What Does Symmetry Say?

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@ Generalized Symmetries in HEP and CMP

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FIVE GUYS®

~~BURGERS and FRIES~~

Overview

1. Symmetry:
The Last Thing First

2. Symmetry Topological Field Theory:
A Tool Looks Unnecessary Until Justified

3. Kinetic Term:
Fun with Topological Operators and Correlators

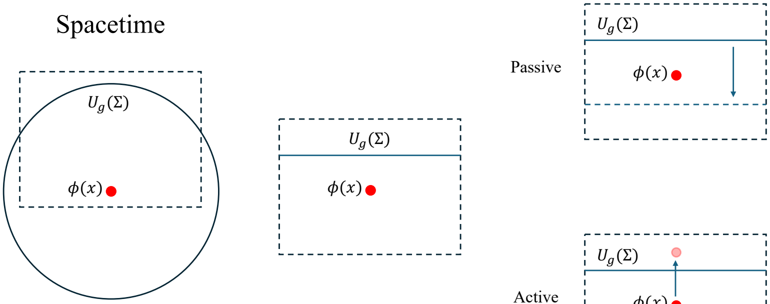
4. Twist Term:
The Need for One Higher Dimension

5. Outlook:
A Road to Category

Symmetry:
The Last Thing First

What is symmetry?

- Symmetry action as topological operator [Gaiotto-Kapustin-Seiberg-Willett]:



What is symmetry?

- Path integral + Noether's theorem:

$$\langle U_g(\Sigma)\ \phi(x)\rangle = \int \mathcal{D}\phi\ e^{iS[\phi]}\ U_g(\Sigma)\ \phi(x) = \int \mathcal{D}\phi\ e^{iS[\phi]+i\int_\Sigma \star J}\phi(x)$$

- A particular simple scenario:

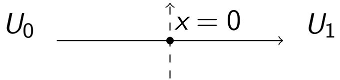
$$\int_\Sigma \star J = \int A \star J \ \Rightarrow\ \overset{A\sim PD(\Sigma)}{\overset{\uparrow}{\overline{\hspace{4em}}}}{}_{\Sigma} \ \Rightarrow\ \text{Active} = \text{Transport by } e^{i\int_\ell A}$$

- Consistency of active view:

$$e^{i\int_\ell A} = e^{i\int_{\ell'} A} \ \Rightarrow\ \text{Flat } A$$

What is symmetry?

- A concrete example



Leads to:

$$A_{\mathcal{L}} = i\alpha\delta(x)dx = e^{-i\alpha H(x)}de^{i\alpha H(x)}$$

or:

$$A_{\mathcal{L}}|_{U_0} = iede, \;\; A_{\mathcal{L}}|_{U_1} = ie^{-i\alpha}de^{i\alpha}\,.$$

A convention

The defect rotates left-handedly to the direction of gauge transformation from U_0 to U_1.

What is symmetry?

- Now we get a theory stands by itself with partition function:

$$Z[A] = \int \mathcal{D}\phi \; e^{iS[\phi] + \int A \star J} \in \mathrm{Fun}(\mathcal{A}_0)$$

Definition

$\mathrm{Fun}(\mathcal{A}_0)$ is the space of function on the space of inequivalent flat A's.

Question

How to obtain/describe $\mathrm{Fun}(\mathcal{A}_0)$?

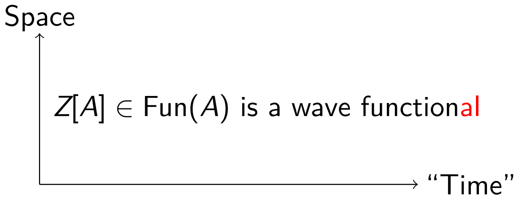
What is symmetry?

- Luckily, with the help of [Baez], we have:

Fact

The canonical quantization of *BF*-theory leads to the state space $\mathrm{Fun}(\mathcal{A}_0)$.

- And with the help of [Any reasonable QM textbook], canonical quantization is (figure simplified from [Kaidi-Ohmori-Zheng]):



What is symmetry?

