Gauging Non-invertible Symmetries in 2+1d

Generalized Symmetries in HEP and CMP

KITS and Peking University

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ICTP

Based on 2507.01142 with

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(QMUL)

Clement Delcamp
(IHES)

Motivation

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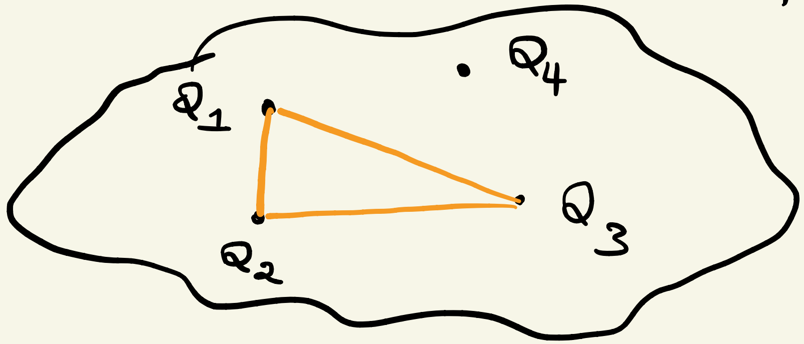

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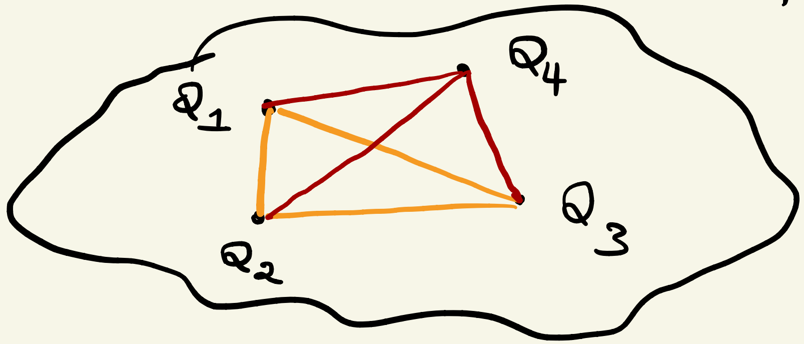

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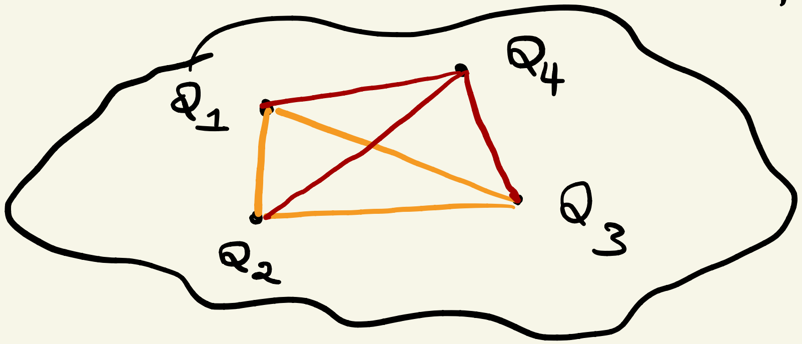

* Gauging invertible symmetries can be used to prepare non-abelian anyons from abelian anyons.
  Eg. $D(S_3)$ anyons from $D(\mathbb{Z}_3)$ anyons.
  [Lyons, Bowen Lo, Tantivasadakarn, Vishwanath, Verresen 2024]

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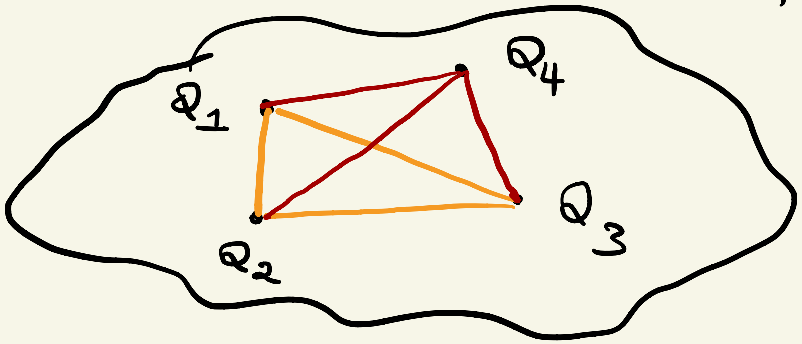

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* What about gauging non-invertible symmetries?

Context

* Topological lines are described by fusion categories and in 1+1d gauging them is described by morita theory.

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Context

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  (See Hao Zhang's talk)

* In 2+1d gauging topological lines is again well understood.

  [Bais, Slingerland, 2008][Kong 2013]

* In 2+1d, there are various approaches to gauging surface operators:

  1) Higher condensation theory. [Gaiotto, Johnson-Freyd, 2019]
     [Kong, Zhang, Zhao, Zheng 2024]
     (See Zhihao Zhang's talk)

  2) Morita theory of fusion 2-categories. [Décoppet 2022]

  3) Orbifold data in 2+1d TQFTs. [Carqueville, Runkel, Schaumann, 2018]

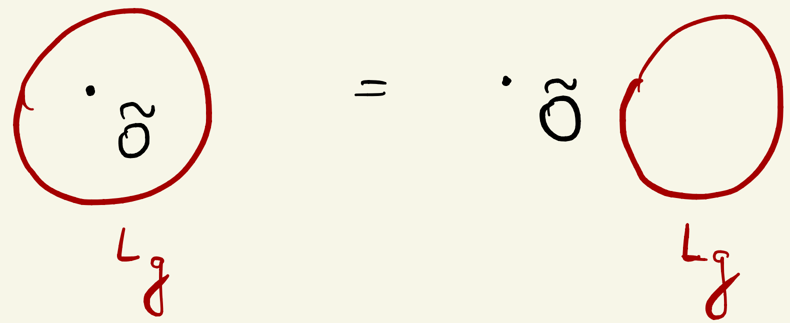
  ⋮

This talk: Take the intuition from orbifolding in 1+1d CFTs and run with it.

Gauging in 1+1d

The QFT Q/G obtained from gauging G have G-invariant local operators.

$\tilde{O}$ is a local operator in Q/G if



$\forall g \in G$

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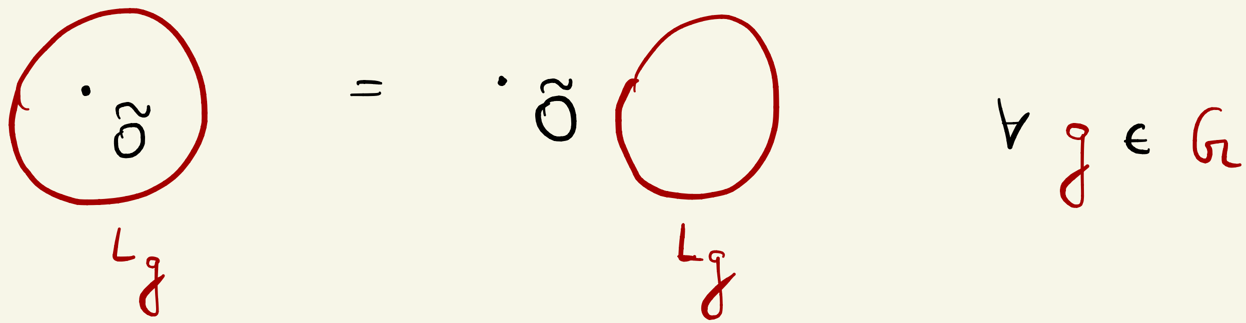
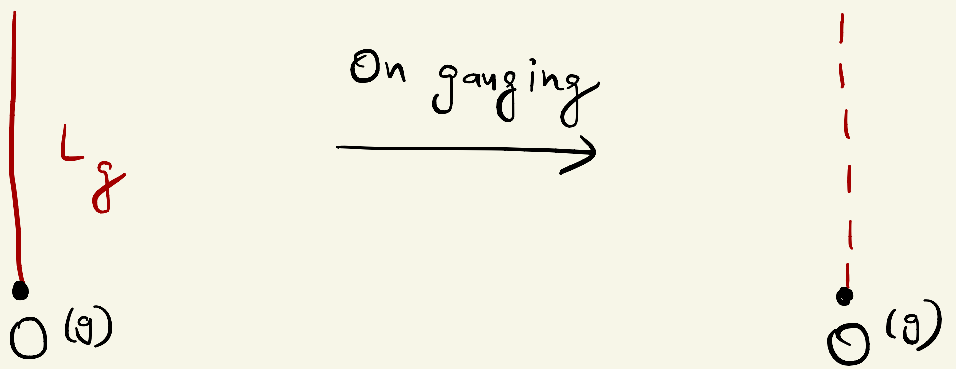


Moreover,

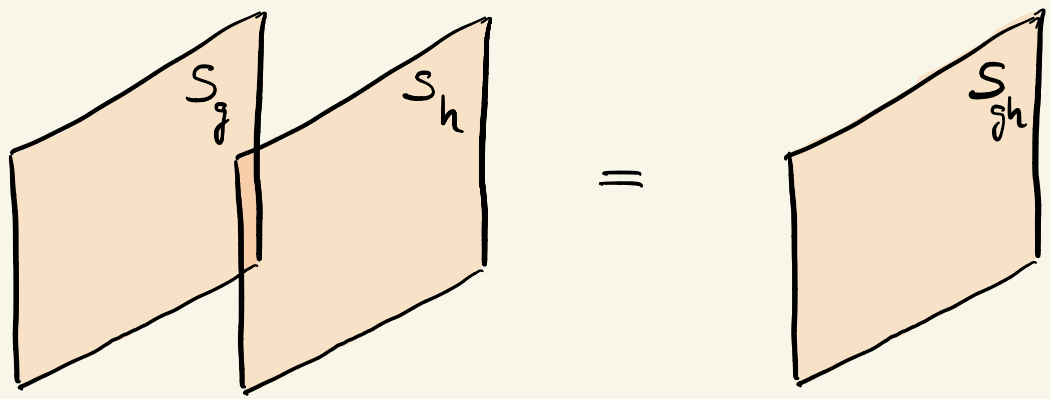


Non-genuine local operator

Genuine local operator

Gauging in 2+1d

* In 2+1d QFTs, consider a symmetry implemented by topological surface operators S_g, $g \in G$.



