Topological Responses of the Standard Model Gauge Group

Zheyan Wan

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

arXiv: 2412.21196 with Juven Wang and Yi-Zhuang You

@Generalized Symmetries in HEP and CMP on July 29, 2025

Well-known: The Lie algebra of the Standard Model gauge group is

$$\mathcal{G}_{\mathrm{SM}} \equiv su(3) \times su(2) \times u(1)_{\tilde{Y}}, \tag{1}$$

Unknown: There are 4 possible versions of the Standard Model gauge group:

$$G_{\mathrm{SM_q}} \equiv \frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_{\mathrm{q}}}, \quad \text{with } \mathrm{q} = 1, 2, 3, 6. \tag{2}$$

Goal: Discern the Standard Model gauge group via measurable fractional topological responses akin to the Hall conductance. Motivated by [Hsin-Gomis, arXiv:2411.18160].

Major difference between our work and [Hsin-Gomis, arXiv:2411.18160]

• [Hsin-Gomis] only considers $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$ (baryon minus lepton symmetry), while our work considers $\mathrm{U}(1)_{X_n}$ where

$$X_n \equiv n(\mathbf{B}-\mathbf{L}) + (1-\frac{n}{3})\tilde{Y}.$$

Thanks to Yunqin Zheng's insight.
• [Hsin-Gomis] considers $\mathrm{Spin} \times \mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$ (spin) as the 0-form symmetry of the gauged SM, however, it should be $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$, and their topological response result differs from our result. We consider instead

$$\begin{cases} \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} = \mathrm{Spin}^c \text{ (non-spin)}, & n \in \mathbb{Z} \text{ odd} \\ \mathrm{Spin} \times \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} = \mathrm{Spin} \times \mathrm{U}(1), & n \in \mathbb{Z} \text{ even} \end{cases}$$

as the 0-form symmetry of the gauged SM.

SM data

	$\bar{d}_R = d_L$	l_L	q_L	$\bar{u}_R = u_L$	$\bar{e}_R = e_L^+$	$\bar{\nu}_R = \nu_L$	ϕ_H
$\mathrm{SU}(3)$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$
$\mathrm{U}(1)_Y$	$1/3$	$-1/2$	$1/6$	$-2/3$	1	0	$1/2$
$\mathrm{U}(1)_{\tilde{Y}}$	2	-3	1	-4	6	0	3
$\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$	$-1/3$	-1	$1/3$	$-1/3$	1	1	0
$\mathrm{U}(1)_{\mathbf{Q}-N_c\mathbf{L}}$	-1	-3	1	-1	3	3	0
$\mathrm{U}(1)_X$	-3	-3	1	1	1	5	-2
$\mathrm{U}(1)_{X_n}$	$2-n$	-3	1	$-4+n$	$6-n$	n	$3-n$

$$
\begin{aligned}
X_n &\equiv \frac{n}{N_c}(\mathbf{Q} - N_c\mathbf{L}) + (1 - \frac{n}{N_c})\tilde{Y} \\
&\equiv n(\mathbf{B} - \mathbf{L}) + (1 - \frac{n}{3})\tilde{Y},
\end{aligned} \tag{3}
$$

New global symmetry X_n

$X_{n=3} = N_c(\mathbf{B} - \mathbf{L}) = \mathbf{Q} - N_c\mathbf{L}$ symmetry,
$X = X_{n=5} = 5(\mathbf{B} - \mathbf{L}) - \frac{2}{3}\tilde{Y} \equiv \frac{5}{3}(\mathbf{Q} - N_c\mathbf{L}) - \frac{2}{3}\tilde{Y}$ is Wilczek-Zee's symmetry,
$X_{n=1} = \mathbf{B} - \mathbf{L} + \frac{2}{3}\tilde{Y}$ is a symmetry for the flipped $u(5)$ model.

