# A 2D-CFT Factory: Critical Lattice Models from Competing Anyon Condensation in SymTO

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**29th, July 2025**
**Generalized Symmetries in HEP and CMP**

Based on works with :
**Karin Ji, Ce Shen, Yidun Wan, Yu Zhao** 2506.05324
**Gong Cheng, Lin Chen, Zheng-Cheng Gu** 2311.18005 Phys.Rev.X 15 (2025) 1, 011073
**Lin Chen, Kaixin Ji, Haochen Zhang, Ce Shen, Ruoshui Wang**
2210.12127 Physical Review X 14 (4), 041033


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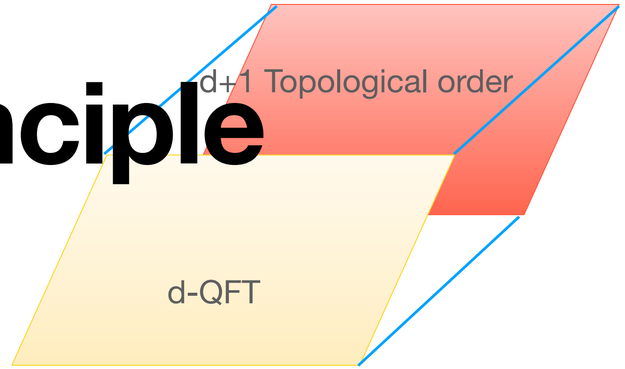
# Overview

- Topological Holographic Principle
- Integrable lattice models as ``strange correlators''
- 2D CFT Factor: an ansatz of unit cell,  and competing anyon condensation
    - Choice of modules to remove 1st order phase transitions
    - Examples: Ak series
- Phase diagram and phase boundaries
- Symmetries Preserved — from refined condensation tree
- Haagerup symmetries and novel CFTs
- Summary and Outlook

# Topological Holographic Principle




- Topological Holographic Principle states that a symmetric d-dimensional QFT can be expressed as a d+1 dimensional TQFT
  path-integral over a “sandwich”
- One boundary is painted a topological boundary condition
- The other is a “dynamical boundary condition”
- The bulk gauges the global symmetries and impose them explicitly, and it is usually called the SymTFT/SymTO.
- Topological defects in the d-QFT corresponds to topological excitations (e.g. anyon lines) in the SymTO.
- e.g. Verlinde lines in 2D CFT ⟶ Wilson lines/anyon lines in a corresponding 3D Chern-Simons theory.

**Sandwich and SymTFT**:
Kong, Zheng 2019;
Ji, Wen 2019;
Gaiotto, Kulp 2020;
Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki 2021;
Freed, Moore, Teleman 2022;

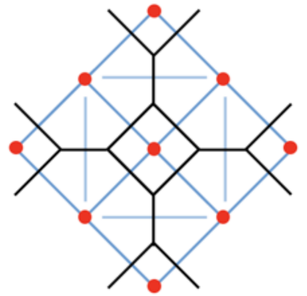
The Strange correlator is an explicit realisation.

# Integrable Lattice Models and Strange Correlators

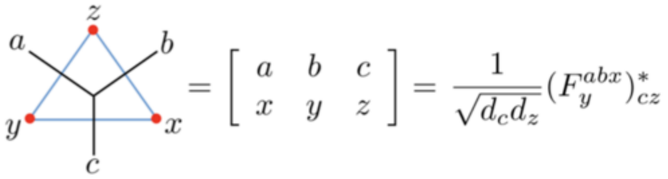
Path-integral
of a 3-ball with a two dimensional surface

# RSOS integrable models and Minimal models and Levin Wen models

Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020;

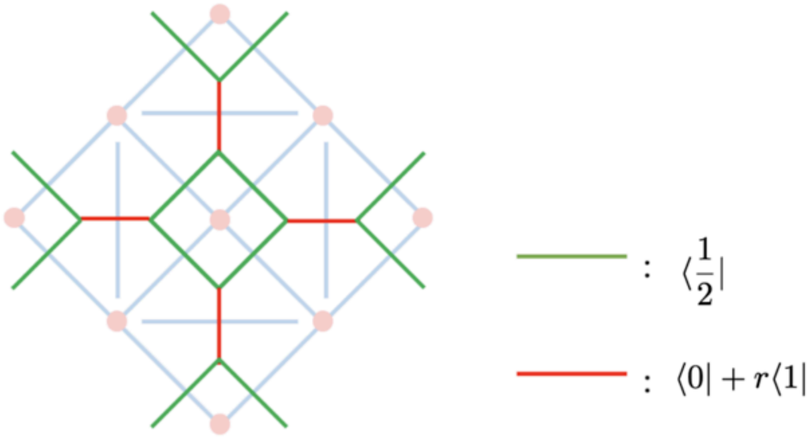


- PEPS representation of Levin-Wen models/Turaev-Viro ground state $|\Psi^{LW}\rangle$
  Gu, Levin, Swingle, Wen PRB 2009; Buershaper, Aguado, Vidal PRB 2009;




$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

- Then pick some mysterious state $\langle \Omega_N |$ and take the overlap with $|\Psi^{LW}\rangle$ i.e. $\langle \Omega_N | \Psi_a^{LW} \rangle$. $\langle \Omega_N |$ is chosen such that the overlap matches exactly the partition function of well known families of integrable models — this is a realisation of the sandwich.

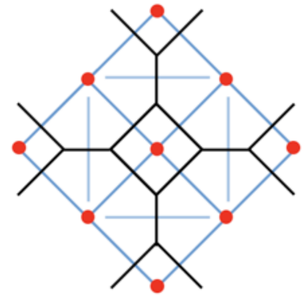



This observation is related to the anyon chain proposed in Feigen et al 2006

Path-integral
of a 3-ball with a two dimensional surface

# RSOS integrable models and Minimal models and Levin Wen models

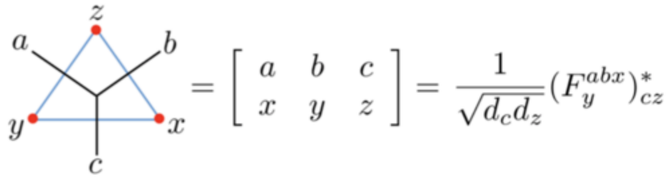
Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020;



r is related to the temperature in the classical Ising spin model:

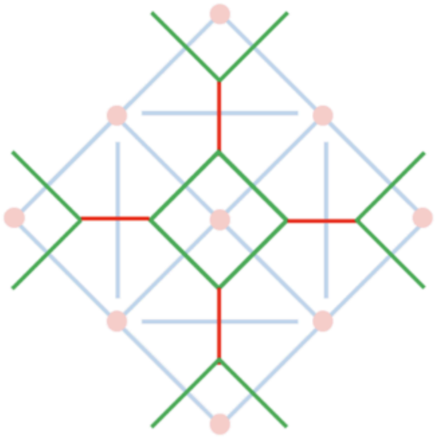
$$r = e^{-2\beta}$$

$$r_c = e^{-2\beta_c}, \qquad \beta_c = 1/2 \ln(1+\sqrt{2})$$




$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

The critical temperature is known for many years. Now expressed as the boundary condition of the Levin-Wen model.



