Chiral Tube Algebra

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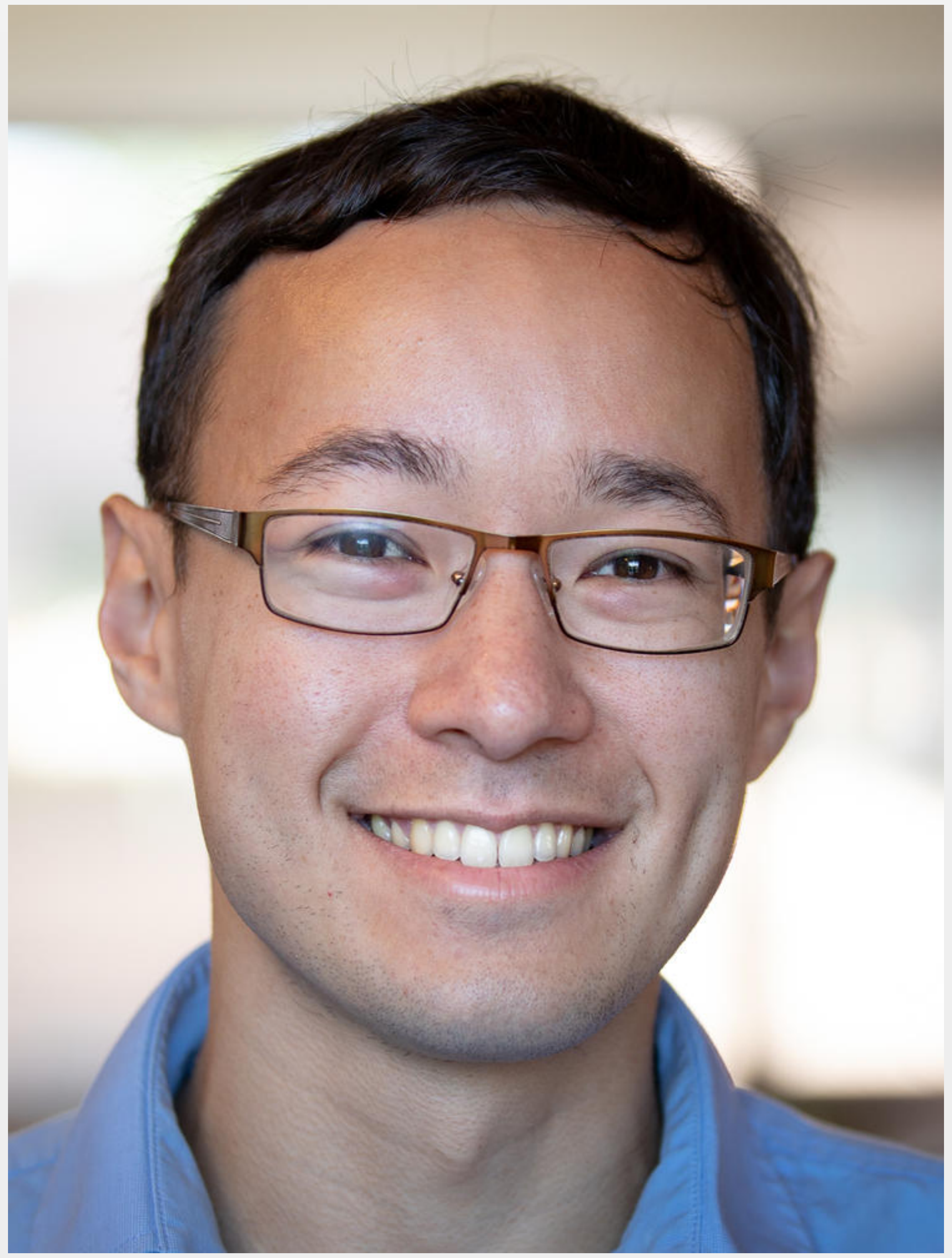

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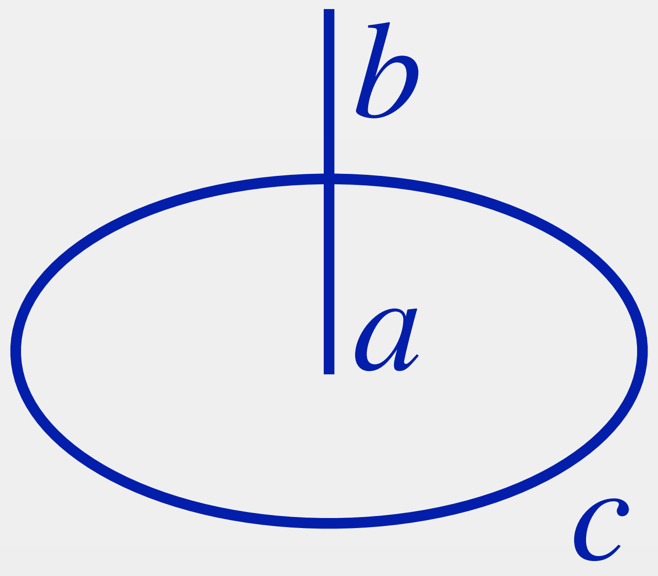
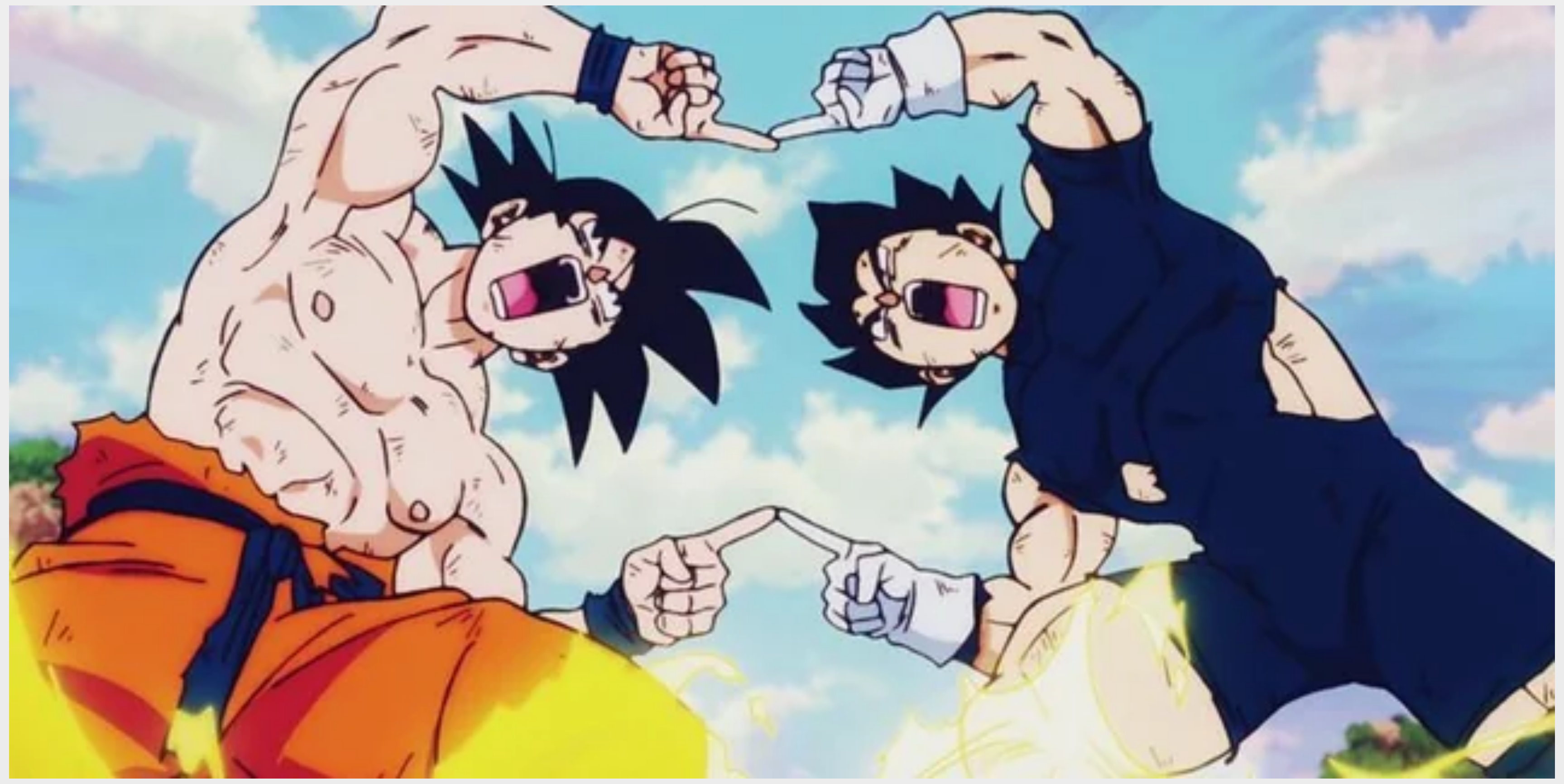
Chiral Tube Algebra

