Higher condensation theory and gauging of symmetries

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1. Category of topological defects
2. Higher condensation theory
3. Gauging the symmetries in topological orders

The higher condensation theory is a mathematical theory of condensation in (potentially anomalous) topological orders and SPT/SET/SSB orders that preserve the symmetry, and also a theory of the gauging of categorical symmetries in gapped/gapless QFTs.

Take home message

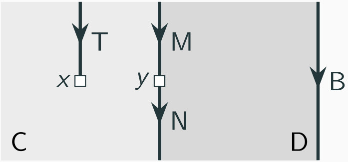
1. The topological defects in an n+1D topological order form a **fusion n-category** $\mathcal{C}$.
2. A codimension-k condensable topological defect is a **condensable E_k-algebra** in $\Omega^{k-1}\mathcal{C}$.
3. A codimension-k condensation of $A \in \mathrm{Alg}^{\mathrm{c}}_{\mathrm{E}_k}(\Omega^{k-1}\mathcal{C})$ can be reduced to a codimension-1 condensation of $\Sigma^{k-1}A \in \mathrm{Alg}^{\mathrm{c}}_{\mathrm{E}_1}(\mathcal{C})$.
4. A symmetry on an n+1D topological order is a monoidal n-functor $\phi \colon \mathcal{T} \to \mathcal{C}$ from the symmetry n-category $\mathcal{T}$ to the fusion n-category $\mathcal{C}$ of topological defects. Gauging this symmetry is the same as first choosing a condensable algebra $A \in \mathcal{T}$ and then condensing $\phi(A) \in \mathcal{C}$.

Category of topological defects

Topological order and topological defect

Topological orders are the simplest quantum phases or quantum field theories. They have no symmetry and are gapped, so the correlation functions exponentially decay. In the long wave length limit, there is no observable.

In the spirit of category theory, we should not study a single topological order, but the relationip between topological orders. An obvious relationship between topological orders is given by the domain walls, including boundary topological orders, junctions between domain walls, . . .

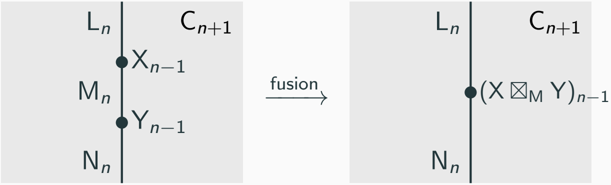


A **topological defect** in a topological order C is a lower-dimensional topological order embedded into C. More generally, we may also talk about topological defects in a general quantum field theory.

Mathematical structure of topological defects

The topological defects in an $n+1$D topological order naturally form a monoidal n-category:

- The 0-morphisms (objects) are codimension-1 topological defects.
- The 1-morphisms are codimension-2 topological defects.
- ...
- The k-morphisms are codimension-$(k+1)$ topological defects.
- The composition of morphisms, incuding the tensor product of objects, is given by the fusion of topological defects.



Mathematical structure of topological defects

The category of topological defects is not arbitrary. For example, it should satisfy the **Condensation Completion Principle** [Carqueville-Runkel: 1210.6363, Douglas-Reutter: 1812.11933, Gaiotto-Johnson-Freyd:1905.09566, Johnson-Freyd: 2003.06663, Kong-Lan-Wen-Z.-Zheng: 2003.08898]. This principle says that the category of topological defects is closed under condensation. I will explain the precise meaning of "condensation" later.

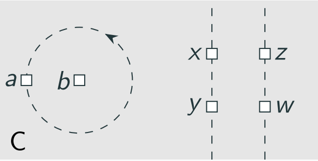
Mathematically, this principle motivates the definition of multi-fusion n-category [Douglas-Reutter: 1812.11933, Johnson-Freyd: 2003.06663, Kong-Zheng: 2011.02859]. Briefly speaking, a multi-fusion n-category is a monoidal n-category satisfying some properties, in which the most important one is condensation completeness.

The topological defects in an n+1D topological order C form a fusion n-category, denoted by $\mathcal{C}$.

Looping construction

The **looping** of $\mathcal{C}$ is the hom $(n-1)$-category on the tensor unit: $\Omega\mathcal{C} := \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$. Its physical meaning is the $(n-1)$-category topological defects of codimension 2 and higher.

The codimension-2 topological defects can be fused in two directions. By Eckmann-Hilton argument, these two fusions are the same, and their compatibility data is equivalent to a braiding structure. Therefore, $\Omega\mathcal{C}$ is a braided fusion $(n-1)$-category, also called a E_2-fusion $(n-1)$-category.



Similarly, the topological defects of codimension k can be fused in k directions, so they form the E_k-fusion $(n-k+1)$-category $\Omega^{k-1}\mathcal{C}$.

Delooping construction

Given a E_2-fusion $(n-1)$-category $\mathcal{B}$, there is a monoidal n-category $\mathrm{B}\mathcal{B}$ with only one object $\bullet$ and the hom space $\mathrm{Hom}(\bullet,\bullet) = \mathcal{B}$. The condensation completion of $\mathrm{B}\mathcal{B}$, denoted by $\Sigma\mathcal{B}$, is a fusion n-category, called the **delooping** of $\mathcal{B}$. We always have $\Omega\Sigma\mathcal{B} \simeq \mathcal{B}$.

For a fusion n-category $\mathcal{C}$, there is an embedding $\Sigma\Omega\mathcal{C} \to \mathcal{C}$. It is an equivalence iff $\mathcal{C}$ is **connected**: $\mathrm{Hom}(x,y) \neq 0$ for any $x, y \in \mathcal{C}$ (so that they can condense to each other).

Theorem [Johnson-Freyd: 2003,06663]

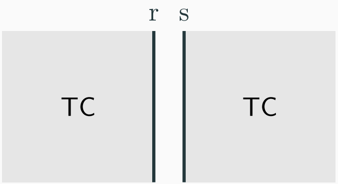
A fusion n-category $\mathcal{C}$ with trivial center is connected.

