Deconfined Quantum Critical Points and the Dual Generalized Symmetries

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Generalized Symmetry in HEP and CMP

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Collaborators: Hanlin Lin, Shuo Yang, Hao-Ran Zhang (to appear)

Outline

- Introduction to deconfined quantum critical points
- Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ with mixed anomaly $\mathrm{Vec}^{\nu_3}_{\mathbb{Z}_2\times\mathbb{Z}_2} \sim \mathrm{Vec}^{0}_{\mathbb{Z}_4}$
- Type-I DQCP with $E^{2,1}_{\infty}$-type anomaly
  $\leftrightarrow$ dual non-anomalous invertible symmetry
- Example: D8 with $E^{1,2}_{\infty}$-type anomaly $\mathrm{Vec}^{\nu_3}_{D_8} \sim \mathrm{Rep}(H_8)$
- Type-II DQCP with $E^{1,2}_{\infty}$-type anomaly
  $\leftrightarrow$ dual non-anomalous non-invertible symmetry
- Conclusion

Usual phase transition

- Spontaneous symmetry breaking
- Order parameter and Ginzburg-Landau theory
- Question: Is there a continuous phase transition breaking/restoring two different symmetries?

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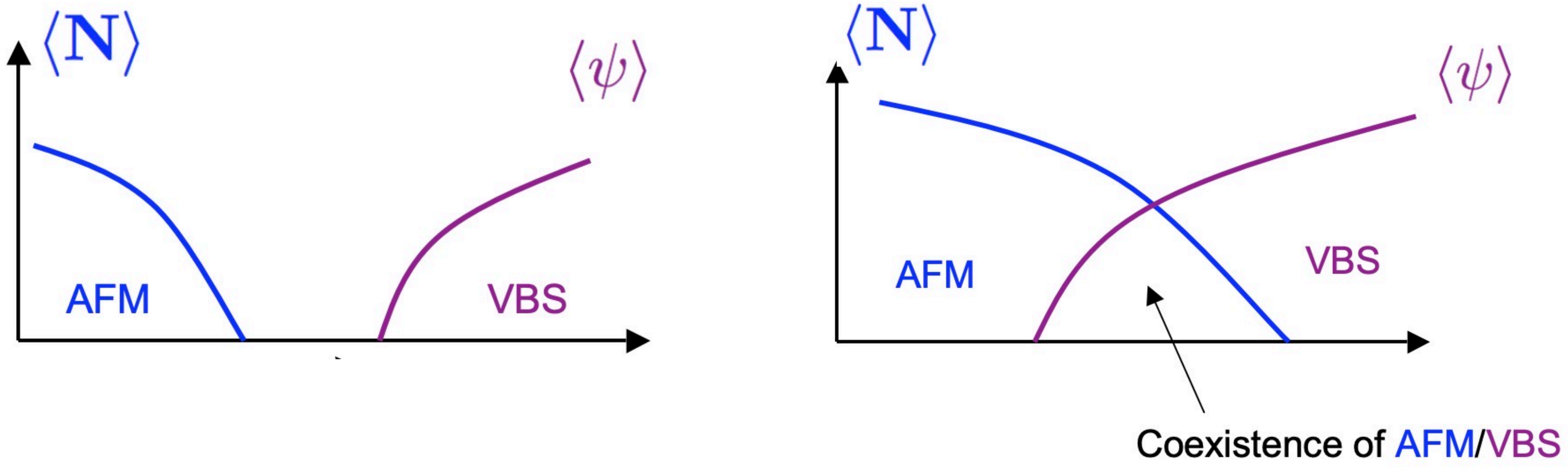
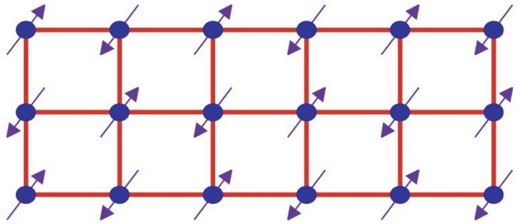
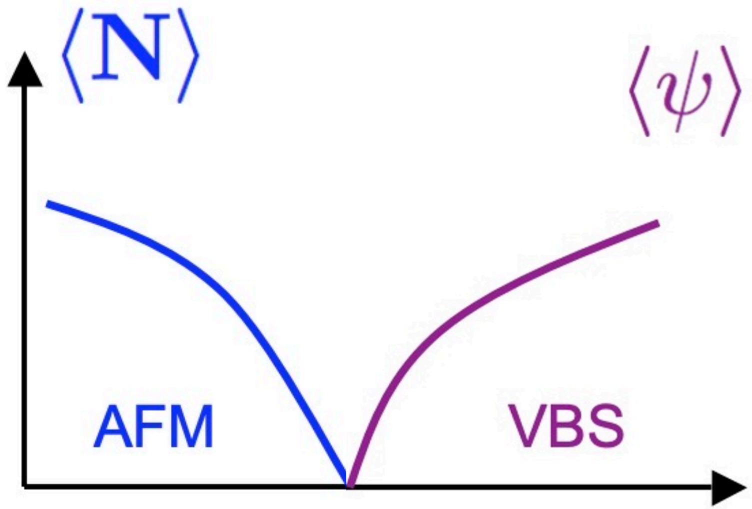


