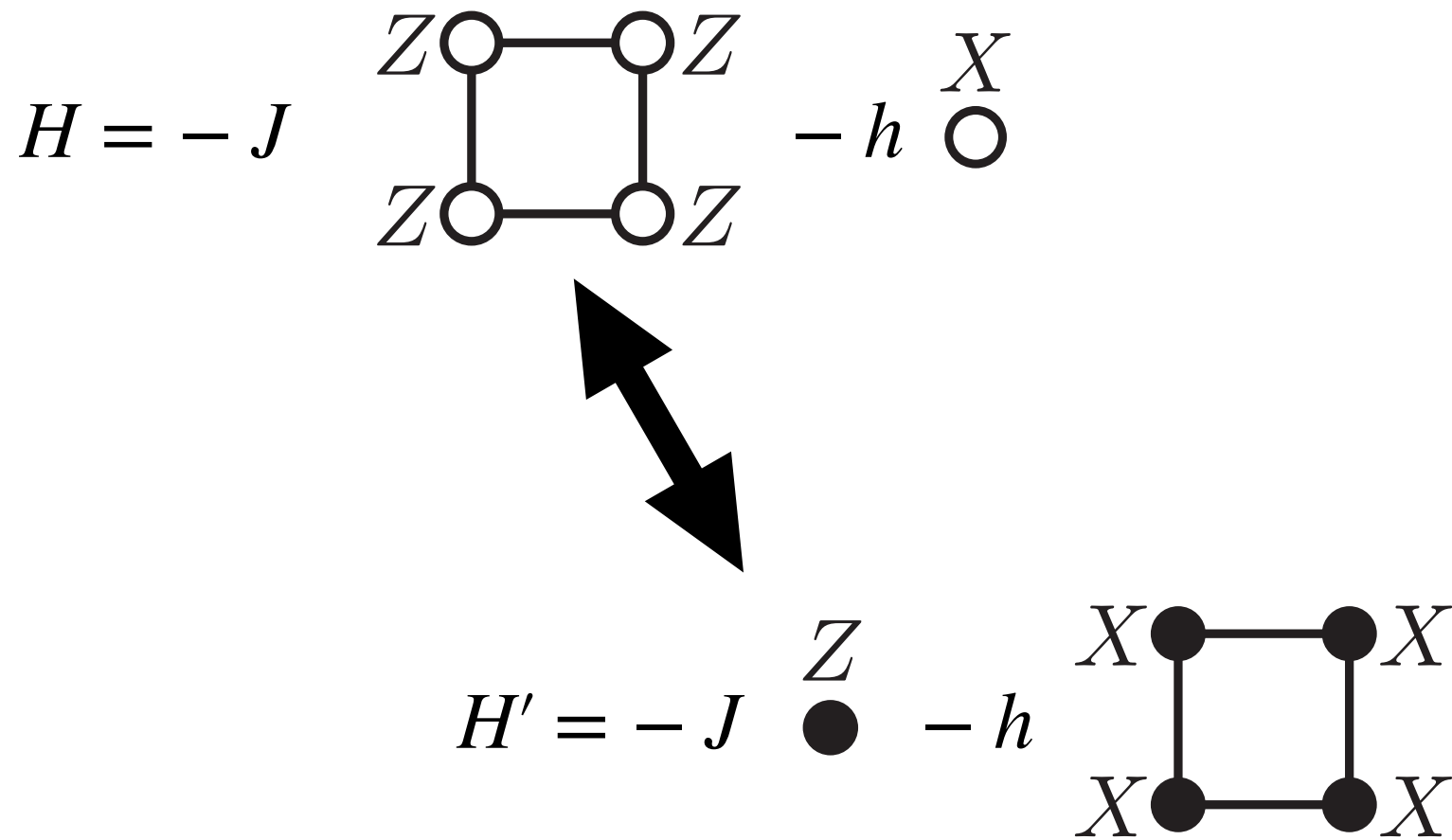
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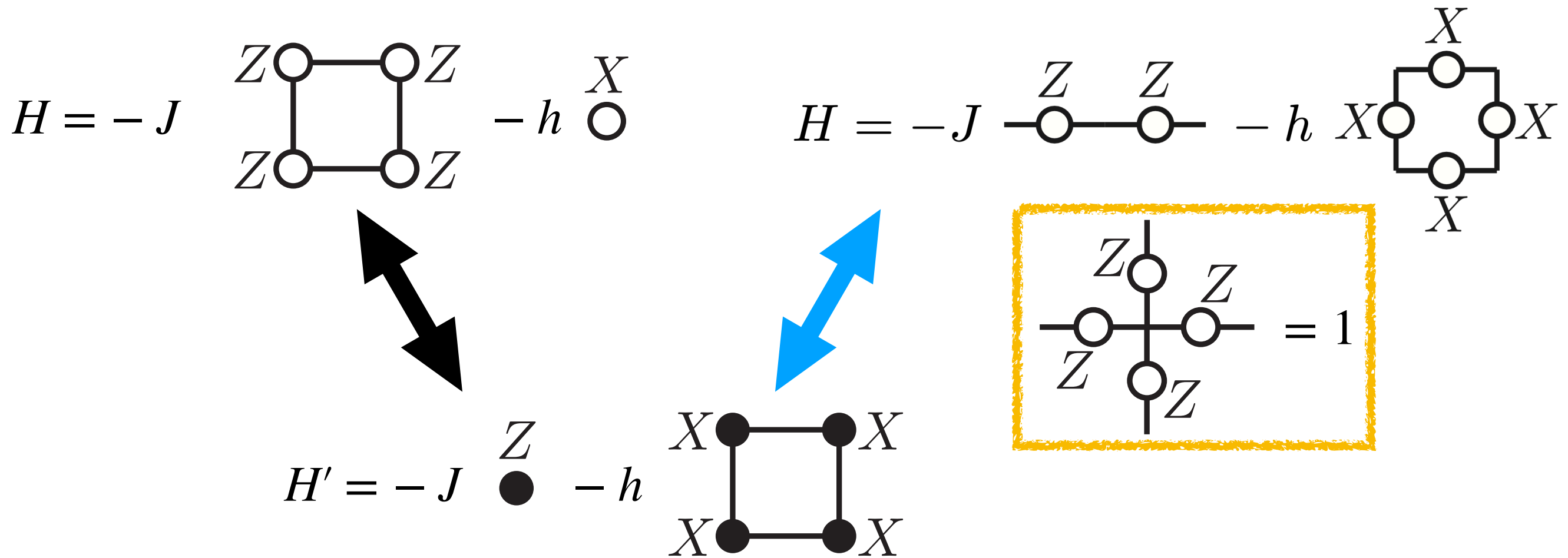
$$H_{\text{bulk}} = - \begin{array}{c} Z \\ \bullet \\ Z \circ \text{---} \circ Z \\ | \quad | \\ Z \circ \text{---} \circ Z \\ \bullet \\ Z \end{array} - \begin{array}{c} X \\ \circ \\ X \bullet \text{---} \bullet X \\ | \quad | \\ X \bullet \text{---} \bullet X \\ \circ \\ X \end{array} \quad \left(\begin{array}{c} z \\ y \\ x \end{array} \right)$$