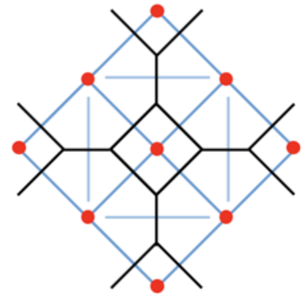
$: \langle \frac{1}{2}|$

$: \langle 0| + r\langle 1|$

# RSOS integrable models and Minimal models and Levin Wen models

Verstraete et al 2017; Aasen, Fendley, Mong 2016, 2020; Chen et al 2022;

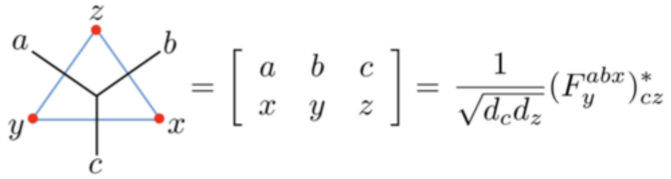
Path-integral
of a 3-ball with a two dimensional surface



More general there is the A series integrable models with the same ansatz :

Expressed in this form, the well known critical coupling is located at :

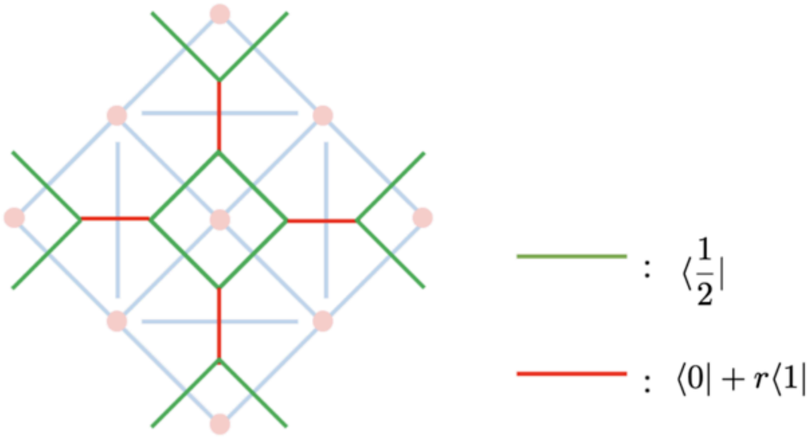
$$r_c = \sqrt{2\cos(2\pi/(k+2))+1}/(2\cos(\pi/(k+2)+1)$$




$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

What is this? Does the Levin Wen/TV model knew about these numbers?

i.e. How exactly is this ansatz chosen in principle????? In particular, which $\langle \Omega_N |$ can produce critical points = CFTs? Is there anything special about these critical couplings???



# Generalised Symmetry Preserving RG and Their Fixed Point

A lattice integrable model can also be written in this form:
R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete;
D. Aasen, P. Fendley, and R. S. K. Mong

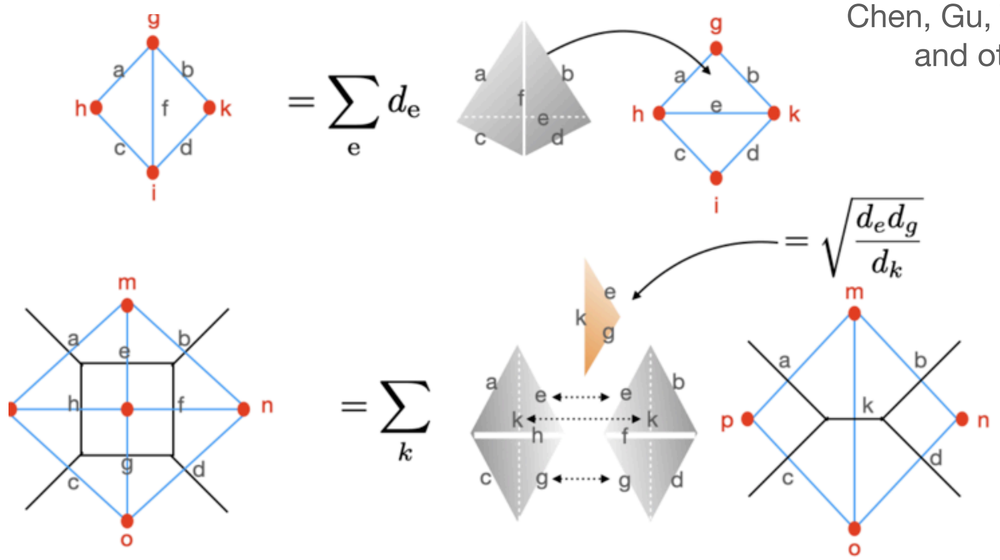
$$\langle \Omega_N | \Psi_a^{LW} \rangle$$

$$|\Psi\rangle_\Lambda = U |\Psi\rangle_{\sqrt{2}\Lambda}$$

$$\langle \Omega_N | FF | \Psi_{ka}^{LW} \rangle = \langle \Omega_{N-1} | \Psi_{ka}^{LW} \rangle$$

$$\sqrt{2}_\Lambda \langle \Omega | =_\Lambda \langle \Omega | U$$

RG of topological order
Chen, Gu, Wen 2010;
and others..




- symmetric topological phase = topological eigenstate of RG operator from Frobenius algebra.
- symmetric CFT is an infinite bond dimension eigenstate of the RG operator from conformal blocks

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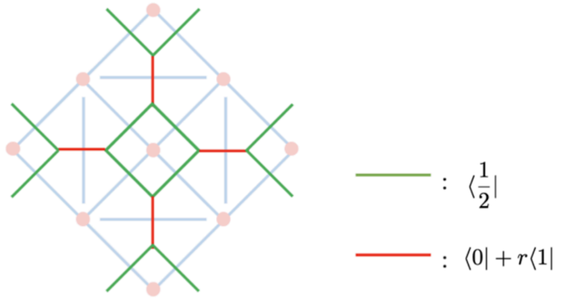
# 2D CFT Factor: an ansatz of unit cell, and competing anyon condensation

# Ansatz of Unit cell

## Module and Frobenius algebra




Example: In the Ising example, we notice that at small beta and large beta, the red edges are coloured by one of the two Frobenius algebra of the Ising model respectively. One can check that in this case they are closely related to topological boundary condition of the 3D TV model.

There are 2 Frobenius algebra:

$A_0 = 0$ Electric condensate: equivalent to Lagrangian condensate = $1 \oplus e$

Low temperature Z2 spontaneously sym breaking

How to connect the two things? There are many ways . See Wan et al. But bottom line is to use the Ribbon operator for the Levin Wen model.