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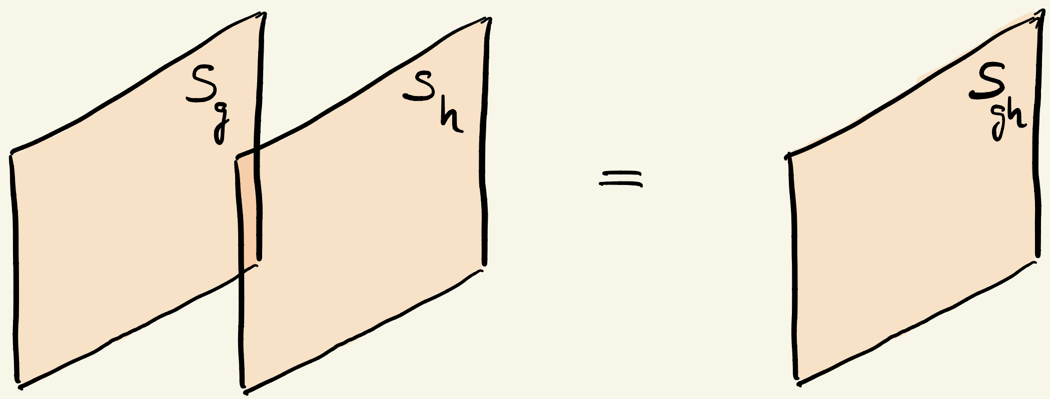
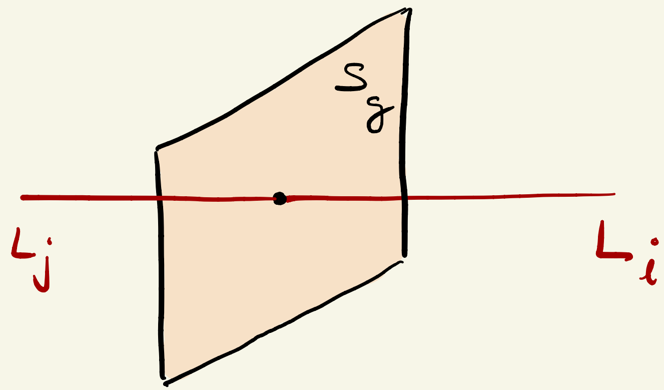
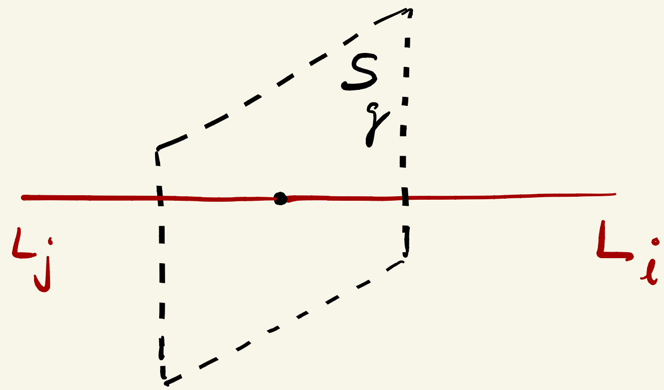


* The action on line operators is



* On gauging S_g, we identify the line operators L_i and L_j.



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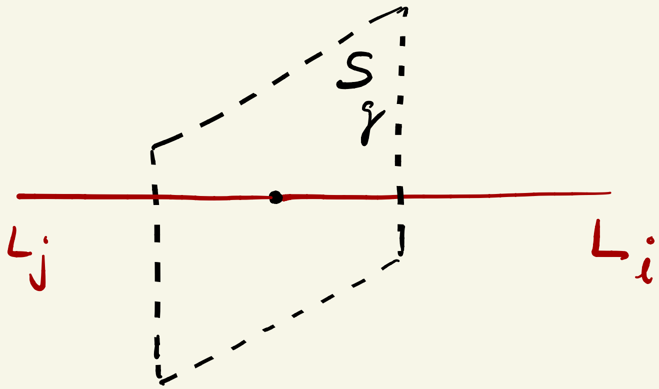
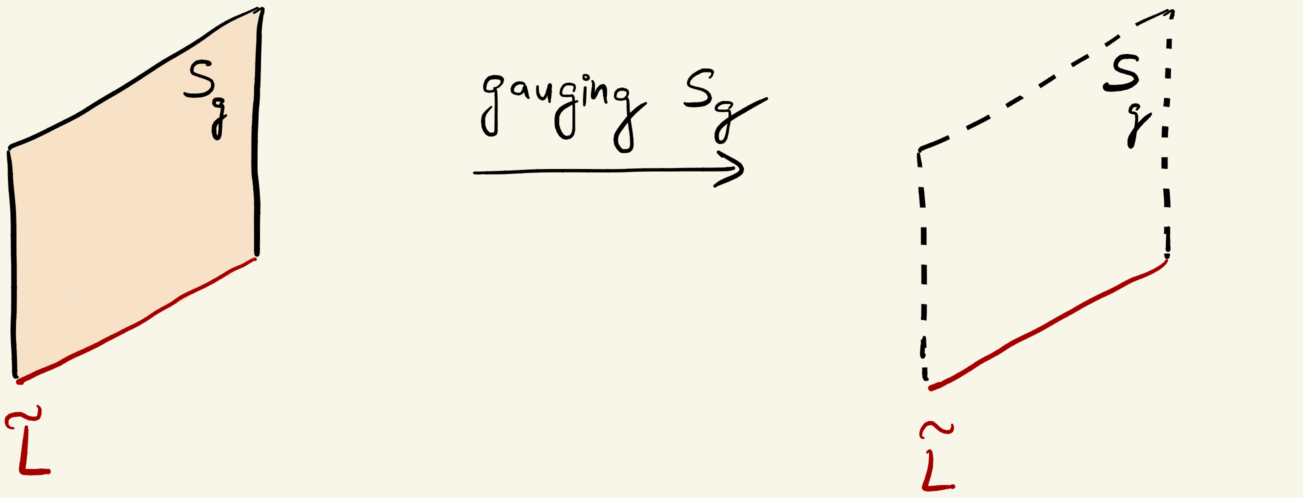


* Genuine line operators from twisted sector.

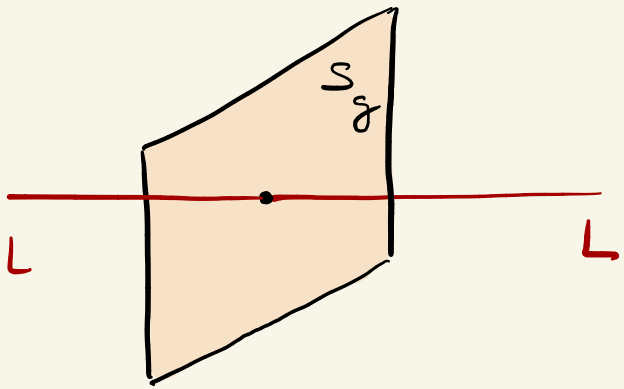


Non-genuine line operator

Genuine line operator

* Fixed points of G-action

$S_g \in H_L$ if



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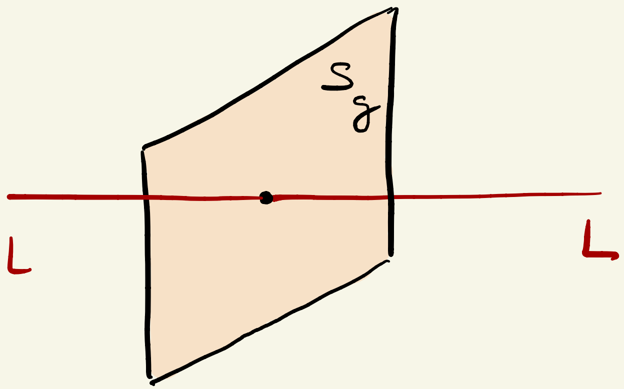
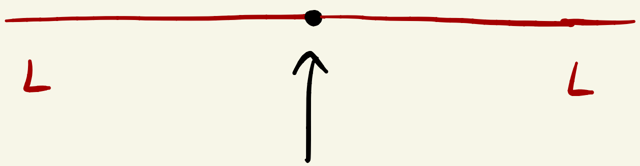


* In the gauged theory, we get



$|H_L|$ local operators.