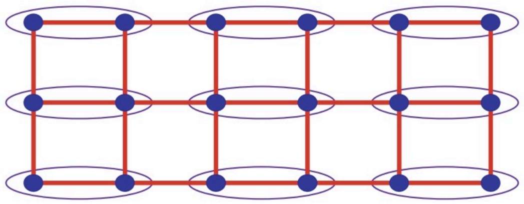
Deconfined quantum critical points (DQCP)

- Consider a spin model on a square lattice $H = J\sum_{\langle rr'\rangle} \vec{S}_r \cdot \vec{S}_{r'} + \ldots$
- Symmetry: spin rotation and lattice rotation
- Possible DQCP in 2+1D: NCCP1 $\mathcal{L} = \sum_{\alpha=1}^{2} |(\partial_\mu - i a_\mu)\, z_\alpha|^2 + s|z|^2 + u(|z|^2)^2 + \kappa(\epsilon_{\mu\nu\kappa}\partial_\nu a_\kappa)^2$
- Application: deconfined excitations, emergent symmetries, dualities, ⋯



spin rotation ✗
lattice rotation ✓





spin rotation ✓
lattice rotation ✗

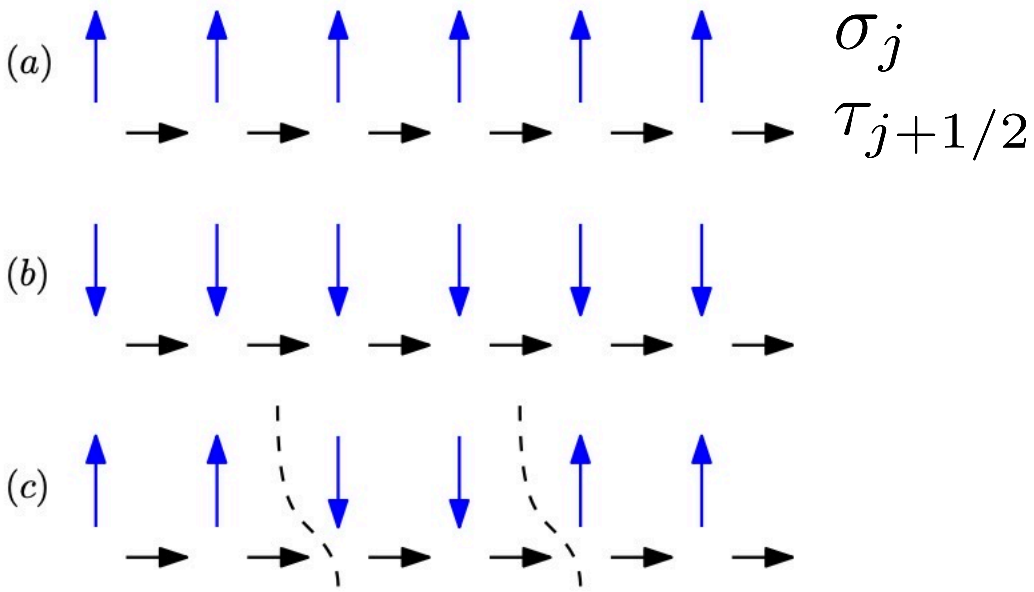
Senthil-Vishwanath-Balents-Sachdev-Fisher 2004

Solvable model example: 1+1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ DQCP

- Anomalous $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symmetry:

$$U_a = \prod_j \sigma_j^x \qquad U_b = \prod_j \tau_{j+1/2}^x i^{\frac{1-\sigma_j^z \sigma_{j+1}^z}{2}}$$

- Two $\mathbb{Z}_2^a$ domain walls carry (fuse into) one $\mathbb{Z}_2^b$ charge $\to \mathbb{Z}_4$



Zhang-Levin 2022

Solvable model example: 1+1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ DQCP