- fusion (tensor unit is simple) = stable = no local ground state degeneracy
- trivial center = bulk is trivial = anomaly-free, by the boundary-bulk relation [Kong-Wen-Zheng: 1502.01690, 1702.00673]
- connected = $\mathcal{C} \simeq \Sigma\Omega\mathcal{C}$ = the topological order can be equivalently described by the braided fusion $(n-1)$-category $\Omega\mathcal{C}$ of codimension-2 topological defects

Example: 2+1D toric code model

It is well-known that there are 4 simple anyons $\mathbb{1}, e, m, f$ in the 2+1D toric code model. They form a braided fusion category (indeed, modular tensor category) $\mathcal{TC} \simeq \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$. All topological defects in the 2+1D toric code model form a fusion 2-category equivalent to $\Sigma\mathcal{TC}$. There are 6 simple objects in $\Sigma\mathcal{TC}$: 2 invertible 1+1D domain walls, including the trivial domain wall and the e-m-exchange domain wall, and 4 non-invertible domain walls obtained by composing two gapped boundaries. [Lan-Wang-Wen: 1408.6514, Kong-Zhang: 2205,05565]

$\otimes$	unit	dual	ss	sr	rs	rr
unit	unit	dual	ss	sr	rs	rr
dual	dual	unit	rs	rr	ss	sr
ss	ss	sr	2ss	2sr	ss	sr
sr	sr	ss	ss	sr	2ss	2sr
rs	rs	rr	2rs	2rr	rs	rr
rr	rr	rs	rs	rr	2rs	2rr

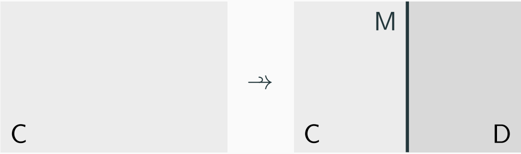


Higher condensation theory

Condensation and condensable algebra

Now we explain the meaning of condensation. Suppose there are two $n+1$D topological orders C, D and a condensation process from C to D. If we only do the condensation in a half of a plane, then we obtain an nD domain wall M between C and D.

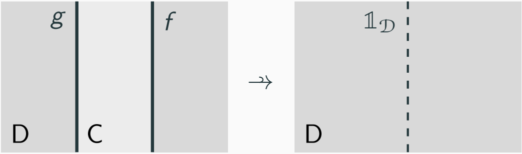
Assumption: M is also a topological order.



We denote this domain wall equipped with two orientations by $f : \mathrm{C} \to \mathrm{D}$ and $g : \mathrm{D} \to \mathrm{C}$, respectively. By composing the domain walls we obtain two codimension-1 topological defects $f \circ g \in \mathcal{D}$ and $A := g \circ f \in \mathcal{C}$.

Condensation and condensable algebra

The same condensation process produces a one-dimension-lower condensation process from $f \circ g \in \mathcal{D}$ to the trivial domain wall $\mathbb{1}_{\mathcal{D}} \in \mathcal{D}$.



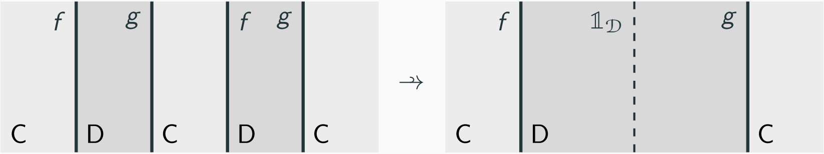
This motivates the following definition of condensation in higher categories.

Definition [Gaiotto-Johnson-Freyd: 1905.09566]

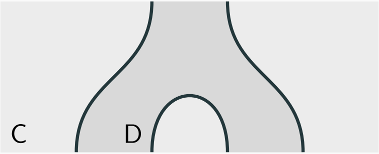
For two objects x, y in an n-category, an n-**condensation** $x \twoheadrightarrow y$ consists of two 1-morphisms $f : x \to y$, $g : y \to x$ and an $(n-1)$-condensation $f \circ g \twoheadrightarrow 1_y$. A 0-condensation is an equality.

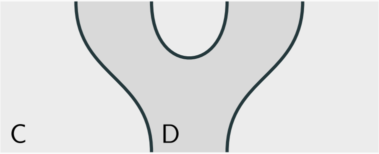
Condensation and condensable algebra

The same condensation process also produces a condensation from $A \circ A = g \circ f \circ g \circ f$ to $A = g \circ f = g \circ \mathbb{1}_{\mathcal{D}} \circ f$.



This condensation produces two lower-dimensional topological defects (morphisms) $\mu \colon A \otimes A \to A$ and $\delta \colon A \to A \otimes A$ in $\mathcal{C}$:

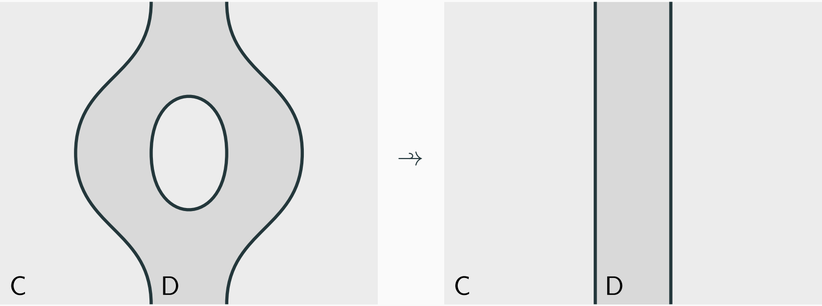




and obviously they satisfy the associativity and Frobenius condition. So $A \in \mathcal{C}$ is an algebra.