L splits into $|H_L|$ line operators in the gauged theory.

(Mathematically, add twisted sector lines to the category of lines C, to form a G-crossed braided category C_G and then equivariantize the G-action.
[Barkeshli, Bonderson, Cheng, Wang, 2014])

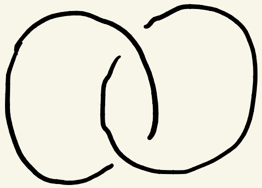
Symmetries of $D(\mathbb{Z}_2)$

Line operators	1	e	m	ψ
Spin θ_a	1	1	1	-1

$$e \times e = m \times m = 1 .$$

$$\psi = e \times m .$$

Braiding

$S_{ab} =$ a  b $= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} .$

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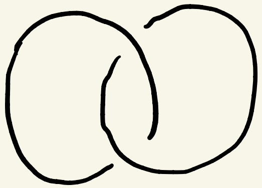
$S_{ab} =$ a  b $= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$

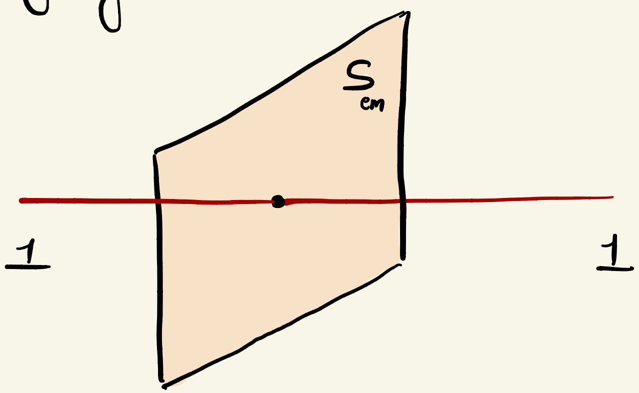
em - symmetry

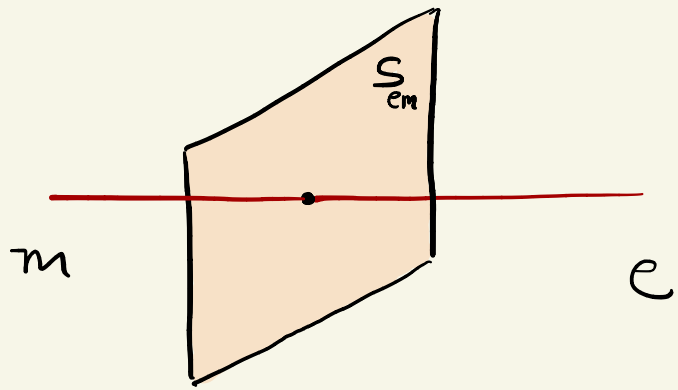
$S_{em}(1) = 1 \quad S_{em}(\psi) = \psi \quad S_{em}(e) = m \quad S_{em}(m) = e$

S_{em} can be gauged.

Gauging S_{em}

[Barkeshli, Bonderson, Cheng, Wang 2014]





$$H_{\underline{1}} = H_{\psi} = \{ \mathbb{1}, S_{em} \} \cong \mathbb{Z}_2$$

$$H_e = H_m = \{ \mathbb{1} \} \cong \mathbb{Z}_{\underline{1}}.$$

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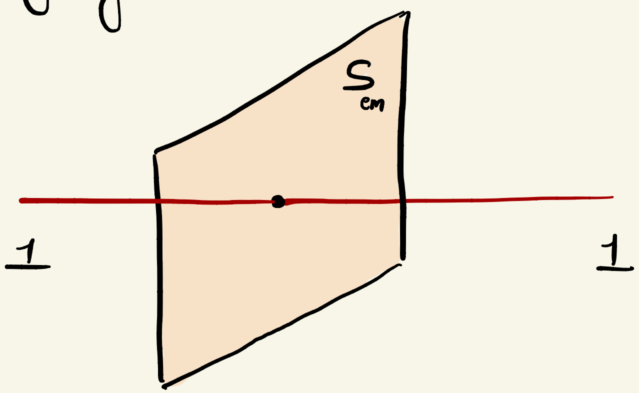

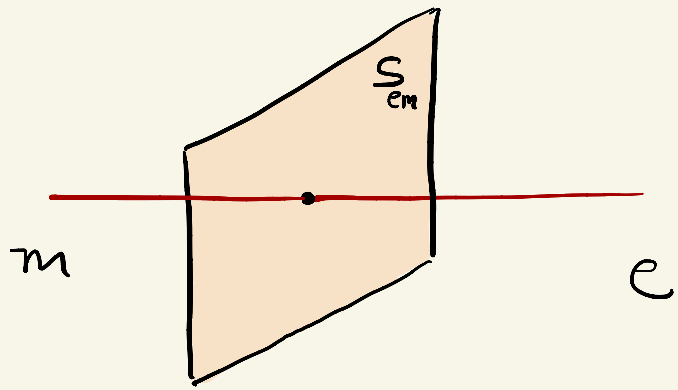

$$H_{1} = H_{\psi} = \{ \mathbb{1}, S_{em} \} \cong \mathbb{Z}_2 \qquad H_e = H_m = \{ \mathbb{1} \} \cong \mathbb{Z}_1.$$

After gauging:

$$1 \to 1_1, 1_2, \quad \psi \to \psi_+, \psi_-, \quad [e, m]$$

Gauging S_{em}

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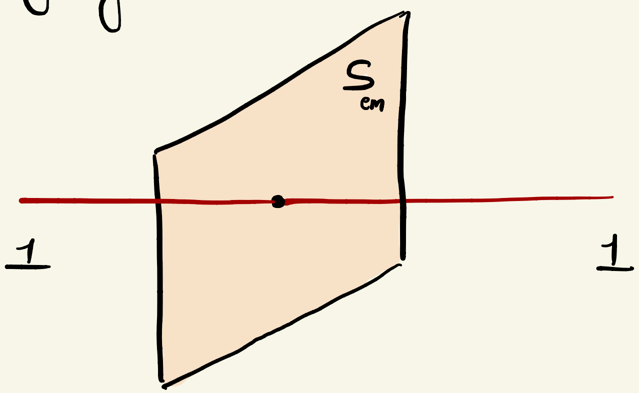

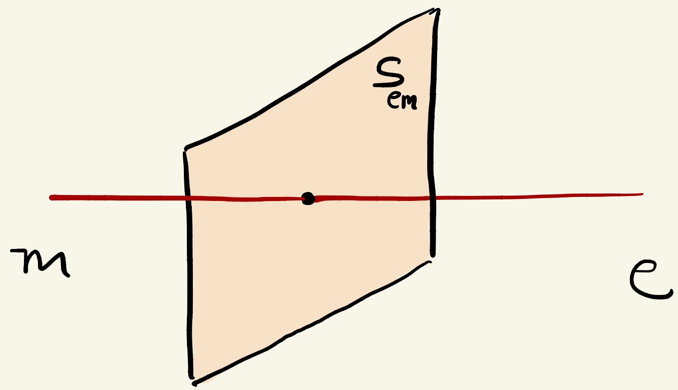

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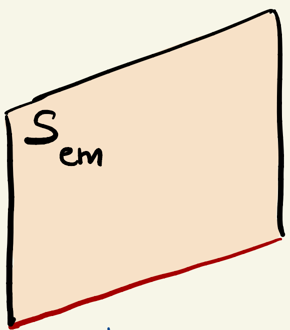

σ^+, σ^-

$$H_\sigma = \{ \mathbb{1}, S_{em} \} \longrightarrow \sigma^+_1, \sigma^+_2, \sigma^-_1, \sigma^-_2.$$

Gauging S_{em}

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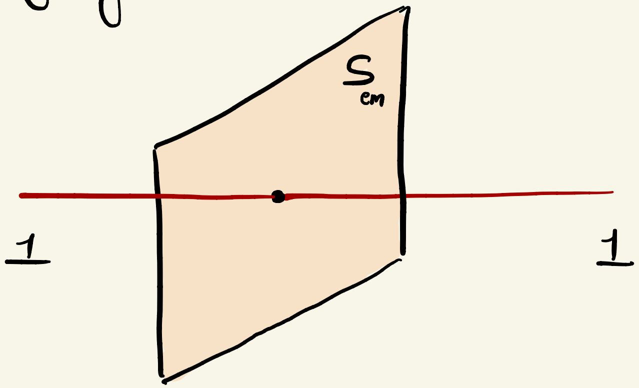

