Super-Strip Algebra in Gapped Fermionic Models

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based on a upcoming work with Qiang Jia
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1. Motivation & Introduction

Symmetry has been playing a prominent role in physics (e.g. Noether & conserved current). In recent years, the concept of symmetry has been greatly evolved:

Ordinary Sym. $\Longleftrightarrow$ Topological Surface of codimension 1

Equiped with this observation, study of symmetry conveniently generalized to

* high form/high group symmetries
* Subsystem Symmetries
* Non-invertible Symmetries.

In this talk, I'll mostly focus on the non-invertible symmetries (in the context of CM/math, it's also know as defect/categorical symmetries)

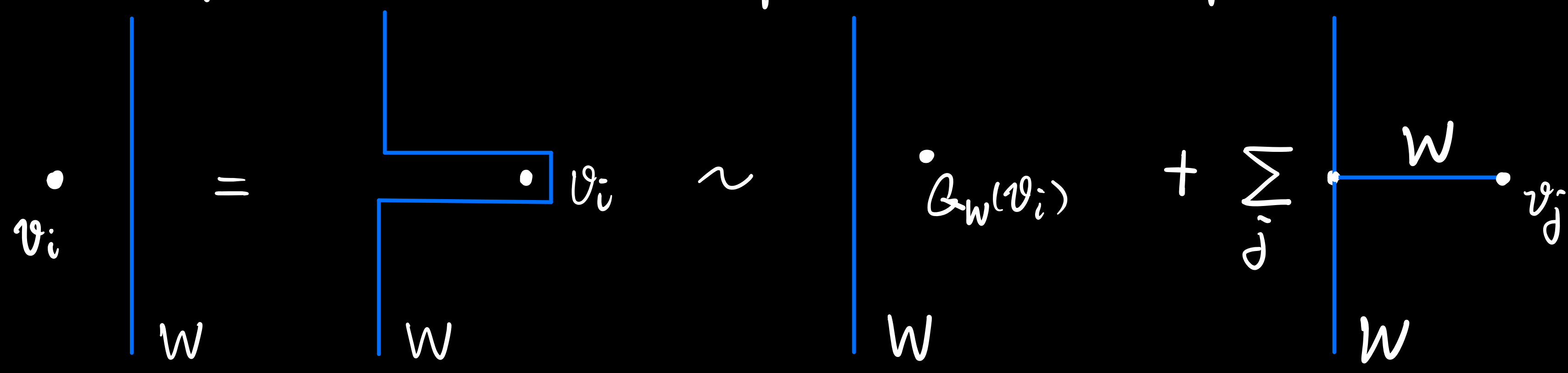
Possibly, the simplest example of non-invertible sym. is the Fibonacci category, $\mathrm{Fib} = \{1, W\}$ with fusion rule

$$W \cdot W = 1 + W$$

The Fib Cat exists ubiquitously in both Gapless/Gaped 2D models: e.g. (Rational) CFTs like

Lee-Yang, Tri-critical Ising, etc.

The action of W-line on local operator can be expressed pictorically,



On the other hand, when the model is Gaped, we need to first understand how the non-invertible symmetry acts on states in the Gaped model.

- Ground states:

In 2D, the Ground states in a Gaped model are classified by 2D TFTs: Putting the Gaped theory on S^1

For $R(S^1) \to \infty$, we end up with a bunch of degenerate states of zero energy, the vacua of the system.

$$|v_i\rangle \qquad i = 1, 2, \cdots, n.$$

We consider the Gaped theory with Categorical sym. $\mathcal{C}$.

In the far infra-red, $\mathcal{C}$ is Spontaneously Broken (SSB)

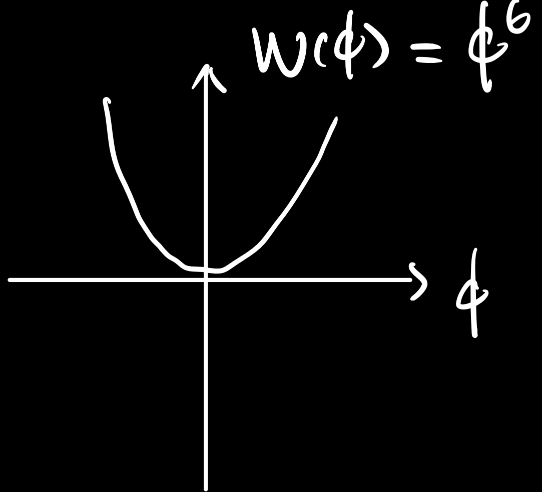
It implies that the vacua can be labeled by elements in $\mathcal{C}$, and symmetries in $\mathcal{C}$ acting on vacua shuffle them,

$$v_i |v_j\rangle = \sum_k N_{ij}^k |v_k\rangle$$

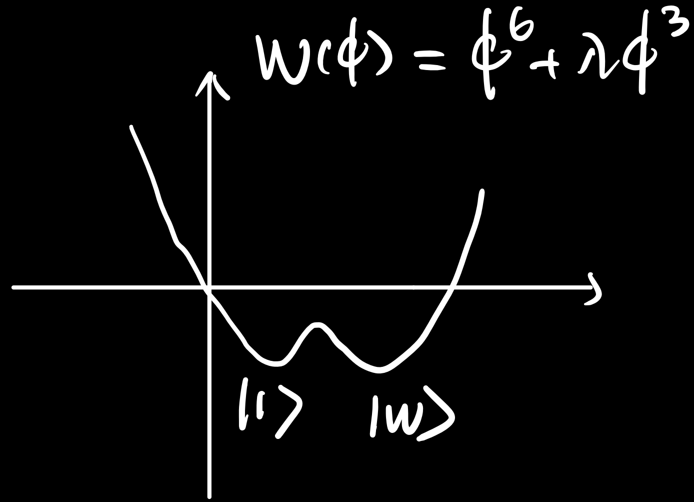
with $N_{ij}^k \in \mathbb{Z}_{\geq 0}$. $(N_i)_j^k \equiv N_{ij}^k$ known as NIM-Rep of $\mathcal{C}$.