$A_1 = 0 \oplus 1$ Magnetic condensate: equivalent to Lagrangian condensate = $1 \oplus m$

High temperature dual Z2 spontaneously sym breaking

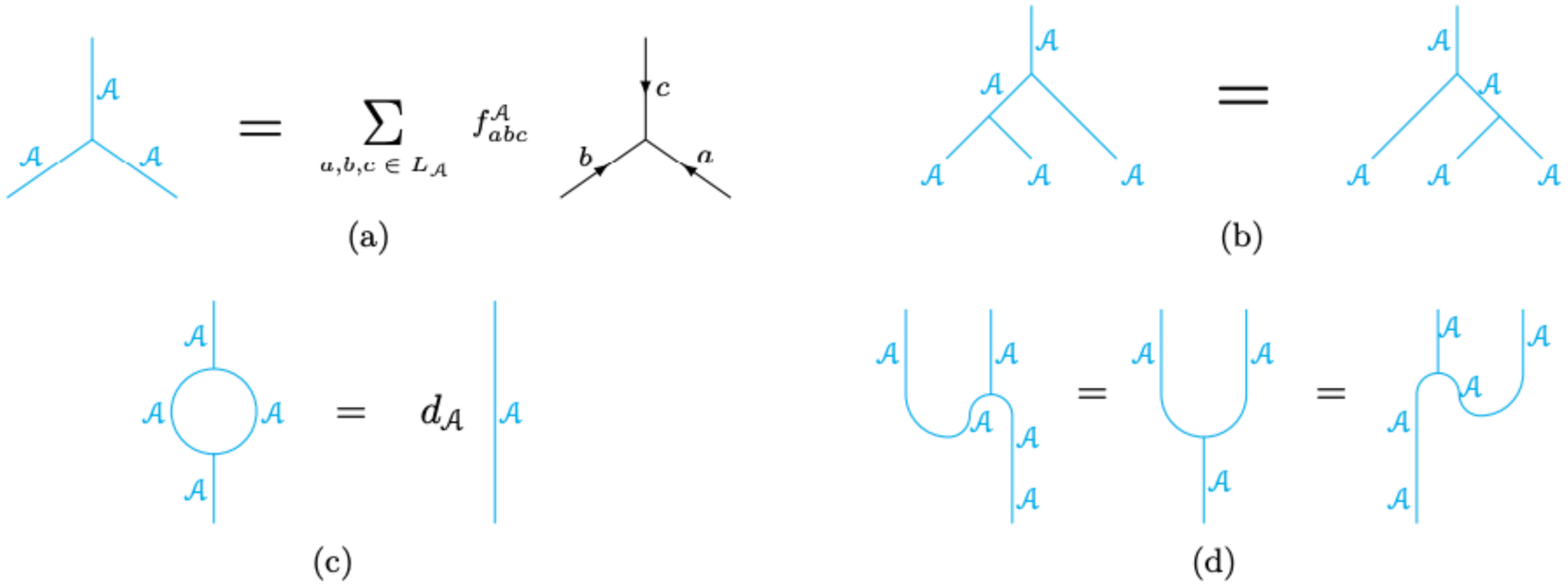
Two anyons with non-trivial braiding cannot condense together. CFT is an equilibrium point in this competition!

# Ansatz of Unit cell

## Module and Frobenius algebra

Fuchs, Runkel, Schweigert 2000s ; Hu, Wan, Wu  1706.00650;

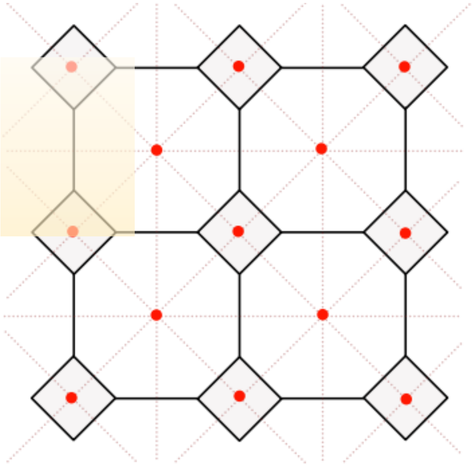
- A Frobenius algebra satisfies the following conditions:




Each Frobenius algebra describes a *Lagrangian algebra* which is a maximal set of anyons that condense. When a maximal set of anyons are condensed — this produces a trivial state. Each such trivial state is a topological boundary condition of the 3D TQFT.

# Ansatz of Unit cell

## Module and Frobenius algebra



Consider a octagon-square lattice. (Inspired by the Ising model)

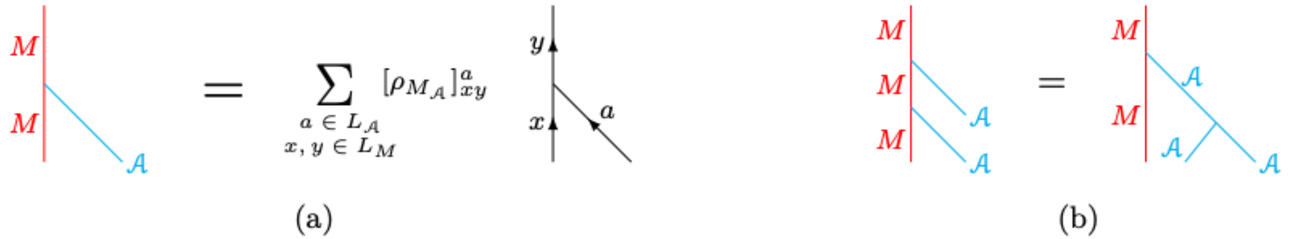
We consider  a unit cell in this lattice as highlighted above. We want to define a state corresponding to pouring the condensate into this cell, while it is well “isolated” from the other cells.

This state is a topological boundary condition of the  TQFT!

This is given by:

$$\langle \; \mathcal{A} \;\; M_{\mathcal{A}} \; | = \sum_{a \in L_{\mathcal{A}}} \sum_{x,y,u,v \in L_{M_{\mathcal{A}}}} \left\langle \begin{matrix} x & & y \\ & a & \\ u & & v \end{matrix} \right| \, [\rho_{M_{\mathcal{A}}}]^{a}_{xy} ([\rho_{M_{\mathcal{A}}}]^{a}_{uv})^{*}.$$

Here, M is a module of A. It satisfies:

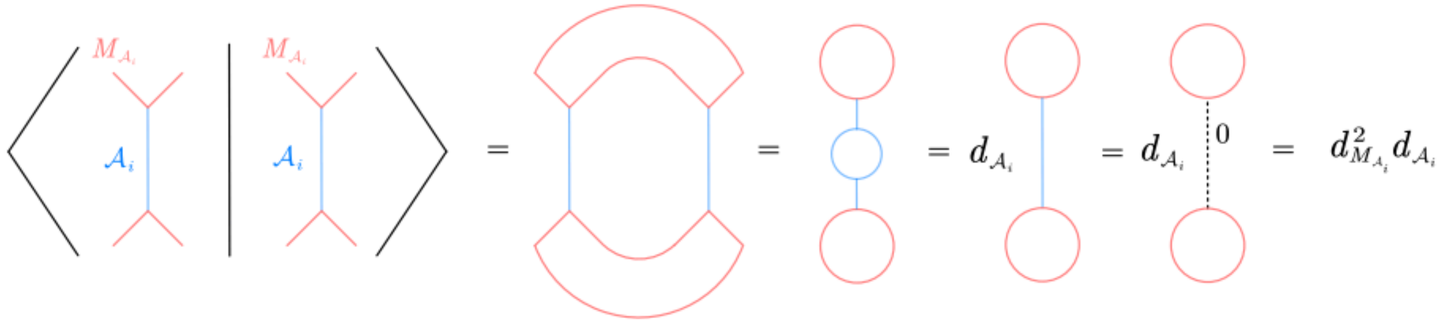


# Ansatz of Unit cell

## Module and Frobenius algebra

- Normalising a Frobenius algebra

$$N^2_{(\mathcal{A}_i, M_{\mathcal{A}_i})} =$$

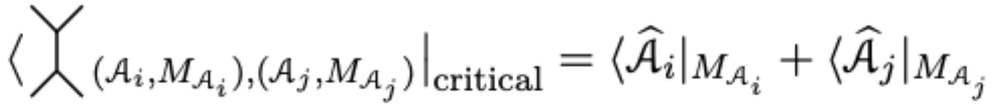


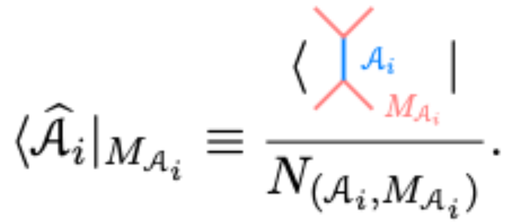

$$d_{M_{\mathcal{A}_i}} \equiv \sum_{c \in M_{\mathcal{A}_i}} m_c \, d_c, \qquad d_{\mathcal{A}_i} \equiv \sum_{a \in \mathcal{A}_i} n_a \, d_a.$$