Restricted mobility along x and y directions.

Bulk Construction: Example 3




Bulk Construction: Example 3



Bulk Construction: Example 3

$$H = -J \begin{array}{c} Z \circ - \circ Z \\ | \quad | \\ Z \circ - \circ Z \end{array} - h \begin{array}{c} X \\ \circ \end{array} \quad H = -J \begin{array}{c} Z \quad Z \\ \circ - \quad - \circ \end{array} - h \begin{array}{c} X \\ \circ \\ X \circ \quad \circ X \\ | \\ X \end{array}$$



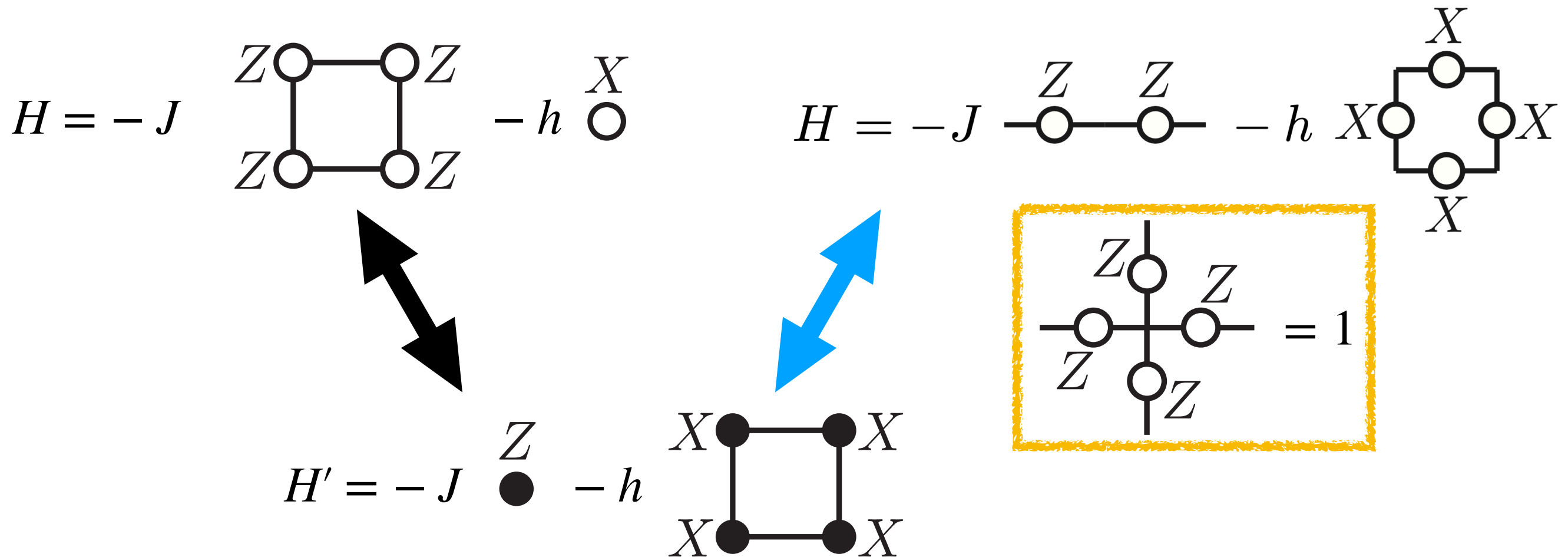
$$H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \bullet \quad \bullet X \\ | \quad | \\ X \bullet \quad \bullet X \end{array}$$

$$\begin{array}{c} Z \circ \quad \circ Z \\ | \quad | \\ \circ - \quad - \circ \\ | \quad | \\ Z \quad Z \end{array} = 1$$