Chiral Algebra

$$[J^a_n, J^b_m] = kn\delta^{ab}\delta_{n+m,0} + if^{ab}{}_c J^c_{n+m}$$

Tube Algebra





Outline

1. Introduction
    - Chiral algebra
    - Tube algebra
    - Motivation
2. Chiral tube algebra
    - Examples: $SU(2)_1$ and orbifolds
3. Conclusion

Chiral Algebra

Chiral algebras pervade the study of 2d CFT.

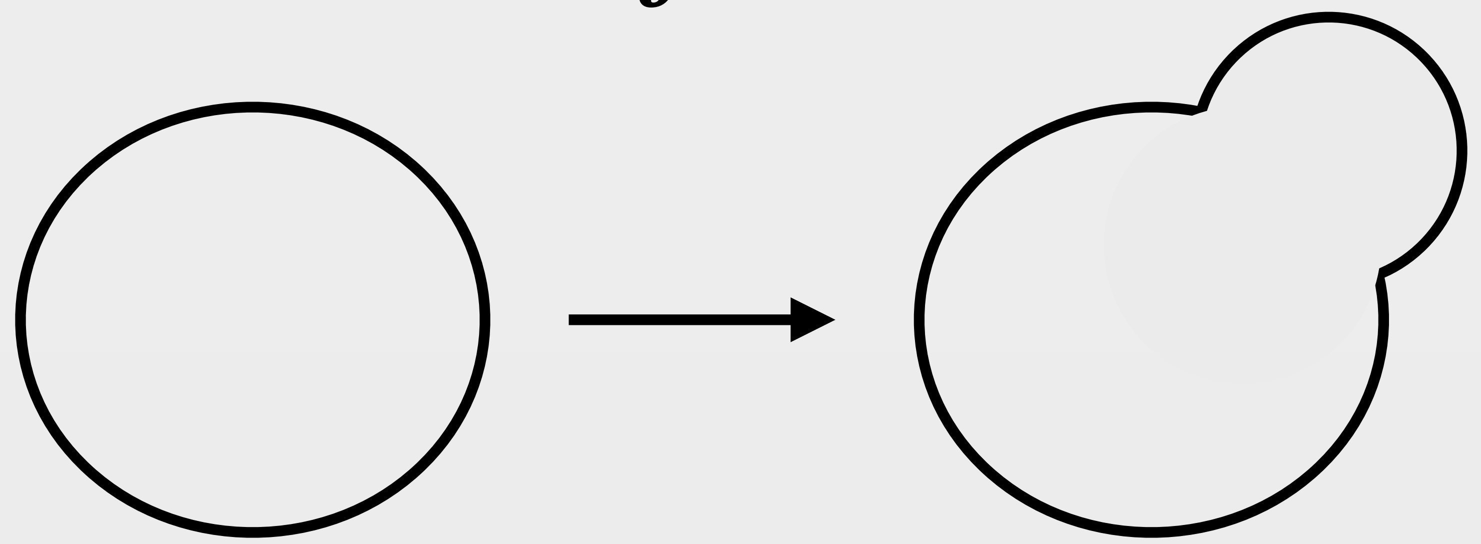
$$\text{Chiral algebra} = \{O^1(z), O^2(z), \cdots\}$$

$$\text{OPE:} \quad O^a(z)O^b(0) \sim \sum \frac{C^{ab}{}_c}{z^{h_a+h_b-h_c}} O^c(0)$$

The Laurent modes of the chiral operators define a family of topological lines (with position-dependent couplings but independent of choices of curves, similar to boost)

$$O^a(z) = \sum_{n\in\mathbb{Z}} O^a_n \, z^{-n-h_a}$$

$$O^a_n = \frac{1}{2\pi i} \oint dz \, O^a \, z^{n+h_a-1}$$



$$\text{Lie algebra:} \quad [O^a_m, O^b_n] = \sum_c f^{ab}{}_c O^c_{n+m}$$

Examples

- Stress tensor $T(z)$

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

Virasoro algebra: $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$

- Conserved current $J(z)$ (chiral if the CFT is unitary and compact)

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + \frac{if^{ab}{}_c J^c(0)}{z}, \qquad T(z)J^a(0) \sim \frac{J^a(0)}{z^2} + \frac{\partial J^a(0)}{z}$$

Affine Kac-Moody algebra: $[J^a_m, J^b_n] = km\delta^{ab}\delta_{m+n,0} + if^{ab}{}_c J^c_{m+n}$, $[L_m, J^a_n] = -nJ^a_{m+n}$

- Superconformal algebra, W-algebra

Representation and Character

Chiral algebras are spectrum generating symmetries

$$L_0 - \frac{c}{24} = \text{(chiral part of) Hamiltonian in radial quantization}$$

$$[L_0, L_n] = -nL_n$$

Operators form representations under the chiral algebra:

$$\text{highest weight state: } L_0|h\rangle = h|h\rangle,\ L_n|h\rangle = 0$$

$L_0 = h$	$L_0 = h+1$	$L_0 = h+2$
$\|h\rangle$	$L_{-1}\|h\rangle$	$L_{-2}\|h\rangle$
		$L_{-1}^2\|h\rangle$

$$\text{character: } \chi_h(q) = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\right]$$

example: non-degenerate rep at $c > 1$

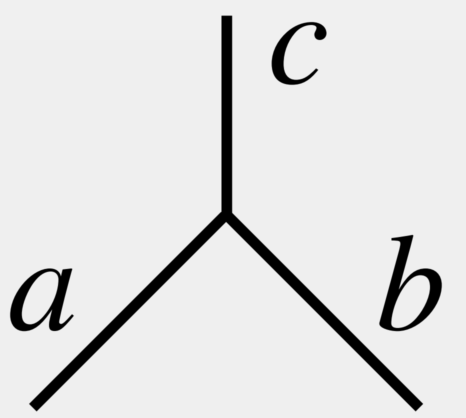
$$\chi_h(q) = \frac{q^{h-\frac{c-1}{24}}}{\eta(q)}$$