- The last thing first:

SymTFT [Jia-Luo-JT-Wang-Zhang]

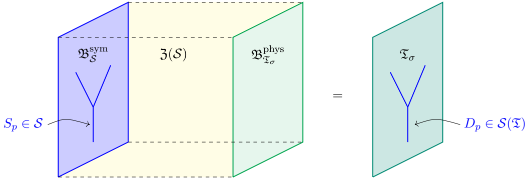
SymTFT of a QFT with G-flavor symmetry on M_d is a BF-theory on $M_d \times I$.

Thanks

Symmetry Topological Field Theory:
A Tool Looks Unnecessary Until Justified

SymTFT

- But what is SymTFT?
- See answers in [Apruzzi, Bhardwaj, Cvetič, Freed, Gaiotto, Heckman, Kaidi, Kapustin, Moore, Ohmori, Schäfer-Nameki, Seiberg, Teleman, Wang, Witten, Zheng, ...]
- Minimally:
  - SymTFT as "sandwich":



  - SymTFT action:

$$S = \int_{M^{d+1}} \sum_{r=0}^{p-1} a_{r+1} \cup \delta b_{d-r-1} + \mathcal{A}^* \omega(a_1, \cdots, a_p)$$

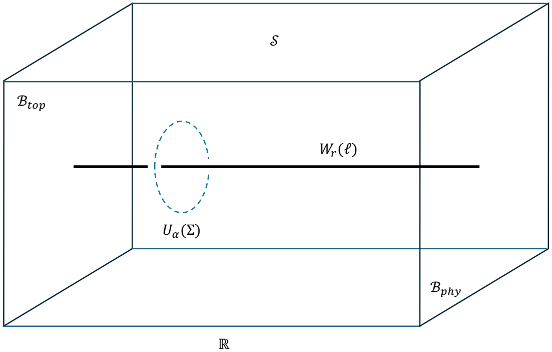
Glossary (aka words)

- Topological field theory in one higher dimension. (But why?)
- Non-commutativity from BdA. (How to see the physical symmetry?)
- Dynamical and topological boundaries. (What are them?)
- Anomaly from AdA or boundary conditions. (How to get it from spacetime manipulations?)
- SymTFT is an auxiliary tool. (Make it less auxiliary? [JT-Wang]&[JT-Wang])
- And many more categorical unpleasants.

Kinetic Term:
Fun with Topological Operators and Correlators

Kinetic term

- What one might attempt:



This is deadly wrong.

Kinetic term

- But why?
- Typically one may expect $U_\alpha(\Sigma) \sim e^{i\alpha}$, hence:

$$U_\alpha(\Sigma) U_\beta(\Sigma) \neq U_\beta(\Sigma) U_\alpha(\Sigma) \, .$$

But this CANNOT be true for any codim > 1 operators.

Because:

Theorem

Higher homotopy commutes.

- Q: So what can we do?

Definition

$U_\alpha(\Sigma) = \exp\left(i\int_\Sigma(\alpha, B)\right)$ where $B \in \Omega^{d-1}(M_{d+1}, \mathrm{ad}(P))$ where $\mathrm{ad}(P)$ is the adjoint bundle over M_{d+1} associated to the principal G-bundle and Σ is a $(d-1)$-dimensional submanifold of M_{d+1}.

Therefore we can compute the correlator:

$$\begin{aligned}\langle U_\alpha(\Sigma)\ W_{\mathbf{R}}(\ell)\rangle &= \int \mathcal{D}B\,\mathcal{D}A\ \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge F)\right)\ \exp\left(i\int_\Sigma(\alpha, B)\right)\ \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right)\\ &= \int \mathcal{D}B\,\mathcal{D}A\ \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge(F+\alpha\delta_\Sigma))\right)\ \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right)\end{aligned}$$

Kinetic term

- The result, at least classically, is:

$$\langle U_\alpha(\Sigma)\ W_{\mathbf{R}}(\ell)\rangle = \int \mathcal{D}B\,\mathcal{D}A\ \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge F)\right)\ \exp\left(i\int_\Sigma (\alpha, B)\right)\ \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right)$$

$$= \int \mathcal{D}B\,\mathcal{D}A\ \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge(F+\alpha\delta_\Sigma))\right)\ \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right)$$