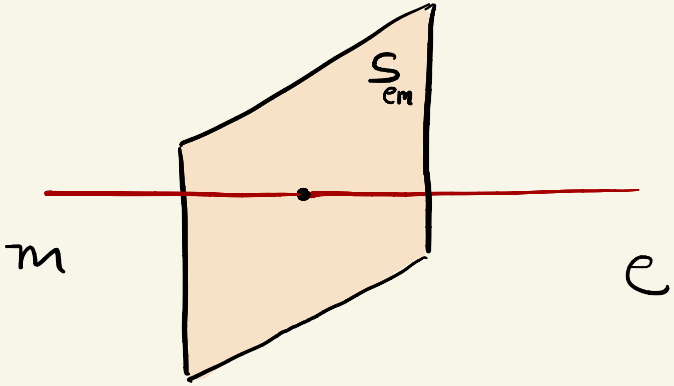

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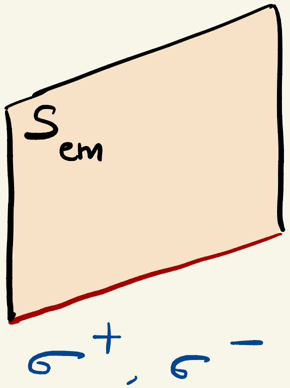


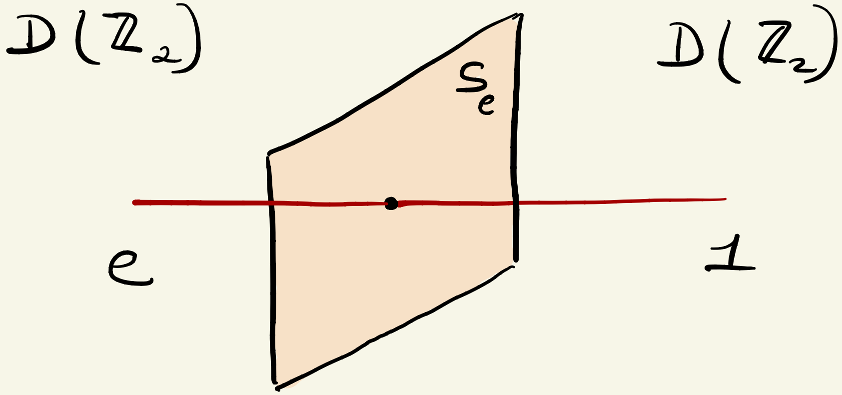
$H_\sigma = \{\mathbb{1}, S_{em}\} \longrightarrow \sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-$.

$S_{em}(\sigma) = \sigma$ for $\Theta_\sigma = {}_\sigma\phi$ to be well-defined

A non-invertible symmetry of $\mathcal{D}(\mathbb{Z}_2)$

$$S_e(1) = S_e(e) = 1 + e \qquad S_e(m) = S_e(\psi) = 0 .$$

[Roumpedakis, Seifnashri, Shao 2022]



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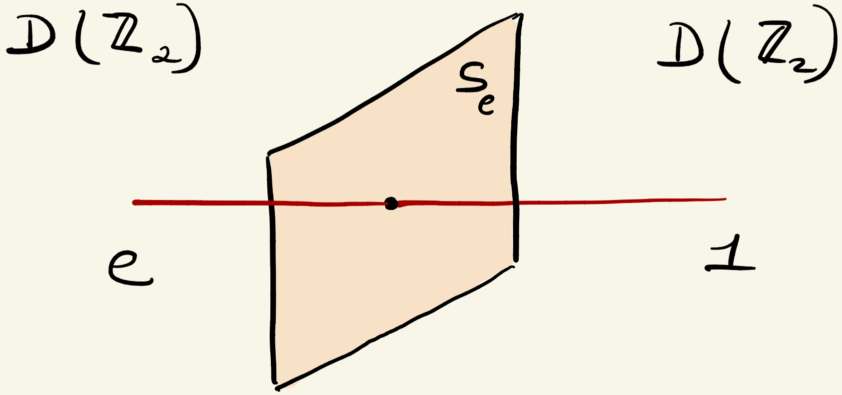

On gauging S_e

$$1 \to 1 \,, \quad e \to 1 \qquad \text{and } m, \psi \text{ are confined.}$$

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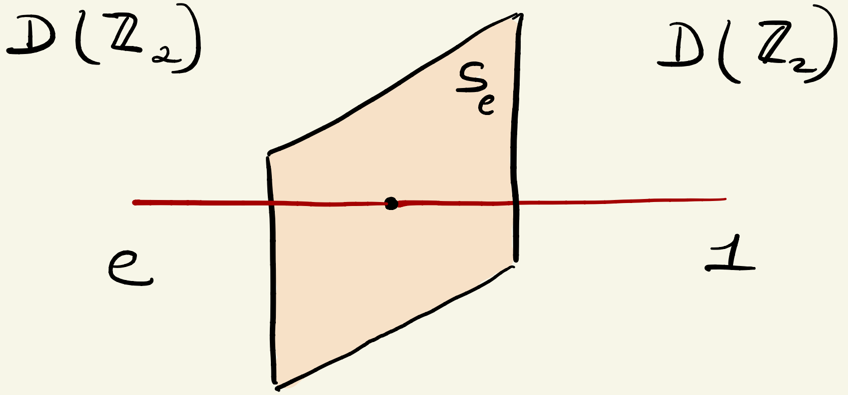


On gauging S_e

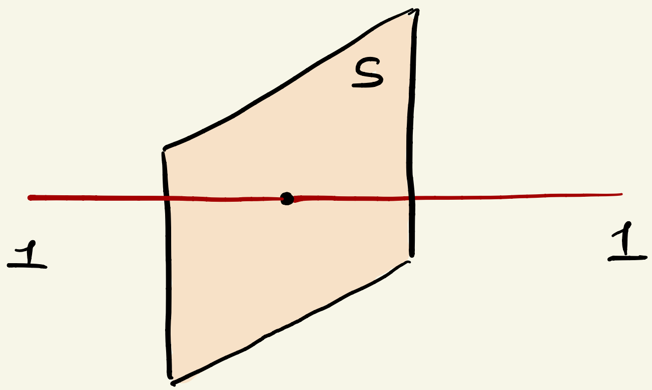
$1 \to 1$, $e \to 1$ and m, ψ are confined.

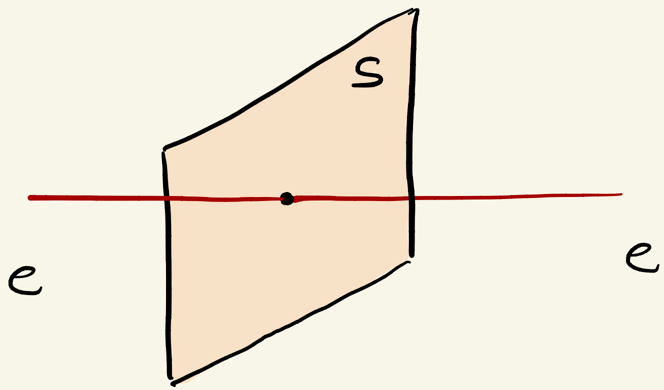
Gauging the 1-form symmetry e = gauge e on a 2-manifold to get S_e + gauge S_e in full spacetime.

(S_e is an F_2 algebra. See Zhihao Zhang's talk.)

Gauging $S = \mathbb{1} + S_e$

Consider action on $\underline{1}$, e





$H_{\underline{1}} = \{ \mathbb{1}, s \}$ $\qquad$ $H_e = \{ \mathbb{1}, s \}$

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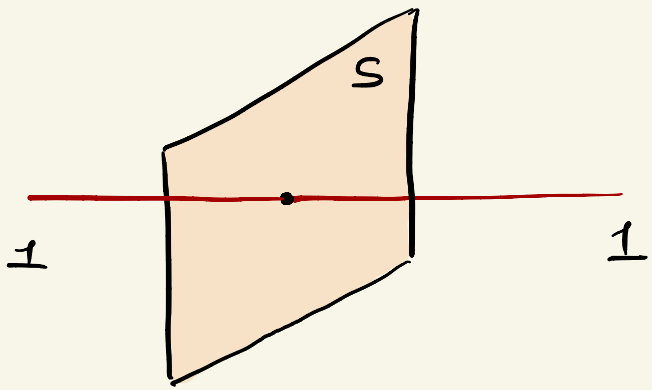
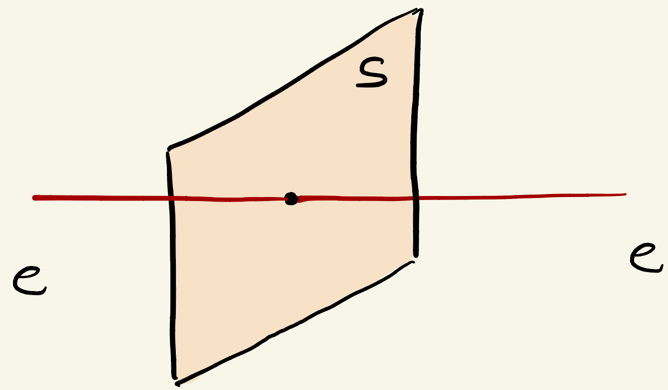
Consider action on $\underline{1}$, e