Condensation and condensable algebra

Moreover, there is a condensation $\mu \circ \delta \twoheadrightarrow 1_A$:



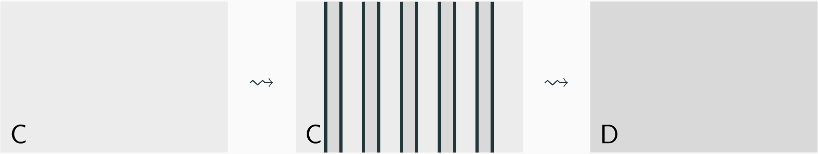
This means that $A \in \mathcal{C}$ is a **separable algebra**, also called a **condensable algebra**.

Similarly, one can show that the domain walls between C and D are A-modules.

Condensation and condensable algebra

The physical intuition of the condensation $\mathsf{C} \twoheadrightarrow \mathsf{D}$:

1. Proliferate many condensable algebra (defect) A in the topological order C.
2. Use the condensation $A \otimes A \twoheadrightarrow A$ to condense (or project) these condensable defects.



Delooping construction, revisit

Let $\mathcal{B}$ be a braided fusion $(n-1)$-category, viewed as the category of codimension-2 topological defects in an n+1D topological order. Recall that $\Sigma\mathcal{B}$ is the condensation completion of $\mathrm{B}\mathcal{B}$.

The objects in $\Sigma\mathcal{B}$ are codimension-1 topological defects that can be condensed from the trivial domain wall. Therefore, every condensable algebra in $\Omega\Sigma\mathcal{B} = \mathcal{B}$ labels an object in $\Sigma\mathcal{B}$.

This idea leads to two 'models' or 'coordinate systems' of $\Sigma\mathcal{B}$.

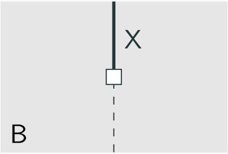
- $\mathrm{Mor}^{\mathrm{c}}_{\mathrm{E}_1}(\mathcal{B})$: the objects are condensable algebras in $\mathcal{B}$, the 1-morphisms are bimodules, the 2-morphisms are bimodule 1-morphisms, ... the tensor product is the tensor of $\mathcal{B}$.
- $\mathrm{RMod}_{\mathcal{B}}(n\mathrm{Vec})$: the objects are right $\mathcal{B}$-module $(n-1)$-categories, the 1-morphisms are right $\mathcal{B}$-module functors, the 2-morphisms are right $\mathcal{B}$-module natural transformations, ... the tensor product is the relative tensor $\boxtimes_{\mathcal{B}}$.

There are equivalences of fusion n-categories $\Sigma\mathcal{B} \simeq \mathrm{Mor}^{\mathrm{c}}_{\mathrm{E}_1}(\mathcal{B}) \simeq \mathrm{RMod}_{\mathcal{B}}(n\mathrm{Vec})$.

Delooping construction, revisit

The physical realization of the equivalences $\Sigma\mathcal{B} \simeq \mathrm{Mor}^{\mathrm{c}}_{\mathrm{E}_1}(\mathcal{B}) \simeq \mathrm{RMod}_{\mathcal{B}}((n-1)\mathrm{Vec})$:

- The equivalence $\mathrm{Mor}^{\mathrm{c}}_{\mathrm{E}_1}(\mathcal{B}) \to \Sigma\mathcal{B}$ maps a condensable algebra A to the codimension-1 domain wall obtained by condensing A.
- The equivalence $\Sigma\mathcal{B} \to \mathrm{RMod}_{\mathcal{B}}(n\mathrm{Vec})$ maps a codimension-1 domain wall X to the $(n-1)$-category of topological defects living on the codimension-2 domain wall between X and the trivial domain wall.



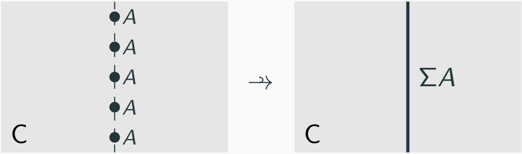
Let X be a codimension-1 domain wall X obtained by condensing a condensable algebra $A \in \mathcal{B}$. In the coordinate system $\Sigma\mathcal{B} \simeq \mathrm{RMod}_{\mathcal{B}}(n\mathrm{Vec})$ it is $\mathrm{LMod}_A(\mathcal{B})$. The fusion $(n-1)$-category of defects on X is $\mathrm{Hom}_{\Sigma\mathcal{B}}(\mathrm{X},\mathrm{X}) = \mathrm{Hom}_{\mathrm{RMod}_{\mathcal{B}}(n\mathrm{Vec})}(\mathrm{LMod}_A(\mathcal{B}), \mathrm{LMod}_A(\mathcal{B})) = \mathrm{BMod}_{A|A}(\mathcal{B})$.

Condensation of higher-codimensional defects

Theorem [Kong-Z.-Zheng-Zhao:2403.07813]: Condensing a codimension-k topological defect A in an n+1D topological order C amounts to a k-step process.

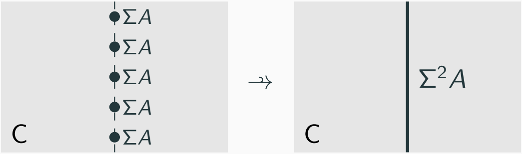
(0) $A \in \Omega^{k-1}\mathcal{C}$ is a condensable E_k-algebra, that is, a condensable algebra equipped with compatible multiplications in k directions.

(1) In the first step, we condense A along one transversal direction. This condensation process produces a codimenion-$(k-1)$ topological defects $\Sigma A \in \Sigma\Omega^{k-1}\mathcal{C} \subseteq \Omega^{k-2}\mathcal{C}$. Mathematically, ΣA is $\mathrm{RMod}_A(\Omega^{k-1}\mathcal{C})$.



Condensation of higher-codimensional defects

(2) There is a natural condensable E_{k-1}-algebra structure on $\Sigma A \in \Omega^{k-2}\mathcal{C}$. So the second step is to condense ΣA along another transversal direction. Then we obtain a codimension $(k-2)$ topological defects $\Sigma^2 A := \mathrm{RMod}_{\Sigma A}(\Omega^{k-2}\mathcal{C}) \in \Sigma\Omega^{k-2}\mathcal{C} \subseteq \Omega^{k-3}\mathcal{C}$.