- Fixing $z_p = 0$, we have:

Classically

$$U_\alpha(\Sigma) W_{\mathbf{R}}(\ell) = \mathcal{P}\left[\exp\left(i\int_0^1 A_{\mathbf{R}}\right)\exp\left(-i\alpha_{\mathbf{R}}(0)\right)\right] = W_{\mathbf{R}}(\ell)e^{-i\alpha_{\mathbf{R}}(0)}$$

- This IS the G-symmetry action on the endpoint of ℓ.

Kinetic term

Warning

But I lied...

- Because then $U_\alpha(\Sigma)$ is NOT topological.
- Q: What operator is REALLY topological?

The guys is [Jia-Luo-JT-Wang-Zhang]:

Definition

$$\widetilde{U}_{[g]}(\Sigma) = \int dh \ \exp\left(i \int_\Sigma (\mathrm{ad}_{h_{\gamma_{xp}}} \mathrm{ad}_h \alpha_0(p), B) \right)$$

Kinetic term

- We would like to calculate:

Correlator

$$\int \mathcal{D}A\mathcal{D}B\; e^{iS_{BF}}\; \widetilde{U}_\alpha(\Sigma) W_\lambda(\ell)$$

- The key observation is [Witten, Beasley]:

Definition

$$W_\lambda(\ell) := \lim_{\eta\to\infty} \int \mathcal{D}U\; e^{i\int_{M_{d+1}} \delta_\ell \wedge (L_\sigma(\eta,U) + C_\lambda(U))}$$

- And the result is [Cordova-Ohmori-Rudelius, Jia-Luo-JT-Wang-Zhang]:

Quantum

$$\langle \widetilde{U}_\alpha(\Sigma) W_\lambda(\ell)\rangle \propto \frac{\chi_{r_\lambda}(e^{i\alpha})}{\dim r_\lambda} := \text{Normalized character}\,.$$

Twist Term:
The Need for One Higher Dimension

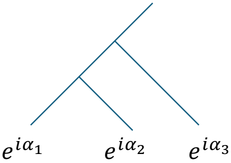
Twist term

- What else can we do than repeating the slogans "SymTFT lives in one-higher dimension" and "twist term encodes 't Hooft anomaly"?

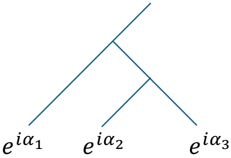
Take-home message

The two aspects are related.

- Starting point: F ()-move (and we will focus on 2D from now on) [Moore-Seiberg, Bhardwaj-Tachikawa, Chang-Lin-Shao-Wang-Yin]



$$= \; e^{-i\alpha_1([\alpha_2+\alpha_3]-\alpha_2-\alpha_3)} \times$$



Twist term

- A claim never been concretized:

Claim [Bhardwaj et al.]

Rearrangements of topological defects $\Leftrightarrow$ Gauge transformations

- Let us do what should have been done:

Suppose:

$$A_{\curlywedge} = \Lambda^{-1} A_{\curlywedge} \Lambda + \Lambda^{-1} d\Lambda$$

We have:

$$\langle \curlywedge \rangle = \int \mathcal{D}\psi \; e^{iS[\psi]} \curlywedge = \int \mathcal{D}\psi \; e^{iS[\psi] + i\,\mathrm{tr} \int_{M_2} A_{\curlywedge} \star J[\psi]} ,$$