A simple example is Tri-Ising + deformation $\phi_{(2,1)}$:

In LG-formalism,

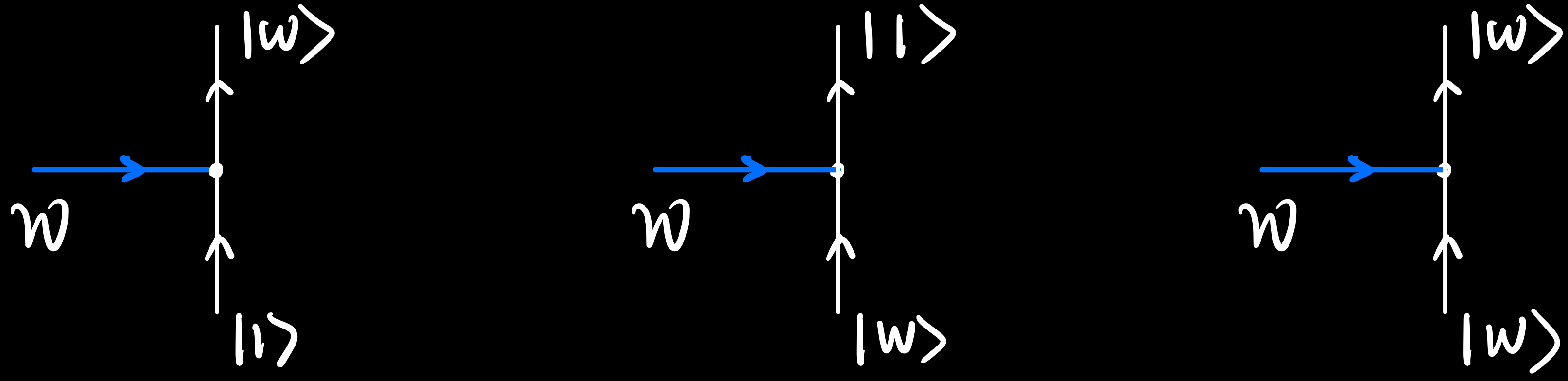


$\xrightarrow{\text{Rel. def.}}$



The deformed model flows to a Gapped phase while preserving Fib-Cat, which is SSB with a pair of asymmetric Vacua.

Pictorially, W-line acts on the two vacua as



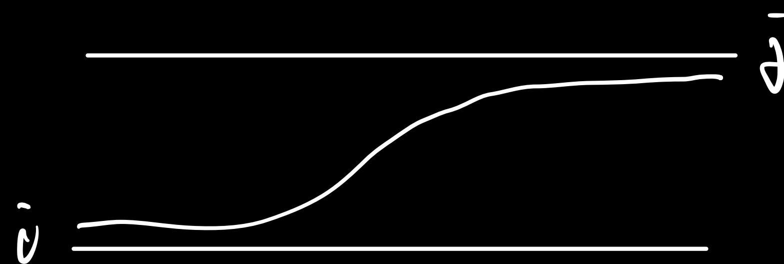
For a 2D Gapped model, in large volume limit $R(S^1)\to\infty$, one can fix different Boundary conditions on Spatial direction $x=\pm\infty$, So the whole Hilbert Space is divided into various superselection sectors H_{ij}

$$H = \bigoplus_{i,j} H_{ij} \qquad \text{with fixed vacua } i \,\&\, j.$$

These superselection can be classied into two classes:

i particle excitations:

ii soliton excitations:

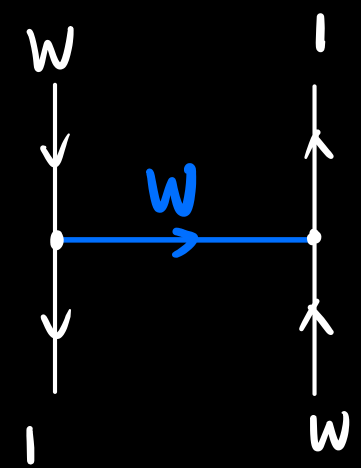


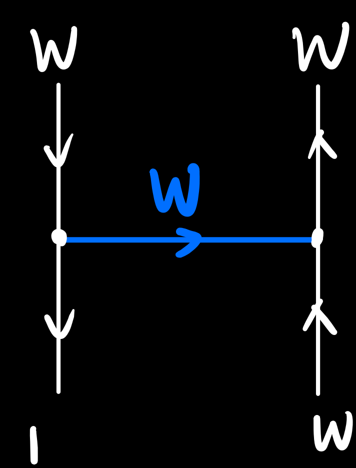
Ordinary Sym. can only map in between sectors of

particle to particle or soliton to soliton

However, non-invertible symmetries can interpolate sectors of particle to soliton. In the case at hand,

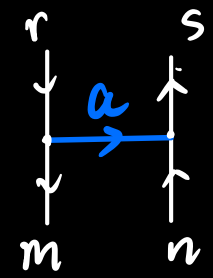
$$W: \quad H_{1W} \longrightarrow H_{W1}, \qquad H_{1W} \longrightarrow H_{WW}$$