- Mapping to non-anomalous $\mathbb{Z}_4$ symmetry

$$\prod_j S_j \leftrightarrow U_a \qquad \prod_j C_j^\dagger C_{j+1} \leftrightarrow U_b.$$

$$\hat{\mathbb{Z}}_2^b \to \mathbb{Z}_4 \to \mathbb{Z}_2^a$$

$\mathbb{Z}_4$ ✗ Usual SSB $\mathbb{Z}_4$ ✓

$\mathbb{Z}_2^a$ ✗ $\mathbb{Z}_2^a$ ✓

DQCP

$\mathbb{Z}_2^b$ ✓ $\mathbb{Z}_2^b$ ✗

Zhang-Levin 2022

Categorical understanding

- From $\mathrm{Vec}^{\nu_3}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ with $\nu_3(g,h,k)=(-1)^{g_1 h_2 k_2}\in H^3(\mathbb{Z}_2\times\mathbb{Z}_2,U(1))=\mathbb{Z}_2^3$
- Gauge a non-anomalous subgroup $\mathbb{Z}_2$
- (Tachikawa 2017) Resulting in the extension

$$\hat{\mathbb{Z}}_2\to\mathbb{Z}_4\to\mathbb{Z}_2$$

with 2-cocycle

$$e_2(h,k)=hk\in H^2(\mathbb{Z}_2,\hat{\mathbb{Z}}_2)=\mathbb{Z}_2$$

- Dual symmetry is $\mathrm{Vec}^0_{\mathbb{Z}_4}$
- Morita equivalent fusion categories:

$$\mathrm{Vec}^{\nu_3}_{\mathbb{Z}_2\times\mathbb{Z}_2}\sim\mathrm{Vec}^0_{\mathbb{Z}_4}\qquad Z(\mathrm{Vec}^{\nu_3}_{\mathbb{Z}_2\times\mathbb{Z}_2})=Z(\mathrm{Vec}^0_{\mathbb{Z}_4})$$

Categorical understanding

- From $\mathrm{Vec}^{\nu_3}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ with $\nu_3(g,h,k) = (-1)^{g_1 h_2 k_2} \in H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2^3$
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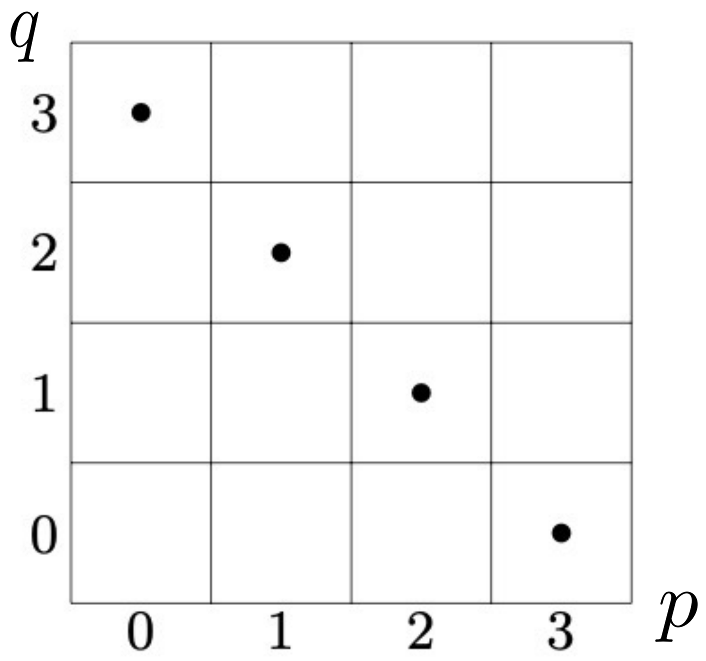
Generalization to other groups with other anomaly

- Generalization of Tachikawa 2017: $\mathrm{Vec}_{A\rtimes_{e_2}Q}^{\nu_3=a\smile\tilde{e}_2(q)-\omega(q)} \sim \mathrm{Vec}_{\hat{A}\rtimes_{\tilde{e}_2}Q}^{\hat{a}\smile e_2(q)-\omega(q)}$
- But the 3-cocycle is of special form $\nu_3 = a \smile \tilde{e}_2(q)$

$$H^3(A\times Q, U(1)) = \oplus_{p+q=3}E_2^{p,q} = \oplus_{p+q=3}H^p(Q, H^q(A, U(1)))$$