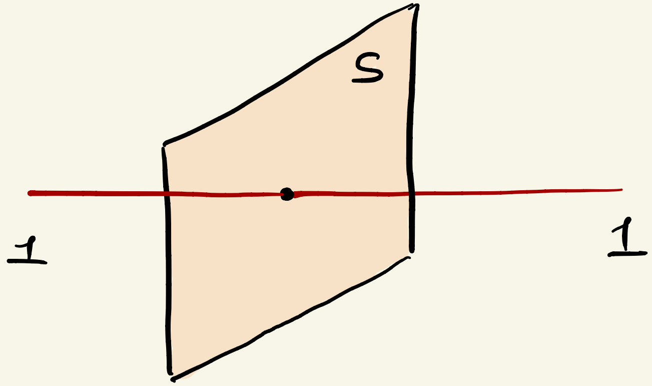
$H_{\underline{1}} = \{ \mathbb{1}, S \}$ $\qquad$ $H_e = \{ \mathbb{1}, S \}$

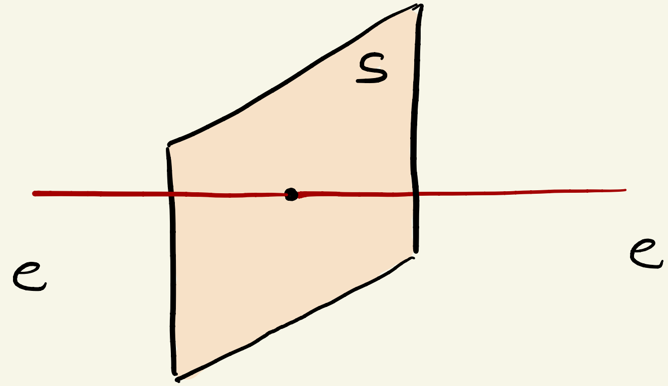
On gauging S,

$$\underline{1} \to \underline{1}_{\underline{1}} + \underline{1}_2 \qquad e \to e_{\underline{1}} + e_2$$

Gauging $S = \mathbb{1} + S_e$

Consider action on 1, e



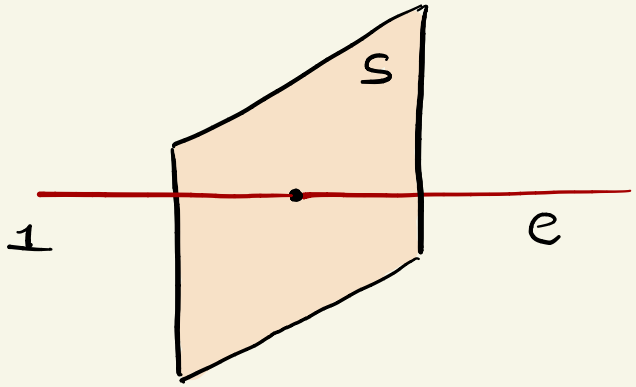


$H_1 = \{\mathbb{1}, S\}$ $\qquad$ $H_e = \{\mathbb{1}, S\}$

On gauging S,

$$1 \to 1_1 + 1_2 \qquad e \to e_1 + e_2$$

But,



$\Rightarrow \quad [1_2, e_2]$

Some Line operators in $D(\mathbb{Z}_2)/S$

In $D(\mathbb{Z}_2)/S$, we get the line operators

$$\underline{1}_1 , \quad e_{\underline{1}} , \quad [1_2, e_2]$$

$$e_1 \times e_1 = \underline{1}_{\underline{1}} \qquad [1_2, e_2] \times [1_2, e_2]$$
$$= \underline{1}_{\underline{1}} + e_{\underline{1}} + [1_2, e_2] .$$

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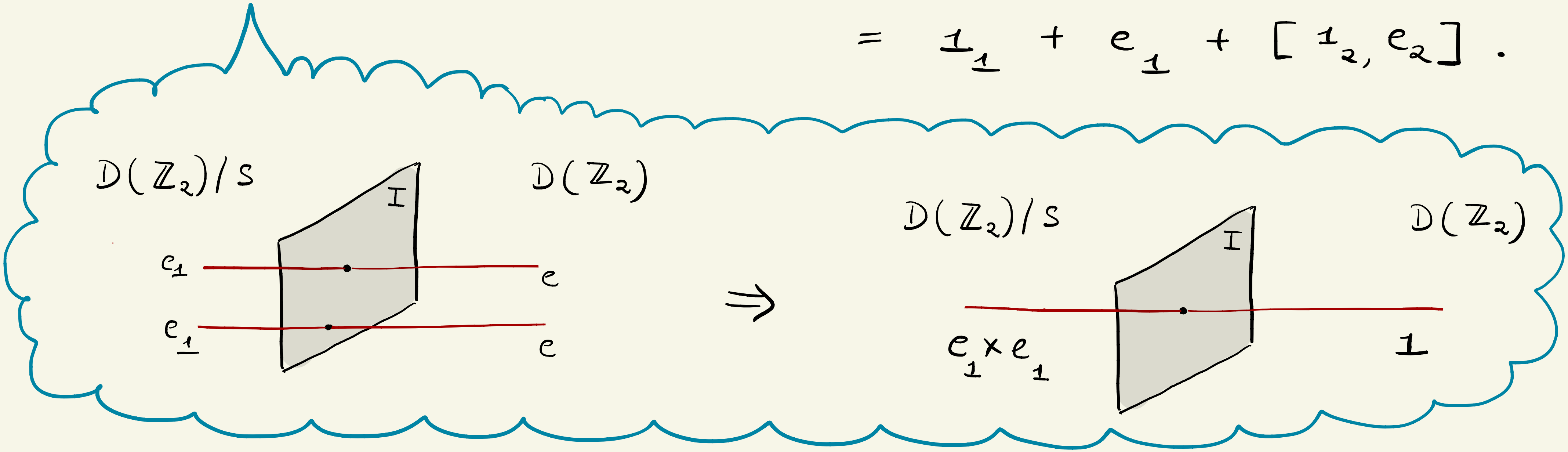

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$$\mathbb{1}_1\ ,\ e_1\ ,\ [\mathbb{1}_2, e_2]$$

$$e_1 \times e_1 = \mathbb{1}_1 \qquad [\mathbb{1}_2, e_2] \times [\mathbb{1}_2, e_2]$$

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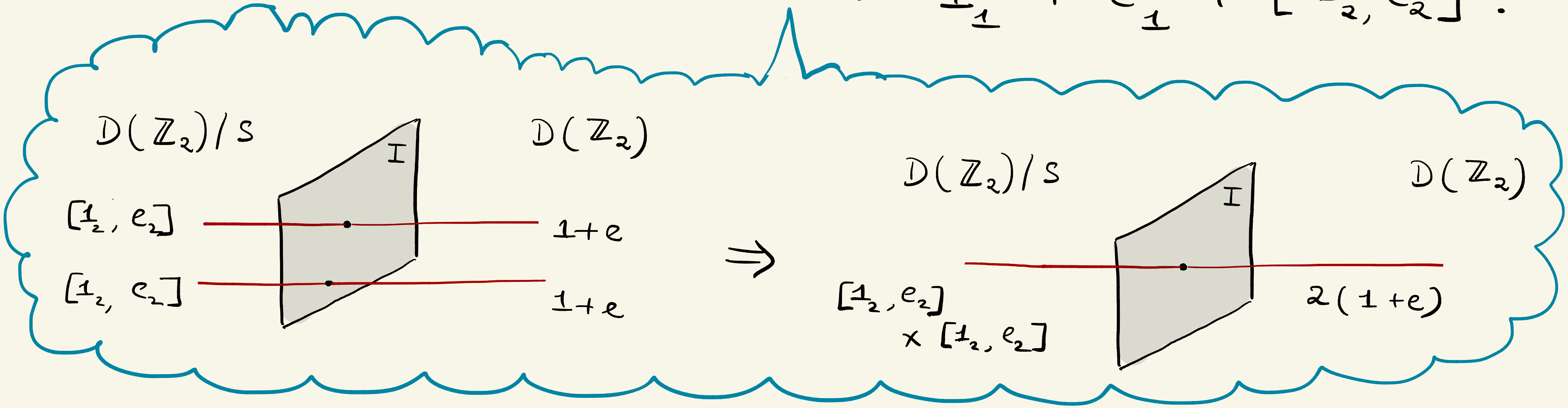


So, $\{\mathbb{1}_1\ ,\ e_1\ ,\ [\mathbb{1}_2, e_2]\} \cong \mathrm{Rep}(S_3)$!

Identifying $D(\mathbb{Z}_2)\,/\,S$

* Rep(S_3) is a subcategory of $D(\mathbb{Z}_2)/S$.