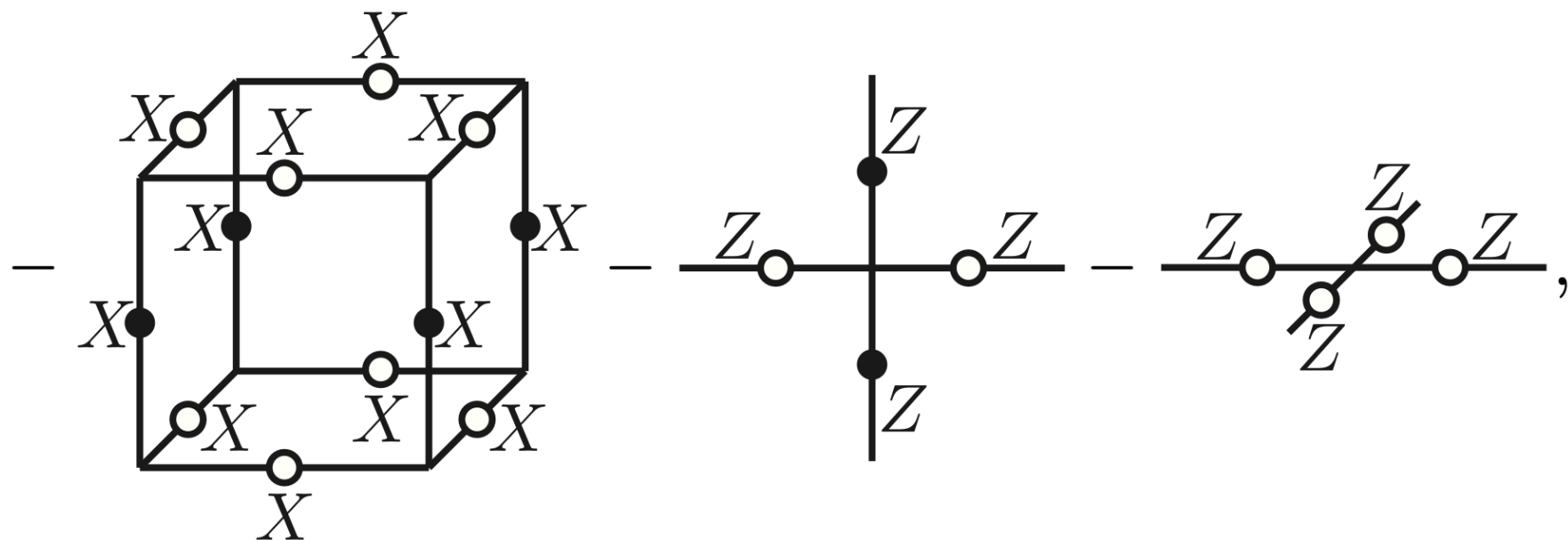
$H_{\text{bulk}} =$

$$- \begin{array}{c} \begin{array}{c} X \circ \quad \circ X \\ | \quad | \\ X \bullet \quad \bullet X \\ | \quad | \\ X \circ \quad \circ X \end{array} \\ \begin{array}{c} X \bullet \quad \bullet X \\ | \quad | \\ X \circ \quad \circ X \end{array} \end{array} - \begin{array}{c} Z \\ | \\ \circ - \quad - \circ \\ | \\ Z \end{array} - \begin{array}{c} Z \quad Z \\ \circ - \quad - \circ \\ | \quad | \\ Z \quad Z \end{array},$$

Bulk Construction: Example 3

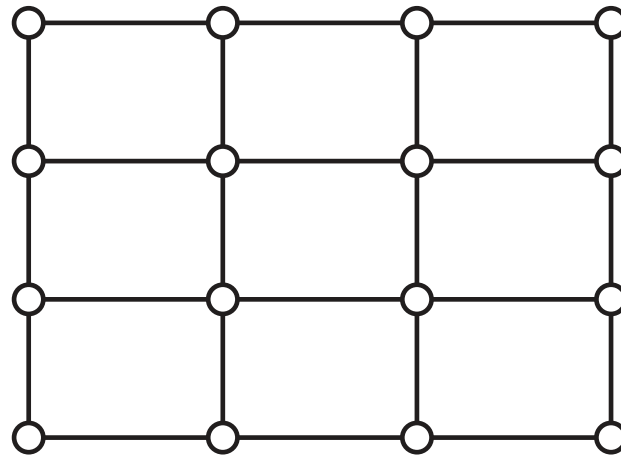


$H_{\text{bulk}} =$



Same boundary theory for two distinct bulks

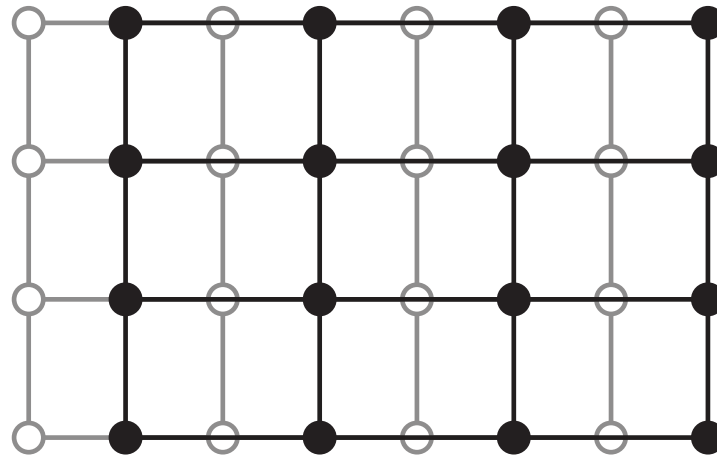
Bulk Construction: Example 4



$$H = -J \left(\begin{array}{cc} IX & II \\ XX & XI \end{array} + \begin{array}{cc} IZ & ZZ \\ II & ZI \end{array} \right) - h \left(\begin{array}{c} XI \\ IX \end{array} + \begin{array}{c} ZI \\ IZ \end{array} \right)$$

The equation defines the Hamiltonian H for the bulk construction. It consists of two terms. The first term is $-J$ multiplied by the sum of two 2x2 matrices of Pauli matrices. The first matrix has elements IX , II , XX , and XI . The second matrix has elements IZ , ZZ , II , and ZI . The second term is $-h$ multiplied by the sum of two vertical pairs of Pauli matrices. The first pair consists of XI (top) and IX (bottom). The second pair consists of ZI (top) and IZ (bottom).