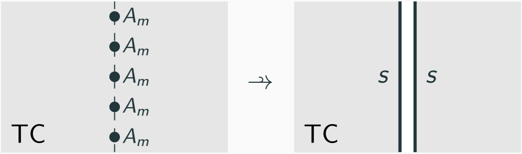


(3) In the k-th step, we condense the codimension-1 topological defect $\Sigma^{k-1}A$, which is a condensable E_1-algebra in $\mathcal{C}$. The topological defects in the condensed phase D form the fusion n-category $\mathcal{D} := \mathrm{Mod}_A^{\mathrm{E}_1}(\mathcal{C}) := \mathrm{BMod}_{\Sigma^{k-1}A|\Sigma^{k-1}A}(\mathcal{C})$. The domain walls between C and D form the n-category $\mathrm{RMod}_{\Sigma^{k-1}A}(\mathcal{C})$, and the domain wall produced by this condensation is exactly $\Sigma^{k-1}A$ itself.

A condensable E_k-algebra is called **Lagrangian** if the condensed phase D is trivial.

Example: codimension-2 condensation in 2+1D toric code model

1. $A_m := \mathbb{1} \oplus m$ is a Lagrangian condensable E_2-algebra in $\mathcal{TC}$. Condensing A_m along a line produces the codimension-1 domain wall $\mathrm{ss} \in \Sigma\mathcal{TC}$. Then by condensing ss we obtain the trivial topological order and the smooth boundary.



2. $A_e := \mathbb{1} \oplus e$ is a Lagrangian condensable E_2-algebra in $\mathcal{TC}$. Condensing A_m along a line produces the codimension-1 domain wall $\mathrm{rr} \in \Sigma\mathcal{TC}$. Then by condensing rr we obtain the trivial topological order and the rough boundary.

Example: anyon condensation in 2+1D

Recall the mathematical theory of anyon condensation in 2+1D topological orders [Moore-Seiberg: 1988–1989, Bais-Slingerland: 2002–2008, Kapustin-Saulina: 1008.0654, Levin: 1301.7355, Barkeshli-Jian-Qi: 1305.7203, ..., Böckenhauer-Evans-Kawahigashi: math/9904109, 0002154, Kirillov-Ostrik: math/0101219, Frölich-Fuchs-Runkel-Schweigert: math/0309465, Kong: 1307.8244]

Let $\mathcal{B}$ be the E_2-fusion category (modular tensor category) of anyons in a 2+1D topological order and $A \in \mathcal{C}$ be a condensable E_2-algebra.

(1) The anyons in the condensed phase (de-confined anyons) form the E_2-fusion category $\mathrm{Mod}_A^{\mathrm{E}_2}(\mathcal{B})$ of local A-modules in $\mathcal{C}$.

(2) The anyons on the domain wall (confined anyons) produced by this condensation form the E_1-fusion category $\mathrm{RMod}_A(\mathcal{B})$ of right A-modules in $\mathcal{B}$.

Example: anyon condensation in 2+1D

On the other hand, we know that condensing A along a line produces a codimension-1 domain wall $\mathrm{LMod}_A(\mathcal{B}) \in \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec}) \simeq \Sigma\mathcal{B}$, which should be a condensable E_1-algebra.

Theorem [Brochier-Jordan-Synder: 1804.07538, Décoppet: 2107.11037, 2208.08722]

The condensable E_1-algebras in $\Sigma\mathcal{B} \simeq \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec})$ are multi-fusion categories $\mathcal{P}$ equipped with a braided functor $\mathcal{B} \to \mathfrak{Z}_1(\mathcal{P})$. Moreover, we have

$$\mathrm{RMod}_{\mathcal{P}}(\Sigma\mathcal{B}) \simeq \Sigma\mathcal{P}, \quad \mathrm{BMod}_{\mathcal{P}|\mathcal{P}}(\Sigma\mathcal{B}) \simeq \Sigma\overline{\mathfrak{Z}_2(\mathcal{B} \to \mathfrak{Z}_1(\mathcal{P}))}.$$

When $\mathcal{P} \simeq \mathrm{LMod}_A(\mathcal{B})$, there is an equivalence [Davydov-Müger-Nikshych-Ostrik: 1009.2117]

$$\overline{\mathfrak{Z}_2(\mathcal{B} \to \mathfrak{Z}_1(\mathcal{P}))} \simeq \mathrm{Mod}_A^{\mathrm{E}_2}(\mathcal{B}).$$

So the result of the codimension-1 condensation agrees with the anyon condensation theory.

Example: 2+1D double Ising topological order and toric code model

Let $\mathcal{I}s$ be the Ising modular tensor category. It has 3 simple objects $\mathbb{1}, \psi, \sigma$ and the fusion rules

$$\psi \otimes \psi = \mathbb{1}, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi, \quad \psi \otimes \sigma = \sigma \otimes \psi = \sigma.$$

The codimension-2 topological defects in the 2+1D double Ising topological order form the E_2-fusion category $\mathfrak{Z}_1(\mathcal{I}s) \simeq \mathcal{I}s \boxtimes \overline{\mathcal{I}s}$. It is well-known that $A := \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathcal{I}s \boxtimes \overline{\mathcal{I}s}$ is a condensable E_2-algebra, and condensing A leads to the toric code [Bais-Slingerland: 0808.0627]:

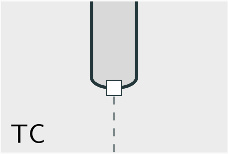
$$\mathrm{Mod}_A^{\mathrm{E}_2}(\mathcal{I}s \boxtimes \overline{\mathcal{I}s}) \simeq \mathcal{TC}.$$

There are 6 simple topological defects living on the domain wall produced by this condensation:

$$\mathbb{1}, e, m, f, \chi_+, \chi_- \in \mathrm{RMod}_A(\mathcal{I}s \boxtimes \overline{\mathcal{I}s}).$$

Example: 2+1D double Ising topological order and toric code model

There is no anyon condensation process from the toric code model to the double Ising topological order. However, there is a codimension-1 condensation from toric code to the double Ising topological order. The condensable E_1-algebra in $\Sigma\mathcal{TC}$ is obtained by composing two domain walls.