Topological Defects

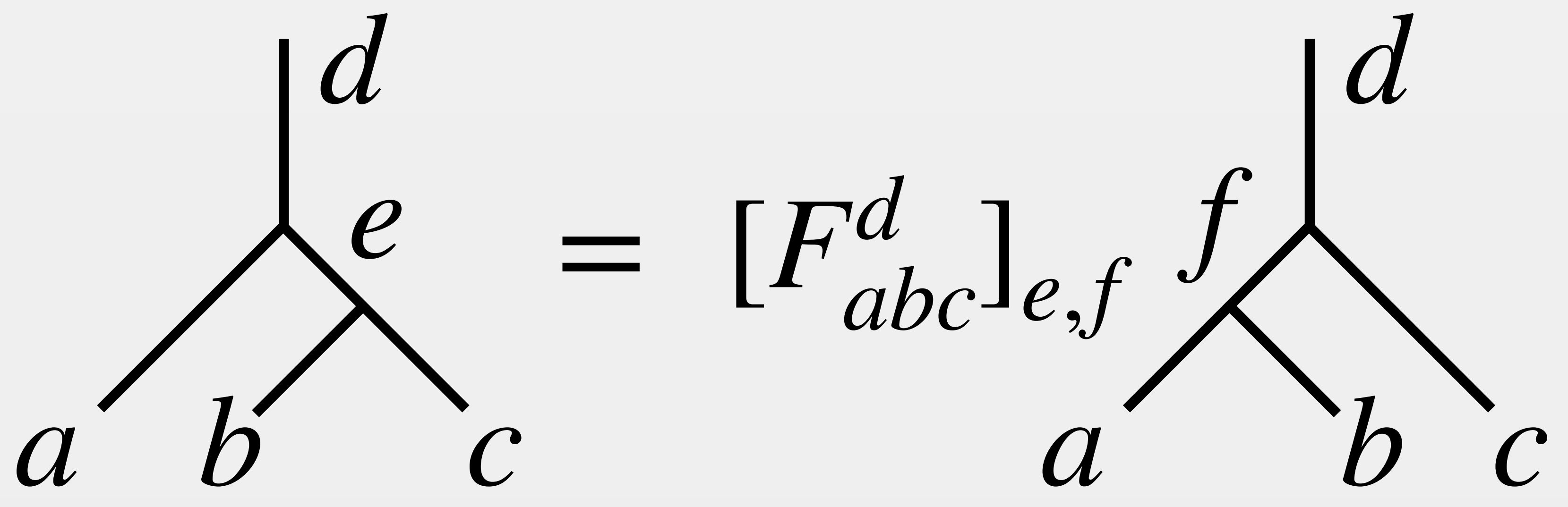
Tube algebras are built out of topological defects that commute with stress-tensor

Topological defects = Fusion category

Fusion: $a \times b = c + \cdots$

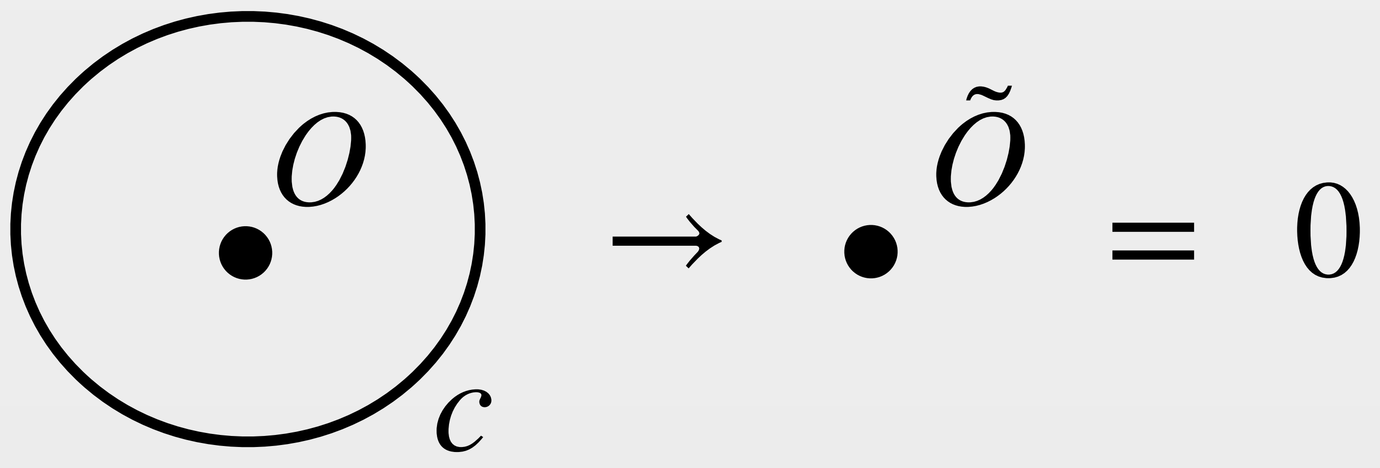


Associator



Topological defects generally define a non-invertible symmetry

- Non-invertible fusion: $a \times \bar{a} = 1 + \cdots$
- Can annihilate local operators