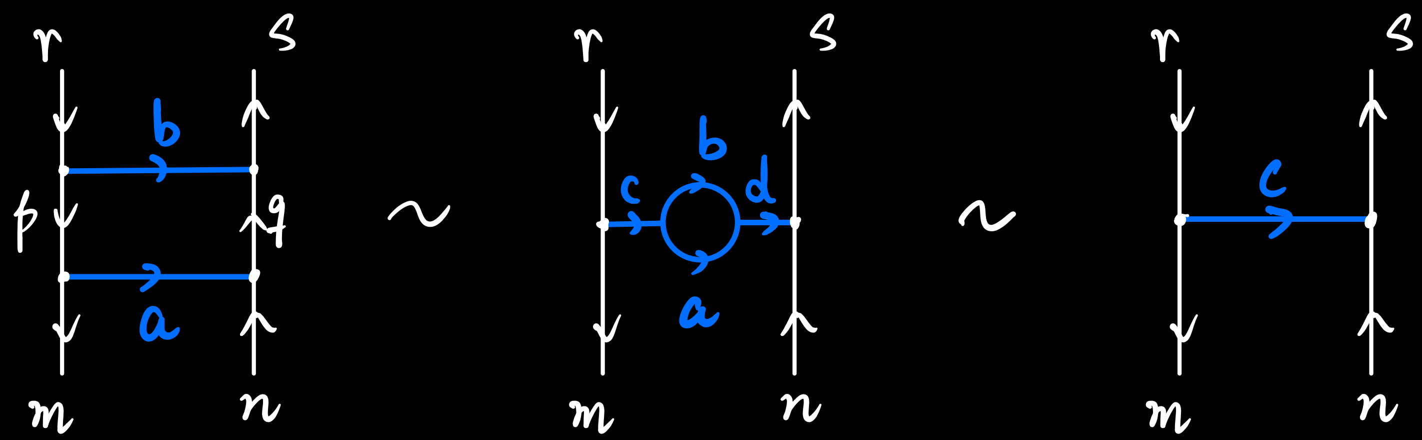


W-line implies kink $k \in H_{1W}$, anti-kink $\bar{k} \in H_{W1}$, and particle $p \in H_{WW}$ furnish a triplet degenerates $(k, \bar{k}, p)$

— Strip Algebra: To understand this novel degeneracy, one considers the so-called "Strip Algebra" structure

$\mathrm{Str}_{\mathcal{C}}(\mathcal{M}) = \{$  $\mid a \in \mathcal{C},\ m, n, r, s \in \mathcal{M} \}$

equiped with fusion



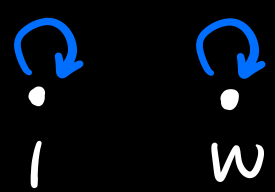
The $\mathrm{Rep}(\mathrm{Str}_{\mathcal{C}}(\mathcal{M}))$ encodes the non-trivial degeneracy between particles and solitons.

In the case of $\mathcal{C}$ being **SSB**, $\mathrm{Rep}(\mathrm{Str}_{\mathcal{C}}(\mathcal{M}))$ is one-to-one correspondence to elements of $\mathcal{C}$.

$$\mathrm{Rep}(\mathrm{Str}_{\mathcal{C}}(\mathcal{M}))\Big|_{\mathcal{M}=\mathcal{C}} \cong \mathcal{C}$$

One can use **Quiver Diagrams** to label these Reps

R_1:



2 Dim Rep (unstable)

R_w:

3 Dim Rep. ($k, \bar{k}, p$)

$Str_{\mathcal{C}}(\mathcal{M})$ is a Weak Hopf C^*-Algebra. One can find idempotes (projectors) from the coproduct structure in $Str_{\mathcal{C}}(\mathcal{M})$

Analogue to Group Algebra, idempotes projects elements in $Str_{\mathcal{C}}(\mathcal{M})$ into different Representations, satisfying.

$$\#\left|Str_{\mathcal{C}}(\mathcal{M})\right| = \sum_{a\in\mathcal{C}} \mathrm{Dim}(R_a)^2$$

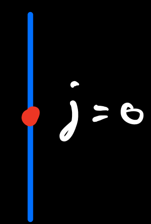
e.g. $Str_{Fib}(\mathcal{M})$: $\#|Str_{Fib}(\mathcal{M})| = 13 = 2^2 + 3^2$

2. Super-Fusion Category & Vacua

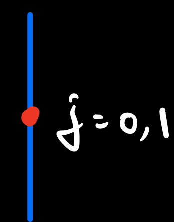
For a fermionic theory $\mathcal{T}_f$, its categorical symmetries are characterized by the so-called super-fusion category $\mathcal{C}_f$.

The morphisms between objects in $\mathcal{C}_f$ is $\mathbb{Z}_2$-graded, and thus two types of simple lines:

* m-type: $\mathrm{Hom}_{\mathcal{C}_f}(m, m) \simeq \mathbb{C}^{1|0}$

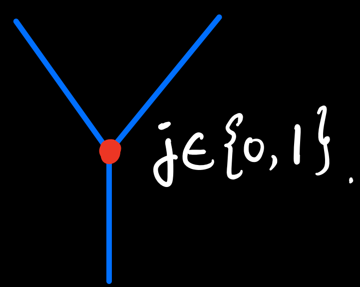


* q-type: $\mathrm{Hom}_{\mathcal{C}_f}(q, q) \simeq \mathbb{C}^{1|1}$



The endomorphism algebra of q-type line is the Clifford Alg. Cl_1, implying there are odd number of 1D Majorana fermion reside on the q-line.

3-way junctions are also $\mathbb{Z}_2$-graded, for both m-/q-lines



— $(-1)^F$: In $\mathcal{C}_f$, there is always a m-type object, $(-1)^F$, the fermionic parity symmetry, and the full category is factorized as $\mathcal{C}_f = \widehat{\mathcal{C}_f} \boxtimes \mathbb{Z}_2^{(-1)^F}$.

Throughout the talk, we assume that

α. $(-1)^F$ is non-anomalous

β. $(-1)^F$ is the only symmetry not SSB.