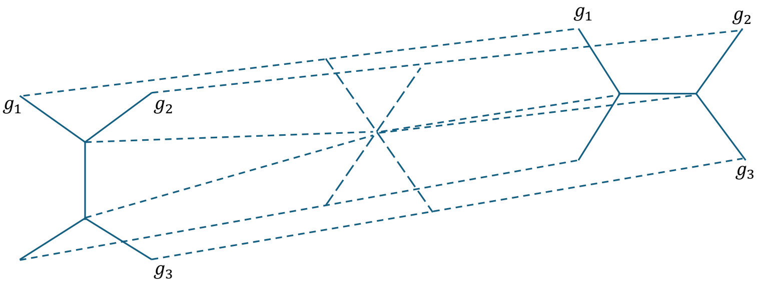
$$\langle \curlywedge \rangle = \int \mathcal{D}\psi \; e^{iS[\psi]} \curlywedge = \int \mathcal{D}\psi \; e^{iS[\psi] + \int_{M_d} A_{\curlywedge} \wedge \star J} = e^{i\mathcal{A}[A_{\curlywedge};\Lambda]} \langle \mathcal{W} \rangle$$

Twist term

- Aim is to compute [Yonekura-Witten]:

$$\mathcal{A}[A_{\pitchfork}; \Lambda] = \mathrm{tr} \int_{M_2 \times \gamma} cs(A)$$

- A path in $\mathcal{G}$:



- A topological magic:

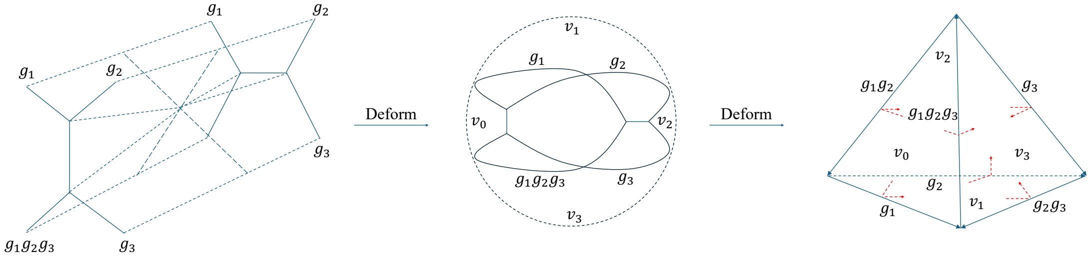


Figure: The deformations of the combined  and  configuration. The group multiplications in this figure are all taken from the right.

Twist term

- Key point: One can extrapolate from  to  via field.
- Therefore $A = -ig^{-1}dg$ on each local patch U.
- Therefore:

$$\int_U cs[A] = \int_U \langle A, [A, A]\rangle = \int_U g^*(\omega^3) = \int_{g(U)} \omega^3$$

with left-invariant 3-form ω^3.

Gluing up:

The anomaly is

$\mathcal{A}[A$$; \Lambda] = \int_{g(B^3)} \omega^3$

Twist term

- What's more? [Houard, Huerta]
- Dualize: $g(v_0) = e,\ g(v_1) = g_1,\ g(v_2) = g_1 g_2,\ g(v_3) = g_1 g_2 g_3$
- We arrive at:

$$\mathcal{A}(A_{\mathbb{A}}; \Lambda) = \int_{g(\Delta)} \omega^3 = \int_{\langle e g_1 (g_1 g_2)(g_1 g_2 g_3) \rangle} \omega^3$$

Fact

$$\int \omega^3 : H^3(\mathfrak{g}, \mathbb{R}) \cong H^3(G, \mathbb{R})$$

- $H^3(\mathfrak{g}, \mathbb{R})$: Lie algebra cohomology
- $H^3(G, \mathbb{R})$: Continuous Lie group cohomology (and many other names)

Twist term

- Main claim I:

What is (or how to compute) anomaly?

't Hooft anomaly is characterized by $H^{2k+1}(\mathfrak{g}, \mathbb{R})$.

What I do not like about it:

Good but all too abstract and too remote from what's known.

Twist term

- Some mathematics [Stasheff, Morita, Rudolph-Schmidt]:

Fact I

$w : H^*(\mathfrak{g}) \to H^*(B\overline{G})$, $B\overline{G}$ classifies flat G-bundle with trivialization.

Fact II

$BG^\delta \xrightarrow{B\overline{G}} BG$. In human language: $B\overline{G}$ extrapolates between BG^δ and BG.