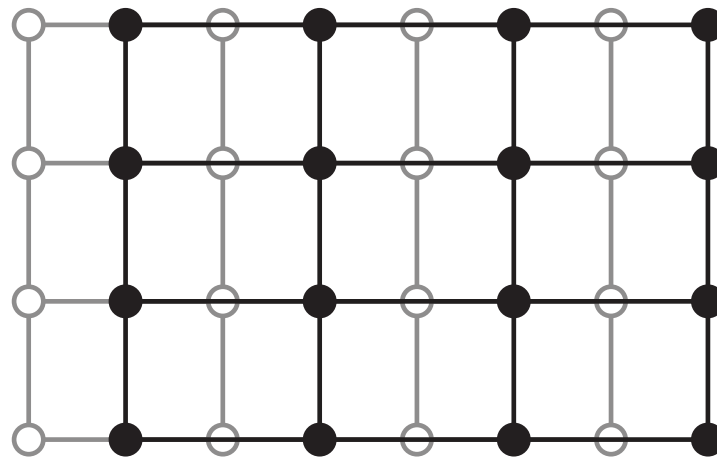
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$$H'(J, h) = H(h, J)$$

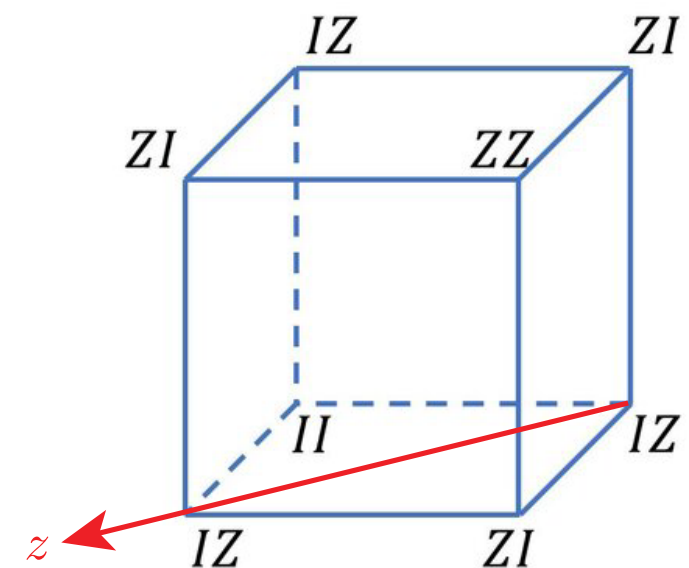
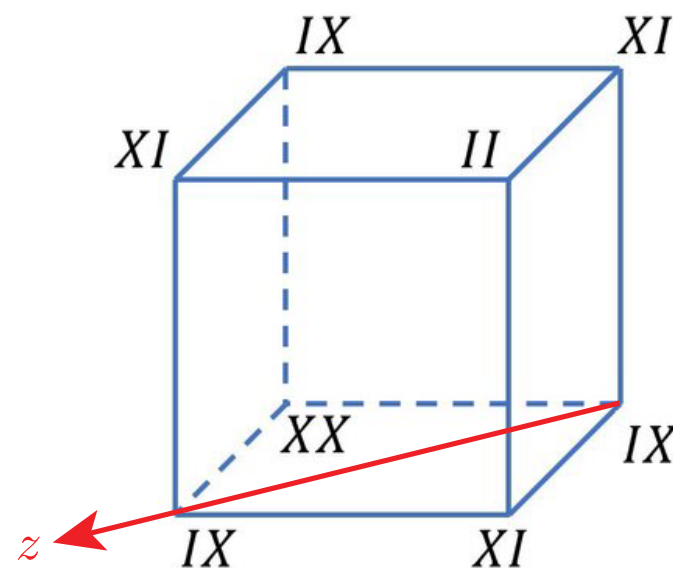
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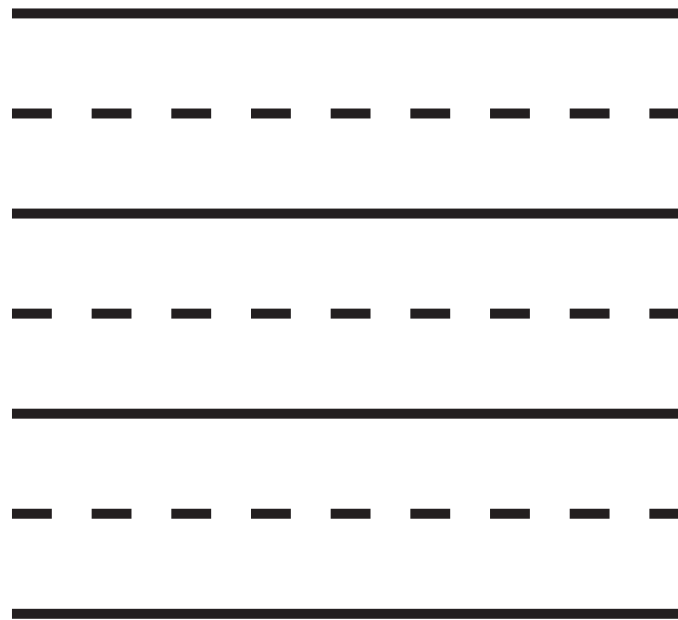
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Bulk Stabilizers:

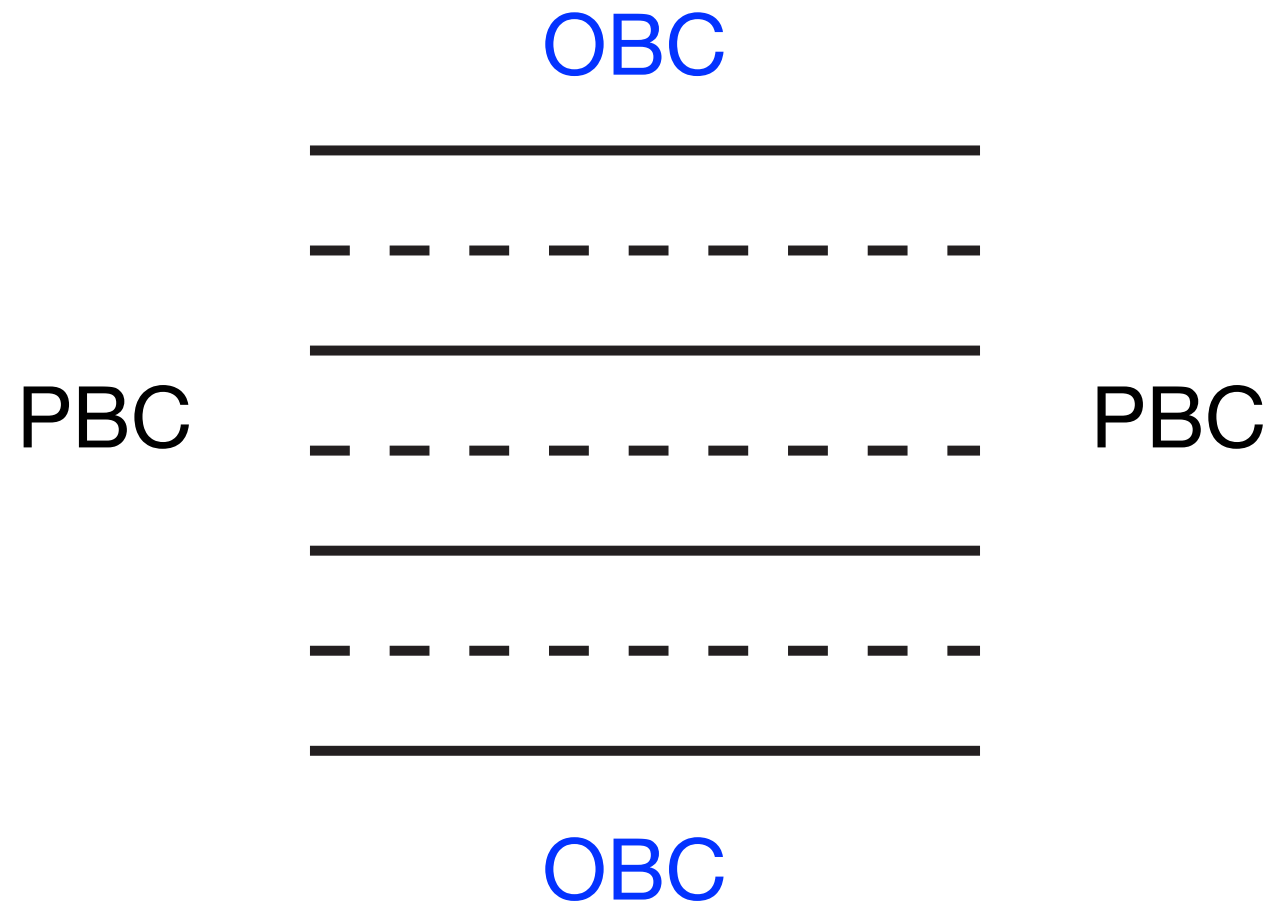
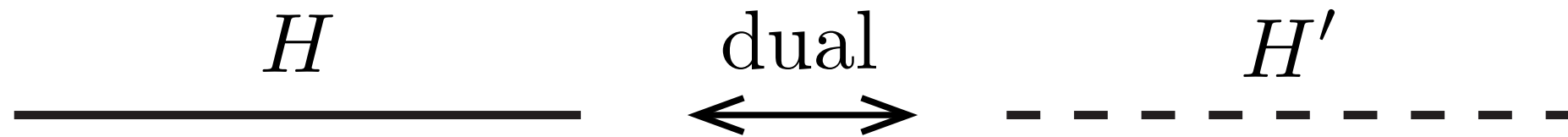


Bulk-Boundary Correspondence

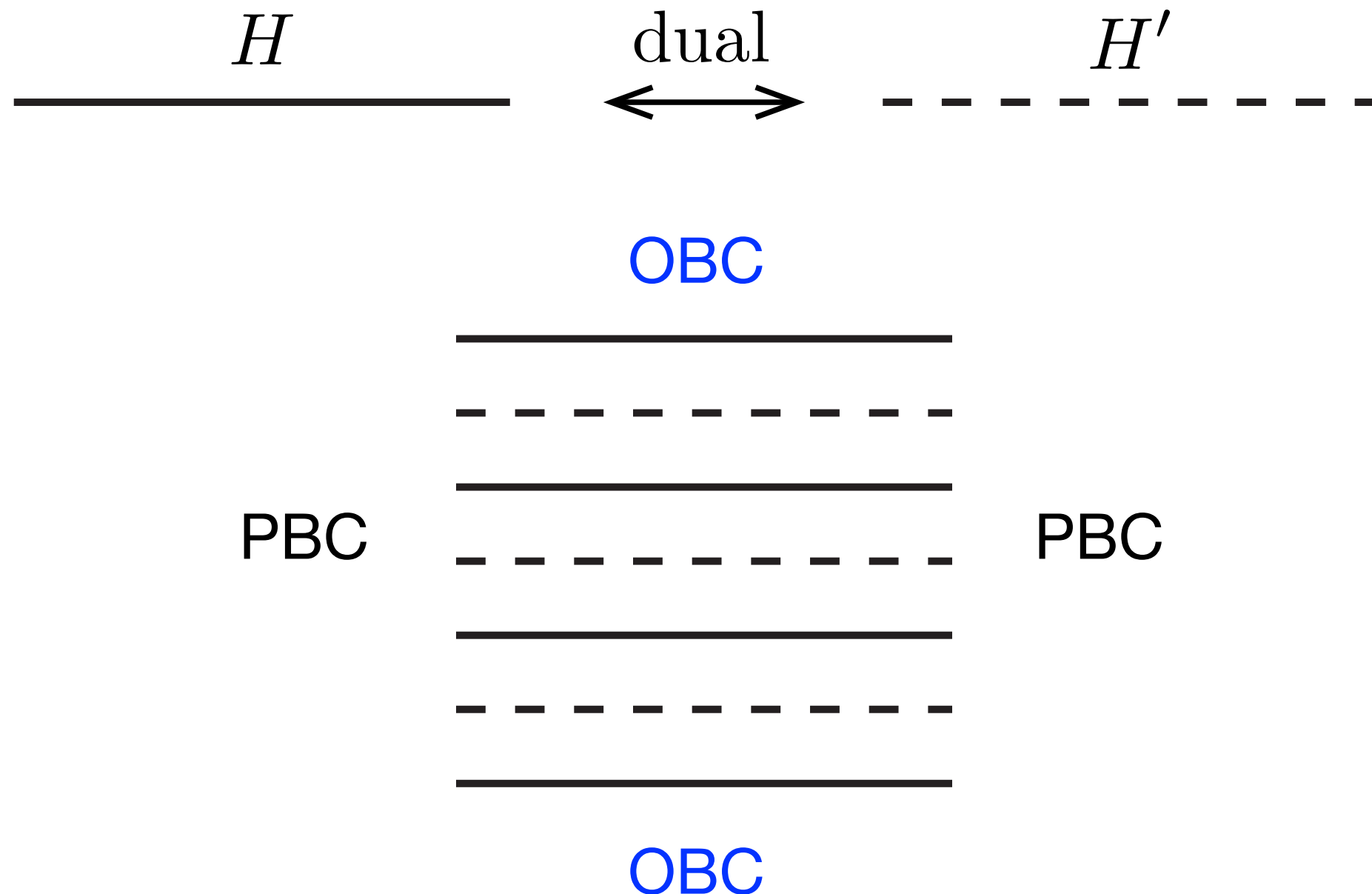
$$\overline{\quad\quad\quad}^H \quad \overset{\text{dual}}{\longleftrightarrow} \quad \text{---}^{H'}\text{---}$$