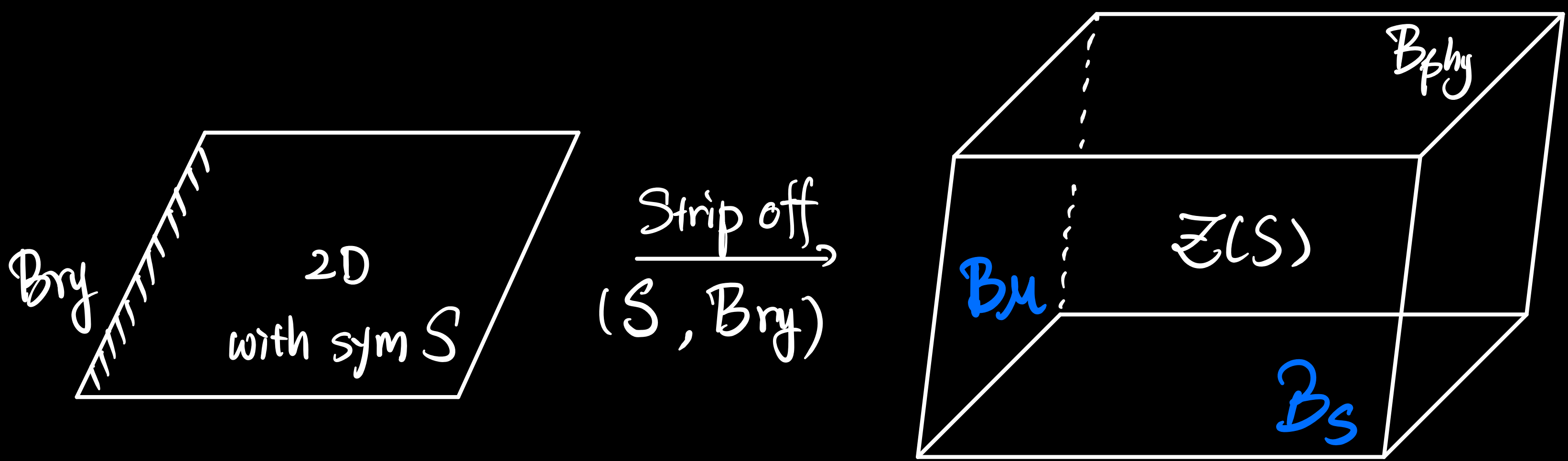
Cond. β implies that only $\widehat{\mathcal{C}}_f$ is completely SSB, and $\#\,\mathrm{Vac} = |\widehat{\mathcal{C}}_f|$

Cond. α guarantees that we can gauge $(-1)^F$ to bosonize $\widetilde{T}_f$ to T_b with an ordinary fusion category $\mathcal{C}$. In $\mathcal{C}$, there is always a non-anomalous $\widehat{\mathbb{Z}}_2$ that can be used to fermionize T_b back to $\widetilde{T}_f$.

$$\underset{\widehat{\mathcal{C}}_f \boxtimes \mathbb{Z}_2^{(-1)^F}}{\widetilde{T}_f} \quad \underset{\text{gauge } \widehat{\mathbb{Z}}_2 \text{ with Arf twist}}{\overset{\text{gauge } (-1)^F}{\rightleftarrows}} \quad \underset{\mathcal{C} \supset \widehat{\mathbb{Z}}_2}{T_b}$$

— SymTFT mechinary:

symmetries and gaped boundary in both bosonic/fermionic theories can be fit in the framework of SymTFT in 3D



$\mathcal{B}_S$ encodes Sym. DATA,

$\mathcal{B}_{\mathcal{U}}$ for DATA of boundry multiplets respect to Sym S.

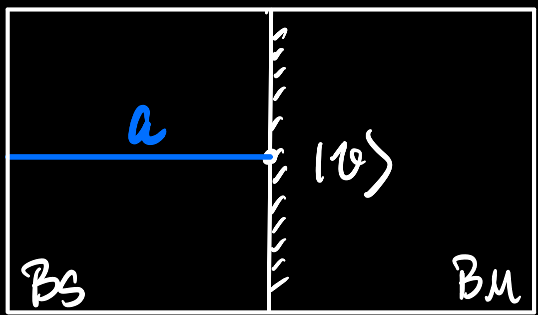
Both B_S and B_M are specified by Lagrangian Algebras that determine which sym-lines can end on B_S / B_M.

The rules are:

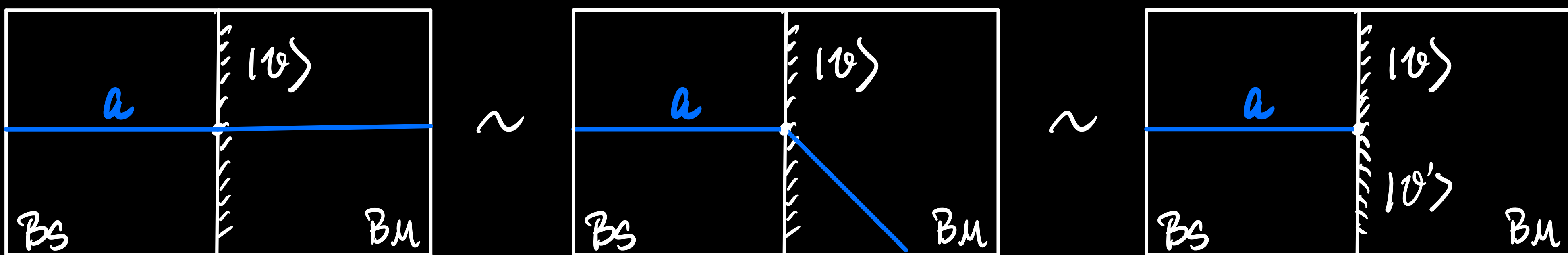
a. # Vacua $= \dim \mathrm{Hom}_{Z(S)}(B_S, B_M)$

e.g. if $B_S = B_M \Rightarrow$ the full symmetry S is SSB.