# Ansatz of Unit cell

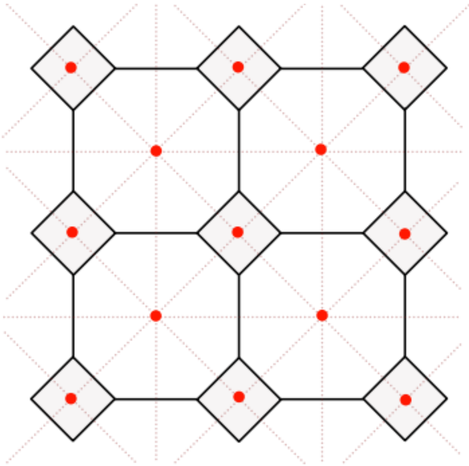
## Module and Frobenius algebra

Interpolating between condensates:




Here, we require that Ai and Aj share the module objects in M, even if the module function  may not be the same.
This is crucial to preserving the isolation of the condensate between unit cells in the full lattice.



# Ansatz of Unit cell

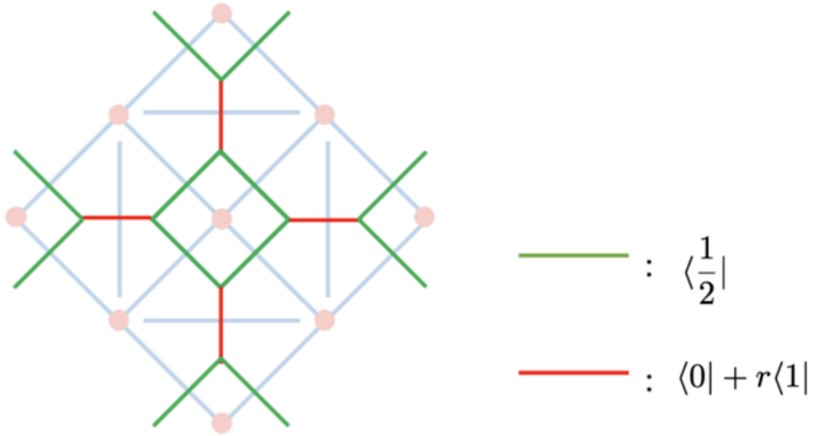
## Module and Frobenius algebra

Example:
Going back to
the A-series

Interpolating between condensates:

$$\langle \; \rangle_{(\mathcal{A}_i,M_{\mathcal{A}_i}),(\mathcal{A}_j,M_{\mathcal{A}_j})}\big|_{\text{critical}} = \langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} + \langle\widehat{\mathcal{A}}_j|_{M_{\mathcal{A}_j}}$$

$$\langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} \equiv \frac{\langle \; \mathcal{A}_i \; | }{N_{(\mathcal{A}_i,M_{\mathcal{A}_i})}}.$$




For all k we have the following Frobenius algebra:

$$A_0 = 0$$

$$A_1 = 0 \oplus 1$$

$$\mathcal{M} = \frac{1}{2}$$

$$r_c = \sqrt{2\cos(2\pi/(k+2)) + 1}/(2\cos(\pi/(k+2) + 1)$$

# Why does it work? — why is it a second order phase transition?

## Key: cut down on degrees of freedom in RG space

- For chosen M such that it is shared by all the interpolating algebras (for simplicity, including module function), the RG attraction basins would be reduced to these algebras.

number of sharing A = $n_M$

  - This happens where $\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k$

Number of couplings in a unit cell:

$$D_{\text{unit}}(\cdots \subset \mathcal{A}_i \subset \mathcal{A}_j \subset \mathcal{A}_k) = N_{\mathcal{A}_k}. \qquad D_c = D_{\text{unit}} - 1$$

This is a sufficient but not *necessary* condition!

***All phase transitions in a phase diagram are forced to be second order if***

$$n_M > D_c.$$

i.e. all directions correspond to moving towards an incompatible phase.
No directions describe degeneracy between two phases.

# Why does it work? — why is it a second order phase transition?

## Key: cut down on degrees of freedom in RG space

$$\langle \curlyvee_{(\mathcal{A}_i,M_{\mathcal{A}_i}),(\mathcal{A}_j,M_{\mathcal{A}_j})}|_{\text{critical}} = \langle\widehat{\mathcal{A}}_i|_{M_{\mathcal{A}_i}} + \langle\widehat{\mathcal{A}}_j|_{M_{\mathcal{A}_j}}$$

This expression reproduces the correct phase transition points in all the infinite set of examples below.

- Ak series examples : nM = 2, Dc = 1 $n_M > D_c.$ The entire series is second order phase transitions !

- N-state Potts model ( $A_e = 0 \qquad A_m = 0 \oplus \cdots N-1 \qquad M = 0 \oplus \cdots N-1$ )
  - N=2 nM = 2, Dc = 1 => 2nd order = Ising
  - N=3 nM = 2, Dc = 2 => 2nd order = 3 state Potts — seems to be an accident?
  - N=5 nM = 2, Dc= 4 => (weak) first order

Beyond N>5 well known that they are all first order transitions.

# Phase Diagrams and Phase Boundaries

# 3 phase competition: Ashkin-Teller model from A5

## phase boundaries and tri-critical point

- The module: $M = 1$ $A_0 = 0 \subset A_1 = 0 \oplus 4 \subset A_3 = 0 \oplus 4 \oplus 2$