$\nu_3 \in E_2^{0,3} = H^0(Q, H^3(A, U(1)))$ A is anomalous

$\nu_3 \in E_2^{1,2} = H^1(Q, H^2(A, U(1)))$

$\nu_3 \in E_2^{2,1} = H^2(Q, H^1(A, U(1)))$ Tachikawa's case

$\nu_3 \in E_2^{3,0} = H^3(Q, H^0(A, U(1)))$ Anomaly of Q

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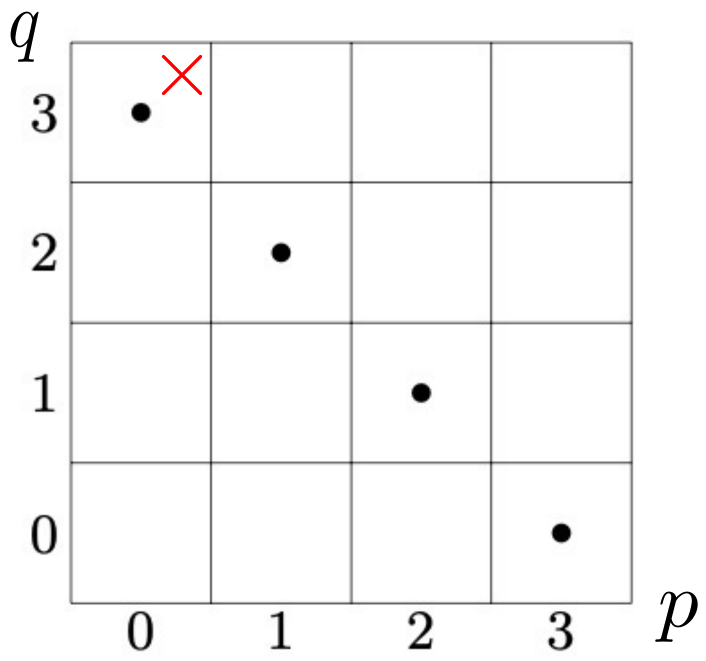

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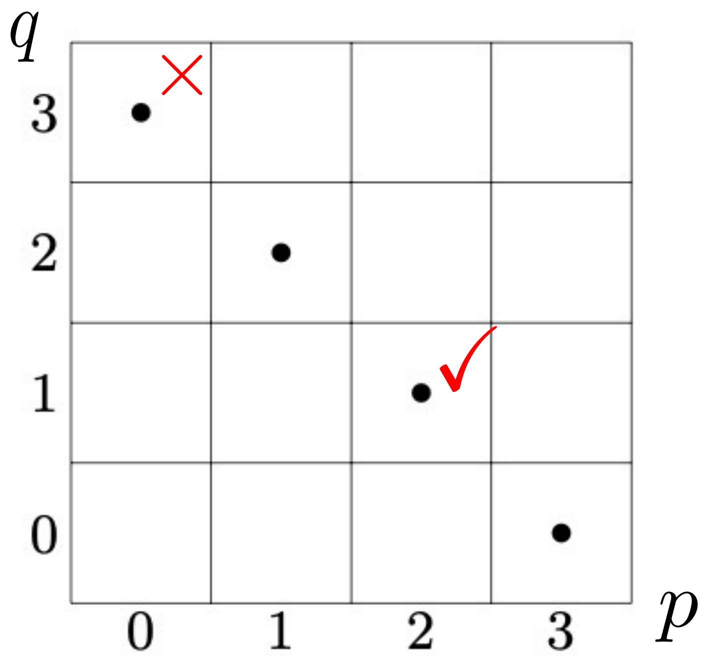

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Generalization to other groups with other anomaly

- Generalization of Tachikawa 2017: $\mathrm{Vec}_{A\rtimes_{e_2} Q}^{\nu_3 = a\smile\tilde{e}_2(q) + \omega(q)} \sim \mathrm{Vec}_{\hat{A}\rtimes_{\tilde{e}_2} Q}^{\hat{a}\smile e_2(q) - \omega(q)}$
- But the 3-cocycle is of special form $\nu_3 = a \smile \tilde{e}_2(q)$

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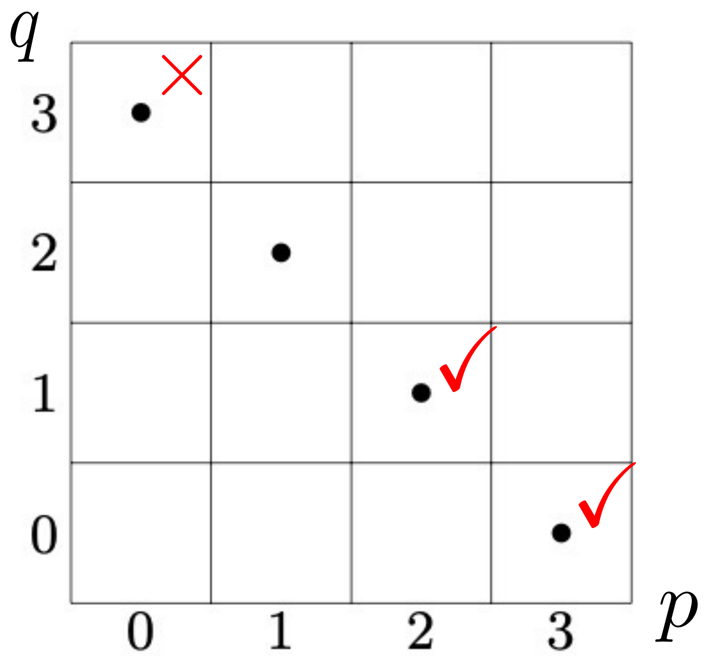