$$q_{X_n} = q_{\tilde{Y}} \mod n \tag{4}$$

$\Rightarrow$ There is a shared common subgroup $\mathbb{Z}_n$ between $\mathrm{U}(1)_{X_n}$ and $\mathrm{U}(1)_{\tilde{Y}}$.
The shared common subgroup $\mathbb{Z}_n$ is generated by

$$\psi \mapsto \exp(\mathrm{i}\frac{2\pi}{n}q)\psi.$$

Ungauged:

$$G_{\mathrm{SM_q}}^{\mathrm{U}(1)_{X_n}} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \left(\frac{\mathrm{U}(1)_{X_n} \times G_{\mathrm{SM_q}}}{\mathbb{Z}_n}\right) \text{ for odd } n,$$
$$G_{\mathrm{SM_q}}^{\mathrm{U}(1)_{X_n}} \equiv \mathrm{Spin} \times \left(\frac{\mathrm{U}(1)_{X_n} \times G_{\mathrm{SM_q}}}{\mathbb{Z}_n}\right) \text{ for even } n. \tag{5}$$

For odd n, there is a shared common subgroup $\mathbb{Z}_2^{\mathrm{F}}$ between Spin and $\mathrm{U}(1)_{X_n}$.
For even n, there is no shared common subgroup $\mathbb{Z}_2^{\mathrm{F}}$ between Spin and $\mathrm{U}(1)_{X_n}$.
Therefore, this spacetime-internal symmetry (5) is faithful.

Higher form symmetries and higher classifying spaces

In general, for an n-dimensional theory, there is an n-group $\mathbb{G}$ of invertible defects in this theory whose objects are a single object, and k-morphisms are codimension k invertible defects.
Here,
$\{$codimension k invertible defects$\} = (k-1)$-form symmetry for $k = 1, 2, \ldots, n$.

A p-form symmetry G on a manifold M is characterized by a map $M \to \mathrm{B}^{p+1}G$. Throughout this talk, all fields of a p-form symmetry G (defined as a differential form or a cohomology class) are pulled back to M via the classifying map $M \to \mathrm{B}^{p+1}G$.

For a topological group G, recursively define higher classifying spaces:

$$\mathrm{B}^n G = \mathrm{B}(\mathrm{B}^{n-1}G), \quad \mathrm{B}^0 G = G.$$

$\mathrm{B}^n G$ corresponds to the n-fold iterated classifying space of G.

Spacetime-internal symmetries of SM

For a pure G gauge theory,
electric 1-form symmetry $= Z(G)$ (the center of G),
magnetic 1-form symmetry $= \text{Hom}(\pi_1(G), \mathrm{U}(1))$ (characters of $\pi_1(G)$).

	$Z(G_{\mathrm{SM_q}})$	$\pi_1(G_{\mathrm{SM_q}})^\vee$	1-form e sym $G^e_{[1]}$	1-form m sym $G^m_{[1]}$
$G_{\mathrm{SM_q}} \equiv \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_{\mathrm{q}}}$	$\mathbb{Z}_{6/\mathrm{q}} \times \mathrm{U}(1)$	$\mathrm{U}(1)$	$\mathbb{Z}^e_{6/\mathrm{q},[1]}$	$\mathrm{U}(1)^m_{[1]}$

, $\mathrm{q} = 1, 2, 3, 6.$ (

Gauged:

$$\begin{aligned} G'^{\mathrm{U}(1)_{X_n}}_{\mathrm{SM_q}} &\equiv \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} \times \mathbb{Z}^e_{6/\mathrm{q},[1]} \times \mathrm{U}(1)^m_{[1]} \text{ for odd } n, \\ G'^{\mathrm{U}(1)_{X_n}}_{\mathrm{SM_q}} &\equiv \mathrm{Spin} \times \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} \times \mathbb{Z}^e_{6/\mathrm{q},[1]} \times \mathrm{U}(1)^m_{[1]} \text{ for even } n. \end{aligned} \tag{7}$$

The 0-symmetry and 1-symmetry of the SM are given by

$$\begin{aligned}(G_{[0]}, G_{[1]}) &= (G_{[0]}, G^e_{[1]} \times G^m_{[1]}) \\ &= (\begin{cases} \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} = \mathrm{Spin}^c, & n \in \mathbb{Z} \text{ odd} \\ \mathrm{Spin} \times \frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_n} = \mathrm{Spin} \times \mathrm{U}(1), & n \in \mathbb{Z} \text{ even} \end{cases}, \\ &\quad \mathbb{Z}^e_{6/\mathrm{q},[1]} \times \mathrm{U}(1)^m_{[1]}). \end{aligned} \tag{8}$$