β. pair (B_s, B_u) specify what sym-line can end on Bdy

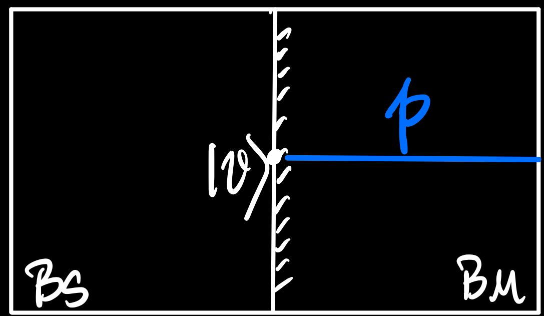


$\Rightarrow$ $|v\rangle$ is invariant under a

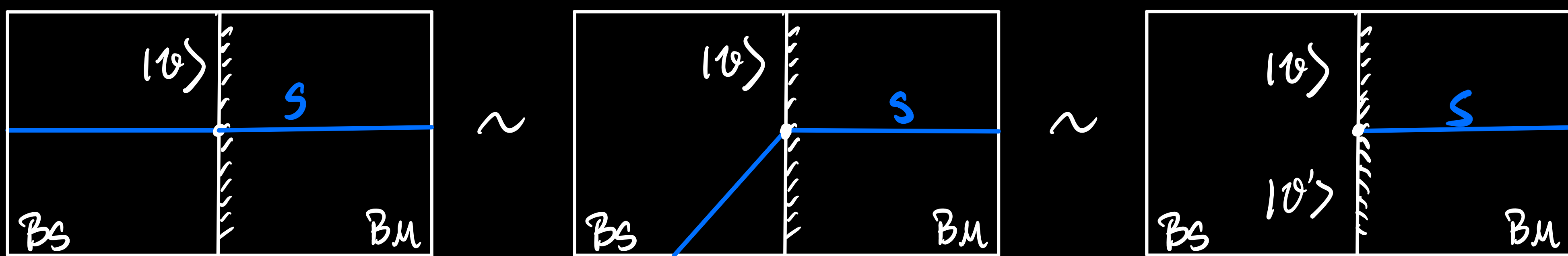


$\Rightarrow$ a is SSB resp to $|v\rangle$ and $|v'\rangle$

γ. pair (B_s, B_u) specify what multiplet-line can end on Bdy



$\Rightarrow$ p is a particle multiplet in $\mathcal{H}_{vv}$



$\Rightarrow$ s is a soliton-multiplet in $\mathcal{H}_{vv'}$

Our set-up is to choose $(\mathcal{B}_S, \mathcal{B}_u) = (\mathcal{B}_{\mathcal{C}_f}, \mathcal{B}_{\mathcal{C}})$

One can show in $\mathcal{T}_b$, $\mathcal{C}$ is SSB

while in $\mathcal{T}_f$, $\mathcal{C}_f = \hat{\mathcal{C}}_f \boxtimes \mathbb{Z}_2^{(-1)^F}$ and $\hat{\mathcal{C}}_f$ is SSB.

There is actually a simple relation between $\mathcal{C}$ and $\hat{\mathcal{C}}_f$,

as well as vacua in $\mathcal{T}_b$ and $\mathcal{T}_f$.

consider $\hat{\mathbb{Z}}_2$-action in $\mathcal{C}$:

- 2-orbit: (a, b) $\hat{\mathbb{Z}}_2$ acts transtively.
- fixed pt: c invariant under $\hat{\mathbb{Z}}_2$-action

2-orbit set $(a,b) \mapsto [a]$ m-type object in $\hat{\mathcal{C}}_f$

fixed point $c \mapsto [c]$ q-type object in $\hat{\mathcal{C}}_f$

In CM, it's known as fermionic anyon condensation.

For vacua in $\widetilde{\mathcal{T}}_b$ & $\widetilde{\mathcal{T}}_f$:

- $\widetilde{\mathcal{T}}_b$: $\mathcal{C}$ is completely SSB, in the large radius limit

$$Z_b[\square] = |\mathcal{C}| = 2m + q$$

m : # of 2-orbits
q : # of fixed pts

$$Z_b[\square] = Z_b[\square] = Z_b[\square] = q$$

- $\widetilde{\mathcal{T}}_f$: $Z_f[S_1, S_2] = m + (-1)^{S_1 S_2}\, q$

m is # of m-type objects, q is # of q-type obj. in $\hat{\mathcal{C}}_f$

$S_1, S_2 = 0, 1$ are spin structures along spatial and temporal directions.

$Z_f[S_1, S_2] = m + (-1)^{S_1 S_2} q$ implies that

q-type vacua are sensitive to the spin structure.

In both NS/R sector there are $|\hat{\mathcal{C}}_f| = m + q$ number of vacua. However, in R-sector q-type vacua are fermionic

Especially

$$Z_f[R, R] = m - q$$

In fermionic model with SUSY, Witten index I_W can be computed from a super-fusion Cat.

— SPT Vacua (q-type vacua)

Starting from a vacua $|0\rangle$, acting a q-line on it,

$$|q\rangle \equiv q|0\rangle$$

must be different from $|0\rangle$ by an Arf-invertible phase, to match with the 1D Majorana fermion anomaly on the boundary $|q\rangle$. Put it differently, the Mod 2-Anomaly requires that solition configuration between m-type & q-type Vacua need ODD numbers of fermionic zero modes to off-set the anomaly on Bry.