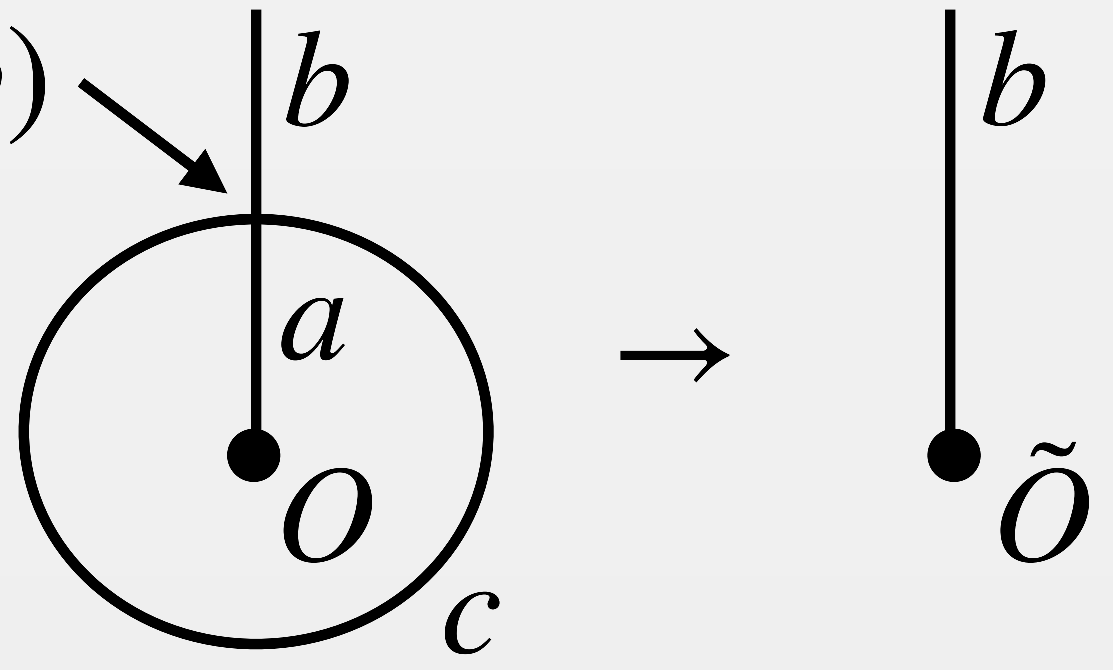


Tube Algebra

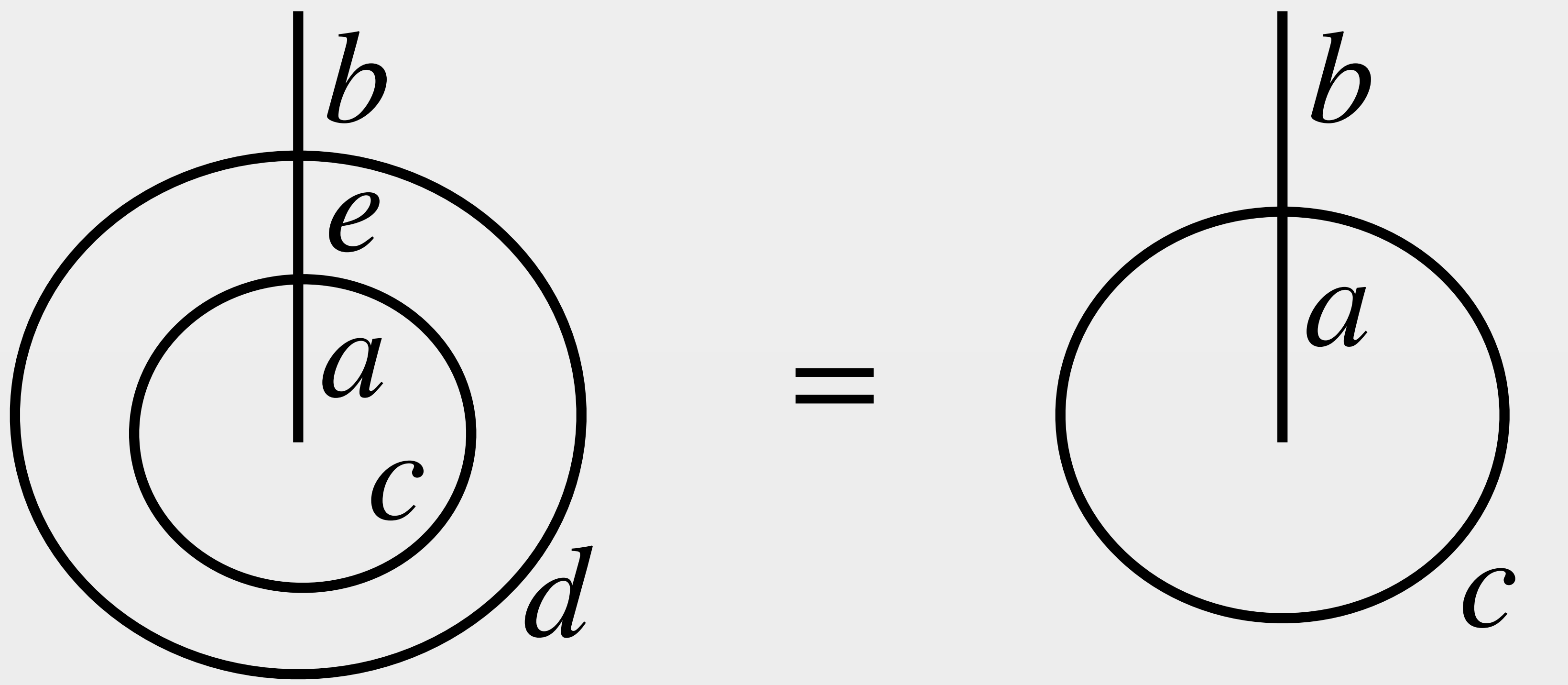
Generally, non-invertible symmetries map local operators to defect operators

$x \in \mathrm{Hom}(a \times c, c \times b)$

Lasso action:



Tube algebra = the algebra formed by lasso operators

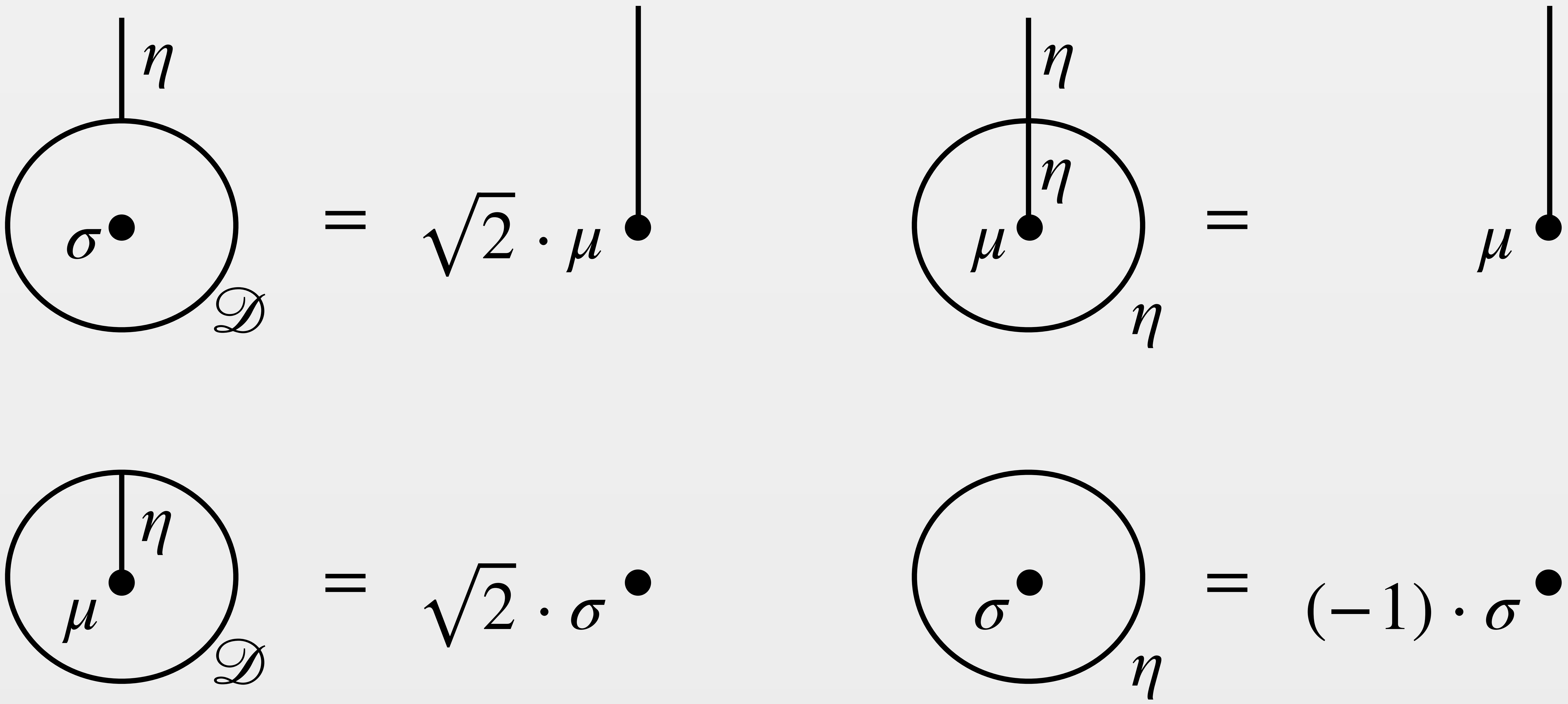


Operators form representations under the tube algebra.