Note that this choice of module kills $A_2 = 0 \oplus 2$

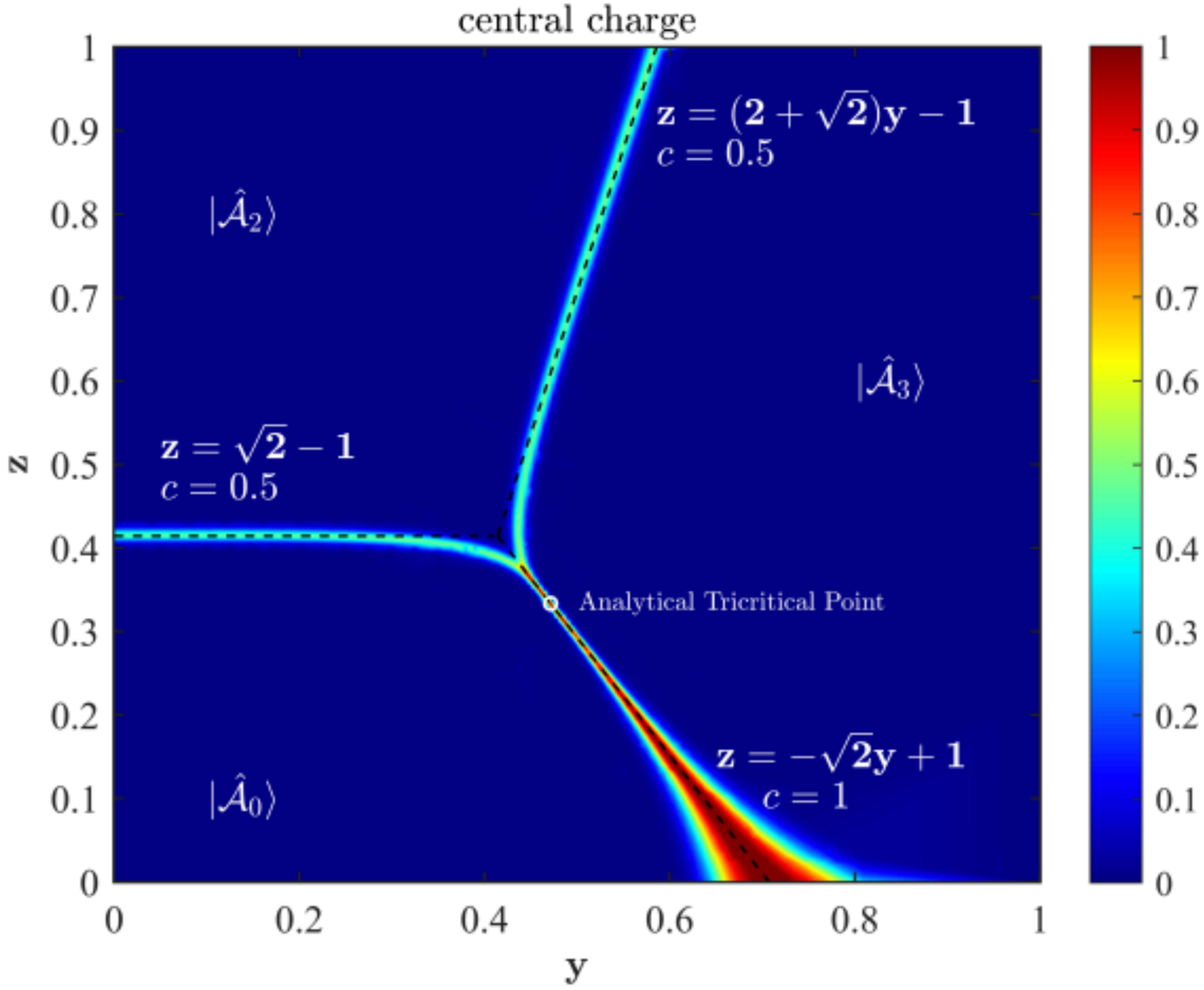
nM = 3, Dc = 2 => All phase transitions are second order.

**phase bou**

- The module

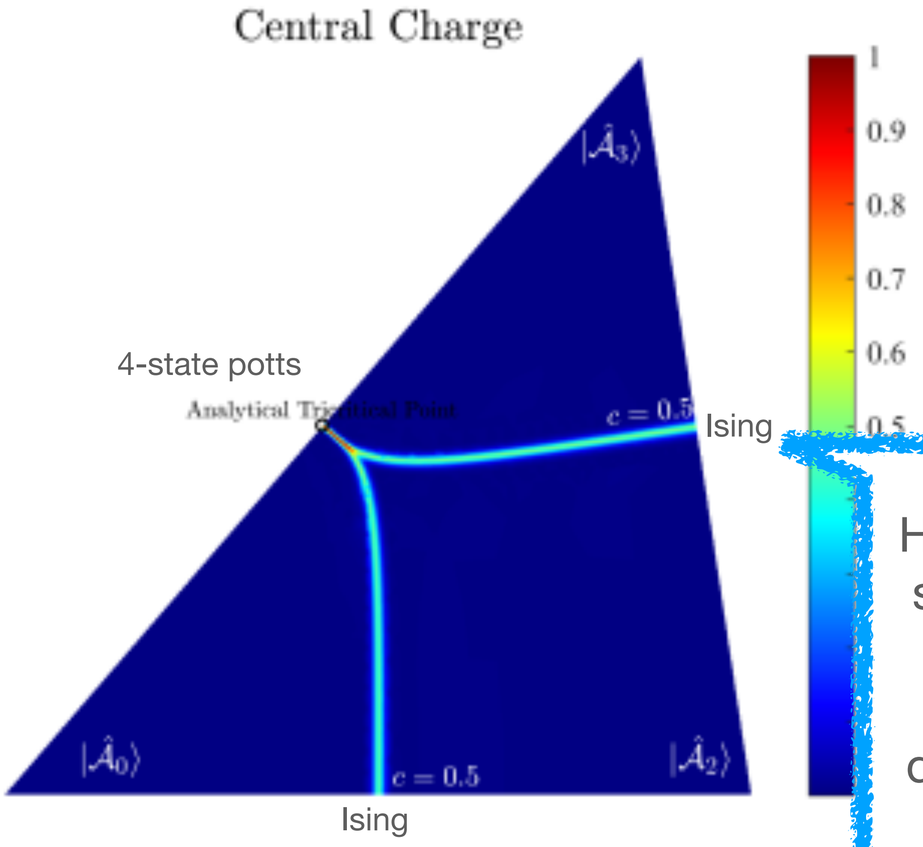
Note that this

nM = 3, Dc =
second order.



# 3 phase competition: Ashkin-Teller model from A5

## phase boundaries and tri-critical point




Phase boundaries:

$$\langle C_{ij}| = \frac{1}{2}(\langle \hat{\mathcal{A}}_i| + \langle \hat{\mathcal{A}}_j|)$$

$$\langle C_{ij}(p)| = \langle C_{ij}| + p\left(\langle \hat{\mathcal{A}}_k| - \frac{\langle \hat{\mathcal{A}}_k|B_{ij}\rangle}{\langle B_{ij}|B_{ij}\rangle}\langle B_{ij}|\right)$$

How do we understand which CFT would show up here? Particularly they seem to preserve different amount of (non)-invertible symmetries. How do we determine what is the *minimal* amount of symmetries preserved?

# Symmetries preserved — refined condensation tree

# Refined condensation tree

## Refining the ``Hasse diagram"

Chatterjee, Wen 22;  Bhardwaj, Pajer, Schafer-Nemaki, Warman 24;