Bulk-Boundary Correspondence



Bulk-Boundary Correspondence



- The boundary theory is **certain charge and boundary condition sectors** of **two copies** of the model H .

Bulk-Boundary Correspondence: Example 1

$$H = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i \quad \Leftrightarrow \quad H' = -J \sum_i Z_{i+1/2} - h \sum_i X_{i-1/2} X_{i+1/2}$$

Ising

Dual Ising

Bulk = Toric Code

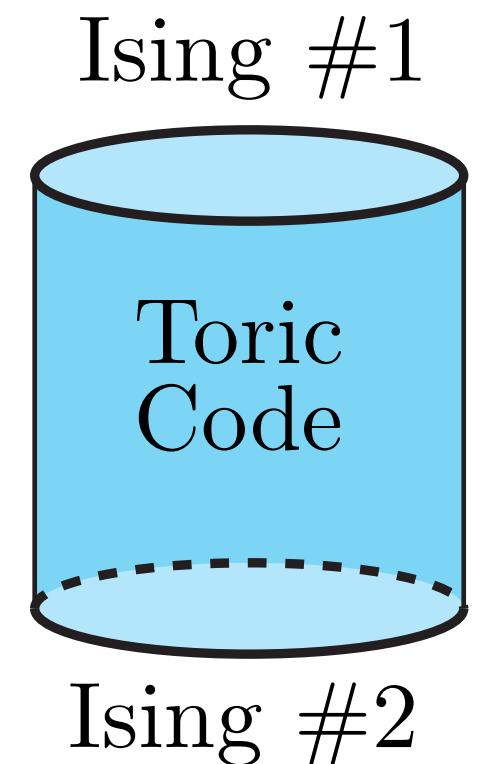
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Ising Dual Ising

Bulk = Toric Code

- The boundary theory \approx two copies of Ising:



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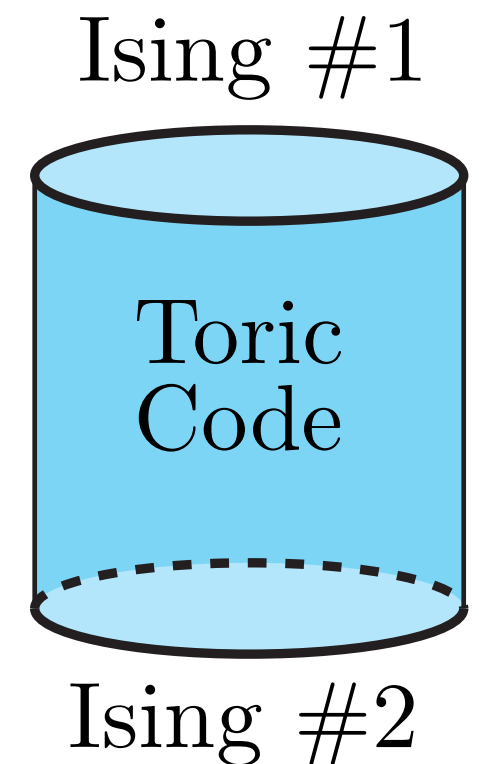
Ising Dual Ising

Bulk = Toric Code

- The boundary theory \approx two copies of Ising:

(1) Each copy can have states with **either** \mathbb{Z}_2 **charge**, but the **total charge must be trivial**.

$$U^X \otimes U^X = 1$$



Bulk-Boundary Correspondence: Example 1

$$H = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i \quad \Leftrightarrow \quad H' = -J \sum_i Z_{i+1/2} - h \sum_i X_{i-1/2} X_{i+1/2}$$

Ising Dual Ising

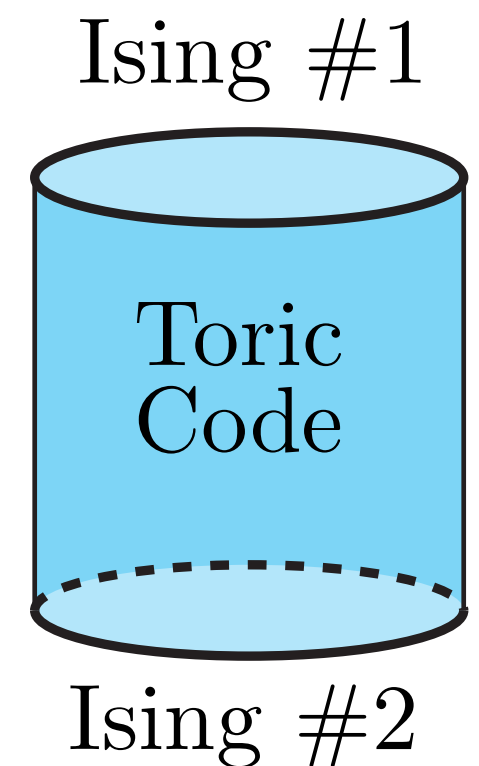
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Ising Dual Ising