Example

Krammer-Wannier Duality Symmetry (TY($\mathbb{Z}_2$) category) in Ising CFT

$$\eta \times \eta = 1 \qquad \eta \times \mathscr{D} = \mathscr{D} \times \eta = \mathscr{D} \qquad \mathscr{D} \times \mathscr{D} = 1 + \eta$$



Motivation

Questions:

- How do chiral algebras interact with topological defects?

$$\text{Chiral Algebra} \quad \overset{?}{\longleftrightarrow} \quad \text{Topological Defect}$$

- How are chiral algebras affected by discrete gauging?

$$\text{Chiral Algebra} \quad \longrightarrow \quad ?$$

- Can one construct a chiral algebra from non-local currents?

$$\overset{?}{\longrightarrow} \quad \text{Chiral Algebra}$$

To answer to these questions, we naturally arrive at the concept of chiral tube algebra.

Motivation

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$$J \quad \overset{?}{\longrightarrow} \quad \text{Chiral Algebra}$$

To answer to these questions, we naturally arrive at the concept of chiral tube algebra.

$\mathfrak{su}(2)_1$ Chiral Algebra

$\mathfrak{su}(2)_1$ chiral algebra

$$J^a(z)J^b(0) \sim \frac{\delta^{ab}}{2z^2} + \frac{i\epsilon^{abc}J^c(0)}{z}$$

$\mathfrak{su}(2)_1$ Kac-Moody algebra

$$[J^3_n, J^3_m] = \frac{n}{2}\delta_{n+m,0}$$

$$[J^+_n, J^-_m] = n\delta_{n+m,0} + 2J^3_{n+m}$$

$$[J^3_n, J^\pm_m] = \pm J^\pm_{n+m}$$

$SU(2)$ topological defects

$$U_g = \exp\left(2\pi i \sum_a \theta^a J^a_0\right) \quad g \in SU(2)$$

- preserve Virasoro symmetry
- generally breaks extended chiral algebra except for U_g with $g \in \mathbb{Z}_2$ center

Is the chiral algebra completely broken?

Twisting

Q: Is the chiral algebra completely broken?

A: No. Instead, in every defect Hilbert space, we can define a twisted Kac-Moody algebra.

$J^3 \longrightarrow J^3$

$J^+ \longrightarrow e^{2\pi i\theta} J^+$

$J^- \longrightarrow e^{-2\pi i\theta} J^-$

$$U_g = \exp(2\pi i \theta J_0^3)$$

$$\theta \sim \theta + 2$$

$$J^3 = \sum_{n\in\mathbb{Z}} J_n^3\, z^{-n-1} \qquad J_n^3 = \frac{1}{2\pi}\oint dz\, J^3(z)\, z^n$$

$$J^+ = \sum_{n\in\mathbb{Z}} J_{n+\theta}^+\, z^{-n-\theta-1} \qquad J_{n+\theta}^+ = \frac{1}{2\pi}\oint dz\, J^+(z)\, z^{n+\theta}$$

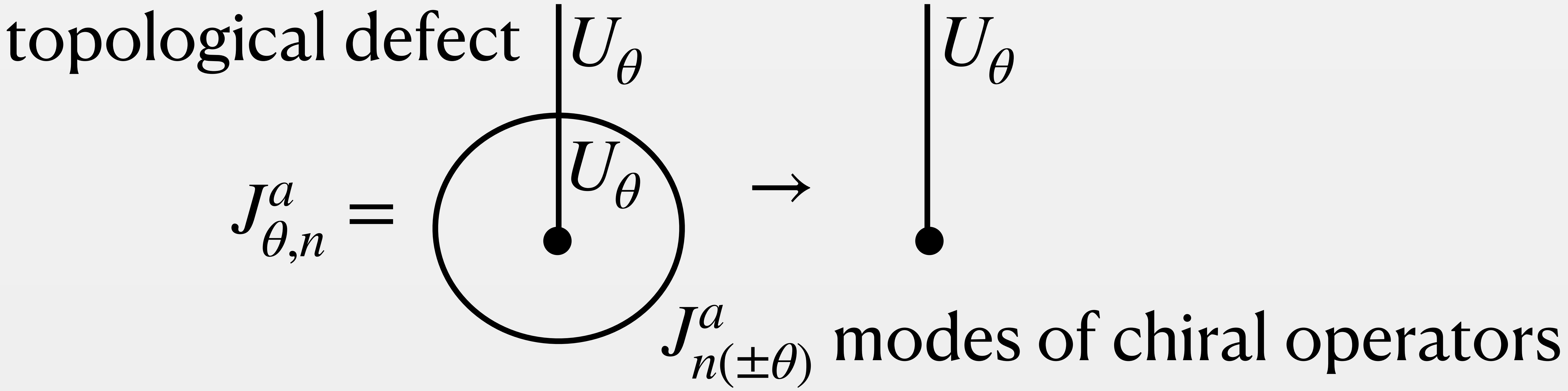
$$J^- = \sum_{n\in\mathbb{Z}} J_{n-\theta}^-\, z^{-n+\theta-1} \qquad J_{n-\theta}^- = \frac{1}{2\pi}\oint dz\, J^-(z)\, z^{n-\theta}$$

Laurent expansion is twisted by the defect.

* Every $SU(2)$ element are related to a Cartan element by conjugation

$\mathfrak{su}(2)_1$ Chiral Tube Algebra

The action of twisted Kac-Moody algebra is naturally captured by the Lasso operator.



These Lasso operators form a chiral tube algebra (isomorphic to $\mathfrak{su}(2)_1$ for each θ)

$$[J^3_{\theta,n}, J^3_{\theta,m}] = \frac{n}{2}\delta_{n+m,0}$$

$$[J^+_{\theta,n}, J^-_{\theta,m}] = (n+\theta)\delta_{n+m,0} + 2J^3_{\theta,n+m}$$

$$[J^3_{\theta,n}, J^\pm_{\theta,m}] = \pm J^\pm_{\theta,n+m}$$

Representations

What are the irreducible representations (irreps) of this chiral tube algebra?

$$[J^3_{\theta,n}, J^3_{\theta,m}] = \frac{n}{2}\delta_{n+m,0} \quad [J^+_{\theta,n}, J^-_{\theta,m}] = (n+\theta)\delta_{n+m,0} + 2J^3_{n+m} \quad [J^3_{\theta,n}, J^\pm_{\theta,m}] = \pm J^\pm_{\theta,n+m}$$

Same Hilbert space structure as the reps of the original Kac-Moody algebra

highest weight state: $J^a_{n>0}|\phi\rangle = 0$

highest weight of the subalgebra isomorphic to $SU(2)$

$$[J^+_{\theta,0}, J^-_{\theta,0}] = 2\left(J^3_0 + \frac{\theta}{2}\right) \quad [J^3_{\theta,0}, J^\pm_{\theta,0}] = \pm J^\pm_{\theta,0}$$

$J^3_0 \in \mathbb{Z} - \theta/2$

$U(1)$ Cartan anomaly

only 2 irreps: $|\phi\rangle = |j,m\rangle \quad j = 0, 1/2$

$$\mathscr{H}_j = \{|\phi\rangle, (J^-_0|\phi\rangle), J^a_{-1}|\phi\rangle, J^a_{-2}|\phi\rangle \ldots\}$$

Sugawara Construction

What are the characters $\chi_{\theta,j}(q) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}}\right]$?