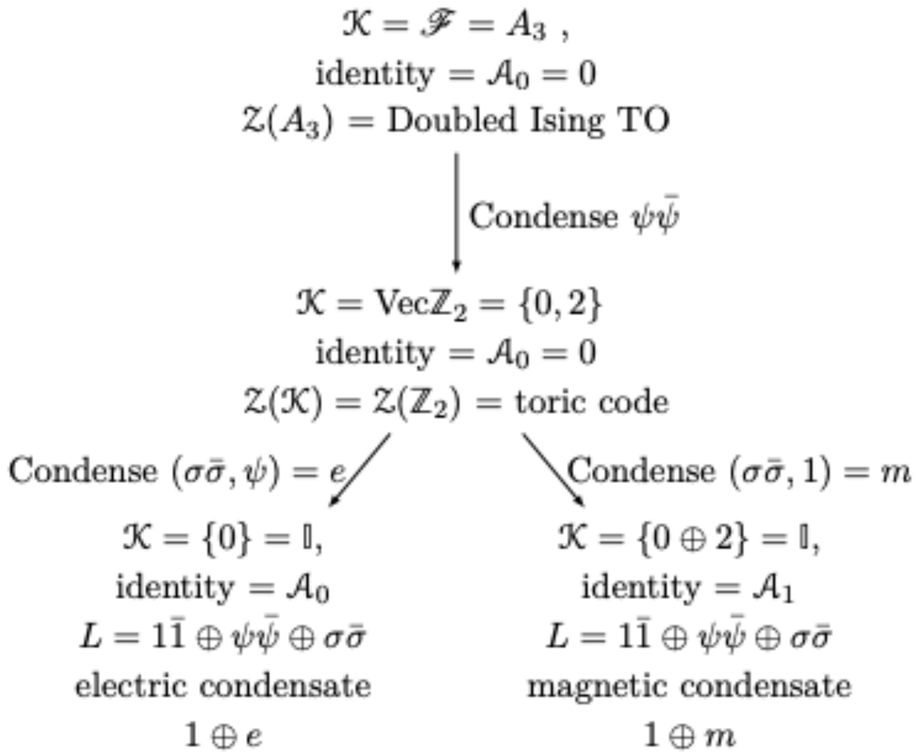
- Each node corresponds to a topological order obtained from partial anyon condensation of the parent phase - one draws a line connecting the child and parent phase
- From the perspective of the ``input-category" C, each node correspond to a sub-category K of C
- Grow the tree from the top node corresponding to K = C (no condensation). The bottom nodes of the diagram are ``maximal" condensation corresponding to Lagrangian algebra L —now K becomes trivial with only 1 object, this object being a Frobenius algebra A .

# Refined condensation tree

- Several Frobenius algebra A corresponds to the ``same" L
- But these L are actually different microscopically ! Different A does correspond to different boundary conditions. Two different A's correspond to the ``same" L can have non-trivial phase transitions between them.
- Our tree contains all these different A's that may correspond to the same L.

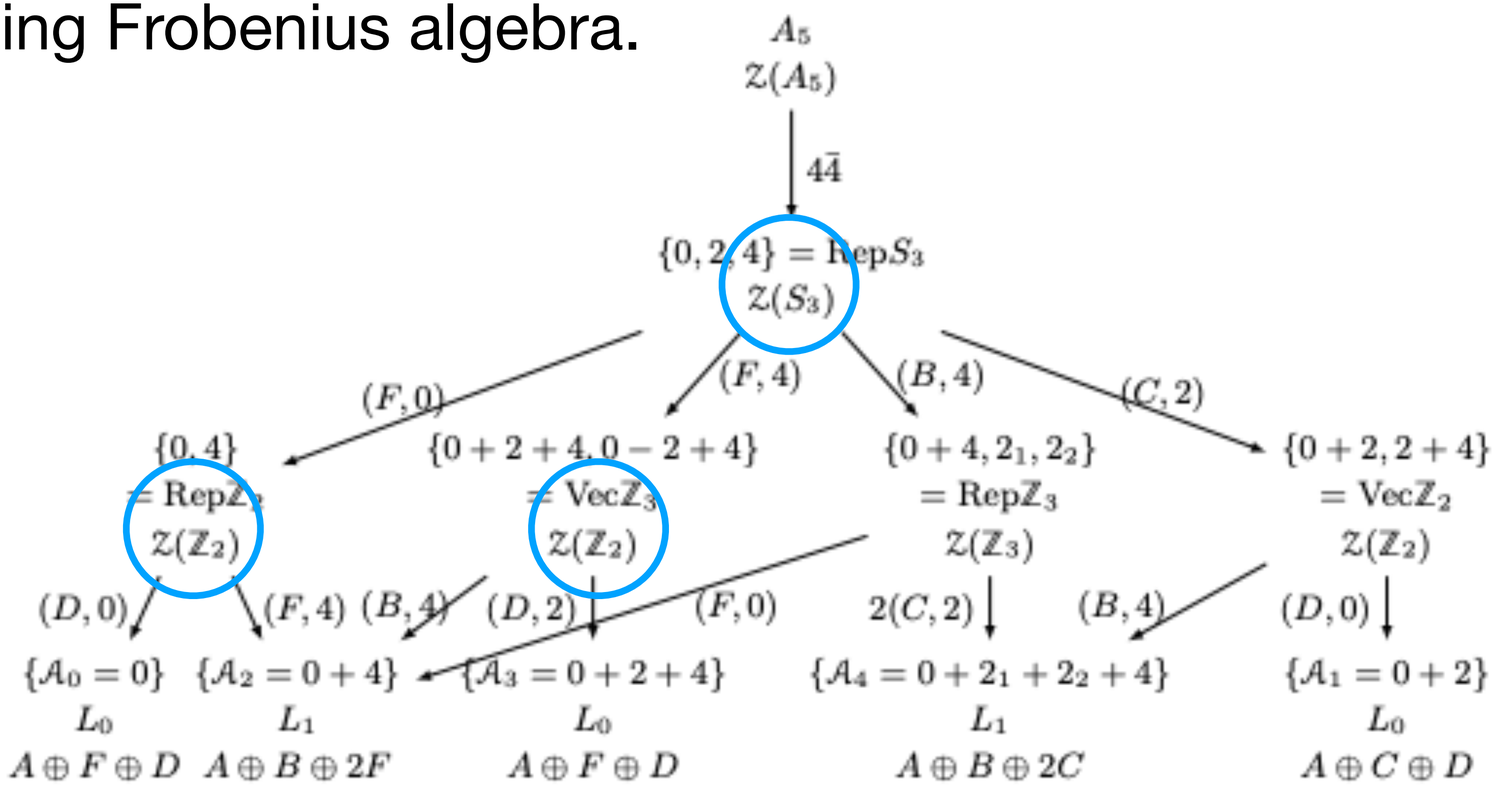
# Refined condensation tree

- Examples: Ising



# Refined condensation tree

- Examples: A5
  The minimal symmetry is determined by the first shared parent between to competing Frobenius algebra.



# Novel Examples - Haagerup TQFT

# Haagerup Model

$$1, \quad \alpha, \quad \alpha^2, \quad \rho, \quad \alpha\rho, \quad \alpha^2\rho,$$

The fusion rules are

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $\alpha$ | $\alpha^2$ | $\rho$ | $\alpha\rho$ | $\alpha^2\rho$ |
| $\alpha$ | $\alpha^2$ | $1$ | $\alpha\rho$ | $\alpha^2\rho$ | $\rho$ |
| $\alpha^2$ | $1$ | $\alpha$ | $\alpha^2\rho$ | $\rho$ | $\alpha\rho$ |
| $\rho$ | $\alpha^2\rho$ | $\alpha\rho$ | $1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ |
| $\alpha\rho$ | $\rho$ | $\alpha^2\rho$ | $\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ |
| $\alpha^2\rho$ | $\alpha\rho$ | $\rho$ | $\alpha^2 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $\alpha \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ | $1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$ |

$$d_1 = d_\alpha = d_{\alpha^2} = 1, \qquad d_\rho = d_{\alpha\rho} = d_{\alpha^2\rho} = \frac{3+\sqrt{13}}{2}.$$