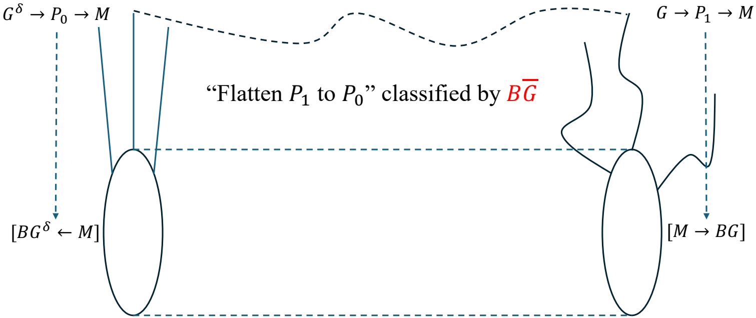
- Five guys' job:

Task

Can we geometrize such extrapolation?

Twist term

- Visualizing the abstract non-sense:



Twist term

- Some high school math:
  - Trivial bundle: $g_{ij} \equiv 1$ on $U_{ij} := U_i \cap U_j$ from U_i to U_j
  - Flat field: $A = \frac{\alpha}{2\pi} d\theta$
  - Gauge transformation: $u_k = e^{-\frac{i\alpha}{2\pi}(\theta_k + \frac{2k\pi}{3})}$ on each U_k

$\Rightarrow A_k \equiv 0$ (everywhere zero)

However, $g'_{ij} = u_i^{-1} g_{ij} u_j = u_i^{-1} u_j$:

$$g'_{01}(U_{01}) = g'_{12}(U_{12}) = 1, \; g'_{20}(U_{20}) = e^{i\alpha}$$

On $S^1 \times \mathbb{R} \times I$, define:

- $\overline{u}_k = e^{-\frac{i\alpha}{2\pi}(\theta_k + \frac{2k\pi}{3}) f(t)}$ with $f(0) = 1$ and $f(1) = 0$ as map $\overline{U}_k := U_k \times \mathbb{R} \times I \to G$
- Transition function: $\overline{g}_{ij} = \overline{u}_i^{-1} \overline{u}_j$ on $\overline{U}_{ij}$
- Flat field: $\overline{A}_k = \overline{u}_k^{-1} A_k \overline{u}_k - i\overline{u}_k^{-1} d\overline{u}_k$ on $\overline{U}_k = -i\overline{g}_k^{-1} d\overline{g}_k$ with $\overline{g}_k = u_k^{-1} \overline{u}_k$

- Observation: $\overline{g}_{20} = e^{i\alpha f(t)}$ is an element $(e^{i\alpha}, \ell)$ of $\overline{G}$

Twist term

- Putting everything together.
  - $\bar{f}: M_2 \times I \to B\bar{G}$
  - $w: H^3(\mathfrak{g}, \mathbb{R}) \to H^3(B\bar{G}, \mathbb{R})$

$$\Rightarrow \bar{f}^* \circ w: H^3(\mathfrak{g}, \mathbb{R}) \to H^3(M_2 \times I, \mathbb{R})$$

- and recall:

$$\int_U cs[A] = \int_U g^*(\omega^3) = \int_{g(U)} \omega^3$$

Naturally leads to

Abstract made concrete

$$\bar{f}^* \circ w \equiv g^*$$

Twist term

- Some comments:
  - Existence of $w : H^*(\mathfrak{g}) \to H^*(B\overline{G})$ is a fact
  - $\bar{f}$ is rather abstract, but it leads to emergence of extra dimension, in a very concrete way.
  - $\bar{f}^* \circ w$ is abstract, but is secretely g^*, leading to flat gauge fields
- Main claim II:

Extra dimension

The emergence of extra dimension is dictated by the math structure of 't Hooft anomaly.

Outlook:
A Road to Category

A physicists' eye chart

- # Biggest Q: Why category?

- ## Bigger Q: Why "SymTFT = Drinfeld center"? ← Hmm.

- ### Big Q: What are the things in the category? ← Almost.

- Q: Is FIVE GUYS ~~BURGERS and FRIES~~ working on that? ← Yes!

Thanks