In the coordinate system $\Sigma\mathcal{TC} \simeq \mathsf{RMod}_{\mathcal{TC}}(2\mathrm{Vec})$, this condensable E_1-algebra is the category $\mathsf{RMod}_A(\mathcal{I}\mathrm{s} \boxtimes \overline{\mathcal{I}\mathrm{s}})$ of topological defects living on the domain wall. We can check that

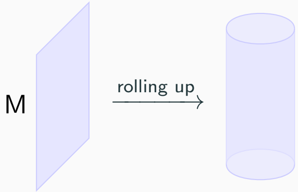
$$\mathsf{Mod}^{\mathrm{E}_1}_{\mathsf{RMod}_A(\mathcal{I}\mathrm{s}\boxtimes\overline{\mathcal{I}\mathrm{s}})}(\Sigma\mathcal{TC}) \simeq \Sigma(\mathcal{I}\mathrm{s} \boxtimes \overline{\mathcal{I}\mathrm{s}}).$$

Example: condense a modular tensor category

Theorem [Davydov-Nikshych: 2006.08022]

Every modular tensor category $\mathcal{B}$ is a Lagrangian condensable E_2-algebra in $2\mathrm{Vec}$.

- $2\mathrm{Vec}$ is the E_2-fusion 2-category of codimension-2 topological defects in the trivial 3+1D topological order.
- An anomaly-free 2+1D topological order M can be viewed as a boundary of the trivial 3+1D topological order. Let $\mathcal{B}$ be the modular tensor category of anyons in M.
- This boundary can be produced by a codimension-2 condensation. The condensable E_2-algebra can be obtained by rolling up M to a string. This string is exactly $\mathcal{B} \in 2\mathrm{Vec}$.



Gauging the symmetries in topological orders

SET orders and gauging

A topological order with a symmetry is usually called a symmetry enriched topological (SET) order. Here we focus on finite onsite symmetries. An n+1D SET orders could be described by a linear monoidal n-functor $\phi\colon \mathcal{T} \to \mathcal{C}$. [Lan-Yue-Wang: 2312.15958, preceded by Frölich-Fuchs-Runkel-Schweigert: cond-mat/0404051, hep-th/0607247, 0909.5013, Davydov-Kong-Runkel: 1107.0495, Bhardwaj-Tachikawa: 1704.02330, Chang-Lin-Shao-Wang-Yin: 1802.04445, Thorngren-Wang: 1912.02817, ...]

- $\mathcal{T}$ is an abstract fusion n-category describing the symmetry:
  - $\mathcal{T} = n\mathrm{Vec}_G$ for a finite group G symmetry;
  - $\mathcal{T} = n\mathrm{Vec}_{\mathcal{G}}$ for a higher form symmetry described by an n-group $\mathcal{G}$;
  - $\mathcal{T}$ is a general fusion n-category for non-invertible symmetries;
- $\mathcal{C}$ is the fusion n-category of topological defects without the symmetry.

To gauge the symmetry, first we need to choose a condensable algebra $A \in \mathcal{T}$ (or equivalently, a $\mathcal{T}$-module). Then gauging the symmetry is the same as condensing the algebra $\phi(A) \in \mathcal{C}$.

Remark: The gauged theory has a dual symmetry described by the monoidal n-functor $\phi\colon\ \mathrm{BMod}_{A|A}(\mathcal{T}) \to \mathrm{BMod}_{\phi(A)|\phi(A)}(\mathcal{C})$.

Example: G SPT orders

Let G be a finite group. An n+1D G SPT order is described by a monoidal n-functor $\phi\colon n\mathrm{Vec}_G \to n\mathrm{Vec}$. For $n \leq 3$, such monoidal n-functors are classified by the group cohomology $H^{n+1}(G;\mathbb{C}^\times)$. For $n \geq 4$, there may be more SPT orders. For simplicity, we consider the trivial SPT order described by the trivial monoidal n-functor $n\mathrm{Vec}_G \to n\mathrm{Vec}$.

- The usual way of gauging the symmetry is the same as condensing $(n-1)\mathrm{Vec}_G \in n\mathrm{Vec}_G$. For $n \geq 2$, the result is the G gauge theory:

$$\mathrm{BMod}_{(n-1)\mathrm{Vec}_G|(n-1)\mathrm{Vec}_G}(n\mathrm{Vec}) \simeq \mathfrak{Z}_0(\Sigma(n-1)\mathrm{Vec}_G) \simeq \Sigma\mathfrak{Z}_1((n-1)\mathrm{Vec}_G).$$

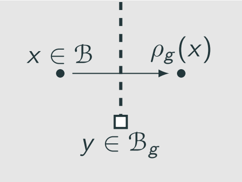
- The 'twisted gauging' is the same as condensing the algebra $(n-1)\mathrm{Vec}_G^\omega \in n\mathrm{Vec}_G$ where $\omega \in H^{n+1}(G;\mathbb{C}^\times)$. The condensed phase is the twisted G gauge theory.
- The partial gauging is the same as condensing the algebra $(n-1)\mathrm{Vec}_H \in n\mathrm{Vec}_G$ where H is a subgroup of G.