To answer this, we need to construct the Virasoro generators L_n. For Kac-Moody algebra, they are given by the Sugawara construction

$$T(z) = \frac{1}{6} :J^a(z)J^a(z): \longrightarrow L_{\theta=0,n} = \frac{1}{6}\left(\sum_{m\leq -1} J^a_{0,m}J^a_{0,n-m} + \sum_{m\geq 0} J^a_{0,n-m}J^a_{0,m}\right)$$

However, because the chiral tube algebra is twisted, we get additional contributions from the normal ordering ambiguity fixed by $[L_n, J_0^+] = -\theta J^+_{n+\theta}$, $[L_1, L_{-1}] = 2L_0$

$$L_{\theta,n} = \frac{1}{6}\left(\sum_{m\leq -1} J^a_{\theta,m}J^a_{\theta,n-m} + \sum_{m\geq 0} J^a_{\theta,n-m}J^a_{\theta,m}\right) - \frac{2}{3}J^3_{\theta,n} - \frac{\theta^2}{6}\delta_{n,0}$$

Spectral Flow

$$L_{\theta,0} = \frac{1}{6}\left(\sum_{m\leq -1} J^a_{\theta,m}J^a_{\theta,-m} + \sum_{m\geq 0} J^a_{\theta,-m}J^a_{\theta,m}\right) - \frac{2}{3}J^3_{\theta,0} - \frac{\theta^2}{6}$$

Spin 0 irrep

- Dimension of the highest weight state is shifted

$$\langle 0,0 | L_{\theta,0} | 0,0 \rangle = \theta^2/4$$

- Dimension of the raising operators is shifted

$$[L_{\theta,0}, J^3_{\theta,-n}] = nJ^3_{\theta,-n} \quad [L_{\theta,0}, J^\pm_{\theta,-n}] = (n \pm \theta)J^\pm_{\theta,-n}$$

- The raising operators also change the $U(1)$ Cartan charges

$$[J^3_{\theta,0}, J^3_{\theta,-n}] = 0 \quad [J_{\theta,0}, J^\pm_{\theta,-n}] = \pm J^\pm_{\theta,-n}$$

- The shifts in dimension of the raising operators are correlated with the $U(1)$ charges Q

Spectral flow: $h_\theta = h_0 + \theta Q + \theta^2/4$

Character

Spin 0 irrep

Spectral flow: $h_\theta = h_0 + \theta Q + \theta^2/4$

$\mathfrak{su}(2)_1$ character (w/ chemical potential)

$$\chi_{\theta=0,j=0}(q,y) = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}y^Q\right] = \frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}} q^{m^2}y^m$$

$\longrightarrow$

$\mathfrak{su}(2)_1$ tube algebra character

$$\chi_{\theta,j=0}(q) = \chi_{\theta=0,j=0}(q,q^\theta)\,q^{\theta^2/4} = \frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}} q^{(m+\theta/2)^2}$$

Spin 1/2 irrep

$$\chi_{\theta,j=1/2}(q) = \frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}} q^{(m+1/2+\theta/2)^2}$$

$SU(2)_1$ WZW Model

$SU(2)_1$ WZW model

Global symmetry: $(g, \bar{g}) \in \dfrac{SU(2) \times \overline{SU(2)}}{\mathbb{Z}_2}$ ⟵ diagonal $\mathbb{Z}_2$ center

Chiral tube algebra: $\mathfrak{su}(2)_1 \times \overline{\mathfrak{su}(2)_1}$

The (defect) Hilbert space splits into irreps of the chiral tube algebra

Defect partition function depends on only the conjugacy class

$$(g, \bar{g}) = (h e^{i\theta\sigma^3} h^{-1}, \bar{h} e^{i\bar{\theta}\sigma^3} \bar{h}^{-1})$$

$$Z_{g,\bar{g}} = \frac{1}{\eta(q)^2} \sum_{e,m\in\mathbb{Z}} q^{\frac{1}{4}(e+m+\theta)^2} \bar{q}^{\frac{1}{4}(e-m+\bar{\theta})^2}$$

$$= \chi_{\theta,0}(q)\chi_{\bar{\theta},0}(\bar{q}) + \chi_{\theta,1/2}(q)\chi_{\bar{\theta},1/2}(\bar{q})$$

$$\chi_{\theta,j}(q) = \frac{1}{\eta(q)} \sum_{m\in\mathbb{Z}} q^{(m+j+\theta/2)^2}$$

Motivation

Questions:

- How do chiral algebras interact with topological defects?

$$\text{Chiral Algebra} \xleftrightarrow{?} \text{Topological Defect}$$

- How are chiral algebras affected by discrete gauging?

$$\text{Chiral Algebra} \longrightarrow ?$$

- Can one construct a chiral algebra from non-local currents?

$$J \bullet\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- \xrightarrow{?} \text{Chiral Algebra}$$

To answer to these questions, we naturally arrive at the concept of chiral tube algebra.

Non-local Chiral Operators

Gauge $\mathbb{Z}_N \in U(1)$ diagonal Cartan of $[SU(2) \times \overline{SU(2)}]/\mathbb{Z}_2$ in $SU(2)_1$ WZW model.

Some $\mathfrak{su}(2) \times \overline{\mathfrak{su}(2)}$ chiral currents become non-local

Q: What happens to chiral algebra? Can we construct one from non-local currents?

A: There should be an analog of chiral algebras after gauging because gauging reshuffles states between defect Hilbert spaces. After gauging, the states in the same irreps are simply redistributed into different defect Hilbert spaces. Similar to tube algebra!

Defect Hilbert Space after Gauging

After gauging, we get a dual $\mathbb{Z}_N$ symmetry from the Wilson lines W.

The defect Hilbert space of W^Q is

$$\widetilde{\mathscr{H}}_Q = \text{charge } Q \text{ states in } \mathscr{H}_k$$

where $\mathscr{H}_k$ is the defect Hilbert space before gauging

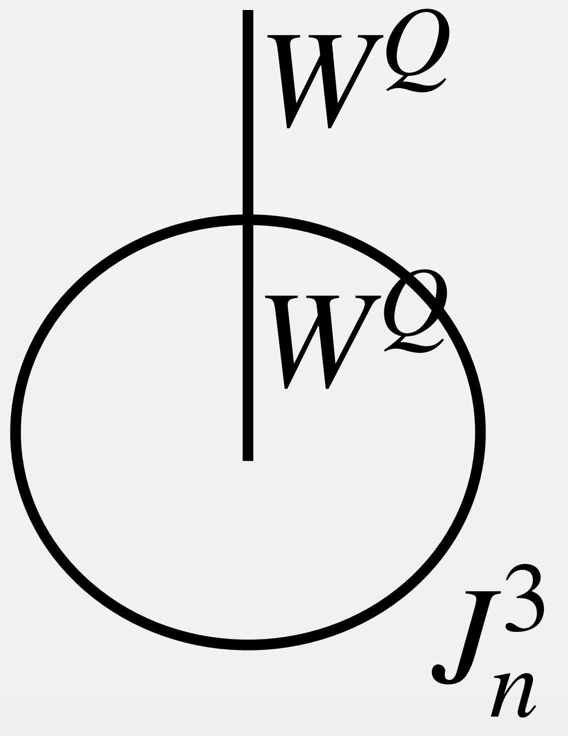
$$\mathscr{H}_k = (e^{\frac{2\pi i k \sigma^3}{N}}, e^{\frac{2\pi i k \sigma^3}{N}}) \text{ defect Hilbert space}$$

States from $\mathscr{H}_k$ carries charge k under the dual $\mathbb{Z}_N$ symmetry

What are the chiral tube algebra acting on these defect Hilbert spaces?