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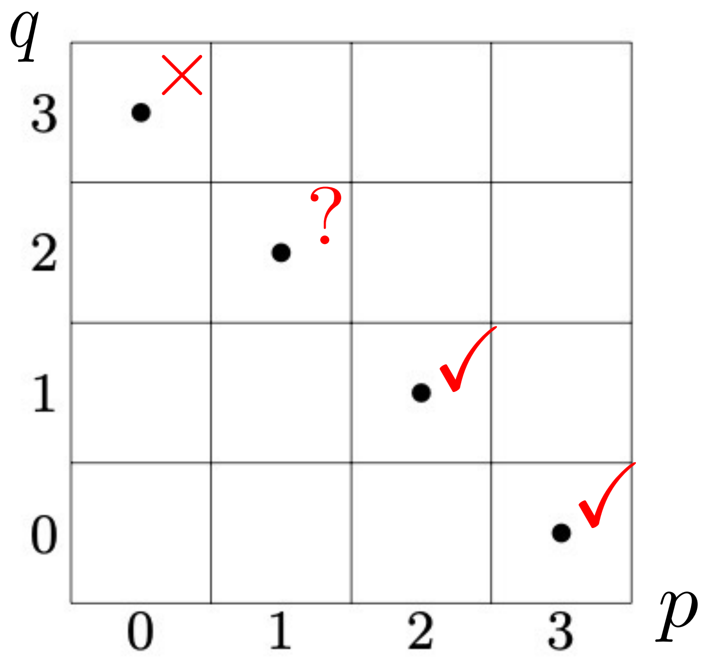

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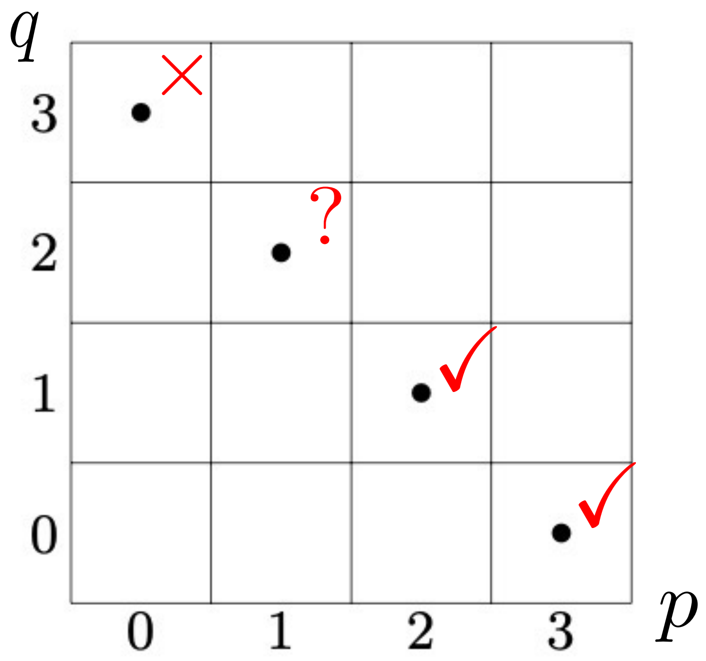

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$\nu_3 \in E_2^{1,2} = H^1(Q, H^2(A, U(1)))$ Dual to non-invertible symmetry

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$\nu_3 \in E_2^{3,0} = H^3(Q, H^0(A, U(1)))$ Anomaly of Q

Physical meanings of $E_\infty^{2,1}$ and $E_\infty^{1,2}$ anomalies

- Bulk SPT/topological order/symTFT
- Domain wall decoration picture (Chen-Lu-Vishwanath 2013, Wang-Ning-Cheng 2021)
- Remarks: can be used to classify SPT/anomalies also in fermionic systems (Gu-Wen 2012, Wang-Gu 2017 & 2018) (Yi Zhang's talk)