— Example 1: $SM^{N=1}(3,5)$ + least relevant Deformation $\mathcal{O}_{(1,3)}$

$\Big\downarrow$ Bosonization

Tri-critical Ising + $\left(\phi_{\frac{3}{5},\frac{3}{5}}\right.$ + lighter Operator $\left.\cdots\right)$

The preserved fusion Cat $C = \{1, \eta, N\}$ (Ising-like Cat)

$\Big\uparrow$ Fermionic condensation $C_f = \{1, (-1)^{F_L}\}$, $(-1)^{F_L}$ q-type object.

$$Z_f[R,R]\big|_{UV\ CFT} = Z_f[R,R]\big|_{IR} = 1-1 = 0$$

as $I_W = Z_f[R,R]$ is RG-flow invariant resp. to local operator Deformation.

We actually have a Lagrangian for this: in LG-formulation

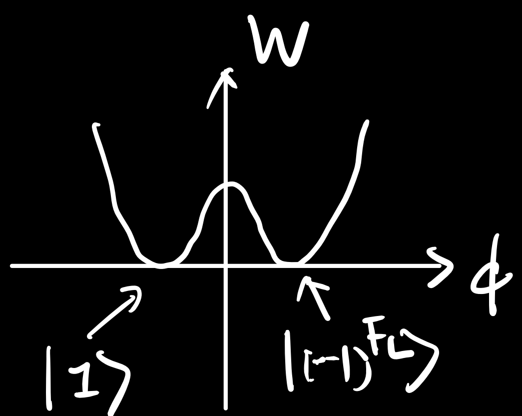
$W(X) = \frac{1}{3}X^3 - \lambda X$ in $\mathcal{N}=(1,1)$ superspace

$X = \phi + \theta\psi + \theta^2 F$

$$W(\phi,\psi) = \int d^2\theta \, W(X) = (\phi^2-\lambda)^2 + \frac{1}{2}\phi\psi\psi$$

$(-1)^{F_L}$: $\psi \to \gamma_5 \psi$ chiral $\mathbb{Z}_2$ on Majorana fermion.

$\phi \to -\phi$



soliton configuration ϕ_{sol} preserves half-SUSY $\hat{Q}$

$\hat{Q}\phi_{sol}$ is the fermionic zero modes

Example 2: $SM^{\mathcal{N}=1}(4,6)$ + least relevant deformation $\mathcal{O}_{(1,3)}$

$\Big\downarrow$ Bosonization

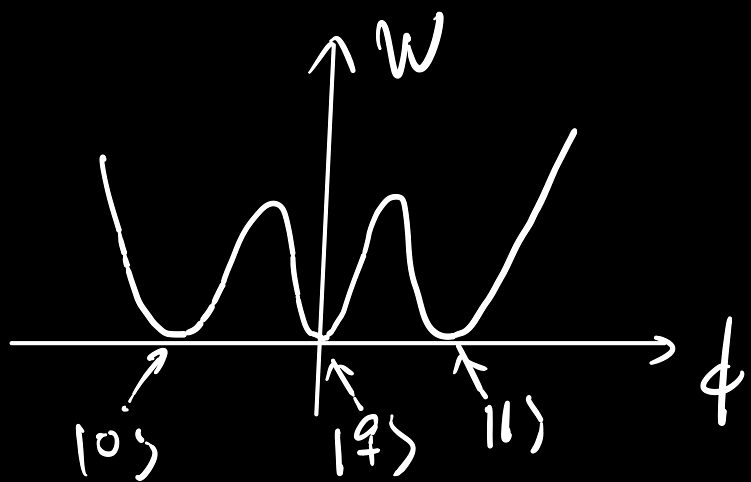
Compact boson on orbifold branch + deformations

$$\mathcal{C} = TY(\mathbb{Z}_2^b \times \mathbb{Z}_2^f) \Rightarrow \hat{\mathcal{C}}_f = \mathbb{Z}_2^b \cup \{q\} \qquad q^2 = \mathbb{C}^{1|1}(0+1)$$

In LG-formulation

$$W(X) = \frac{1}{4}X^4 - \lambda X^2$$

$$\Rightarrow W(\phi, \psi) = \phi^2(\phi^2 - \lambda)^2 + \text{fermions}.$$



$$I_W = Z_f[R,R]\Big|_{IR} = 2-1 = 1$$

Interesting test is to orbifold $SM^{\mathcal{N}=1}_{(4,6)}$ to its D-type variant.

$$SM^{\mathcal{N}=1}_{(4,6)}/\mathbb{Z}_2^{orb} = A_2^{\mathcal{N}=2}$$

In LG-formalism, $\mathbb{Z}_2^{orb}: X \to -X$

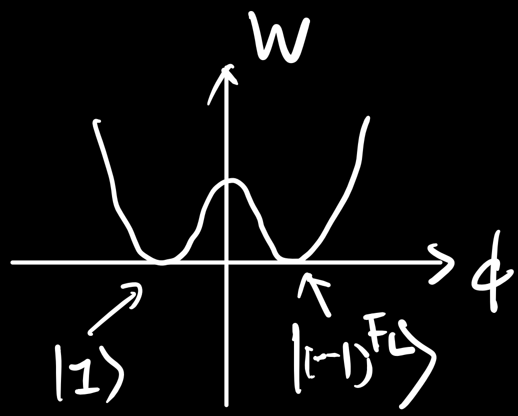
$$\left(SM^{\mathcal{N}=1}_{(4,6)} + \lambda \int d^2\theta\, X^2\right)/\mathbb{Z}_2^{orb} = A_2^{\mathcal{N}=2} + \text{least rel. deform.}$$

So $\hat{\mathcal{C}}_{f,\mathcal{N}=2}$ can be obtained from $\hat{\mathcal{C}}_{f,\mathcal{N}=1} = \mathbb{Z}_2^b \cup \{q\}$ via a