Topological response of the SM

We shall focus on the topological response in the form

$$\int_{M^4} \frac{\sigma_n}{2\pi} B_m \mathrm{d}A_{X_n} \tag{9}$$

and the fractional $\sigma_n(\mathrm{q}, k)$ with (q, k)-dependence of the SM gauge group. Here, M^4 is a closed 4-manifold, $A_{X_n} \in \Omega^1(\mathrm{B}(\frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_{\mathrm{lcm}(2,n)}}), \mathbb{R})$ is the background gauge field of the $\frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_{\mathrm{lcm}(2,n)}}$ symmetry, $B_m \in \Omega^2(\mathrm{B}^2\mathrm{U}(1), \mathbb{R})$ is the background gauge field of the magnetic 1-form $\mathrm{U}(1)^m_{[1]}$ symmetry of the SM, and $k = k^e \in \mathbb{Z}_{6/\mathrm{q}}$ is the **symmetry fractionalization class** of the 0-form symmetry $G_{[0]}$ and the electric 1-form symmetry $G^e_{[1]}$.

Topological response of the SM-1st contribution

The 1st contribution $\frac{\mathrm{q}(1-n)\gcd(2,n)}{2n}$ of the SM's σ_n response is obtained from $\int_{M^4} B_m F$ where F is the fractional part of

$$\star J_m^{(2)} = \mathrm{q}\frac{\mathrm{d}a_{\tilde{Y}}}{2\pi} \tag{10}$$

[Gaiotto-Kapustin-Seiberg-Willett, arXiv: 1412.5148] due to the coupling $\int_{M^4} B_m \star J_m^{(2)}$ where $J_m^{(2)}$ is the 2-form current of the magnetic 1-form symmetry $\mathrm{U}(1)_{[1]}^m$ without turning on the background gauge field B_e of the electric 1-form symmetry $\mathbb{Z}_{6/\mathrm{q},[1]}^e$.
Here, $a_{\tilde{Y}}$ is the dynamical gauge field of the $\mathrm{U}(1)_{\tilde{Y}}$ symmetry.

Topological response of the SM-1st contribution

Here, we have used the gauge bundle constraint of the ungauged SM (5):

$$\frac{\mathrm{d}a_{\tilde{Y}}}{2\pi} = \frac{1}{\mathrm{lcm}(2,n)}\frac{\mathrm{d}A_{X_n}}{2\pi} - \frac{\mathrm{gcd}(2,n)}{2}w_2(TM) + \frac{1}{\mathrm{q}}w_2^{(\mathrm{q})} \mod 1 \tag{11}$$

and the gauge bundle constraint of the gauged SM for odd n (7):

$$w_2(TM) = \frac{\mathrm{d}A_{X_n}}{2\pi} \mod 2. \tag{12}$$

Here, $w_2(TM)$ is the second Stiefel-Whitney class of the tangent bundle which is the obstruction class to lifting a $\frac{\mathrm{Spin}}{\mathbb{Z}_2^{\mathrm{F}}}$ bundle to a Spin bundle, and $w_2^{(\mathrm{q})}$ is the obstruction class to lifting a $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathbb{Z}_{\mathrm{q}}}$ bundle to a $\mathrm{SU}(3)\times\mathrm{SU}(2)$ bundle.

Topological response of the SM-2nd contribution

By turning on the background gauge field B_e for the electric 1-form $\mathbb{Z}^e_{6/\mathrm{q},[1]}$ symmetry, there is a mixed anomaly [Gaiotto-Kapustin-Seiberg-Willett, arXiv: 1412.5148]

$$\int_{M^5} -\frac{1}{2\pi}\tilde{B}_e \, \mathrm{d}B_m \tag{13}$$

of the electric $\mathbb{Z}^e_{6/\mathrm{q},[1]}$ 1-form symmetry and the magnetic $\mathrm{U}(1)^m_{[1]}$ 1-form symmetry of the Standard Model for $\mathrm{q}=1,2,3,6$ where $\tilde{B}_e = \frac{2\pi}{6/\mathrm{q}} B_e$ and M^4 is the boundary of the 5-manifold M^5.