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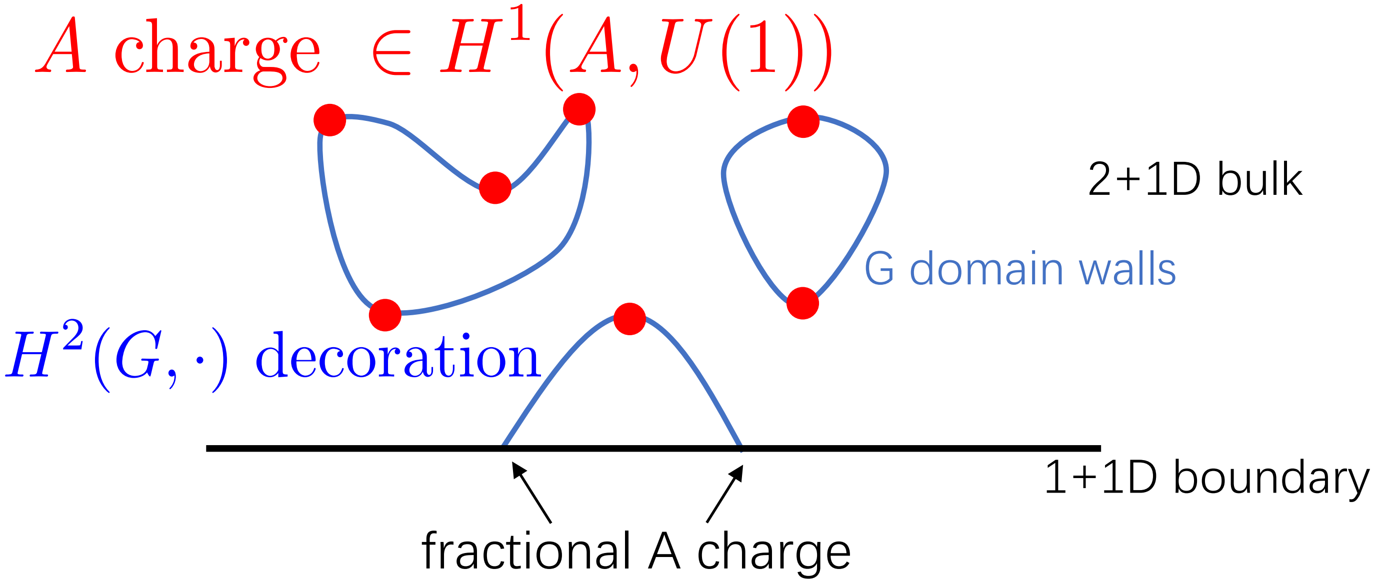

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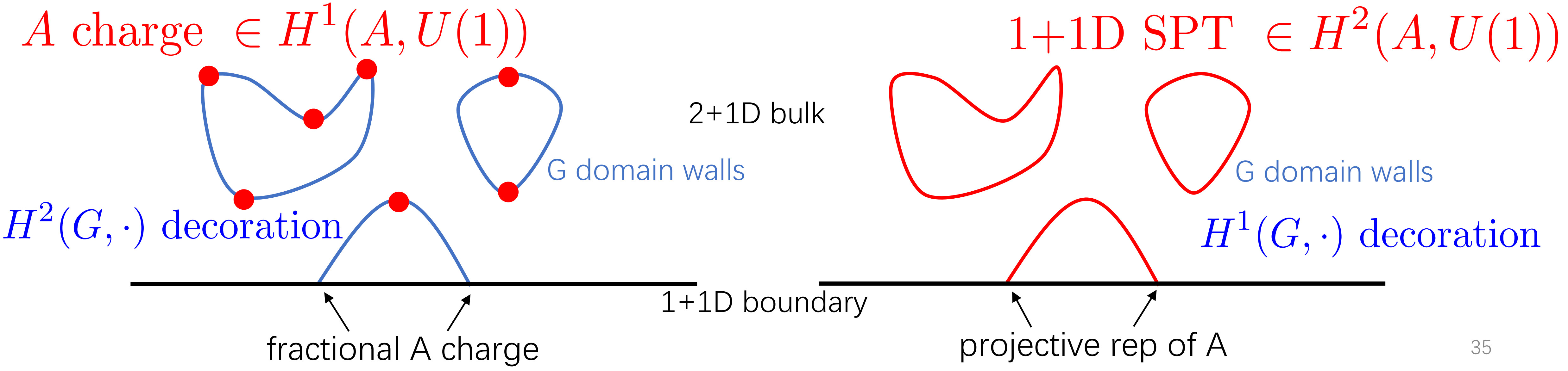


From bulk SPT to boundary anomalous symmetry

- Given arbitrary anomaly 3-cocycle $\nu_3 \in H^3(G, U(1))$
- Bulk 2+1D group cohomology SPT models (Chen-Gu-Liu-Wen 2011)
- Anomalous symmetry action on 1+1D spin chain:

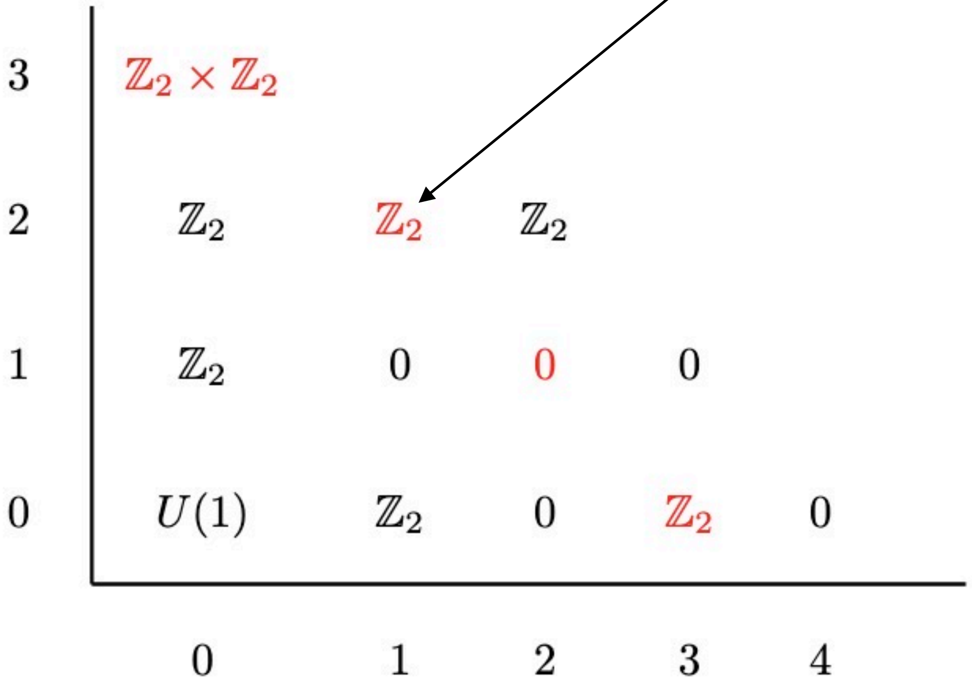
$$U(g)|\{g_i\}\rangle|\{gg_i\}\rangle = \prod_{\langle ij\rangle} \nu_3^{s_{ij}}(g_i^{-1}g_j, g_j^{-1}, g^{-1})|\{gg_i\}\rangle$$

- Multiplication rule: $$U(g)U(h) = \prod_{\langle ij\rangle} \frac{\nu_3^{s_{ij}}(g_i^{-1}, h^{-1}, g^{-1})}{\nu_3^{s_{ij}}(g_j^{-1}, h^{-1}, g^{-1})} U(gh)$$
- Symmetry localization/fractionalization
- Associator = Anomaly: $[U(g)U(h)]\,U(k)\big|_{\text{left}} = \nu_3(g,h,k)U(g)\,[U(h)U(k)]\,\big|_{\text{left}}$
- Similar to extracting F symbol from local unitaries in Yu-An Chen’s talk

Example: D8 group

$$\mathbb{Z}_2^{a^2} \times \mathbb{Z}_2^{x} \to D_8 \to \mathbb{Z}_2^{\bar{a}} \qquad H^3(D_8, \mathrm{U}(1)) = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\nu_3(a^{i_1}x^{j_1}, a^{i_2}x^{j_2}, a^{i_3}x^{j_3}) = (-1)^{i_1\left[\delta_{j_2}\left\lfloor \frac{i_2+i_3}{4}\right\rfloor - \delta_{j_2,1}\delta_{i_3\neq 0}\left(1-\left\lfloor \frac{i_2+[-i_3]_4}{4}\right\rfloor\right)\right]}$$



$(G, \nu_3) = (D_8, 2\zeta)$	
$A = (H, \gamma)$	${}_A\mathcal{C}_A$
$(1, 0)$	$\mathrm{Vec}_{D_8}^{2\zeta}$
$(\langle x\rangle, 1)$ or $(\langle xa^2\rangle = \mathbb{Z}_2, 1)$	$\mathrm{Rep}H_8$
$(\langle xa\rangle, 1)$ or $(\langle xa^3\rangle = \mathbb{Z}_2, 1)$	$\mathrm{Rep}H_8$
$(\langle a^2\rangle = \mathbb{Z}_2, 1)$	$\mathrm{Vec}_{D_8}^{2\zeta}$
$(\langle x, a^2\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, 1)$	$\mathrm{Rep}H_8$
$(\langle x, a^2\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, \gamma)$	$\mathrm{Rep}H_8$
$(\langle xa, a^2\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, 1)$	$\mathrm{Rep}H_8$
$(\langle xa, a^2\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, \gamma)$	$\mathrm{Rep}H_8$

Recall Hao Zhang's talk on A-bimodules

Example: D8 group

$$\mathbb{Z}_2^{a^2} \times \mathbb{Z}_2^{x} \to D_8 \to \mathbb{Z}_2^{\bar{a}}$$

- Anomalous symmetry actions:

$$U(a)|\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle = \prod_i X_{\bar{a}_i} \mathrm{CNOT}_{\bar{a}_i, \tilde{a}_i} \prod_{\langle ij \rangle} (-1)^{(\bar{a}_j - \bar{a}_i)\left[1+\delta_{g_j \in \{000,111\}}\right]} |\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle,$$

$$U(a^2)|\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle = \prod_i X_{\tilde{a}_i} \prod_{\langle ij \rangle} (-1)^{(\bar{a}_j - \bar{a}_i)\delta_{g_j \in \{001,010,011,100\}}} |\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle,$$

$$U(x)|\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle = |\{\tilde{a}_i + \delta_{\bar{a}_i,1}, \bar{a}_i, x_i + 1\}\rangle = \prod_i \mathrm{CNOT}_{\bar{a}_i, \tilde{a}_i} X_{x_i} |\{\tilde{a}_i, \bar{a}_i, x_i\}\rangle.$$