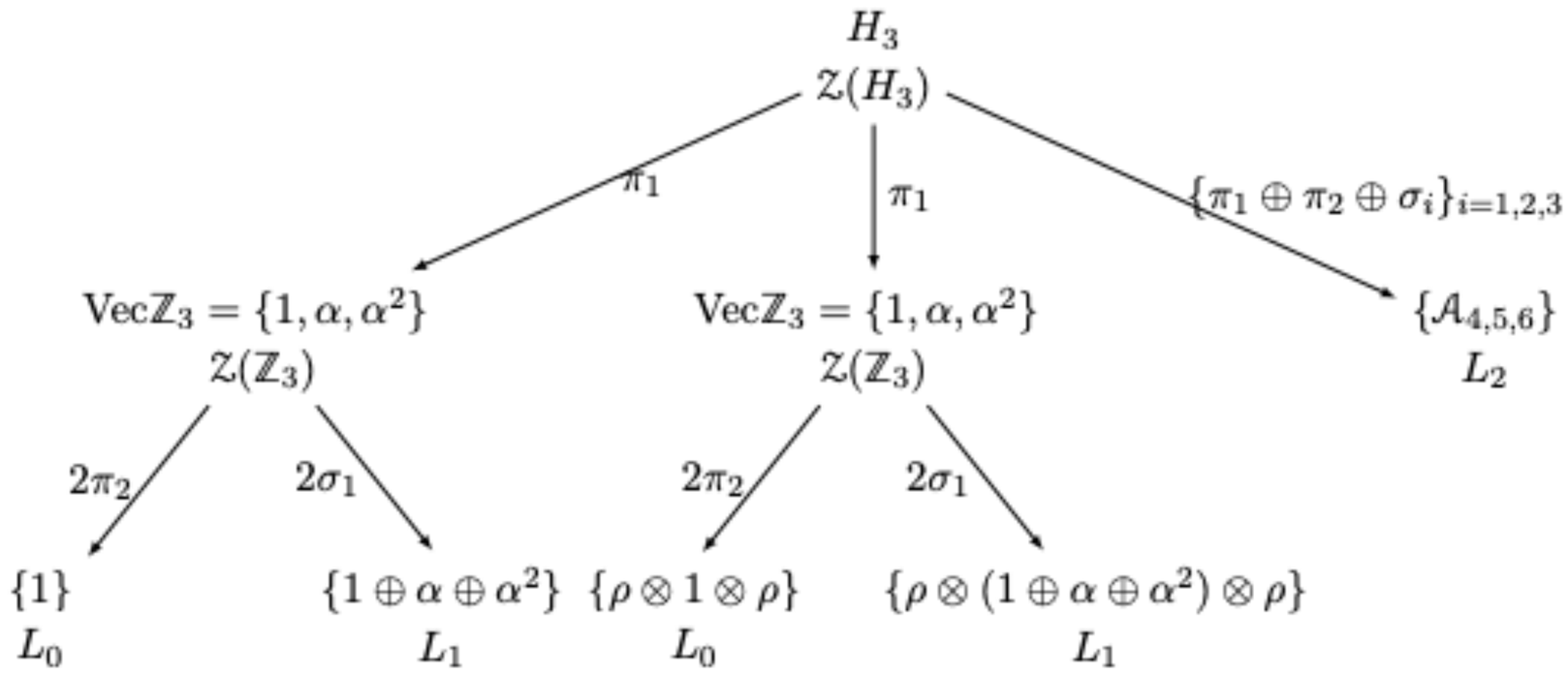
$$L_0 = 1 \oplus \pi_1 \oplus 2\pi_2$$
$$L_1 = 1 \oplus \pi_1 \oplus 2\sigma_1$$
$$L_2 = 1 \oplus \pi_1 \oplus \pi_2 \oplus \sigma_1.$$

$$\mathcal{A}_0 = 1$$
$$\mathcal{A}_1 = \rho \otimes \mathcal{A}_0 \otimes \rho = 1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$$
$$\mathcal{A}_2 = 1 \oplus \alpha \oplus \alpha^2$$
$$\mathcal{A}_3 = \rho \otimes \mathcal{A}_2 \otimes \rho = (1 \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho) \otimes (1 \oplus \alpha \oplus \alpha^2).$$
$$\mathcal{A}_4 = 1 \oplus \rho \oplus \alpha\rho,$$
$$\mathcal{A}_5 = \alpha \otimes \mathcal{A}_4 \otimes \alpha^2 = 1 \oplus \alpha\rho \oplus \alpha^2\rho,$$
$$\mathcal{A}_6 = \alpha^2 \otimes \mathcal{A}_4 \otimes \alpha = 1 \oplus \rho \oplus \alpha^2\rho.$$

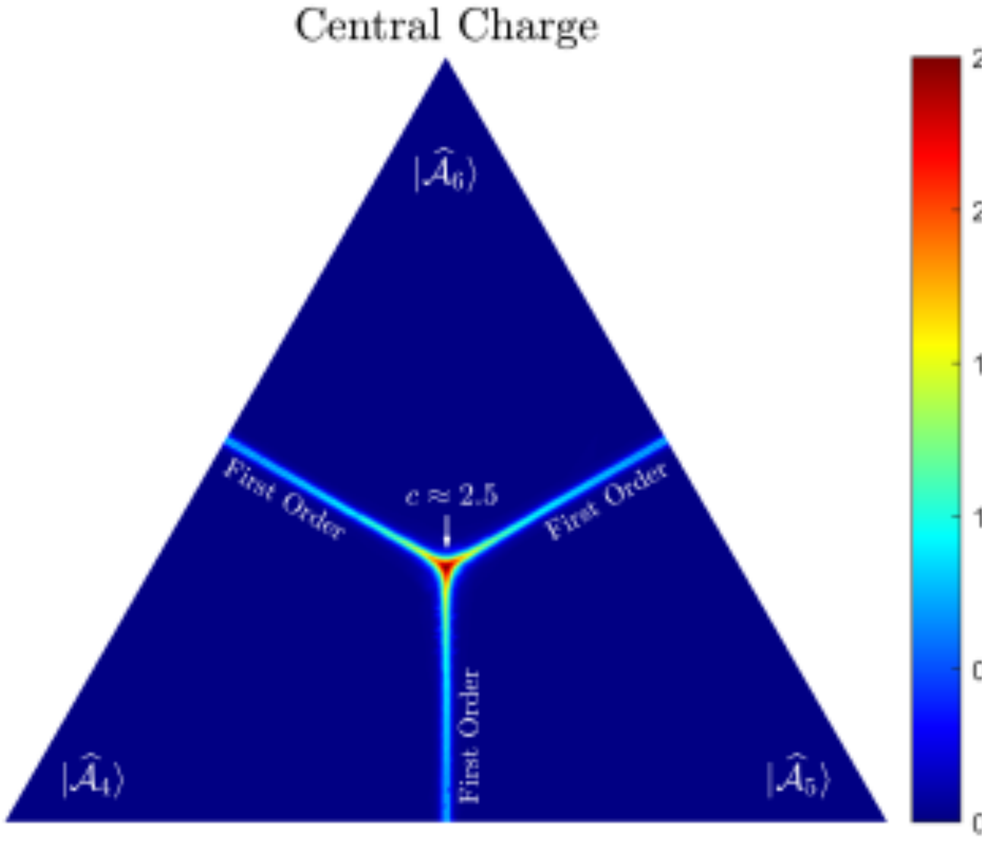
$$L_0 : \ \mathcal{A}_0, \mathcal{A}_1$$
$$L_1 : \ \mathcal{A}_2, \mathcal{A}_3$$
$$L_2 : \ \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6.$$