Topological response of the SM-2nd contribution

We will derive the constraint

$$f_k^* \tilde{B}_e = \frac{k}{6/\mathrm{q}} \mathrm{d}A_{X_n} \tag{14}$$

for the symmetry fractionalization of the 0-form $G_{[0]}$ symmetry and the 1-form electric $G^e_{[1]} = \mathbb{Z}^e_{6/\mathrm{q},[1]}$ symmetry of the gauged Standard Model.
Here, $k \in \mathbb{Z}_{6/\mathrm{q}} = \mathrm{H}^2(\mathrm{B}G_{[0]}, G^e_{[1]})$ is represented by the map $f_k : \mathrm{B}G_{[0]} \to \mathrm{B}^2 G^e_{[1]}$.
To cancel the anomaly $\int_{M^5} -\frac{1}{2\pi} \tilde{B}_e \mathrm{d}B_m$, we need to add $\int_{M^4} \frac{1}{2\pi} \frac{k}{6/\mathrm{q}} B_m \mathrm{d}A_{X_n}$ in the Standard Model Lagrangian. The total topological response is $\int_{M^4} \frac{\sigma_n}{2\pi} B_m \mathrm{d}A_{X_n}$ where

$$\sigma_n = \frac{\mathrm{q}(1-n)\mathrm{gcd}(2,n)}{2n} + \frac{k\mathrm{q}}{6} \mod 1. \tag{15}$$

Symmetry fractionalization

For the 4d SM, there is a 2-group $\mathbb{G}$ of invertible defects whose objects are a single object, 1-morphisms are the 0-form symmetry, and 2-morphisms are the 1-form symmetry.

The 0-form symmetry $G_{[0]}$ of the 4d SM can be viewed as a 1-group whose objects are a single object, and 1-morphisms are the 0-form symmetry.

Symmetry fractionalization means that there is a homomorphism from $G_{[0]}$ to $\mathbb{G}$, or there is a map from $\mathrm{B}G_{[0]}$ to $\mathrm{B}\mathbb{G}$.

Thanks to Liang Kong and Zhi-Hao Zhang for their insightful discussions.

Symmetry fractionalization

We consider the following symmetry extension of the 0-form symmetry $G_{[0]}$ by the 1-form symmetry $G_{[1]}$ in terms of $\mathrm{B}\mathbb{G}$ fibration over $\mathrm{B}G_{[0]}$ with fiber $\mathrm{B}^2 G_{[1]}$:

$$\mathrm{B}^2 G_{[1]} \xrightarrow{i} \mathrm{B}\mathbb{G} \xrightarrow{p} \mathrm{B}G_{[0]} \tag{16}$$

where $\mathbb{G}$ is the total symmetry.

In general, there is a homomorphism $\rho : \pi_1(\mathrm{B}G_{[0]}) \to \mathrm{Aut}(G_{[1]})$ as a twist. In our SM case, since $\pi_1(\mathrm{B}G_{[0]}) = \pi_0(G_{[0]}) = 0$, our specific $\rho : \pi_1(\mathrm{B}G_{[0]}) \to \mathrm{Aut}(G_{[1]})$ is trivial here. We may denote $\rho = 0$.

Symmetry fractionalization

For topological spaces X and Y, $[X, Y]$ denotes the set of homotopy classes of maps from X to Y.
Since (16) is a fibration with cofiber $\mathrm{B}^3 G_{[1]}$, it induces a long exact sequence of groups

$$\cdots \to [\mathrm{B} G_{[0]}, \mathrm{B}^2 G_{[1]}] \xrightarrow{i_*} [\mathrm{B} G_{[0]}, \mathrm{B}\mathbb{G}] \xrightarrow{p_*} [\mathrm{B} G_{[0]}, \mathrm{B} G_{[0]}] \xrightarrow{[\beta]_*} [\mathrm{B} G_{[0]}, \mathrm{B}^3 G_{[1]}] \to$$

where $[\beta]$ is the Postnikov class classifying the symmetry extension (16).
So the lifting (dashed arrow) in the following diagram

$$\begin{array}{ccccc} \mathrm{B}^2 G_{[1]} & \xrightarrow{i} & \mathrm{B}\mathbb{G} & & \\ & \nearrow & \downarrow p & & \\ \mathrm{B} G_{[0]} & \xrightarrow{\mathrm{id}} & \mathrm{B} G_{[0]} & \xrightarrow{[\beta]} & \mathrm{B}^3 G_{[1]} \end{array} \tag{18}$$

exists if and only if $[\beta] = 0$.