Example: 2+1D SET orders and G-crossed braided fusion category

Let $\mathcal{B}$ be the E_2-fusion category (modular tensor category) of anyons in a 2+1D topological order. A finite group G symmetry on this topological order can permute anyons, thus defines a monoidal functor $\rho\colon G \to \mathrm{Aut}^{\mathrm{br}}(\mathcal{B})$. Moreover, a symmetry enrichment on this topological order is desribed by a G-crossed braided fusion category [Barkeshli-Bonderson-Cheng-Wang: 1410.4540]:

$$\mathcal{B}_G^\times = \bigoplus_{g\in G} \mathcal{B}_g , \quad \mathcal{B}_e = \mathcal{B},$$

where the category $\mathcal{B}_g$ consists of the g-defects living on the end of an invertible 1+1D domain wall realizing the braided auto-equivalence ρ_g.



Example: 2+1D SET orders and G-crossed braided fusion category

This G-crossed extension of $\mathcal{B}$ is equivalent to a 3-group homomorphism $G \to \mathrm{Pic}(\mathcal{B})$ [Etingof-Nikshych-Ostrik: 0909.3140], where $\mathrm{Pic}(\mathcal{B})$ is the 3-group of invertible $\mathcal{B}$-modules. This 3-group homomorphism can be constructed layer by layer:

- The map $g \mapsto \mathcal{B}_G$ is a group homomorphism $G \to \pi_1(\mathrm{Pic}(\mathcal{B}))$.
- To lift it to a 2-group homomorphism, we need to specify the monoidal structure, which defines the tensor product of $\mathcal{B}_G^\times$:

$$\mathcal{B}_g \boxtimes_{\mathcal{B}} \mathcal{B}_h \simeq \mathcal{B}_{gh}, \quad \forall g, h \in G.$$

  This lifting exists iff certain obstruction class $O_3 \in H^3(G; \mathcal{B}^\times)$ vanishes.
- To lift it to a 3-group homomorphism, we need to specify some $\mathcal{B}$-module natural isomorphisms as the monoidal structure, which define the associator and G-crossed braiding of $\mathcal{B}_G^\times$. This lift exists iff certain obstruction class $O_4 \in H^4(G; \Bbbk^\times)$ vanishes.

Example: 2+1D SET orders and G-crossed braided fusion category

Remark: There is an equivalence of 2-groups $\Pi_2(\mathrm{Pic}(\mathcal{B})) \simeq \mathrm{Aut}^{\mathrm{br}}(\mathcal{B})$. So the first two layers of this 3-group homomorphism is the monoidal functor $\rho\colon G \to \mathrm{Aut}^{\mathrm{br}}(\mathcal{B})$.

By gauging the G-symmetry in this SET order, we obtain a 2+1D topological order. The modular tensor category of codimension-2 topological defects in the gauged theory is the equivariantization $(\mathcal{B}_G^{\times})^G$ [Barkeshli-Bonderson-Cheng-Wang: 1410.4540].

Example: There is an e-m-exchange $\mathbb{Z}_2$-symmetry on the 2+1D toric code model. It can lifts to a 3-group homomorphism $\mathbb{Z}_2 \to \mathrm{Pic}(\mathcal{TC})$, and the corresponding $\mathbb{Z}_2$-crossed extension of $\mathcal{TC}$ is $\mathcal{P} := \mathrm{RMod}_A(\mathcal{I}\mathrm{s} \boxtimes \overline{\mathcal{I}\mathrm{s}})$. By gauging this $\mathbb{Z}_2$-symmetry we obtain the double Ising topological order:

$$\mathcal{P}^{\mathbb{Z}_2} \simeq \mathcal{I}\mathrm{s} \boxtimes \overline{\mathcal{I}\mathrm{s}}.$$

Example: G-crossed braided fusion categories are condensable algebras

Since $\mathrm{Pic}(\mathcal{B}) \simeq \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec})^\times$, the 3-group homomorphism $\phi\colon G \to \mathrm{Pic}(\mathcal{B})$ is equivalent to a linear monoidal 2-functor $\phi\colon 2\mathrm{Vec}_G \to \Sigma\mathcal{B}$.

This monoidal 2-functor $\phi\colon 2\mathrm{Vec}_G \to \Sigma\mathcal{B}$, it maps the condensable E_1-algebra $\mathrm{Vec}_G \in 2\mathrm{Vec}_G$ to a condensable E_1-algebra $\phi(\mathrm{Vec}_G) \in \Sigma\mathcal{B} \simeq \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec})$. What is this algebra?

- The underlying object is

$$\phi(\mathrm{Vec}_G) = \phi\Big(\bigoplus_{g\in G} g\Big) = \bigoplus_{g\in G} \phi(g) = \bigoplus_{g\in G} \mathcal{B}_g = \mathcal{B}_G^\times.$$

- The multiplication is induced by the group multiplication of G and the monoidal structure of ϕ:

$$\mathcal{B}_g \boxtimes_{\mathcal{B}} \mathcal{B}_h = \phi(g) \boxtimes_{\mathcal{B}} \phi(h) \to \phi(gh) = \mathcal{B}_{gh}, \quad g, h \in G.$$

Therefore, $\phi(\mathrm{Vec}_G)$ is exactly the G-crossed braided fusion category $\mathcal{B}_G^\times$.

Example: Condense a G-crossed braided fusion category

Mathematicall, the algebra structure of $\mathcal{B}_G^\times \in \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec})$ (i.e., the braided functor $\mathcal{B} \to \mathfrak{Z}_1(\mathcal{B}_G^\times)$) is induced by the G-crossed braiding. By condensing this algebra, the fusion 2-category of topological defects in the condensed phase is $\Sigma\overline{\mathfrak{Z}_2(\mathcal{B} \to \mathfrak{Z}_1(\mathcal{B}_G^\times))}$.