- Projective representation of A on G domain wall

$$U(a^2)U(x) = U(x)U(a^2) \prod_{\langle ij \rangle} (-1)^{(\bar{a}_j - \bar{a}_i)}$$

- Gauging A -> dual non-invertible symmetry $\mathrm{Vec}_{D_8}^{\nu_3} \sim \mathrm{Rep}(H_8)$

Different types of DQCP

- Type-I DQCP with (G, ν_3, A): The dual symmetry after gauging A is invertible

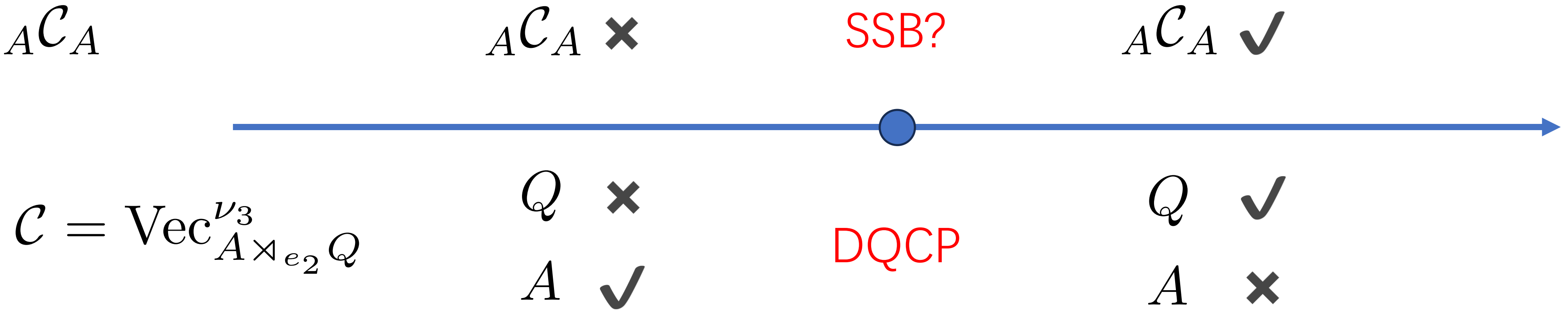
  - iff A is an Abelian normal subgroup of G, and $\nu_3 \in E_\infty^{2,1} + E_\infty^{3,0}$

- Type-II DQCP with (G, ν_3, A): The dual symmetry after gauging A is non-invertible

  - A is non-Abelian
  - A is non-normal
  - A is Abelian normal, but $\nu_3 \in E_\infty^{1,2}$

Spontaneous symmetry breaking of non-invertible symmetry?

- To obtain DQCP of A and Q, we need to find SSB of the dual non-invertible symmetry



Dual non-anomalous non-invertible symmetry

- Condition for the existence of a gapped symmetric phase for ${}_A\mathcal{C}_A$
- The dual symmetry is non-anomalous
- The dual symmetry exists fiber functor
- For the case of $\mathcal{C} = \mathrm{Vec}^{\nu_3}_{A \rtimes_{e_2} Q}$, we can convert the problem of #(fiber functors) to a concrete cocycle problem

- The class $\nu_3|_{A'}$ is trivial.
- The number of double cosets $A\backslash G/A'$ is 1.
  $\iff G = AA'$.
  $\iff |A \cap A'| = \frac{|A||A'|}{|G|}$.
- The class $(\psi_2 \cdot \psi_2'^{-1})|_{A\cap A'} \in H^2(A \cap A', \mathrm{U}(1))$ of the subgroup $A \cap A'$ is nondegenerate.
  $\iff$ For every element $g \neq e$ in $A \cap A'$, there exists $h \in A \cap A'$, such that $gh = hg$ and $\psi_2(g,h)/\psi_2'(g,h) \neq \psi_2(h,g)/\psi_2'(h,g)$.
  $\iff A \cap A' = \oplus_{i=1}^{K}(\mathbb{Z}_{N_i} \times \mathbb{Z}_{N_i})$ and $(\psi_2 \cdot \psi_2'^{-1})|_{A\cap A'}$ is cohomologous to the 2-cocycle $\omega(g,h) = \prod_{i=1}^{K} e^{\frac{2\pi i k_i}{N_i} g_i^a h_i^b}$ with $k_i \in \mathbb{Z}_{N_i}^{\times}$.

Summary

- DQCP and SSB of the dual generalized symmetry can be related by gauging
- Type-I / II DQCP depending on the dual symmetry is invertible / non-invertible
- Two types of anomalies in $E_\infty^{2,1}$ and $E_\infty^{1,2}$
- Conditions of dual symmetry being non-anomalous (admits fiber functors)

Thanks you!