Symmetry fractionalization

Two different liftings in the diagram (18) have the same image under p_*, so their difference belongs to the kernel of p_*, hence the image of i_*. Fix a lifting in the diagram (18), other liftings in the diagram (18) differ from the fixed lifting by $[\mathrm{B}G_{[0]}, \mathrm{B}^2 G_{[1]}]$, so the set of liftings in the diagram (18) is a $[\mathrm{B}G_{[0]}, \mathrm{B}^2 G_{[1]}]$-torsor, namely, $[\mathrm{B}G_{[0]}, \mathrm{B}^2 G_{[1]}]$ acts on the set of liftings in the diagram (18) simply transitively.

A **symmetry fractionalization** of the 0-form symmetry $G_{[0]}$ and the 1-form symmetry $G_{[1]}$ is a lifting in the diagram (18), and the symmetry fractionalization is classified by a homotopy class

$$k \in [\mathrm{B}G_{[0]}, \mathrm{B}^2 G_{[1]}]. \tag{19}$$

The obstruction class to the symmetry fractionalization is

$$[\beta] \in [\mathrm{B}G_{[0]}, \mathrm{B}^3 G_{[1]}]. \tag{20}$$

Symmetry fractionalization

The 0-form symmetry and 1-form symmetry are classified by

$$f_{[0]} : M \to \mathrm{B} G_{[0]} \tag{21}$$

and

$$f_{[1]} : M \to \mathrm{B}^2 G_{[1]} \tag{22}$$

respectively. The symmetry fractionalization is a map

$$g : \mathrm{B} G_{[0]} \to \mathrm{B}^2 G_{[1]} \tag{23}$$

such that

$$g \circ f_{[0]} = f_{[1]}. \tag{24}$$

Symmetry fractionalization

The electric Postnikov class

$$[\beta^e] \in [\mathrm{B}G_{[0]}, \mathrm{B}^3 G^e_{[1]}]_\rho = \mathrm{H}^3_\rho(\mathrm{B}G_{[0]}, G^e_{[1]}) = 0 \tag{25}$$

is trivial, namely $[\beta^e] = 0$. Because $[\beta^e] = 0$, we can define the symmetry fractionalization class k^e in the electric sector.

The magnetic Postnikov class

$$[\beta^m] \in [\mathrm{B}G_{[0]}, \mathrm{B}^3 G^m_{[1]}]_\rho = \mathrm{H}^4_\rho(\mathrm{B}G_{[0]}, \mathbb{Z}) = \mathbb{Z}^2 \tag{26}$$

may be a nontrivial obstruction class. If $[\beta^m] \neq 0$ is nontrivial, we will not be able to define the symmetry fractionalization class k^m in the magnetic sector later.

Symmetry fractionalization

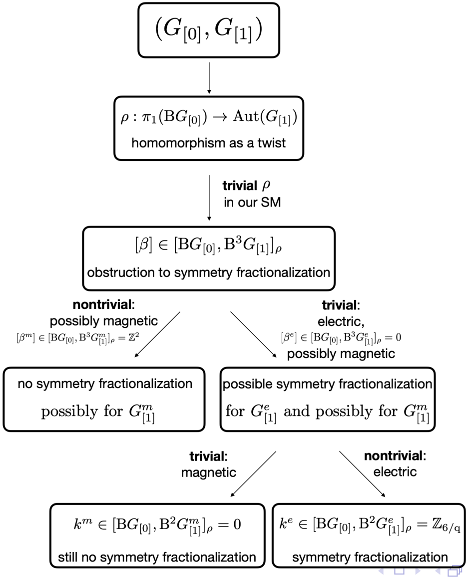
Electric $k^e \in [\mathrm{B}G_{[0]}, \mathrm{B}^2 G^e_{[1]}]_\rho$:
Thus, the symmetry fractionalization in the electric sector

$$k^e \in [\mathrm{B}G_{[0]}, \mathrm{B}^2 G^e_{[1]}]_\rho = \mathsf{H}^2_\rho(\mathrm{B}G_{[0]}, G^e_{[1]}) = \mathbb{Z}_{6/\mathrm{q}} \tag{27}$$

can be defined, because the obstruction $[\beta^e] = 0$.