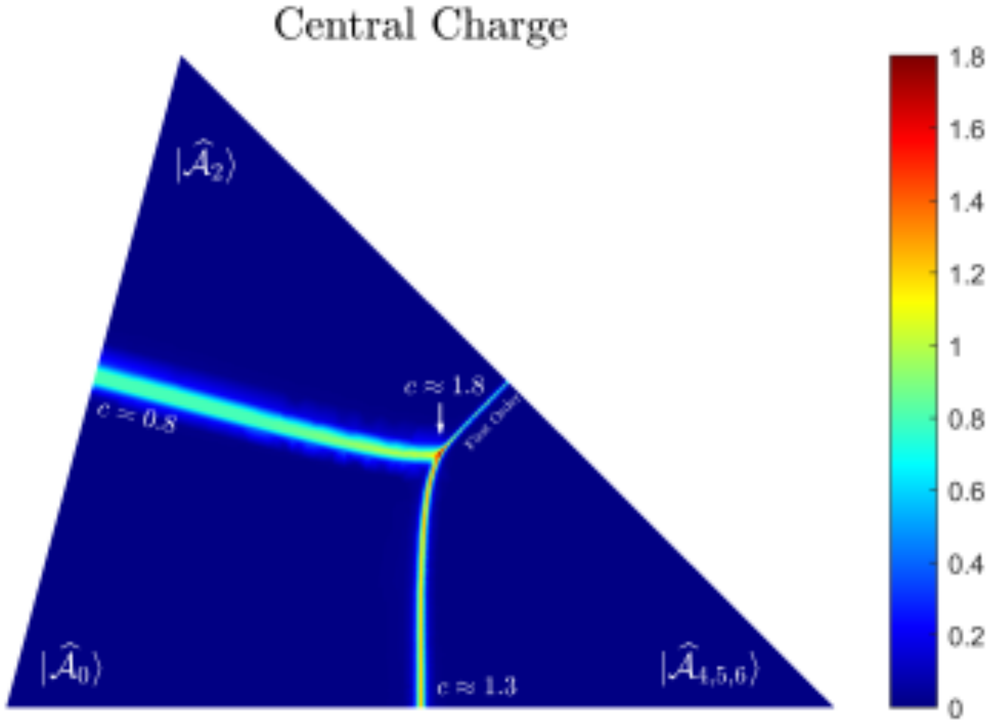
# Haagerup Model

$$1, \quad \alpha, \quad \alpha^2, \quad \rho, \quad \alpha\rho, \quad \alpha^2\rho,$$

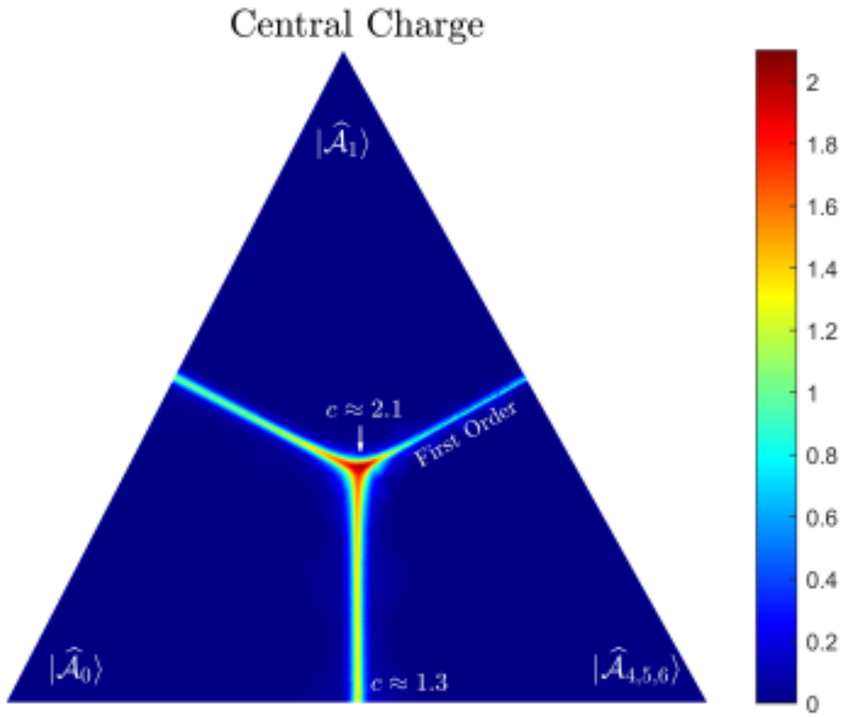


# Haagerup Model



# Haagerup Model



# Haagerup Model - summary of results

| Cat | $H_3$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathcal{A}_i$ | $\mathcal{A}_0$ | | | $\mathcal{A}_1$ | | $\mathcal{A}_2$ | $\mathcal{A}_{4,5,6}$ |
| $\mathcal{A}_j$ | $\mathcal{A}_1$ | $\mathcal{A}_2$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_2$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_{(5,6),(4,6),(4,5)}$ |
| $M$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | | | | | | |
| $D_{\text{unit}}$ | 4 | 3 | 3 | 6 | 6 | 5 | 5 |
| Type | Second order | | | First order | | | |
| CFT | $c \approx 2.0^*$ | 3-Potts $c = 0.8$ | $c \approx 1.3$ | – | | | |

* is possibly the CFT that has been observed in [20, 21]. See section 4 for details.

# Haagerup Model - summary of results

| Cat | $H_3$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}_i$ | $\mathcal{A}_0$ | $\mathcal{A}_0$ | $\mathcal{A}_0$ | $\mathcal{A}_1$ | $\mathcal{A}_1$ | $\mathcal{A}_0$ | $\mathcal{A}_2$ | $\mathcal{A}_4$ |
| $\mathcal{A}_j$ | $\mathcal{A}_1$ | $\mathcal{A}_2$ | $\mathcal{A}_1$ | $\mathcal{A}_2$ | $\mathcal{A}_{4,5,6}$ | | | $\mathcal{A}_5$ |
| $\mathcal{A}_k$ | $\mathcal{A}_2$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_{4,5,6}$ | $\mathcal{A}_{(5,6),(4,6),(4,5)}$ | | | $\mathcal{A}_6$ |
| $M$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | | | | | | | |
| $D_{\text{unit}}$ | 6 | 5 | 6 | 8 | 8 | 5 | 7 | 7 |
| Type | Second order | | | First order | | Second order | | |
| CFT | $c \approx 1.8$ | $c \approx 1.8$ | $c \approx 2.1$ | – | – | $c \approx 1.8$ | $c \approx 2.1$ | $c \approx 2.5$ |

# Summary and Outlook

- We produce a systematic way of constructing *critical* lattice models based on the strange correlator
- This gives the Landau paradigm for generalised symmetries a more precise handle to construct effective field theories — here, the order parameter is precisely the anyon creation operator defined on some choice of unit cells
- We are at the same time constructing UV complete CFTs from these lattice models, providing an alternative route to searching for CFTs using a different strategy from the bootstrap — we have in fact found several candidate novel CFTs with Haagerup symmetries
- We combine loop TNR and this symTRG method to produce these precise phase diagrams - and find confirmation of our prediction of location of critical points and phase boundaries over large swathes of the phase diagram up to a small area close to high critical points
- How to enlarge the unit cell?
  Change the shape of the lattice?
  Connection to integrability?
  generalisation to higher dimensions?????

# Thank you!