This Müger centralizer can be computed as follows. Let $\mathcal{M}$ be the equivariantization of $\mathcal{B}_G^\times$. Then $\mathcal{M}$ contains a fusion subcategory $\mathrm{Rep}(G) \subset \mathcal{M}$. Let $A := \mathrm{Fun}(G) \in \mathrm{Rep}(G) \subset \mathcal{M}$. Then $\mathcal{B}_G^\times \simeq \mathrm{RMod}_A(\mathcal{M})$ is the de-equivariantization of $\mathcal{M}$, and $\mathrm{Mod}_A^{\mathrm{E}_2}(\mathcal{M})$ is the trivial component $\mathcal{B}_e = \mathcal{B}$ of $\mathcal{B}_G^\times$ [Kirillov: math/0110221, Müger: math/0209093]. Since $\mathcal{M}$ is nondegenerate or modular, there is a braided equivalence $\mathfrak{Z}_1(\mathcal{B}_G^\times) \simeq \mathcal{B} \boxtimes \overline{\mathcal{M}}$. Thus $\overline{\mathfrak{Z}_2(\mathcal{B} \to \mathfrak{Z}_1(\mathcal{B}_G^\times))} \simeq \mathcal{M}$.

Hence, condensing the algebra $\mathcal{B}_G^\times = \phi(\mathrm{Vec}_G) \in \mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec}) \simeq \Sigma\mathcal{B}$ is the same as gauging the G-symmetry:

$$\mathrm{BMod}_{\mathcal{B}_G^\times|\mathcal{B}_G^\times}(\Sigma\mathcal{B}) \simeq \Sigma(\mathcal{B}_G^\times)^G.$$

Example: 2+1D SET orders and G-crossed braided fusion category

A 2+1D G SET order is described by the G-crossed braided fusion category $\mathcal{B}_G^\times$ of symmetry defects, and the modular tensor category of anyons in the gauged theory is the equivariantization $(\mathcal{B}_G^\times)^G$ [Barkeshli-Bonderson-Cheng-Wang: 1410.4540].

This G-crossed extension of $\mathcal{B}$ is equivalent to a 3-group homomorphism $G \to \mathrm{Pic}(\mathcal{B})$ [Etingof-Nikshych-Ostrik: 0909.3140], which is the same as a monoidal 2-functor $\phi\colon 2\mathrm{Vec}_G \to \Sigma\mathcal{B}$.

One can verify that ϕ maps the condensable E_1-algebra $\mathrm{Vec}_G \in 2\mathrm{Vec}_G$ to $\mathcal{B}_G^\times \in \mathsf{RMod}_{\mathcal{B}}(2\mathrm{Vec}) \simeq \Sigma\mathcal{B}$, and condensing $\mathcal{B}_G^\times$ is the same as gauging the G symmtry:

$$\mathsf{BMod}_{\mathcal{B}_G^\times|\mathcal{B}_G^\times}(\Sigma\mathcal{B}) \simeq \Sigma(\mathcal{B}_G^\times)^G.$$

Example: 1-form symmetry in 2+1D topological orders

Let A be a finite abelian group. Its delooping $\mathrm{B}A$ is a 2-group, describing a 1-form symmetry. Let $\mathcal{C}$ be the modular tensor category of anyons in a 2+1D topological order. Then a 1-form symmetry $\mathrm{B}A$ on this topological order is described by a monoidal 2-functor $\phi\colon \mathrm{B}A \to \Sigma\mathcal{C}$ or $\phi\colon 2\mathrm{Vec}_{\mathrm{B}A} \simeq 2\mathrm{Rep}(\hat{A}) \to \Sigma\mathcal{C}$. A monoidal 2-functor $\phi\colon \mathrm{B}A \to \Sigma\mathcal{C}$ is the same as a braided functor $\phi\colon A \to \mathcal{C}$. Then $\phi(a) \in \mathcal{C}$ is an abelian boson for all $a \in A$. So $E := \bigoplus_{a\in A} \phi(a) \in \mathcal{C}$ is a condensable E_2-algebra.

Gauging this 1-form symmetry is the same as condensing the algebra $\phi(\mathrm{Vec}_{\mathrm{B}A}) \in \Sigma\mathcal{C}$.

- $\mathrm{Vec}_{\mathrm{B}A}$ is the condensation completion (or Karoubi completion) of $\mathrm{B}\Bbbk[A]$. In other words, $\mathrm{Vec}_{\mathrm{B}A}$ can be obtained by condensing $\Bbbk[A]$ from the tensor unit $\mathrm{Vec} \in 2\mathrm{Vec}_{\mathrm{B}A}$.
- Then $\phi(\mathrm{Vec}_{\mathrm{B}A}) \in \Sigma\mathcal{C}$ can be obtained by condensing $\phi(\Bbbk[A]) = \bigoplus_{a\in A} \phi(a) = E$ from the tensor unit of $\Sigma\mathcal{C}$. In other words, $\phi(\mathrm{Vec}_{\mathrm{B}A}) = \Sigma E$, which is the 1+1D domain wall obtained by condensing the condensable E_2-algebra E along a line.

Hence gauging this 1-form symmetry is the same as the codimension-2 condensation (anyon condensation) of E.

Example: anomalous 1-form symmetry in 2+1D topological orders

It is possible to find a subgroup A of abelian anyons in $\mathcal{C}$, but they are not bosons. In literature this case is also called a 1-form symmetry with 't Hooft anomaly [Gaiotto-Kapustin-Seiberg-Willett: 1412.5148].

These abelian anyons span a fusion subcategory of $\mathcal{C}$. Its associator and braiding determine a class π in the abelian cohomology (E_2 Eilenberg-MacLane cohomology) $H^4_{\mathrm{E}_2}(A;\mathbb{C}^\times)$. On the other hand, one can also twist $2\mathrm{Vec}_{\mathrm{B}A}$ by $\pi \in H^4(\mathrm{B}A;\mathbb{C}^\times) \simeq H^4_{\mathrm{E}_2}(A;\mathbb{C}^\times)$. Then this anomalous 1-form symmetry is a monoidal 2-functor $\phi\colon 2\mathrm{Vec}^\pi_{\mathrm{B}A} \simeq \Sigma\mathrm{Vec}^\pi_A \to \Sigma\mathcal{C}$.