* $\exists$ a 1-form gauging $A = 1_{\underline{1}} + [1_2, e_2]$ such that

$$F_{\underline{1}} : \mathrm{Rep}(S_3) \longrightarrow \mathrm{Rep}(\mathbb{Z}_2)$$

$$1_{\underline{1}} \rightarrow 1$$

$$e_{\underline{1}} \rightarrow e$$

$$[1_2, e_2] \rightarrow 1 + e$$

Identifying $D(\mathbb{Z}_2)/S$

* $\mathrm{Rep}(S_3)$ is a subcategory of $D(\mathbb{Z}_2)/S$.

* $\exists$ a 1-form gauging $A = 1_1 + [1_2, e_2]$ such that

$$F_1 : \mathrm{Rep}(S_3) \longrightarrow \mathrm{Rep}(\mathbb{Z}_2)$$

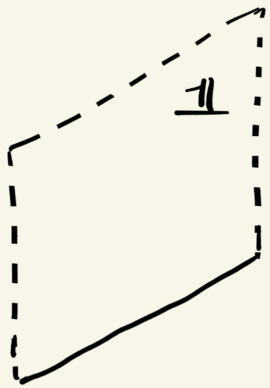
$$\begin{aligned} 1_1 &\to 1 \\ e_1 &\to e \\ [1_2, e_2] &\to 1 + e \end{aligned}$$

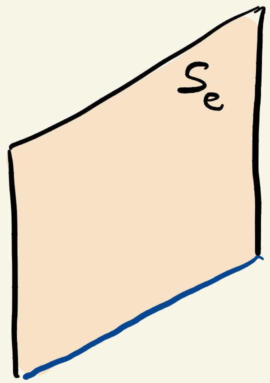
F_1 followed by condensing e gives the trivial theory.

$\Rightarrow$ $\mathrm{Rep}(S_3)$ is a Lagrangian subcategory of $D(\mathbb{Z}_2)/S$.

$$\Rightarrow \quad D(\mathbb{Z}_2)/S \simeq D(S_3)$$

All line operators in $\mathcal{D}(\mathbb{Z}_2)/S$

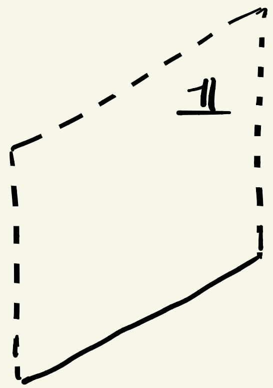


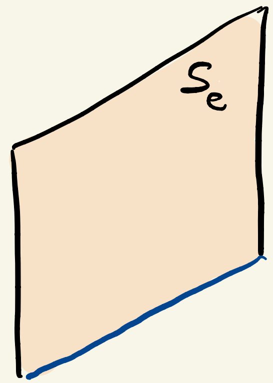


$$C_{\mathbb{1}} := \{\mathbb{1}, e, m, \psi\}$$

$$C_{S_e} := \{a, b\}$$

All line operators in $D(\mathbb{Z}_2)/S$





$C_{\mathbb{1}} := \{1, e, m, \psi\}$

$C_{S_e} := \{a, b\}$

Full action of S_e on $C_{\mathbb{1}} + C_{S_e}$

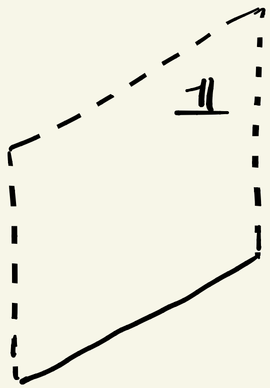
$S_e(1) = S_e(e) = 1 + e$

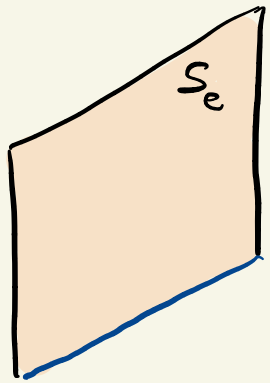
$S_e(\psi) = S_e(m) = b$

$S_e(b) = m + \psi + b$

$S_e(a) = 2 \cdot a$

All line operators in $D(\mathbb{Z}_2)/S$





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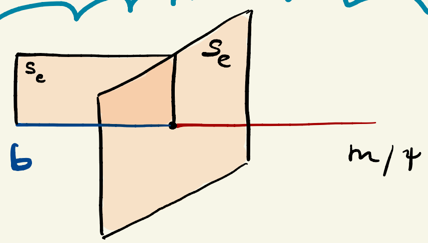

and $b \in S_e(b)$ for Θ_b to be well-defined.

All line operators in $\mathcal{D}(\mathbb{Z}_2)/S$

For $S = \mathbb{1} + S_e$ we have

$$S(1) = S(e) = 2\cdot 1 + e \qquad S(m) = m + b \qquad S(\psi) = \psi + b$$

$$S(b) = m + \psi + 2\cdot b \qquad S(a) = 3\cdot a$$

All line operators in $\mathcal{D}(\mathbb{Z}_2)/S$

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On gauging S, we get

$$1 \to 1_1, 1_2; \quad e \to e_1, e_2; \quad m \to m; \quad \psi \to \psi;$$

$$b \to b_1, b_2; \quad a \to a_1, a_2, a_3$$

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For $S = \mathbb{1} + S_e$ we have

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Simple line operators in $\mathcal{D}(\mathbb{Z}_2)/S$ are

$$\underbrace{1_1 \, , \; e_1 \, , \; [1_2, e_2]}_{\mathrm{Rep}(S_3)} \, , \; [m, b_1] \, , \; [\psi, b_2] \, , \; a_1 \, , \; a_2 \, , \; a_3 \, .$$

$$\Rightarrow \quad \mathcal{D}(\mathbb{Z}_2)/S = \mathcal{D}(S_3) \, .$$

$D(S_3)$

S_3 Dijkgraaf - Witten Theory $\qquad S_3 = \langle r, s \mid r^3 = s^2 = e \,;\, srs = r^{-1} \rangle$

Conjugacy class $[g]$	C_g	$\mathrm{Irr}(C_g)$
$[e]$	S_3	$\mathbb{1}, \pi_1, \pi_2$
$[s] = \{s, sr, sr^2\}$	$C_s = \{e, s\}$	$\mathbb{1}_s, \sigma$
$[r] = \{r, r^2\}$	$C_r = \{e, r, r^2\}$	$\mathbb{1}_r, \omega, \omega^2$

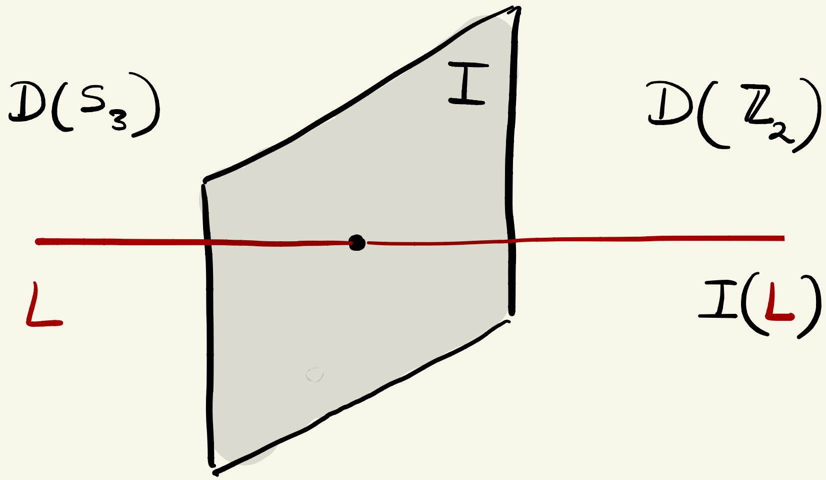
Simple line operators:

$([e], \mathbb{1}) \quad ([e], \pi_1) \quad ([e], \pi_2) \quad ([s], \mathbb{1}_s) \quad ([s], \sigma)$

$([r], \mathbb{1}_r) \quad ([r], \omega) \quad ([r], \omega^2)$

$\mathcal{D}(S_3) \underset{I}{\Big|} \mathcal{D}(\mathbb{Z}_2)$

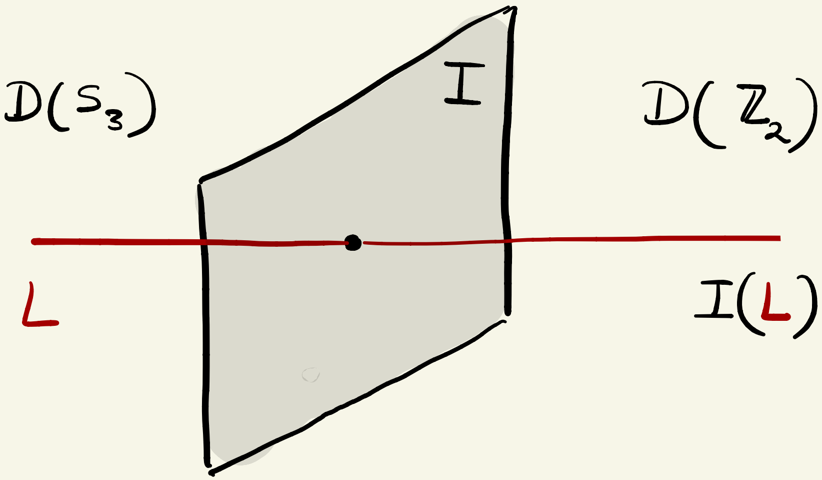
[Bais, Slingerland, 2008]