Bulk = Toric Code

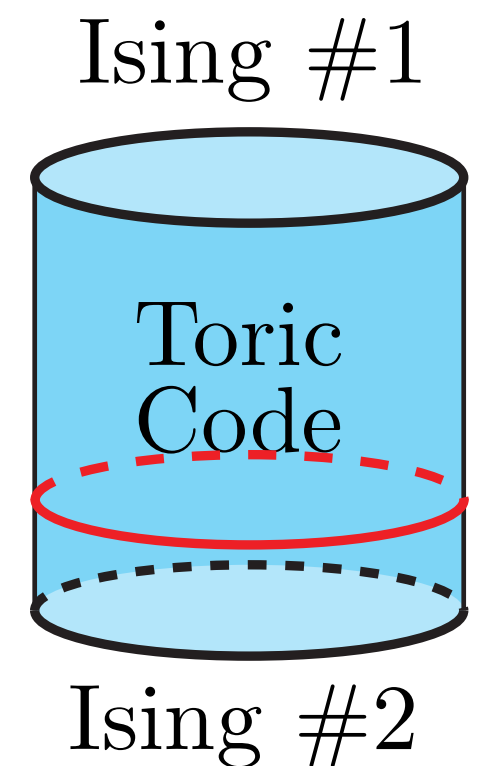
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boundary condition labeled by the bulk string



Bulk-Boundary Correspondence: Example 2

$$H = -J \begin{array}{c} Z \\ \circ \end{array} \begin{array}{c} Z \\ \circ \end{array} - h \begin{array}{c} X \\ \circ \\ \text{---} \circ \text{---} \\ \circ \\ X \end{array} \iff H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \bullet \text{---} \bullet X \\ | \quad | \\ X \bullet \text{---} \bullet X \end{array}$$

$$\begin{array}{c} Z \\ \circ \\ \text{---} \circ \text{---} \\ | \quad | \\ Z \quad \circ \end{array} = 1$$

Bulk = X-Cube Model

Bulk-Boundary Correspondence: Example 2

$$H = -J \begin{array}{c} Z \\ \circ \end{array} \begin{array}{c} Z \\ \circ \end{array} - h \begin{array}{c} X \\ \circ \\ \text{---} \circ \text{---} \\ \circ \\ X \end{array} \Leftrightarrow H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \bullet \text{---} \bullet X \\ \text{---} \\ X \bullet \text{---} \bullet X \end{array}$$

$$\begin{array}{c} Z \\ \circ \\ \text{---} \circ \text{---} \\ Z \end{array} = 1$$

Bulk = X-Cube Model

Boundary \approx Two Copies of the Model H

Bulk-Boundary Correspondence: Example 2

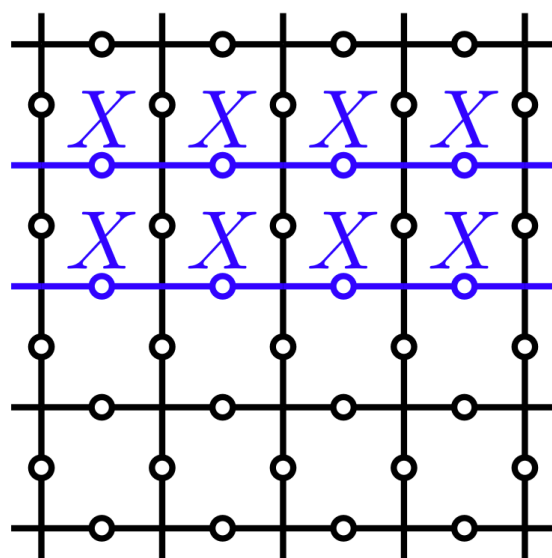
$$H = -J \begin{array}{c} Z \\ \circ \end{array} \begin{array}{c} Z \\ \circ \end{array} - h \begin{array}{c} X \\ \circ \\ \text{---} \circ \text{---} \\ \circ \\ X \end{array} \longleftrightarrow H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \bullet \text{---} \bullet X \\ | \\ X \bullet \text{---} \bullet X \end{array}$$

$$\begin{array}{c} Z \\ \circ \\ \text{---} \circ \text{---} \\ | \\ \circ \\ Z \end{array} = 1$$

Bulk = X-Cube Model

Boundary \approx Two Copies of the Model H

(1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1$.



(boundary layer)

Bulk-Boundary Correspondence: Example 2

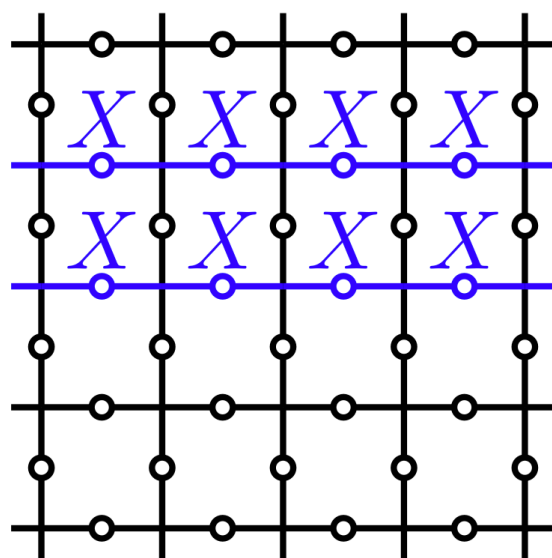
$$H = -J \begin{array}{c} Z \\ \circ \end{array} - \begin{array}{c} Z \\ \circ \end{array} - h \begin{array}{c} X \\ \circ \\ \text{---} \\ \circ \\ X \end{array} \iff H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \\ \bullet \\ \text{---} \\ \bullet \\ X \end{array}$$

A quantum circuit diagram enclosed in a yellow border. It features a CNOT gate with control on the top qubit and target on the bottom qubit. Each of the four qubits (top-left, top-right, bottom-left, and bottom-right) has a Z gate. The circuit is followed by an equals sign and the number 1.