When π is nontrivial, $\mathrm{Vec}_{\mathrm{B}A}$ is not a condensable algebra in $2\mathrm{Vec}^\pi_{\mathrm{B}A}$ because there is no fiber 2-functor $2\mathrm{Vec}^\pi_{\mathrm{B}A} \to 2\mathrm{Vec}$. So one can not gauge the $\mathrm{B}A$-symmetry. We can say that the nontrivial class $\pi \in H^4_{\mathrm{E}_2}(A;\mathbb{C}^\times)$ characterizes the 't Hooft anomaly of this 1-form symmetry.

Example: gauging a $\mathbb{Z}_2$-symmetry in the 3+1D toric code model

The codimension-2 topological defects in the 3+1D toric code model form a E_2-fusion 2-category $\mathcal{C} := \mathfrak{Z}_1(2\mathrm{Rep}(\mathbb{Z}_2))$ [Kong-Tian-Z.: 2009.06564]. Denote $\mathcal{A} := 2\mathrm{Rep}(\mathbb{Z}_2)$. There is an invertible domain wall in the 3+1D toric code model corresponding to the nontrivial 2+1D $\mathbb{Z}_2$ SPT order [Kong-Lan-Wen-Z.-Zheng: 2003.08898]. In the coordinate system $\Sigma\mathcal{C} \simeq \mathrm{BMod}_{\mathcal{A}|\mathcal{A}}(3\mathrm{Vec})$, this invertible domain wall is $2\mathrm{Rep}(\mathbb{Z}_2, \omega) = \Sigma\mathrm{Vec}_{\mathbb{Z}_2}^{\omega}$, where $\omega \in H^3(\mathbb{Z}_2; \mathbb{C}^\times) \simeq \mathbb{Z}_2$ is the nontrivial element.

We want to find a monoidal 3-functor $\phi\colon 3\mathrm{Vec}_{\mathbb{Z}_2} \to \Sigma\mathcal{C}$ which maps the nontrivial simple object to the above invertible domain wall. In the coordinate system $\Sigma\mathcal{C} \simeq \mathrm{BMod}_{\mathcal{A}|\mathcal{A}}(3\mathrm{Vec})$, this is determined by the condensable algebra $\phi(2\mathrm{Vec}_{\mathbb{Z}_2}) \in \Sigma\mathcal{C}$, which is a $\mathbb{Z}_2$-graded fusion 2-category

$$\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1, \quad \mathcal{P}_0 = \mathcal{A} = 2\mathrm{Rep}(\mathbb{Z}_2),\ \mathcal{P}_1 = 2\mathrm{Rep}(\mathbb{Z}_2, \omega).$$

Such $\mathcal{P}$ is unique. There is a 2-group $\mathcal{G}$ defined by $\pi_1(\mathcal{G}) = \pi_2(\mathcal{G}) = \mathbb{Z}_2$ and the Postnikov class (associator) $\omega \in H^3(\mathbb{Z}_2; \mathbb{Z}_2) \simeq H^3(\mathbb{Z}_2; \mathbb{C}^\times) \simeq \mathbb{Z}_2$ such that $\mathcal{P} = 2\mathrm{Rep}(\mathcal{G})$. The gauged theory is $\mathrm{BMod}_{\mathcal{P}|\mathcal{P}}(\Sigma\mathcal{C}) \simeq \mathrm{BMod}_{\mathcal{P}|\mathcal{P}}(3\mathrm{Vec}) \simeq \Sigma\mathfrak{Z}_1(\mathcal{P})$, a 3+1D twisted $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory.

Remark: continuous group symmetry

This framework also works for continuous group G symmetry. It is hard to make sense of a monoidal n-functor $n\mathrm{Vec}_G \to \mathcal{C}$, but its invertible part $G \to \mathcal{C}^\times$ could be understood as a pointed continuous map $\mathrm{B}G \to \mathrm{B}\mathcal{C}^\times$, where B denotes the classifying space.

For example, consider the 2+1D trivial phase equipped with a compact group G symmetry. The space of invertible topological defects in the 2+1D trivial phase is $\mathcal{C}^\times \simeq \mathrm{B}^2\mathbb{C}^\times \simeq \mathrm{B}^3\mathbb{Z} = K(\mathbb{Z},3)$. The set of homotopy classes of pointed maps $\mathrm{B}G \to \mathrm{B}\mathcal{C}^\times \simeq K(\mathbb{Z},4)$ is $H^4(\mathrm{B}G,\mathbb{Z})$, which exactly classifies the 2+1D Chern-Simons G-gauge theory [Dijkgraaf-Witten: 90]. In other words, the 2+1D SPT orders with a compact group G symmetry should be classified by $H^4(\mathrm{B}G,\mathbb{Z})$, and their gauged theories are the Chern-Simons theories.

However, a categorical description of the gauging (or condensation) process in this case is not clear.

Example: condensation as gauging a non-invertible symmetry

Let $\mathcal{C}$ be the fusion n-category of topological defects in an n+1D topological order. The identity functor $\mathrm{id}\colon \mathcal{C} \to \mathcal{C}$ defines a non-invertible symmetry on this topological order.

To gauge this symmetry, we need to choose a condensable algebra $A \in \mathcal{C}$ and then condense $\mathrm{id}(A) = A$. So we see that condensing A is the same as a gauging of this non-invertible symmetry.

Summary

1. The topological defects in an $n+1$D topological order form a fusion n-category $\mathcal{C}$.
2. A codimension-k condensation means taking modules of a condensable E_k-algebra in $\Omega^{k-1}\mathcal{C}$.
3. A symmetry is a monoidal n-functor $\phi\colon \mathcal{T} \to \mathcal{C}$. The higher condensation theory unifies the gauging of all finite type symmetries.

Thanks for listening!