$\mathcal{D}(S_3) \mid \mathcal{D}(\mathbb{Z}_2)$

$\mathcal{I}$

[Bais, Slingerland, 2008]



$$\mathcal{I}([e], \mathbb{1}) = 1$$

$$\mathcal{I}([e], \pi_1) = e$$

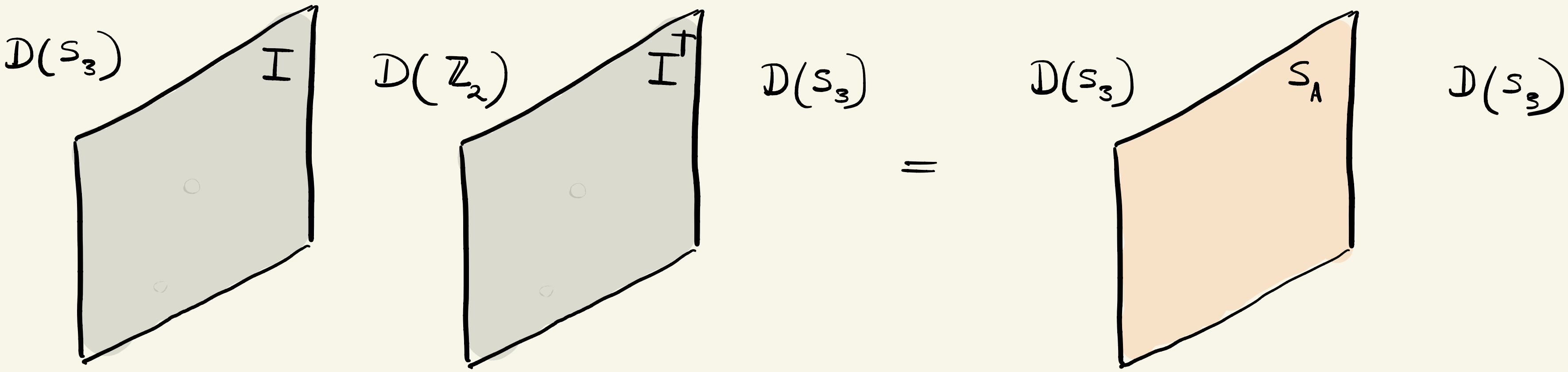
$$\mathcal{I}([e], \pi_2) = 1 + e$$

$$\mathcal{I}([s], \mathbb{1}_s) = m + b$$

$$\mathcal{I}([s], \sigma) = \psi + b$$

$$\left.\begin{array}{l} \mathcal{I}([r], \mathbb{1}_r) \\ \mathcal{I}([r], \omega^2) \\ \mathcal{I}([r], \omega) \end{array}\right\} = a$$

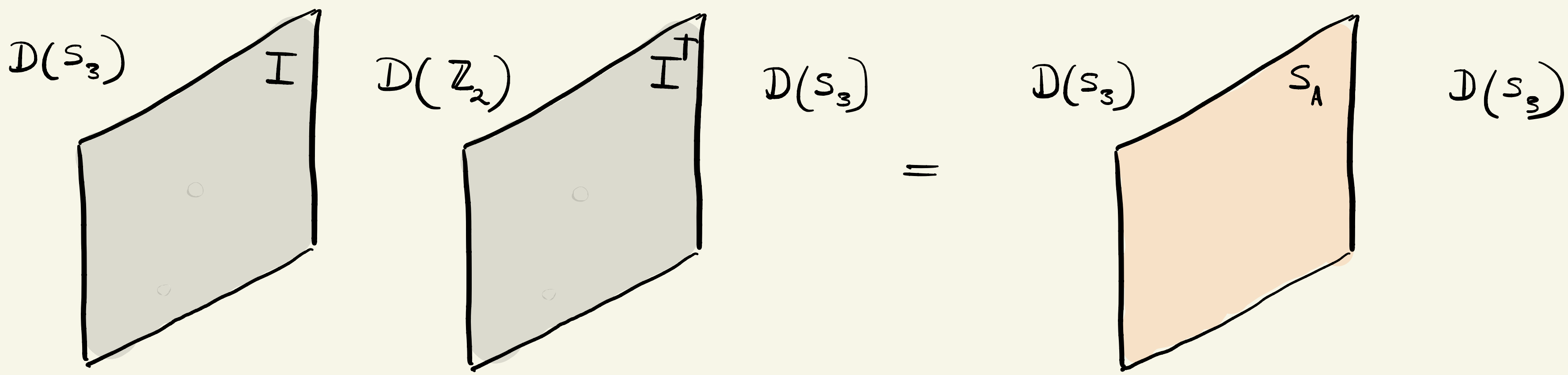
$\mathcal{I} \times \mathcal{I}^\dagger$ and $\mathcal{I}^\dagger \times \mathcal{I}$



$$A = ([e], \mathbb{1}) + ([e], \pi_2)$$

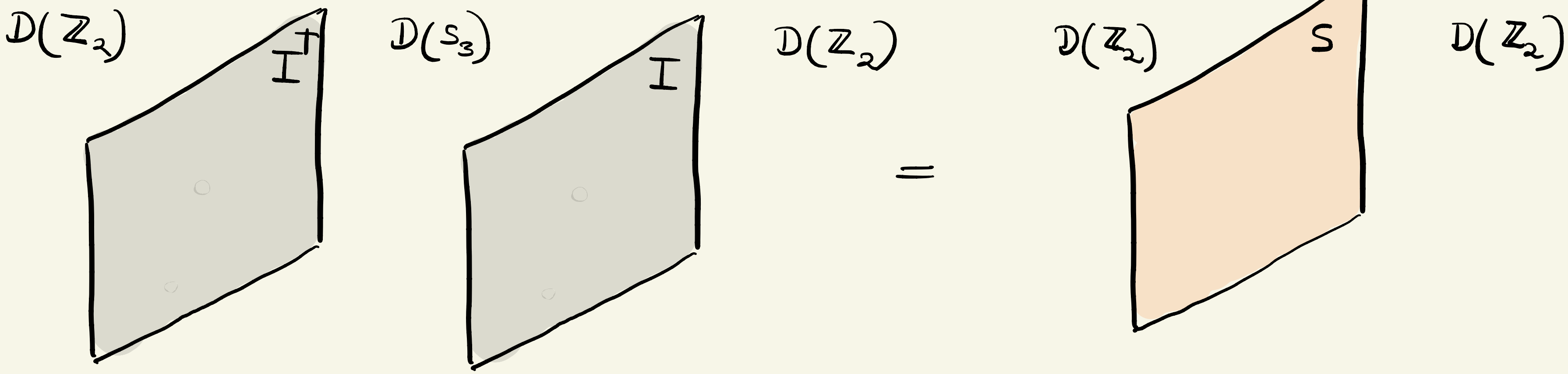
[Buican, Radhakrishnan, 2023]

$\mathcal{I} \times \mathcal{I}^\dagger$ and $\mathcal{I}^\dagger \times \mathcal{I}$

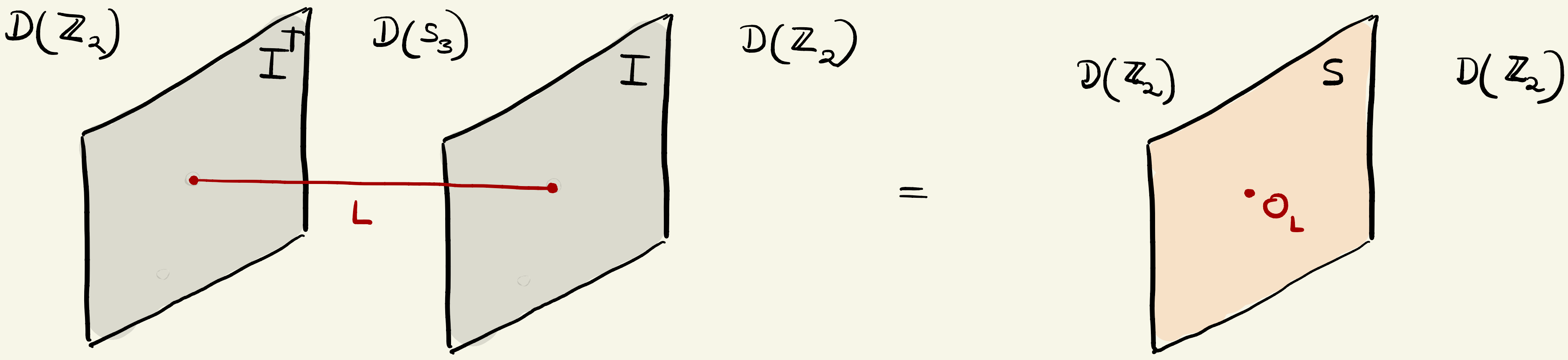


$A = ([e], \mathbb{1}) + ([e], \pi_2)$

[Buican, Radhakrishnan, 2023]