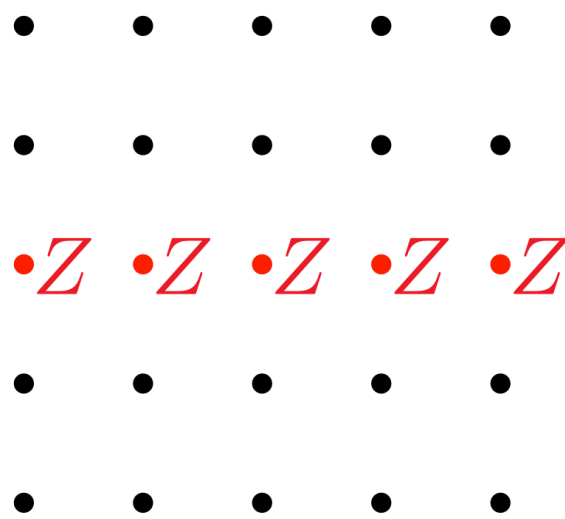
Bulk = X-Cube Model

Boundary \approx Two Copies of the Model H

- (1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1$.
- (2) Correlated boundary conditions.



(boundary layer)



(some internal layer)

Bulk-Boundary Correspondence: Example 2

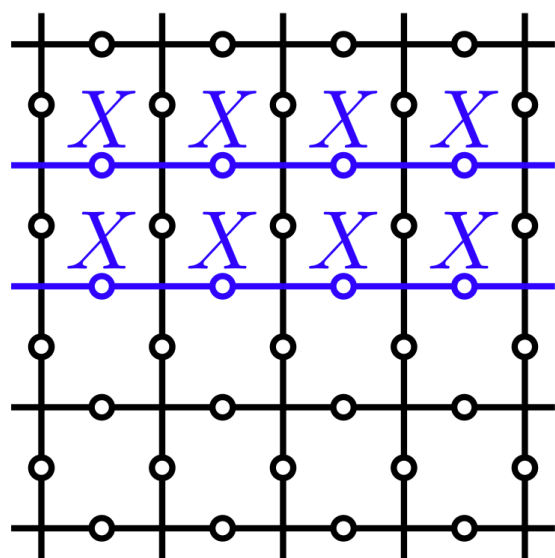
$$H = -J \begin{array}{c} Z \\ \circ \end{array} \begin{array}{c} Z \\ \circ \end{array} - h \begin{array}{c} X \\ \circ \\ \text{---} \circ \text{---} \\ \circ \\ X \end{array} \longleftrightarrow H' = -J \begin{array}{c} Z \\ \bullet \end{array} - h \begin{array}{c} X \bullet \quad \bullet X \\ \text{---} \quad \text{---} \\ X \bullet \quad \bullet X \end{array}$$

$$\begin{array}{c} Z \\ \circ \\ \text{---} \circ \text{---} \\ \circ \\ Z \end{array} = 1$$

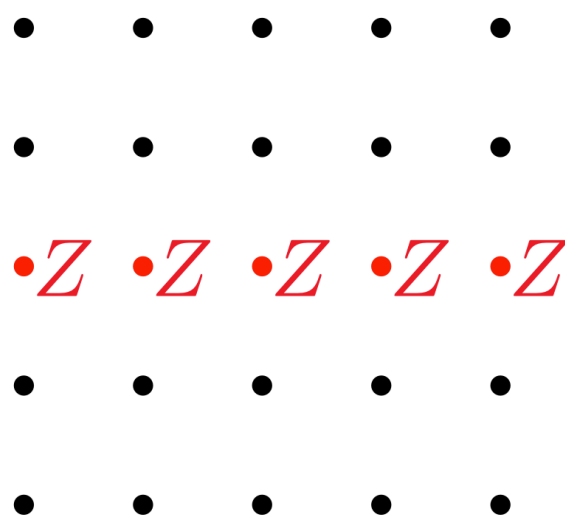
Bulk = X-Cube Model

Boundary \approx Two Copies of the Model H

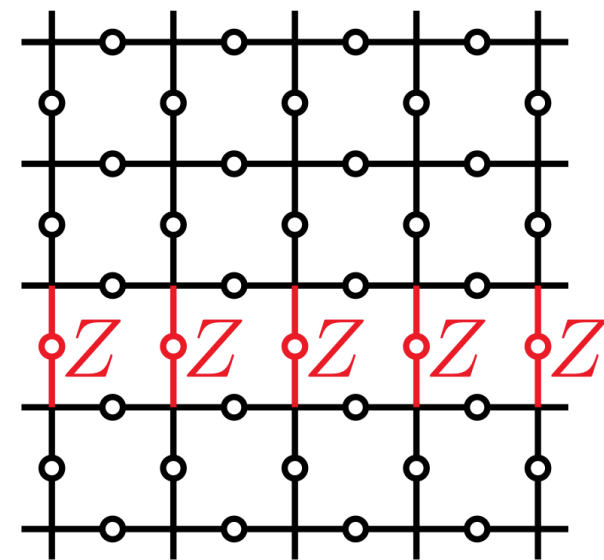
- (1) Symmetry charge projections: $U_k^X \otimes U_k^X = 1$.
- (2) Correlated boundary conditions.
- (3) Extra degeneracy.



(boundary layer)



(some internal layer)



(all internal layers)

Summary & Outlook

A bulk construction approach to the bulk-boundary correspondence of noninvertible topological phases.

- Subsystem symmetry / fracton.

SL, Ji, SciPost Phys (2023).

- Bulk excitations.
- Anomalous symmetries.
- New interesting topological/fracton orders.