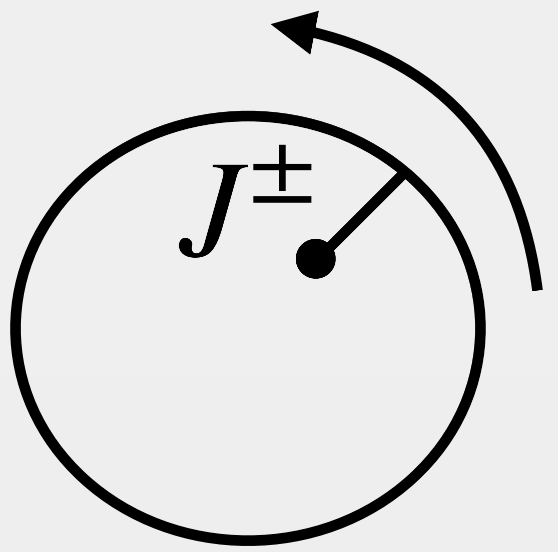
Chiral Tube Algebra

- J^3 is a local current and is not twisted by W^q
- $J^{\pm}$ are non-local currents 

$[J^3_n]^Q_Q =$ 

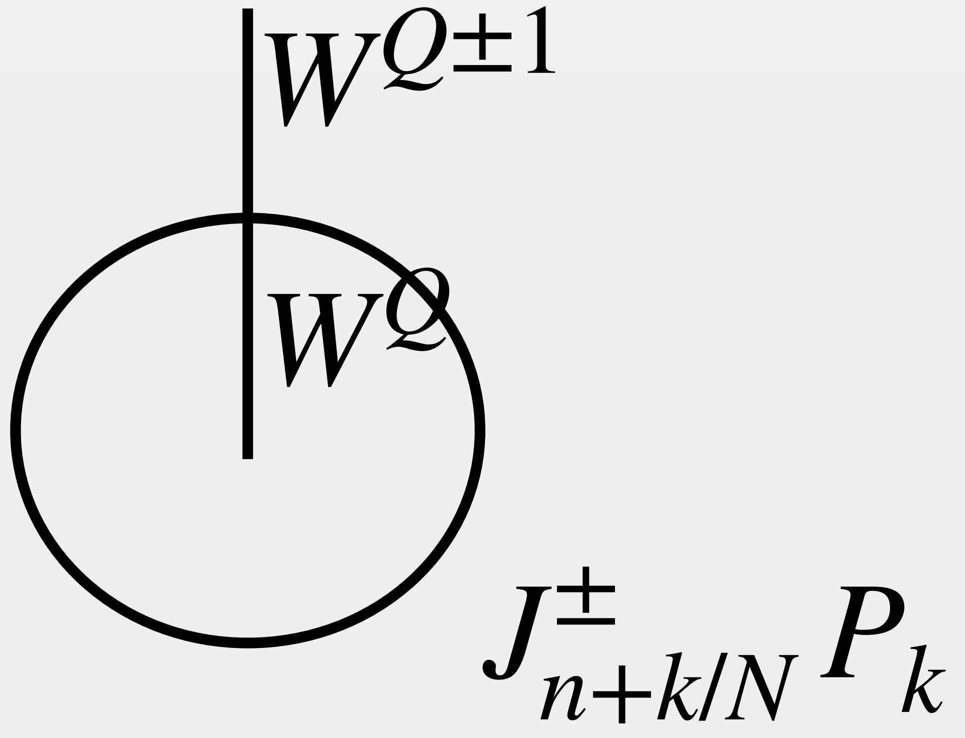
$e^{\pm 2\pi i k/N}$ monodromy

$$W^{\pm} P_k = e^{\pm \frac{2\pi i k}{N}} P_k$$



$$P_k = \sum_{n=1}^{N} e^{-\frac{2\pi i k}{N}} W^n$$

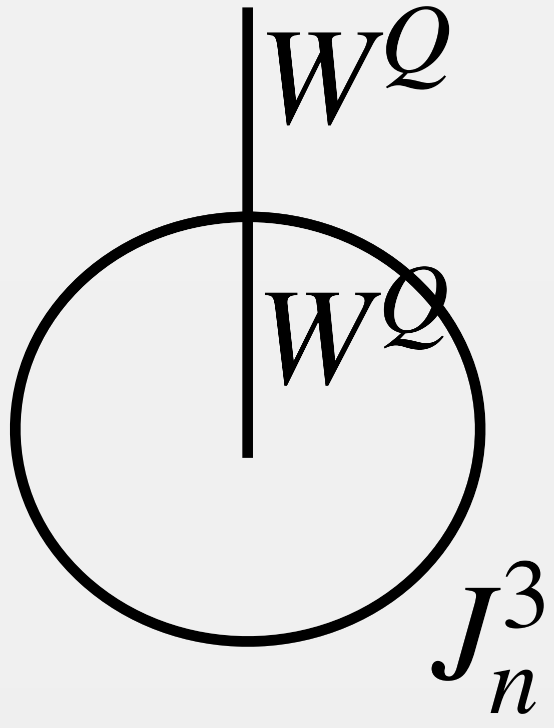
projector to charge k sector

$[J^{\pm}_{k,n}]^{Q\pm 1}_{Q} =$ 

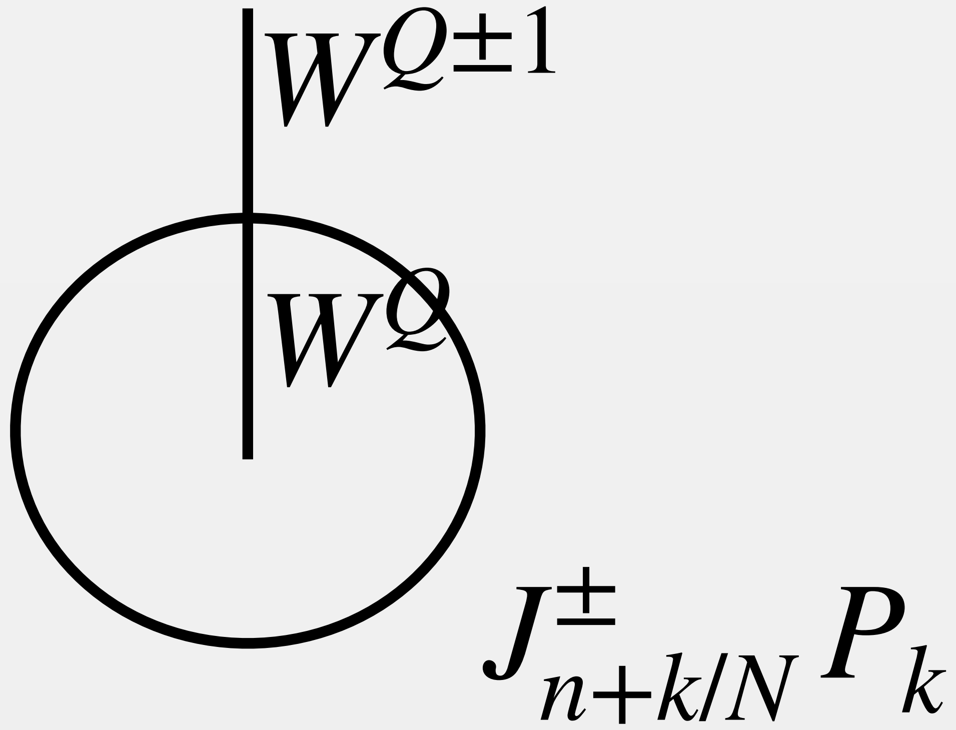
Wilson lines can end on the projector

Chiral Tube Algebra

$$[J^3_n]^Q_Q =$$



$$[J^{\pm}_{k,n}]^{Q\pm 1}_Q =$$



$$\Big[\, [J^3_n]^Q_Q \,, [J^3_m]^Q_Q \Big] = \frac{n}{2}\delta_{n+m,0}$$

$$[J^+_n]^{Q-1}_Q\,[J^-_{k,m}]^{Q-1}_Q - [J^-_{k,m}]^Q_{Q+1}\,[J^+_n]^{Q+1}_Q = (n + 2\pi k/N)\delta_{n+m,0} + 2[J^3_{n+m}]^Q_Q$$

$$[J^3_n]^{Q\pm 1}_{Q\pm 1}\,[J^{\pm}_{k,m}]^{Q\pm 1}_Q - [J^{\pm}_{k,m}]^{Q\pm 1}_Q\,[J^3_n]^Q_Q = \pm\,[J^{\pm}_{k,m}]^{Q\pm 1}_Q$$

Character

$$\left[[J^3_n]^Q_Q \,, [J^3_m]^Q_Q \right] = \frac{n}{2}\delta_{n+m,0}$$

$$[J^+_n]^{Q-1}_Q \, [J^-_{k,m}]^{Q-1}_Q - [J^-_{k,m}]^Q_{Q+1} \, [J^+_n]^{Q+1}_Q = (n + 2\pi k/N)\delta_{n+m,0} + 2[J^3_{n+m}]^Q_Q$$

$$[J^3_n]^{Q\pm1}_{Q\pm1} \, [J^\pm_{k,m}]^{Q\pm1}_Q - [J^\pm_{k,m}]^{Q\pm1}_Q \, [J^3_n]^Q_Q = \pm \, [J^\pm_{k,m}]^{Q\pm1}_Q$$

In an irrep of chiral tube algebra, there are states from different defect Hilbert space.
Let us define a defect character that sums over states from the same defect Hilbert space.

$$\chi_Q = \mathrm{Tr}_{\mathscr{H}_Q}\left[q^{L_0 - \frac{c}{24}}\right]$$

The irrep of the chiral tube algebra is labeled by