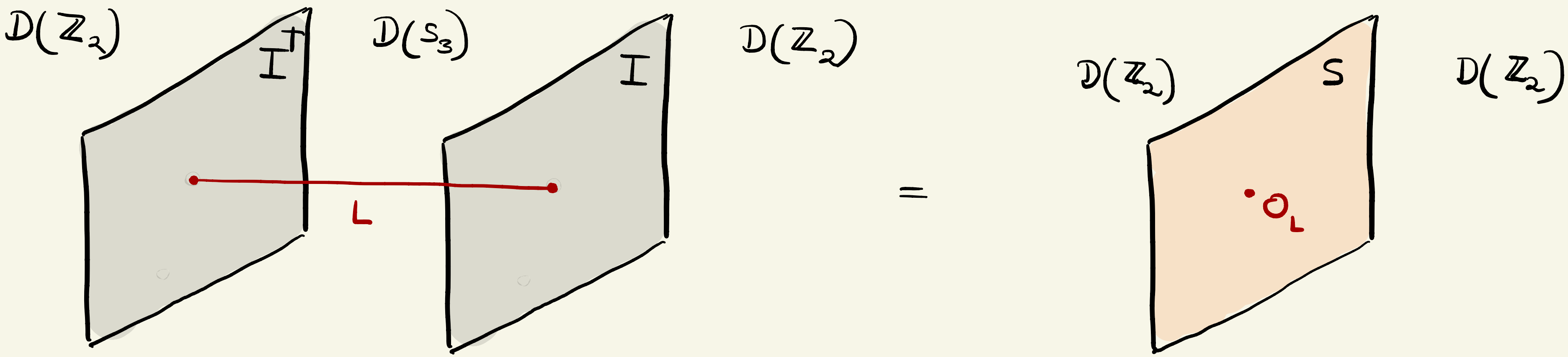
$$I \times I^\dagger = \mathbb{1} + S_e$$



$$L = ([e], \mathbb{1}) \text{ or } ([e], \pi_2)$$

$\Rightarrow$ S is a non-simple surface operator.

$$I \times I^\dagger = \mathbb{1} + S_e$$



$L = ([e], \mathbb{1})$ or $([e], \pi_2)$

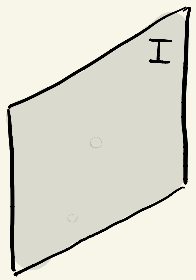
$\Rightarrow$ S is a non-simple surface operator.

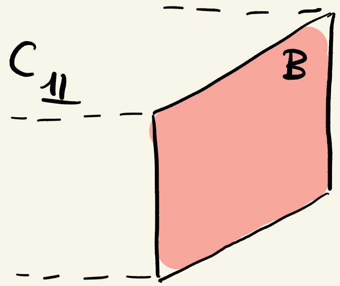
$$S(1) = I \times I^\dagger(1) = I\big(([e], \mathbb{1}) + ([e], \pi_2)\big)$$

$$= 1 + 1 + e.$$

$$\Rightarrow \quad I \times I^\dagger = \mathbb{1} + S_e$$

Gauging general non-invertible symmetry $S = \sum_i S_i$.

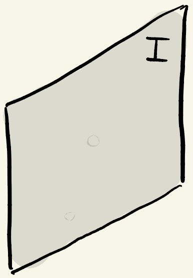
$\mathcal{C}_{\mathbb{1}}/S$ 

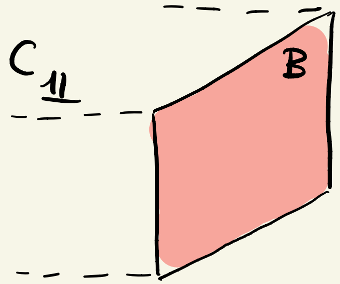
$\mathcal{C}_{\mathbb{1}}$ $\Longrightarrow$ $\mathcal{C}_{\mathbb{1}}/S \boxtimes \mathcal{C}_{\mathbb{1}}$ 

$$\Longrightarrow \quad Z(\mathcal{C}) \simeq \mathcal{C}_{\mathbb{1}}/S \boxtimes \mathcal{C}_{\mathbb{1}} .$$

where $\mathcal{C}$ is the fusion category of line operators on I.

Gauging general non-invertible symmetry $S = \sum_i s_i$.

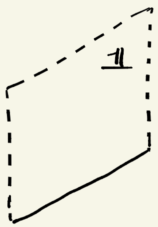
$C_{\mathbb{1}}/S$ 

$C_{\mathbb{1}}$ $\Rightarrow$ $C_{\mathbb{1}}/S \boxtimes C_{\mathbb{1}}$ 

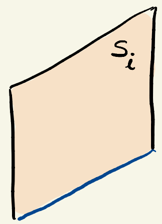
$$\Rightarrow \quad Z(C) \simeq C_{\mathbb{1}}/S \boxtimes C_{\mathbb{1}} \, .$$

where C is the fusion category of line operators on I.

Also,

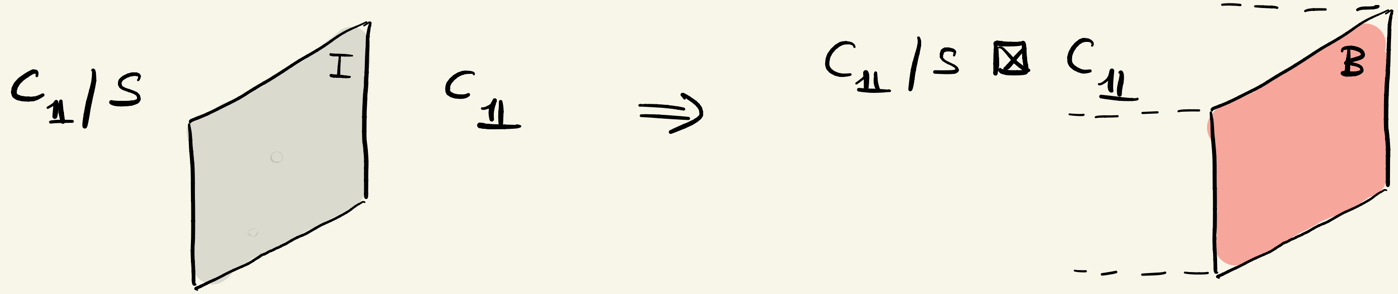
$C =$ 

$C_{\mathbb{1}}$

$\oplus$ 

C_{s_i}

Gauging general non-invertible symmetry $S = \sum_i s_i$.



$$\Longrightarrow \quad Z(\mathcal{C}) \simeq \mathcal{C}_{\mathbb{1}}/S \boxtimes \mathcal{C}_{\mathbb{1}} \; .$$

where $\mathcal{C}$ is the fusion category of line operators on I .

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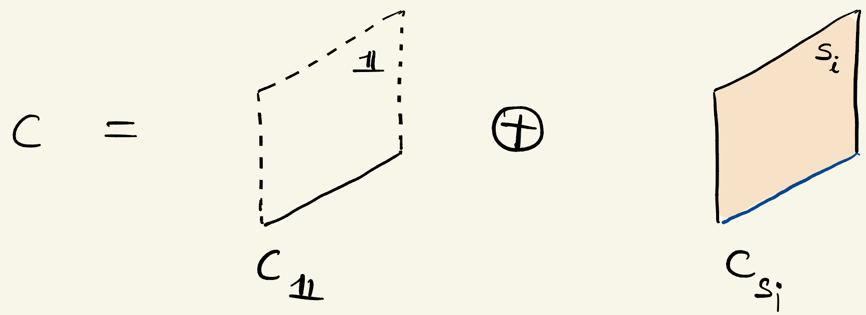

Generalized symmetry fractionalization : choice of fusion rules on $\mathcal{C}$.

Generalized discrete torsion : choice of F symbols on $\mathcal{C}$.

(Agrees with Morita theory of fusion 2-categories. [Décoppet 2023])

Conclusion

* Gauging non-invertible symmetries can relate QFTs which are not related by gauging any invertible symmetry.

* These allow us to write the (complicated) operator content of one QFT in terms of the simpler opertor content of another QFT.

* Non-invertible symmetries cannot be always gauged. This is a generalization of 't Hooft anomaly.

* Preparing $D(S_3)$ anyons starting from $D(\mathbb{Z}_2)$ anyons and gauging.

  Applications to topological quantum computation?

Requires realizing $D(\mathbb{Z}_2) \to D(S_3)$ on the lattice. (Ongoing work!)