Magnetic $k^m \in [\mathrm{B}G_{[0]}, \mathrm{B}^2 G^m_{[1]}]_\rho$:
If $[\beta^m] \neq 0$ is nontrivial in eq. (26), the symmetry fractionalization is not defined.
If $[\beta^m] = 0$ is trivial in eq. (26), the symmetry fractionalization in the magnetic sector $k^m \in [\mathrm{B}G_{[0]}, \mathrm{B}^2 G^m_{[1]}]_\rho = \mathsf{H}^3_\rho(\mathrm{B}G_{[0]}, \mathbb{Z}) = 0$ is trivial.

Symmetry fractionalization



Symmetry fractionalization

The symmetry fractionalization $k = k^e \in \mathbb{Z}_{6/\mathrm{q}}$ requires the following constraint between the background fields of electric 1-form symmetry $G^e_{[1]} = \mathbb{Z}^e_{6/\mathrm{q},[1]}$ and the 0-form symmetry $G_{[0]}$:

$$f_k^* \tilde{B}_e = \frac{k}{6/\mathrm{q}} \mathrm{d} A_{X_n} \tag{28}$$

Here $\tilde{B}_e = \frac{2\pi}{6/\mathrm{q}} B_e$, with $B_e \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_{\frac{6}{\mathrm{q}}}, \mathbb{Z}_{\frac{6}{\mathrm{q}}}) = \mathbb{Z}_{\frac{6}{\mathrm{q}}}$ is the background gauge field of the electric 1-form $\mathbb{Z}^e_{6/\mathrm{q},[1]}$ symmetry of the SM, while $A_{X_n} \in \Omega^1(\mathrm{B}(\frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_{\mathrm{lcm}(2,n)}}), \mathbb{R})$ is the background gauge field of the $\frac{\mathrm{U}(1)_{X_n}}{\mathbb{Z}_{\mathrm{lcm}(2,n)}}$ symmetry, and $k \in \mathbb{Z}_{6/\mathrm{q}} = \mathrm{H}^2(\mathrm{B}G_{[0]}, G^e_{[1]})$ is represented by the map $f_k : \mathrm{B}G_{[0]} \to \mathrm{B}^2 G^e_{[1]}$.

Conclusion

Our topological response result of the SM is

$$\sigma_n = \sigma_n(\mathrm{q}, k) = \frac{\mathrm{q}(1-n)\gcd(2,n)}{2n} + \frac{k\mathrm{q}}{6} \mod 1. \tag{29}$$

For a fixed n specified by the X_n global symmetry, the $\sigma_n(\mathrm{q}, k)$ can uniquely determine the SM gauge group for $\mathrm{q} = 1, 2, 3, 6$ and its fractionalization class k, if and only if

$$n \geq 7 \text{ and } n \neq 10, 12, 15, 30.$$

Moreover, by choosing a pair of $n = 2$ and 3 together, then we can further uniquely discern $\mathrm{SM}_{(\mathrm{q},k)}$ by measuring both of their σ_2 and σ_3. Similarly, pairs of σ_{n_1} and σ_{n_2} with

$$(n_1, n_2) = (2,3),\ (2,5),\ (3,4),\ (3,5),\ (4,5), \dots, \text{ etc.},$$

all such pairs can discern $\mathrm{SM}_{(\mathrm{q},k)}$.