$$\text{eigenvalue of } W = e^{\frac{2\pi i k}{N}}, \qquad \text{spin } j = 0, 1/2$$

$$\chi_{(k,j),Q}(q) = \frac{1}{\eta(q)} \sum_{m \in N\mathbb{Z}+Q} q^{(m+j+k/2N)^2}$$

Compact Boson

$SU(2)_1$ WZW model is the same as compact boson at self-dual radius $R = \sqrt{2}$.

After $\mathbb{Z}_N$ gauging, the theory becomes compact boson at radius $R = N\sqrt{2}$, which has the chiral tube algebra (both chiral and anti-chiral part) we constructed before.

The torus partition function can be decomposed into characters of chiral tube algebra

$$Z\left(R = N\sqrt{2}\right) = \frac{1}{|\eta(q)|^2} \sum_{e,m\in\mathbb{Z}} q^{(e/N+mN)^2/4} \bar{q}^{(e/N-mN)^2/4}$$

$$= \sum_{j=0,1/2} \sum_{k=1}^{N} \sum_{Q=1}^{N} \chi_{(k,j),Q}(q) \bar{\chi}_{(k,j),-Q}(\bar{q})$$

$$\chi_{(k,j),Q}(q) = \frac{1}{\eta(q)} \sum_{m\in N\mathbb{Z}+Q} q^{(m+j+k/2N)^2}$$

correlated defect characters of left and right chiral tube algebra

Conclusions

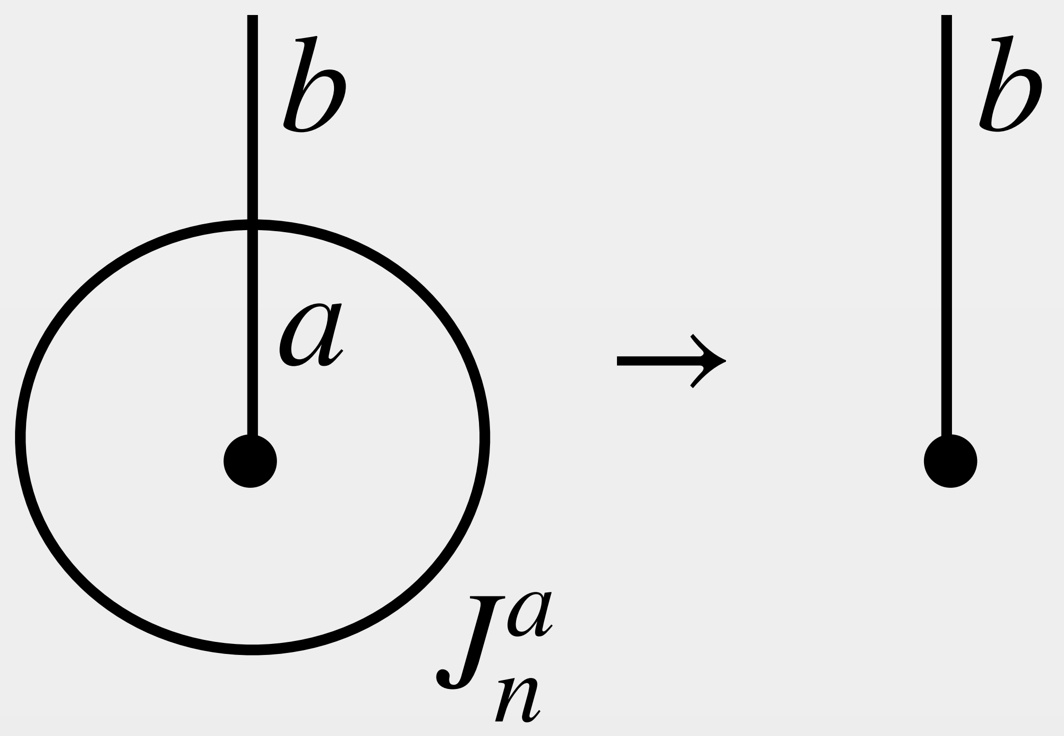
We introduced chiral tube algebras to

- describe the interplay between chiral algebras and topological defects
- discuss what happens to chiral algebras after gauging (non-local currents)

We focused on the example of $SU(2)_1$ tube algebra and its orbifolds.

The structure generalizes to (didn't have time to discuss)

- Superconformal algebras and their bosonization
- W-algebras and their orbifolds



Future directions and open questions:

- Chiral tube algebra with non-invertible topological defects?
- Chiral tube algebra from general chiral defect operators with fractional dimension?
- Application to 4d $\mathcal{N} = 2$ SCFT?