Bosonic condensation of $\mathbb{Z}_2^b$. It gives

$$\hat{\mathcal{C}}_{f,\mathcal{N}=2} = \mathbb{Z}_2 \Rightarrow I_W = Z_f[R,R]\big|_{IR} = 2$$

In LG-formalism $A_2^{N=2}$ + the least relevant deformation gives



$$W(\Phi) = \frac{1}{3}\Phi^3 - \lambda\Phi$$

$$\Rightarrow W(\phi,\psi) = |\phi^2 - \lambda|^2 + \frac{1}{2}\phi\psi^2 + h.c.$$

The $\hat{\mathcal{C}}_{f,N=2} = \mathbb{Z}_2$ is simply the chiral symmetry

$$(-1)^{F_L}: \ \psi \to \gamma_5\psi \ , \ \phi \to -\phi$$

However now ψ is a Dirac fermion $\simeq$ Maj. ferm $\times 2$

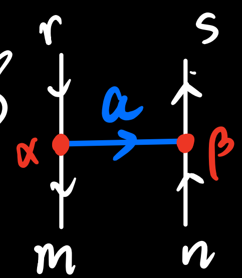
$(-1)^F$ has an Even element of the $\mathbb{Z}_8$ classification of the $\mathbb{Z}_2$ anomaly,

so a m-type.

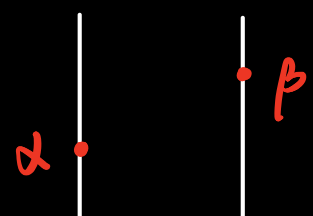
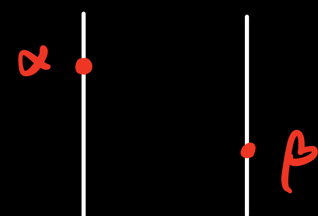
3. Super-Strip Algebra

To study the Reps of particle/soliton multiplets resp. to $\widehat{\mathcal{C}_f}$.

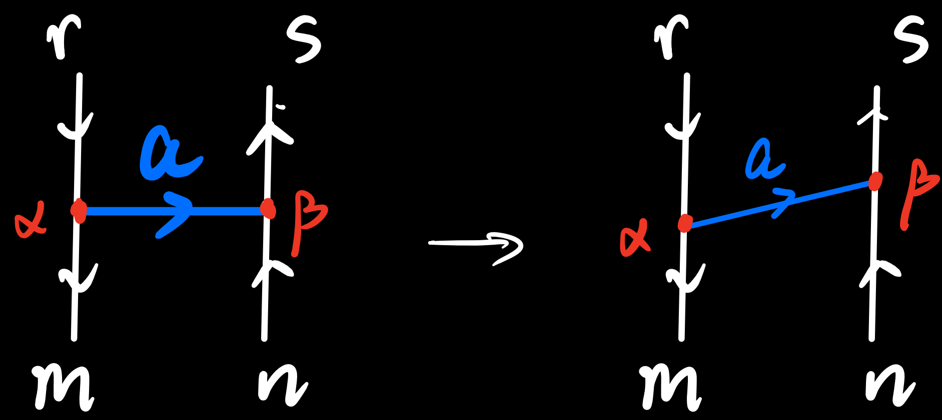
We parallely introduce the super-strip Alg.

$$Str_{\widehat{\mathcal{C}_f}}(\mathcal{M}) = \Big\{$$  $$\Big|\ a \in \widehat{\mathcal{C}_f},\ m, n, r, s \in \mathcal{M},\ \begin{matrix} \alpha, \beta \in \{0,1\} \\ \mathbb{Z}_2\text{-Graded junc.} \end{matrix} \Big\}$$

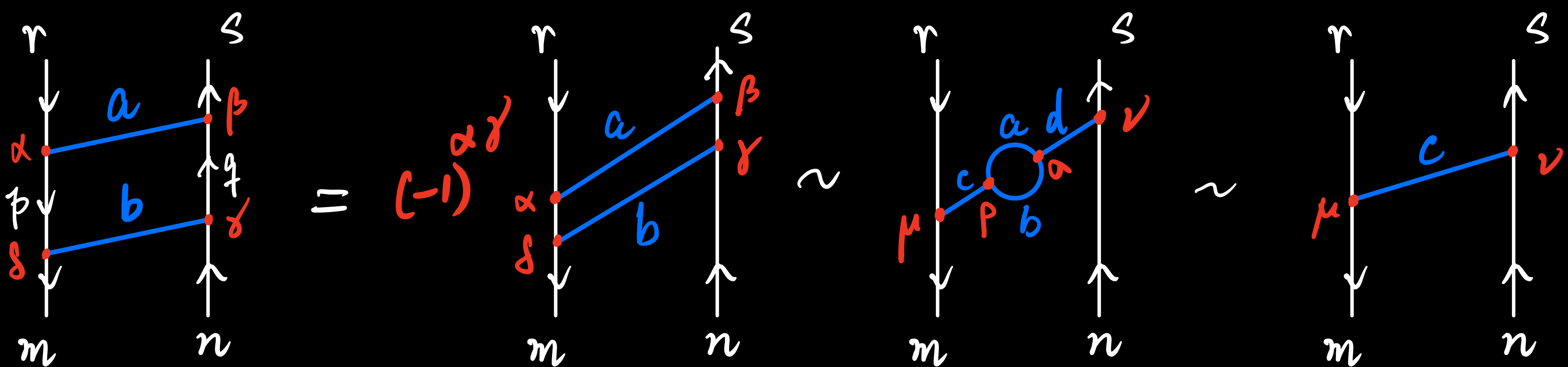
The fusions of elements in $Str_{\mathcal{C}_f}(\mathcal{M})$ matter with the order of junctions:

 $= (-1)^{\alpha\beta}$ 

Our convention is to slightly tilt the sym-line



and thus the fusion of two elements:

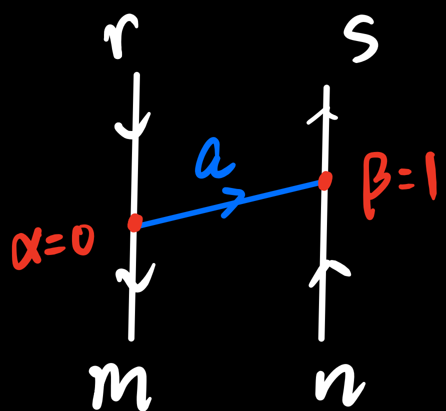


When $\widehat{\mathcal{C}_f}$ is SSB, the Irreducible Rep of $\mathrm{Str}_{\widehat{\mathcal{C}_f}}(\mathcal{M})$ is again one-to-one correspondence to the elements in $\widehat{\mathcal{C}_f}$ and one can use Quiver Diagrams to Label these Reps.

It's notable that, in fermionic theory, the Hilbert Space is further refined to $(-1)^F$-Graded,

$$\mathcal{H} = \bigoplus_{i,j} \left(\mathcal{H}_{i,j}^{b} \oplus \mathcal{H}_{i,j}^{f} \right)$$

The super-strip Algebra will encode some nontrivial degeneracy between boson & fermion secters via non-invertible lines decorated by fermionic zero modes, e.g.



$$: \quad \mathcal{H}^{b}_{m,n} \longrightarrow \mathcal{H}^{f}_{r,s}$$

$$\mathcal{H}^{f}_{m,n} \longrightarrow \mathcal{H}^{b}_{r,s}$$

— Example: 2D massless Adjoint QCD

We consider 2D $SU(N)$ gauge theory with massless Adjoint ψ.

$$\mathcal{L} = -\frac{1}{4g_{YM}^2} \mathrm{Tr} F_{\mu\nu}^2 + \mathrm{Tr}(i\psi^T \not{D} \psi)$$

* This fermionic model is Gapped. $m_{IR} \sim g_{YM}$

* when $M_{UV}=0$, there are many non-invertible sym $\Rightarrow \# \mathrm{Vac} \sim 2^N$

* $\mathbb{Z}_N$ one-form sym, center of $SU(N)$, divides the whole theory into different Universes with Domain wall of infinite tension.

$$\mathcal{H} = \bigoplus_{A=0}^{N-1} \bigoplus_{i,j} \left(\mathcal{H}_{i,j}^{(A),b} \oplus \mathcal{H}_{i,j}^{(A),f} \right)$$

$\mathcal{H}^{(A,i),(B,j)}$ is Forbidden by the one-form sym. super-selection rule.

* At short distance, it's just a bunch of free fermions

N^2-1 free Majorana fermion $\xrightarrow{\text{Bosonization}}$ $Spin(N^2-1)_1$.

In Deep IR, its bosonization is a coset WZW model

$Spin(N^2-1)_1 / SU(N)_N$ with central charge $c=0$

The first non-trivial Example is $SU(4) + \psi_{adj}$.

For simplicity, focus on the universe of 0 1-form charge, $\mathcal{H}^{(0)}$

$$\mathcal{C}^{(0)} = \{1, \eta, A, B\} \quad (\sim SO(3)_6)$$

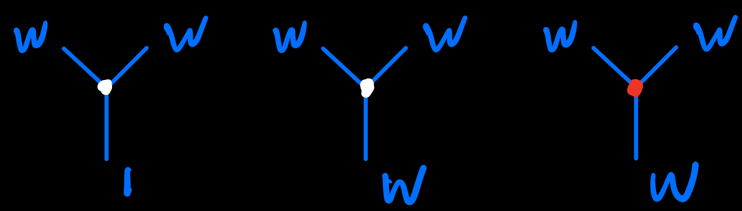
	η	A	B
η	1	B	A
A		$1+A+B$	$\eta+A+B$
B			$1+A+B$

$\Big\downarrow$ fermionic Condensation

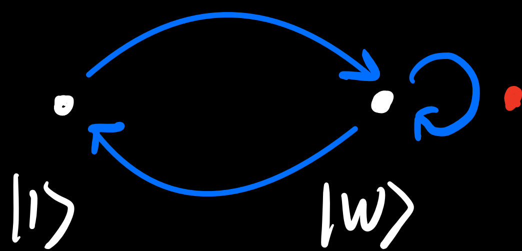
condense $\eta \leftrightarrow 1$

$B \leftrightarrow A \qquad W \equiv [A]$

$$\Rightarrow \hat{\mathcal{C}}^{(0)}_f = \{1, W\} \quad \text{with} \quad W \cdot W = 1 + \mathbb{C}^{1|1} W$$



in $H^{(0)}$, there are two vacua $|1\rangle$, $|w\rangle$, and the Super-Strip Algebra has two irreps: the connected one is

R_w: 

4-Dim Rep.

It implies that $k \in H_{1w}$, $\bar{k} \in H_{w1}$, $p_B, p_F \in H_{ww}$ furnishing a Quadruplet $(k, \bar{k}, p_B, p_F)$ of same mass degeneracy.

$\boxed{\text{There is a Bose-Fermion Degeneracy even without SUSY!}}$

4. Summary & Outlook:

* We introduce "Super-Strip Algebra" to analyze the vacua structures, multiplet degeneracies in Gapped fermionic model.

* Many interesting future directions:

  - Test "Super-Strip Algebra" in Integrable models.
  - Generalized symmetry meets Supersymmetry.
  - Applications in 2D Quantum Chromodynamics

Thank you