Conclusion

$\sigma_n(\mathrm{q}, k \in \mathbb{Z}_{6/\mathrm{q}})$	$\mathrm{q}=1$	$\mathrm{q}=2$	$\mathrm{q}=3$	$\mathrm{q}=6$	discernible
$n=1$	$\left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right\}$	$\left\{0, \frac{1}{3}, \frac{2}{3}\right\}$	$\left\{0, \frac{1}{2}\right\}$	$\{0\}$	No
$n=2$	$\left\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 0, \frac{1}{6}, \frac{1}{3}\right\}$	$\left\{0, \frac{1}{3}, \frac{2}{3}\right\}$	$\left\{\frac{1}{2}, 0\right\}$	$\{0\}$	No
$n=3$	$\left\{\frac{2}{3}, \frac{5}{6}, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\}$	$\left\{\frac{1}{3}, \frac{2}{3}, 0\right\}$	$\left\{0, \frac{1}{2}\right\}$	$\{0\}$	No
$n=4$	$\left\{\frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{1}{12}\right\}$	$\left\{\frac{1}{2}, \frac{5}{6}, \frac{1}{6}\right\}$	$\left\{\frac{3}{4}, \frac{1}{4}\right\}$	$\left\{\frac{1}{2}\right\}$	No
$n=5$	$\left\{\frac{3}{5}, \frac{23}{30}, \frac{14}{15}, \frac{1}{10}, \frac{4}{15}, \frac{13}{30}\right\}$	$\left\{\frac{1}{5}, \frac{8}{15}, \frac{13}{15}\right\}$	$\left\{\frac{4}{5}, \frac{3}{10}\right\}$	$\left\{\frac{3}{5}\right\}$	No
$n=6$	$\left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 0\right\}$	$\left\{\frac{1}{3}, \frac{2}{3}, 0\right\}$	$\left\{\frac{1}{2}, 0\right\}$	$\{0\}$	No
$n=7$	$\left\{\frac{4}{7}, \frac{31}{42}, \frac{19}{21}, \frac{1}{14}, \frac{5}{21}, \frac{17}{42}\right\}$	$\left\{\frac{1}{7}, \frac{10}{21}, \frac{17}{21}\right\}$	$\left\{\frac{5}{7}, \frac{3}{14}\right\}$	$\left\{\frac{3}{7}\right\}$	Yes
$n=8$	$\left\{\frac{1}{8}, \frac{7}{24}, \frac{11}{24}, \frac{5}{8}, \frac{19}{24}, \frac{23}{24}\right\}$	$\left\{\frac{1}{4}, \frac{7}{12}, \frac{11}{12}\right\}$	$\left\{\frac{3}{8}, \frac{7}{8}\right\}$	$\left\{\frac{3}{4}\right\}$	Yes
$n=9$	$\left\{\frac{5}{9}, \frac{13}{18}, \frac{8}{9}, \frac{1}{18}, \frac{2}{9}, \frac{7}{18}\right\}$	$\left\{\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right\}$	$\left\{\frac{2}{3}, \frac{1}{6}\right\}$	$\left\{\frac{1}{3}\right\}$	Yes
$n=10$	$\left\{\frac{1}{10}, \frac{4}{15}, \frac{13}{30}, \frac{3}{5}, \frac{23}{30}, \frac{14}{15}\right\}$	$\left\{\frac{1}{5}, \frac{8}{15}, \frac{13}{15}\right\}$	$\left\{\frac{3}{10}, \frac{4}{5}\right\}$	$\left\{\frac{3}{5}\right\}$	No
$n=11$	$\left\{\frac{6}{11}, \frac{47}{66}, \frac{29}{33}, \frac{1}{22}, \frac{7}{33}, \frac{25}{66}\right\}$	$\left\{\frac{1}{11}, \frac{14}{33}, \frac{25}{33}\right\}$	$\left\{\frac{7}{11}, \frac{3}{22}\right\}$	$\left\{\frac{3}{11}\right\}$	Yes
$n=12$	$\left\{\frac{1}{12}, \frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}\right\}$	$\left\{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right\}$	$\left\{\frac{1}{4}, \frac{3}{4}\right\}$	$\left\{\frac{1}{2}\right\}$	No
...	...	...	...	...	...

Summary and future directions

- Introduced a new integer series of global $\mathrm{U}(1)_{X_n}$ symmetries of the SM.
- Considered the topological response of the SM involving the 0-form symmetry and the magnetic 1-form symmetry of the gauged SM. Two contributions: one from the gauge bundle constraints and another from the symmetry fractionalization of the 0-form symmetry and the electric 1-form symmetry.
- Explained the symmetry fractionalization.
- Discerned the SM gauge group via the topological responses.
- Investigate potential symmetry extensions.
- Consider baryon plus lepton and other discrete symmetries.
- Experimentally measure the topological responses.

Thank You!